RATIONAL CUBIC FOURFOLDS IN HASSETT DIVISORS

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ABSTRACT. We prove that every Hassett's Noether-Lefschetz divisor of special cubic fourfolds contains a union of three subvarieties parametrizing rational cubic fourfolds, of codimension-two in the moduli space of smooth cubic fourfolds.

1. INTRODUCTION

The rationality problem of smooth cubic fourfolds is one of the most widely open problems in algebraic geometry; we refer to the survey [Has16] for a comprehensive progress. It has been known that all smooth cubic surfaces are rational since the 19th century. In 1972, Clemens–Griffiths [CG72] proved that all smooth cubic threefolds are nonrational. For smooth cubic fourfolds, however, the situation is very mysterious. It is expected that a very general smooth cubic fourfold should be nonrational (cf. [Has99, Has00]). Until now, many examples of smooth rational cubic fourfolds are known, but the existence of a smooth nonrational cubic fourfold is still unknown.

Using Hodge theory and lattice theory, Hassett [Has00] introduced the notion of *special cubic fourfolds* (see Definition 2.1). Simultaneously, Hassett [Has00, Theorem 1.0.1] gave a countably infinite list of irreducible divisors $\mathcal{C}_d$ of special cubic fourfolds in the moduli space $\mathcal{C}$ of smooth cubic fourfolds and showed that $\mathcal{C}_d$ is nonempty if and only if $d > 6$ and $d \equiv 0, 2 \pmod 6$. Such a nonempty $\mathcal{C}_d$ is called a *Hassett's Noether-Lefschetz divisor* (for short a *Hassett divisor*).

Currently, there exist two popular point of views toward the rationality of smooth cubic fourfolds and both have associated $K3$ surfaces:

- Hassett's Hodge-theoretic result ([Has00, Theorem 5.1.3]): a smooth cubic fourfold X has a Hodge-theoretically associated $K3$ surface if and only if its moduli point $[X] \in \mathcal{C}_d$ for some *admissible value* d (i.e., $d > 6$, $d \equiv 0, 2 \pmod 6$, $4 \nmid d$, $9 \nmid d$ and $p \nmid d$ for any odd prime $p \equiv 2 \pmod 3$);
- Kuznetsov's derived categorical conjecture ([Kuz10, Conjecture 1.1]): a smooth cubic fourfold X is rational if and only if its Kuznetsov component $\mathrm{Ku}(X)$ is derived equivalent to a $K3$ surface (i.e., $\mathrm{Ku}(X)$ is called *geometric*), where $\mathrm{Ku}(X)$ is the right orthogonal to the exceptional collection $\{\mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2)\}$ in the bounded derived category of coherent sheaves on X.

It is important to notice that Kuznetsov's conjecture implies that a very general cubic fourfold is not rational, since for a very general cubic fourfold its Kuznetsov component

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can not be geometric. Addington–Thomas [AT14, Theorem 1.1] showed that for a smooth cubic fourfold X if $\mathrm{Ku}(X)$ is geometric then $[X] \in \mathcal{C}_d$ for some admissible d, and conversely for any admissible value d, the set of cubic fourfolds $[X] \in \mathcal{C}_d$ for which $\mathrm{Ku}(X)$ is geometric is a Zariski open dense subset; see also Huybrechts [Huy17] for the twisted version and a further study. Recently, based on Bridgeland stability conditions on $\mathrm{Ku}(X)$ constructed in [BLMS17, Theorem 1.2], Bayer–Lahoz–Macrì–Nuer–Perry–Stellari [BLMNPS19, Corollary 29.7] proved that for any admissible value d, $\mathrm{Ku}(X)$ is geometric for *every* $[X] \in \mathcal{C}_d$. So we now know that for a smooth cubic fourfold X its Kuznetsov component $\mathrm{Ku}(X)$ is geometric if and only if $[X] \in \mathcal{C}_d$ for some admissible value d. Then one can restate Kuznetsov's conjecture as the following equivalent form.

Conjecture 1.1. A smooth cubic fourfold X is *rational* if and only if $[X] \in \mathcal{C}_d$ for some admissible value d.

The first three admissible values are $14, 26, 38$. Every cubic fourfold in $\mathcal{C}_{14}$ is rational [Fan43, BRS19]; see also [RS19a, Theorem 2] for a different proof. Based on Kontsevich–Tschinkel [KT19, Theorem 1], Russo–Staglianò [RS19a, Theorems 4, 7] finally showed that every cubic fourfold in $\mathcal{C}_{26}$ and $\mathcal{C}_{38}$ is rational; see also [RS18] for the construction of explicit birational maps. So far "if " part of Conjecture 1.1 has been confirmed only for the three Hassett divisors $\mathcal{C}_{14}, \mathcal{C}_{26}, \mathcal{C}_{38}$. Thus finding rational cubic fourfolds in other Hassett divisors is of interest. The main result of this paper is the following.

Theorem 1.2 (=Theorem 3.3)**.** *Every Hassett divisor $\mathcal{C}_d$ contains a union of three subvarieties parametrizing rational cubic fourfolds, of codimension-two in $\mathcal{C}$.*

The idea of the proof is simple: we first show any two Hassett divisors intersect by Theorem 3.1, which is of independent interest (for considerations of the intersections among Hassett divisors, see [Has99, AT14, ABBVA14, BRS19] etc.), and finally we consider the intersections $\mathcal{C}_d \cap \mathcal{C}_{14}$, $\mathcal{C}_d \cap \mathcal{C}_{26}$ and $\mathcal{C}_d \cap \mathcal{C}_{38}$ for every Hassett divisor $\mathcal{C}_d$.

After completing this paper, Russo–Staglianò [RS19b] announced the rationality of every cubic fourfold in $\mathcal{C}_{42}$. We remark that our method used for the proof of Theorem 1.2 also works in this case (in particular, it can be shown that the four intersections $\mathcal{C}_d \cap \mathcal{C}_{14}$, $\mathcal{C}_d \cap \mathcal{C}_{26}$, $\mathcal{C}_d \cap \mathcal{C}_{38}$, $\mathcal{C}_d \cap \mathcal{C}_{42}$ are mutually distinct).

Throughout this paper, we work over the complex number field $\mathbb{C}$.

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2. Lattice and Hodge theory for cubic fourfolds

In this section, we collect some known results on Hodge structures and lattices associated with smooth cubic fourfolds. We refer to [BD85, Has00, Has16, Huy18] for more detailed discussions, especially for the Hodge-theoretic aspect, and to [Ser73, Nik80] for the basics of abstract lattice theory.

The cubic hypersurfaces in $\mathbb{P}^5$ are parametrized by $\mathbb{P}(H^0(\mathbb{P}^5, \mathcal{O}(3))) \cong \mathbb{P}^{55}$. Moreover, the smooth cubic hypersurfaces form a Zariski open dense subset $\mathcal{U} \subset \mathbb{P}^{55}$. Then the moduli space of smooth cubic fourfolds is the quotient space

$$\mathcal{C} := \mathcal{U}//\mathrm{PGL}(6, \mathbb{C})$$

which is a 20-dimensional quasi-projective variety.

Let X be a smooth cubic fourfold. Then the cohomology $H^*(X, \mathbb{Z})$ is torsion-free and the Hodge numbers for the middle cohomology of X are as follows:

$$0 \quad 1 \quad 21 \quad 1 \quad 0.$$

The Hodge-Riemann bilinear relations imply that $H^4(X, \mathbb{Z})$ is a unimodular lattice under the intersection form $(\,.\,)$ of signature $(21, 2)$. Furthermore, as abstract lattices, [Has00, Proposition 2.1.2] implies the middle cohomology and the primitive cohomology

$$L := E_8^{\oplus 2} \oplus U^{\oplus 2} \oplus I_{3,0} \simeq H^4(X, \mathbb{Z})$$
$$L^0 := (h^2)^\perp \simeq E_8^{\oplus 2} \oplus U^{\oplus 2} \oplus A_2 \simeq H^4_{\mathrm{prim}}(X, \mathbb{Z})$$

where the square of the hyperplane class h is given as $h^2 = (1, 1, 1) \in I_{3,0}$ of which the intersection form is given by the identity matrix of rank 3, $A_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ the hyperbolic plane, E_8 is the unimodular positive definite even lattice of rank 8. Note that L^0 is an even lattice.

Definition 2.1 (Hassett [Has00])**.** A smooth cubic fourfold X is called *special* if it contains an algebraic surface not homologous to a complete intersection.

The integral Hodge conjecture holds for smooth cubic fourfolds ([Voi07, Theorem 18] or see also [BLMNPS19, Corollary 29.8] for a new proof). Thus, a smooth cubic fourfold X is *special* if and only if the rank of the positive definite lattice

$$A(X) := H^4(X, \mathbb{Z}) \cap H^{2,2}(X)$$

is at least 2.

Definition 2.2 (Hassett [Has00])**.** A *labelling* of a special cubic fourfold consists of a positive definite rank two saturated (i.e. the quotient group $A(X)/K$ is torsion free) sublattice

$$K \subset A(X) \text{ such that } h^2 \in K,$$

and its discriminant d is the determinant of the intersection form on K.

In [Has00], Hassett defined $\mathcal{C}_d$ as the set of special cubic fourfolds X with labelling of discriminant d. Moreover, Hassett [Has00, Theorem 1.0.1] showed that $\mathcal{C}_d \subset \mathcal{C}$ is an irreducible divisor and is nonempty if and only if

$$d > 6 \quad \text{and} \quad d \equiv 0, 2 \pmod 6. \tag{$\star$}$$

The following proposition is a generalization of [Has00, Theorems 1.0.1].

Proposition 2.3 ([Has16, Proposition 12 and page 43])**.** *Fix a positive definite lattice M of rank $r \geq 2$ admitting a saturated embedding*

$$M \subset L \text{ such that } h^2 \in M.$$

We denote by $\mathcal{C}_M \subset \mathcal{C}$ the smooth cubic fourfolds X admitting algebraic classes with this lattice structure

$$h^2 \in M \subset A(X) \subset L.$$

Then $\mathcal{C}_M$ has codimension $r-1$ and there exists a cubic fourfold $[X] \in \mathcal{C}_M$ with $A(X) = M$, provided $\mathcal{C}_M$ is nonempty. Moreover, $\mathcal{C}_M$ is nonempty if and only if there exists no sublattice $K \subset M$, $h^2 \in K$, with $K = K_2$ or K_6, where $K_2 = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$ and $K_6 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$.

This proposition is crucial for our purpose, so we sketch a proof for the convenience of readers.

Sketch of proof. Suppose $\mathcal{C}_M$ is nonempty. If $K_6 \subset M$ is a sublattice with $h^2 \in K_6$, then there is a smooth cubic fourfold X such that $A(X) \cap \langle h^2 \rangle^{\perp}$ contains an element r with $(r.r) = 2$ and this contradicts Voisin [Voi86, Section 4, Proposition 1]; furthermore, Hassett [Has00, Theorem 4.4.1] excludes the case when $K_2 \subset M$ is a sublattice with $h^2 \in K_2$.

Conversely, suppose that there exists no rank two sublattice $K \subset M$, $h^2 \in K$, with $K = K_2$ or K_6. Since the signature of L is $(21, 2)$ and $M \subset L$ is positive definite, by a standard argument, one can always find $\omega \in L \otimes_{\mathbb{Z}} \mathbb{C}$ such that

$$(\omega.\omega) = 0, \quad (\omega.\bar{\omega}) < 0 \text{ and } L \cap \omega^{\perp} = M.$$

According to the description of the image of the period map for cubic fourfolds (Laza [Laz10, Theorem 1.1] and Looijenga [Loo09, Theorem 4.1]), one has a smooth cubic fourfold X and an isometry $\phi : H^4(X, \mathbb{Z}) \xrightarrow{\simeq} L$ mapping the square of the hyperplane class to $h^2 \in L$ and a generator of $H^{3,1}(X)$ to ω. Thus $M = A(X)$ and hence $\mathcal{C}_M$ contains $[X]$ and nonempty. □

In the rest of the text, we will frequently use the following lemma to check the nonemptyness condition in the Proposition 2.3.

Lemma 2.4. *Let $M \subset L$ be a positive definite saturated sublattice and $h^2 \in M$. Then the following three conditions are equivalent:*

(i) *there exists no sublattice $K \subset M$, $h^2 \in K$, with $K = K_2$ or K_6;*
(ii) *there exists no $r \in M$ such that $(r.r) = 2$ (i.e., M does not represent 2);*
(iii) *for any $0 \neq x \in M$, $(x.x) \geq 3$.*

In particular, if M satisfies one of the three equivalent conditions, then $\emptyset \neq \mathcal{C}_M \subset \mathcal{C}_{M'}$ for any saturated sublattice $M' \subset M \subset L$ such that $h^2 \in M'$.

Proof. First of all, $(ii) \Rightarrow (i)$ is clear since both K_2 and K_6 represent 2.

Secondly, $(i) \Rightarrow (ii)$. Suppose that there exists $r \in M$ such that $(r.r) = 2$. We denote by $K \subset M$ the sublattice generated by h^2 and r. Hence, the Gram matrix of K with respect to the basis (h^2, r) is

$$\begin{pmatrix} (h^2.h^2) & (h^2.r) \\ (r.h^2) & (r.r) \end{pmatrix} = \begin{pmatrix} 3 & a \\ a & 2 \end{pmatrix}.$$

Replacing r by $-r$ if necessary, we may and will assume $a \geq 0$. Since K is positive definite, we have $a^2 < 6$ and thus $a = 0, 1, 2$. If $a = 0$ (resp. 2) , then K is isometric to K_6 (resp. K_2), contradiction. If $a = 1$, then $h^2 - 3r \in (h^2)^{\perp} = L^0$ and $((h^2 - 3r).(h^2 - 3r)) = 15$, an odd number, contradicting to the fact L^0 is even.

Finally, clearly (iii) implies (ii). Conversely, we show (ii) implies (iii). By hypothesis, we may assume that there is $r \in M$ with $(r.r) = 1$. Then let $K \subset M$ be the sublattice generated by h^2 and r. Hence, the Gram matrix of K with respect to the basis (h^2, r) is

$$\begin{pmatrix} 3 & a \\ a & 1 \end{pmatrix}$$

where $a = (h^2.r)$. Replacing r by $-r$ if necessary, we may and will assume $a \geq 0$. Since K is positive definite, we have $a^2 < 3$ and thus $a = 0, 1$. If $a = 0$, then $r \in (h^2)^{\perp} = L^0$ and $(r.r) = 1$, an odd number, contradicting to the fact L^0 is even. If $a = 1$, then K is isometric to K_2 and K represents 2, contradiction. $\square$

3. Intersections of Hassett divisors

In this section, we prove Theorem 1.2 (=Theorem 3.3) and discuss some related results (Theorem 3.1 and Theorem 3.7).

Firstly, we setup some notations for latter use. Let

$$L = E_8^{\oplus 2} \oplus U_1 \oplus U_2 \oplus I_{3,0},$$

where U_1 and U_2 are two copies of U. The standard basis of U consists of isotropic elements e, f with $(e.f) = 1$. We denote the standard basis of U_i by e_i, f_i, $i = 1, 2$, and denote by h^2 the element $(1, 1, 1) \in I_{3,0} \subset L$.

We will use the following theorem, an interesting result for itself, to prove Theorem 3.3.

Theorem 3.1. *Any two Hassett divisors intersect, i.e.,* $\mathcal{C}_{d_1} \cap \mathcal{C}_{d_2} \neq \emptyset$ *for any two integers* d_1 *and* d_2 *satisfying* $(\star)$. *Moreover, there exists a smooth cubic fourfold* X *and a codimension-two subvariety* $\mathcal{C}_{A(X)} \subset \mathcal{C}$ *such that* $[X] \in \mathcal{C}_{A(X)} \subset \mathcal{C}_{d_1} \cap \mathcal{C}_{d_2}$ *and* $A(X)$ *is a rank 3 lattice with discriminant* $d_1 d_2/3$, *except if both* d_1 *and* d_2 *are* $\equiv 2 \bmod 6$, *in which case the discriminant is* $(d_1 d_2 - 1)/3$.

Proof. By definition, an integer d satisfies $(\star)$ if $d > 6$ and $d \equiv 0, 2 \pmod 6$. Therefore, the proof is divided into three cases:

Case (1)**:** $d_1 \equiv 0 \pmod 6$ **and** $d_2 \equiv 0 \pmod 6$**.** Suppose $d_1 = 6n_1$, $d_2 = 6n_2$ and $n_1, n_2 \geq 2$. We consider the rank 3 lattice

$$M := \langle h^2, \alpha_1, \alpha_2 \rangle \subset L$$

generated by h^2, $\alpha_1 := e_1 + n_1 f_1$ and $\alpha_2 := e_2 + n_2 f_2$. Then the Gram matrix of M with respect to the basis $(h^2, \alpha_1, \alpha_2)$ is

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2n_1 & 0 \\ 0 & 0 & 2n_2 \end{pmatrix}.$$

Therefore, $M \subset L$ is positive definite saturated sublattice such that $h^2 \in M$. In addition, for any nonzero $v = xh^2 + y\alpha_1 + z\alpha_2 \in M$, where x, y, z are integers, we have

$$(v.v) = 3x^2 + 2n_1y^2 + 2n_2z^2 \geq 3$$

since $n_1, n_2 \geq 2$ and at least one of the integers x, y, z is nonzero. Hence, the embedding $M \subset L$ satisfies Lemma 2.4 (*iii*). Thus, by Lemma 2.4 and Proposition 2.3, $\mathcal{C}_M \subset \mathcal{C}$ is nonempty and has codimension 2, and there exists a cubic fourfold $[X] \in \mathcal{C}_M$ with $A(X) = M$. Thus $A(X)$ is a rank 3 lattice of discriminant $\operatorname{disc}(A(X)) = d_1d_2/3$. Moreover, we consider the sublattices

$$K_{d_1} := \langle h^2, \alpha_1 \rangle \subset M$$

with discriminant d_1, and

$$K_{d_2} := \langle h^2, \alpha_2 \rangle \subset M$$

with discriminant d_2. Clearly, both K_{d_1} and K_{d_2} are saturated sublattices of M. Applying Lemma 2.4 and Proposition 2.3 again, we obtain $[X] \in \mathcal{C}_M \subset \mathcal{C}_{K_{d_1}} = \mathcal{C}_{d_1}$ and $[X] \in \mathcal{C}_M \subset \mathcal{C}_{K_{d_2}} = \mathcal{C}_{d_2}$. Consequently, $[X] \in \mathcal{C}_M \subset \mathcal{C}_{d_1} \cap \mathcal{C}_{d_2}$ is what we want.

Case (2)**:** $d_1 \equiv 0 \pmod 6$ **and** $d_2 \equiv 2 \pmod 6$**.** Given $d_1 = 6n_1$ and $d_2 = 6n_2 + 2$ with $n_1 \geq 2, n_2 \geq 1$. We consider the rank 3 sublattice

$$M := \langle h^2, \alpha_1, \alpha_2 + (0,0,1) \rangle \subset L$$

where $(0,0,1) \in I_{3,0}$. Then the Gram matrix of M with respect to the basis $(h^2, \alpha_1, \alpha_2 + (0,0,1))$ is

$$\begin{pmatrix} 3 & 0 & 1 \\ 0 & 2n_1 & 0 \\ 1 & 0 & 2n_2 + 1 \end{pmatrix}$$

Thus, $M \subset L$ is positive definite saturated sublattice with $h^2 \in M$. Furthermore, for any nonzero $v = xh^2 + y\alpha_1 + z(\alpha_2 + (0,0,1)) \in M$, we get

$$(v.v) = 2x^2 + 2n_1y^2 + 2n_2z^2 + (x+z)^2 \geq 3$$

since $n_1 \geq 2$, $n_2 \geq 1$ and at least one of the integers x, y, z is nonzero. Hence, by Lemma 2.4 and Proposition 2.3, we conclude that $\mathcal{C}_M \subset \mathcal{C}$ is nonempty and has codimension

2, and there exists a cubic fourfold $[X] \in \mathcal{C}_M$ with $A(X) = M$. Thus $A(X)$ is a rank 3 lattice of discriminant $\mathrm{disc}(A(X)) = d_1 d_2/3$. Similarly, we consider the sublattices:

$$K_{d_1} := \langle h^2, \alpha_1 \rangle \subset M$$

of discriminant d_1, and

$$K_{d_2} := \langle h^2, \alpha_2 + (0,0,1) \rangle \subset M$$

of discriminant d_2. Again Lemma 2.4 and Proposition 2.3 imply $[X] \in \mathcal{C}_M \subset \mathcal{C}_{K_{d_1}} = \mathcal{C}_{d_1}$ and $[X] \in \mathcal{C}_M \subset \mathcal{C}_{K_{d_2}} = \mathcal{C}_{d_2}$. Consequently, $[X] \in \mathcal{C}_M \subset \mathcal{C}_{d_1} \cap \mathcal{C}_{d_2}$ is what we wanted.

Case (3)**:** $d_1 \equiv 2 \pmod 6$ **and** $d_2 \equiv 2 \pmod 6$**.** Assume $d_1 = 6n_1 + 2$ and $d_2 = 6n_2 + 2$ with $n_1, n_2 \geq 1$. We consider the rank 3 sublattice

$$M := \langle h^2, \alpha_1 + (0,1,0), \alpha_2 + (0,0,1) \rangle \subset L$$

here $(0,1,0) \in I_{3,0}$. Then the Gram matrix of M with respect to the basis $(h^2, \alpha_1 + (0,1,0), \alpha_2 + (0,0,1))$ is

$$\begin{pmatrix} 3 & 1 & 1 \\ 1 & 2n_1+1 & 0 \\ 1 & 0 & 2n_2+1 \end{pmatrix}$$

Thus, $M \subset L$ is positive definite saturated sublattice such that $h^2 \in M$. For any nonzero $v = xh^2 + y(\alpha_1 + (0,1,0)) + z(\alpha_2 + (0,0,1)) \in M$, we obtain

$$(v.v) = x^2 + 2n_1 y^2 + 2n_2 z^2 + (x+y)^2 + (x+z)^2 \geq 3$$

since $n_1, n_2 \geq 1$ and at least one of the integers x, y, z is nonzero. Hence, Lemma 2.4 and Proposition 2.3 concludes that $\mathcal{C}_M \subset \mathcal{C}$ is nonempty and has codimension 2, and there exists a cubic fourfold $[X] \in \mathcal{C}_M$ with $A(X) = M$. Thus $A(X)$ is a rank 3 lattice of discriminant $\mathrm{disc}(A(X)) = (d_1 d_2 - 1)/3$. Moreover, we consider

$$K_{d_1} := \langle h^2, \alpha_1 + (0,1,0) \rangle \subset M$$

with discriminant d_1 and

$$K_{d_2} := \langle h^2, \alpha_2 + (0,0,1) \rangle \subset M$$

with discriminant d_1. By Lemma 2.4 and Proposition 2.3, we obtain $[X] \in \mathcal{C}_M \subset \mathcal{C}_{K_{d_1}} = \mathcal{C}_{d_1}$ and $[X] \in \mathcal{C}_M \subset \mathcal{C}_{K_{d_2}} = \mathcal{C}_{d_2}$. As a consequence, $[X] \in \mathcal{C}_M \subset \mathcal{C}_{d_1} \cap \mathcal{C}_{d_2}$ is what we wanted. This finishes the proof of Theorem 3.1. $\square$

Remark 3.2. Note that it has been known for 20 years that $\mathcal{C}_8 \cap \mathcal{C}_{14} \neq \emptyset$ (Hassett [Has99]) and proved more recently that $\mathcal{C}_8$ intersects every Hassett divisor (Addington–Thomas [AT14, Theorem 4.1]). It is also shown that $\mathcal{C}_8 \cap \mathcal{C}_{14}$ has five irreducible components ([ABBVA14, BRS19]). Moreover, [BRS19, page 166, paragraph 4 line 2] has mentioned that $\mathcal{C}_{14}$ intersects many other divisors $\mathcal{C}_d$, however, it is not obvious to see which Hassett divisors intersect with $\mathcal{C}_{14}$.

Consequently, Theorem 3.1 not only generalizes [AT14, Theorem 4.1] but also implies that $\mathcal{C}_{14}$ intersects all Hassett divisors. Because of the same reason, we may conclude the main result of the current paper.

Theorem 3.3 (=Theorem 1.2)**.** *Every Hassett divisor $\mathcal{C}_d$ contains a union of three subvarieties parametrizing rational cubic fourfolds, of codimension-two in $\mathcal{C}$.*

Proof. Applying Theorem 3.1 to the pairs of integers $(d_1, d_2) = (d, 14)$, $(d, 26)$, $(d, 38)$. Then there exist three smooth cubic fourfolds X_1, X_2 and X_3 such that

$$[X_1] \in \mathcal{C}_{A(X_1)} \subset \mathcal{C}_d \cap \mathcal{C}_{14} \subset \mathcal{C}_d,$$
$$[X_2] \in \mathcal{C}_{A(X_2)} \subset \mathcal{C}_d \cap \mathcal{C}_{26} \subset \mathcal{C}_d,$$
$$[X_3] \in \mathcal{C}_{A(X_3)} \subset \mathcal{C}_d \cap \mathcal{C}_{38} \subset \mathcal{C}_d,$$

where $\mathcal{C}_{A(X_1)}$, $\mathcal{C}_{A(X_2)}$, and $\mathcal{C}_{A(X_3)}$ are subvarieties of codimension-two in $\mathcal{C}$. Here $A(X_1)$, $A(X_2)$ and $A(X_3)$ are three different rank 3 lattices of discriminants:

- if $d \equiv 0 \pmod 6$, then $\mathrm{disc}(A(X_1)) = 14d/3$, $\mathrm{disc}(A(X_2)) = 26d/3$ and $\mathrm{disc}(A(X_3)) = 38d/3$;
- if $d \equiv 2 \pmod 6$, then $\mathrm{disc}(A(X_1)) = (14d-1)/3$, $\mathrm{disc}(A(X_2)) = (26d-1)/3$ and $\mathrm{disc}(A(X_3)) = (38d-1)/3$.

By definition of $\mathcal{C}_{A(X_i)}$ (see Proposition 2.3), a smooth cubic fourfold $[X] \in \mathcal{C}_{A(X_i)}$ only if there exists a saturated embedding $A(X_i) \subset A(X)$. Since $A(X_1)$, $A(X_2)$ and $A(X_3)$ are rank 3 lattices of different discriminants, it follows that there is no saturated embedding $A(X_i) \subset A(X_j)$ if $i \neq j$. Therefore, $[X_i] \notin \mathcal{C}_{A(X_j)}$ if $i \neq j$ and $\mathcal{C}_{A(X_1)}$, $\mathcal{C}_{A(X_2)}$, and $\mathcal{C}_{A(X_3)}$ are three different subvarieties of codimension-two in $\mathcal{C}$.

Moreover, since every smooth cubic fourfold in $\mathcal{C}_{14}$, $\mathcal{C}_{26}$ and $\mathcal{C}_{38}$ is rational ([BRS19, RS19a]), so every smooth cubic fourfold in $\mathcal{C}_{A(X_1)}$, $\mathcal{C}_{A(X_2)}$ and $\mathcal{C}_{A(X_3)}$ is rational. Therefore, $\mathcal{C}_{A(X_1)}$, $\mathcal{C}_{A(X_2)}$ and $\mathcal{C}_{A(X_3)}$ are three different codimension-two subvarieties which parametrize rational cubic fourfolds. This completes the proof of Theorem 3.3. □

Our main result also motivates the following natural question:

Question 3.4. Suppose that d satisfies $(\star)$ and d is not an admissible value. Does the Hassett divisor $\mathcal{C}_d$ contain a union of countably infinite codimension-two subvarieties in $\mathcal{C}$ parametrizing rational cubic fourfolds?

The answer to Question 3.4 has already been known for $\mathcal{C}_8$ and $\mathcal{C}_{18}$ ([Has99, AHTVA16]).

Corollary 3.5. *The answer to Question 3.4 is yes if the "if" part of Conjecture 1.1 holds.*

Returning to Conjecture 1.1, as a by-product of Theorem 3.3 (=Theorem 1.2), we have the following.

Corollary 3.6. *For every admissible value d, the Hassett divisor $\mathcal{C}_d$ contains a union of three subvarieties parametrizing rational cubic fourfolds, of codimension-two in $\mathcal{C}$.*

To obtain more information about the Hassett divisors, it is of importance to notice that Addington–Thomas [AT14, Theorem 4.1] showed that for any d satisfying $(\star)$ there exists a cubic fourfold $[X] \in \mathcal{C}_8 \cap \mathcal{C}_d$ such that $[X] \in \mathcal{C}_{d'}$ for some admissible value d'. Even if it is conjectured to be rational, however, it is still unknown whether such a X

is rational or not. Using the idea of the proof of Theorem 3.1 and Theorem 3.3, we obtain a generalization of [AT14, Theorem 4.1].

Theorem 3.7. *If d_1 and d_2 satisfy $(\star)$, then $\mathcal{C}_{14} \cap \mathcal{C}_{d_1} \cap \mathcal{C}_{d_2}$ contains a codimension-three subvariety in $\mathcal{C}$ parametrizing rational cubic fourfolds.*

Proof. Analogously to the proof of Theorem 3.1, we only need to consider three cases:

Case (1)**:** Given $d_1 = 6n_1$ and $d_2 = 6n_2$ with $n_1, n_2 \geq 2$. We consider the rank 4 sublattice

$$M := \langle h^2, \nu, \alpha_1, \alpha_2 \rangle \subset L$$

where $\nu = (3, 1, 0) \in I_{3,0} \subset L$, $\alpha_1 := e_1 + n_1 f_1$ and $\alpha_2 := e_2 + n_2 f_2$. Then the Gram matrix of M with respect to the basis $(h^2, \nu, \alpha_1, \alpha_2)$ is

$$\begin{pmatrix} 3 & 4 & 0 & 0 \\ 4 & 10 & 0 & 0 \\ 0 & 0 & 2n_1 & 0 \\ 0 & 0 & 0 & 2n_2 \end{pmatrix}$$

Thus, $M \subset L$ is positive definite saturated sublattice with $h^2 \in M$. For any nonzero $v = x_1 h^2 + x_2 \nu + x_3 \alpha_1 + x_4 \alpha_2 \in M$, we have

$$(v.v) = 2(x_1 + 2x_2)^2 + x_1^2 + 2x_2^2 + 2n_1 x_3^2 + 2n_2 x_4^2 \geq 3$$

since $n_1, n_2 \geq 2$ and at least one of the integers x_i is nonzero $(i = 1, 2, 3, 4)$. Hence, Lemma 2.4 and Proposition 2.3 conclude that $\mathcal{C}_M$ is nonempty and has codimension 3. In addition, we consider the lattices $K_{14} = \langle h^2, \nu \rangle$ and $K_{d_i} := \langle h^2, \alpha_i \rangle \subset M$ with discriminant d_i. By Lemma 2.4 and Proposition 2.3, we obtain $\mathcal{C}_M \subset \mathcal{C}_{K_{d_1}} = \mathcal{C}_{d_1}$ and also $\mathcal{C}_M \subset \mathcal{C}_{K_{d_2}} = \mathcal{C}_{d_2}$. Consequently, $\emptyset \neq \mathcal{C}_M \subset \mathcal{C}_{14} \cap \mathcal{C}_{d_1} \cap \mathcal{C}_{d_2}$ is what we wanted, since every cubic fourfold in $\mathcal{C}_{14}$ is rational.

Since **Case** (2) and **Case** (3) are the same as **Case** (1), we just give the main ingredients and left the details to the interested readers.

Case (2)**:** Given $d_1 = 6n_1$ and $d_2 = 6n_2 + 2$ with $n_1 \geq 2, n_2 \geq 1$. We consider the rank 4 sublattice

$$M := \langle h^2, \nu, \alpha_1, \alpha_2 + (0, 0, 1) \rangle \subset L$$

and its sublattices $K_{14} = \langle h^2, \nu \rangle$, $K_{d_1} := \langle h^2, \alpha_1 \rangle \subset M$ and $K_{d_2} := \langle h^2, \alpha_2 + (0, 0, 1) \rangle \subset M$.

Case (3)**:** Given $d_1 = 6n_1 + 2$ and $d_2 = 6n_2 + 2$ with $n_1, n_2 \geq 1$. We consider the rank 4 sublattice

$$M := \langle h^2, \nu, \alpha_1 + (0, 1, 0), \alpha_2 + (0, 0, 1) \rangle \subset L$$

and its sublattices $K_{14} = \langle h^2, \nu \rangle$, $K_{d_1} := \langle h^2, \alpha_1 + (0, 1, 0) \rangle \subset M$ and $K_{d_2} := \langle h^2, \alpha_2 + (0, 0, 1) \rangle \subset M$. ∎

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