Energy Distribution of Radial Solutions to Energy Subcritical Wave Equation with an Application on Scattering Theory*

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Abstract

The topic of this paper is a semi-linear, energy sub-critical, defocusing wave equation $\partial_t^2 u - \Delta u = -|u|^{p-1}u$ in the 3-dimensional space ($3 \leq p < 5$) whose initial data are radial and come with a finite energy. We split the energy into inward and outward energies, then apply energy flux formula to obtain the following asymptotic distribution of energy: Unless the solution scatters, its energy can be divided into two parts: "scattering energy" which concentrates around the light cone $|x| = |t|$ and moves to infinity at the light speed and "retarded energy" which is at a distance of at least $|t|^\beta$ behind when $|t|$ is large. Here β is an arbitrary constant smaller than $\beta_0(p) = \frac{2(p-2)}{p+1}$. A combination of this property with a more detailed version of the classic Morawetz estimate gives a scattering result under a weaker assumption on initial data (u_0, u_1) than previously known results. More precisely, we assume

$$\int_{\mathbb{R}^3} (|x|^\kappa + 1) \left(\frac{1}{2}|\nabla u_0|^2 + \frac{1}{2}|u_1|^2 + \frac{1}{p+1}|u|^{p+1}\right) dx < +\infty.$$

Here $\kappa > \kappa_0(p) = 1 - \beta_0(p) = \frac{5-p}{p+1}$ is a constant. This condition is so weak that the initial data may be outside the critical Sobolev space of this equation. This phenomenon is not covered by previously known scattering theory, as far as the author knows.

1 Introduction

1.1 Background

In this work we consider the Cauchy problem of the defocusing semi-linear wave equation

$$\begin{cases} \partial_t^2 u - \Delta u = -|u|^{p-1}u, & (x,t) \in \mathbb{R}^3 \times \mathbb{R}; \\ u(\cdot, 0) = u_0; \\ u_t(\cdot, 0) = u_1. \end{cases} \qquad (CP1)$$

If u is a solution as above and λ is a positive constant, then the function $u_\lambda = \lambda^{-2/(p-1)} u(x/\lambda, t/\lambda)$ is another solution to (CP1) with initial data

$$u_\lambda(\cdot, 0) = \lambda^{-\frac{2}{p-1}} u_0(\cdot/\lambda); \qquad \partial_t u_\lambda(\cdot, 0) = \lambda^{-\frac{2}{p-1}-1} u_1(\cdot/\lambda).$$

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Both pairs of initial data share the same $\dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^3)$ norm if we choose $s_p = 3/2-2/(p-1)$. As a result, this Sobolev space is called the critical Sobolev space of this equation. We may combine Strichartz estimates, as given in Ginibre-Velo [11], with a fixed-point argument to prove the local well-posedness of this problem for any initial data in the critical Sobolev space. Readers may see Kapitanski [15] and Lindblad-Sogge [23], for instance, for more details of local theory. There is also an energy conservation law for suitable solutions:

$$E(u, u_t) = \int_{\mathbb{R}^3} \left(\frac{1}{2}|\nabla u(\cdot,t)|^2 + \frac{1}{2}|u_t(\cdot,t)|^2 + \frac{1}{p+1}|u(\cdot,t)|^{p+1} \right) dx = \text{Const.}$$

The question about global behaviour of solutions is more difficult. In early 1990's M. Grillakis [12, 13] gave a satisfying answer in the energy critical case $p = 5$: Any solution with initial data in the critical space $\dot{H}^1 \times L^2(\mathbb{R}^3)$ must scatter in both two time directions. In other words, the asymptotic behaviour of any solution mentioned above resembles that of a free wave. We expect that a similar result holds for other exponent p as well.

Conjecture 1.1. *Any solution to (CP1) with initial data $(u_0, u_1) \in \dot{H}^{s_p} \times \dot{H}^{s_p-1}$ must exist for all time $t \in \mathbb{R}$ and scatter in both two time directions.*

This is still an open problem, although we do have progress in many different aspects:

Scattering with radial data Dodson in his recent works [1, 2] gives a proof for the radial case of this conjecture when $3 \leq p < 5$. More precisely, any solution with radial initial data in the critical Soblev space $\dot{H}^{s_p} \times \dot{H}^{s_p-1}$ must exist globally for all time $t \in \mathbb{R}$ and scatter in both two time directions. The radial assumption is essential, because the argument uses not only radial Strichartz estimates but also a conformal transformation for 3D wave equation, which works only for radial solutions.

Scattering result with a priori estimates There are many works proving that if a solution u with a maximal lifespan I satisfies an a priori estimate

$$\sup_{t \in I} \|(u(\cdot,t), u_t(\cdot,t))\|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^3)} < +\infty, \tag{1}$$

then u is defined for all time t and scatters. The proof uses a compactness-rigidity argument, which was first introduced by Kenig-Merle in the works [18, 19] for the study of energy critical wave and Schödinger equations. The compactness part is nowadays a standard procedure in the study of dispersive equations; while the rigidity part does depend on specific situations. To learn more about the energy supercritical case $p > 5$, one may look at Duyckaerts et al. [9], Kenig-Merle [20], Killip-Visan [22] for the radial case and Killip-Visan [21] for the non-radial case. While in the energy subcritical case, please refer to Dodson-Lawrie [3] for $1 + \sqrt{2} < p \leq 3$ and Shen [25] for $3 < p < 5$, both in the radial case, as well as Dodson et al. [4] for $3 < p < 5$ in the nonracial case. One advantage of these results is that their methods work in the focusing case as well. Finally the author would like to mention that (1) is automatically true in the defocusing, energy critical case $p = 5$, as long as initial data are contained in the critical Sobolev space $\dot{H}^1 \times L^2$, thanks to the energy conservation law.

Remark 1.2. *There are also works regarding the global behaviour of type II solutions, i.e. solutions satisfying* (1) *in its maximal lifespan I, to energy critical, focusing wave equations. Please see Duyckaerts-Kenig-Merle [7, 8] in the radial case, and Duychaerts-Jia-Kenig [5] in the non-radial case, for example.*

Strong Assumptions on Initial Data There is also multiple scattering results if we assume that the initial data satisfy stronger regularity and/or decay conditions. These results are usually proved via a suitable global space-time integral estimate.

- In the energy sub-critical case $3 \leq p < 5$, the solutions always scatter if initial data satisfy an additional regularity-decay condition
$$\int_{\mathbb{R}^3} \left[(|x|^2+1)(|\nabla u_0(x)|^2 + |u_1(x)|^2) + |u_0(x)|^2\right] dx < \infty. \tag{2}$$
The main tool is the following conformal conservation law
$$\frac{d}{dt} Q(t,u,u_t) = \frac{4(3-p)t}{p+1} \int_{\mathbb{R}^3} |u(x,t)|^{p+1}\, dx.$$
Here $Q(t,\varphi,\psi) = Q_0(t,\varphi,\psi) + Q_1(t,\varphi)$ is called the conformal charge with
$$Q_0(t,\varphi,\psi) = \|x\psi + t\nabla\varphi\|_{L^2(\mathbb{R}^3)}^2 + \left\|(t\psi + 2\varphi)\frac{x}{|x|} + |x|\nabla\varphi\right\|_{L^2(\mathbb{R}^3)}^2;$$
$$Q_1(t,\varphi) = \frac{2}{p+1}\int_{\mathbb{R}^3} (|x|^2+t^2)|\varphi(x,t)|^{p+1} dx.$$
The assumption (2) guarantees the finiteness of conformal charge $Q(t,u,u_t)$ when $t=0$. It immediately gives a global space-time integral estimate since we have assumed $3 \leq p < 5$:
$$\begin{aligned}\int_{|t|>1}\int_{\mathbb{R}^3} |u(x,t)|^{p+1}\, dxdt &\lesssim_p \int_{|t|>1} \frac{Q_1(t,u(\cdot,t))}{t^2} dt \lesssim_p \sup_{t\in\mathbb{R}} Q_1(t,u(\cdot,t)) \\ &\leq \sup_{t\in\mathbb{R}} Q(t,u,u_t) = Q(0,u_0,u_1) < +\infty,\end{aligned}$$
which then implies the scattering of solutions. For more details please see Ginibre-Velo [10] and Hidano [14].
- The author's previous work [26] proved the scattering result for $3 \leq p < 5$ if initial data $(u_0,u_1) \in \dot{H}^1 \times L^2$ are radial and satisfy
$$\int_{\mathbb{R}^3} (|x|+1)^{1+2\varepsilon}\left(|\nabla u_0|^2 + |u_1|^2\right) dx < \infty$$
for an arbitrary constant $\varepsilon > 0$, by introducing a conformal transformation: If u is a solution as assumed, then for any $t_0 \in \mathbb{R}$, the function
$$v(y,\tau) = \frac{\sinh|y|}{|y|} e^{\tau} u\left(e^{\tau}\frac{\sinh|y|}{|y|}\cdot y, t_0 + e^{\tau}\cosh|y|\right), \quad (y,\tau) \in \mathbb{R}^3 \times \mathbb{R}$$
solves another wave equation
$$v_{\tau\tau} - \Delta_y v = -\left(\frac{|y|}{\sinh|y|}\right)^{p-1} e^{-(p-3)\tau}|v|^{p-1}v.$$
We then apply a Morawetz-type estimate on the solutions v of the second equation and rewrite it in the form of original solutions u. This helps to give a global space-time integral $\|u\|_{L^{2(p-1)}L^{2(p-1)}(\mathbb{R}\times\mathbb{R}^3)} < +\infty$ and finishes the proof.
- In the author's recent work [27] we proved the same scattering result for radial solutions under a weaker assumption on initial data
$$\int_{\mathbb{R}^3} (|x|^{\kappa}+1)\left(\frac{1}{2}|\nabla u_0|^2 + \frac{1}{2}|u_1|^2 + \frac{1}{p+1}|u_0|^{p+1}\right) < +\infty.$$
Here $\kappa > \kappa_1(p) = \frac{3(5-p)}{p+3}$ is a constant. The proof uses a detailed version of the classic Morawetz estimate (see Section 2.3 below) to give a decay rate of the space-time integral
$$\int_{-\infty}^{+\infty}\int_{|x|>R} \frac{|u|^{p+1}}{|x|} dxdt \lesssim R^{-\kappa}.$$
This then gives the same estimate $\|u\|_{L^{2(p-1)}L^{2(p-1)}(\mathbb{R}\times\mathbb{R}^3)} < +\infty$ and implies the scattering.

1.2 Main Results

In this paper we always consider radial solutions to (CP1) with a finite energy. Energy-subcriticality guarantees the global existence of the solutions.The goal of this work is two-fold.

- We want to understand the spatial distribution of the energy as t goes to infinity. This gives plentiful information about the global behaviour of solutions.
- If the energy of initial data satisfies an additional decay assumption, we prove the scattering results.

In this subsection we give two main theorems.

Theorem 1.3. *Assume $3 \leq p < 5$. Let u be a radial solution to (CP1) with a finite energy E. Then there exist a three-dimensional free wave $v^-(x,t)$, with an energy $\tilde{E}_- \leq E$, so that*

- *We have scattering outside any backward light cone ($R \in \mathbb{R}$)*

$$\lim_{t\to-\infty} \left\|(\nabla u(\cdot,t), u_t(\cdot,t)) - \left(\nabla v^-(\cdot,t), v_t^-(\cdot,t)\right)\right\|_{L^2(\{x\in\mathbb{R}^3:|x|>R+|t|\})} = 0.$$

- *If we have $\tilde{E}_- = E$, then the scattering happens in the whole space in the negative time direction.*

$$\lim_{t\to-\infty} \left\|(u(\cdot,t), u_t(\cdot,t)) - \left(v^-(\cdot,t), v_t^-(\cdot,t)\right)\right\|_{\dot{H}^1\times L^2(\mathbb{R}^3)} = 0.$$

- *If $\tilde{E}_- < E$, the remaining energy (also called "retarded energy") can be located: for any constants $c \in (0,1)$ and $\beta < \frac{2(p-2)}{p+1}$ we have*

$$\lim_{t\to-\infty} \int_{c|t|<|x|<|t|-|t|^\beta} \left(\frac{1}{2}|\nabla u(x,t)|^2 + \frac{1}{2}|u_t(x,t)|^2 + \frac{1}{p+1}|u(x,t)|^{p+1}\right) dx = E - \tilde{E}_-.$$

The asymptotic behaviour in the positive time direction is similar.

Remark 1.4. *If $\tilde{E}_- < E$, then the energy distribution is illustrated in figure 1. The "gap" between scattering energy, which concentrates around the light cone and travels at the light speed, and "retarded energy", which travels slightly slower, becomes wider and wider as time goes to infinity.*

Theorem 1.5. *Assume $3 \leq p < 5$. Let $\kappa > \kappa_0(p) = \frac{5-p}{p+1}$ be a constant. If u is a radial solution to (CP1) with initial data (u_0, u_1) so that*

$$\int_{\mathbb{R}^3} (|x|^\kappa + 1)\left(\frac{1}{2}|\nabla u_0|^2 + \frac{1}{2}|u_1|^2 + \frac{1}{p+1}|u_0|^{p+1}\right) dx < +\infty,$$

then u must scatter in both two time directions. More precisely, there exists $(v_0^\pm, v_1^\pm) \in \dot{H}^1 \times L^2(\mathbb{R}^3)$, so that

$$\lim_{t\to\pm\infty} \left\|\begin{pmatrix} u(\cdot,t) \\ \partial_t u(\cdot,t)\end{pmatrix} - \mathbf{S}_L(t)\begin{pmatrix} u_0^\pm \\ u_1^\pm\end{pmatrix}\right\|_{\dot{H}^1\times L^2(\mathbb{R}^3)} = 0.$$

Here $\mathbf{S}_L(t)$ is the linear wave propagation operator.

Remark 1.6. *The assumptions in our main theorems are too weak to guarantee $(u_0, u_1) \in \dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^3)$. For example, we can choose a radial function $u_0 \in C^\infty(\mathbb{R}^3)$ with decay*

$$u_0(x) \simeq |x|^{-\frac{2(p+4)}{(p+1)^2}-\varepsilon}; \qquad |\nabla u_0(x)| \simeq |x|^{-\frac{2(p+4)}{(p+1)^2}-1-\varepsilon}; \qquad |x| \gg 1.$$

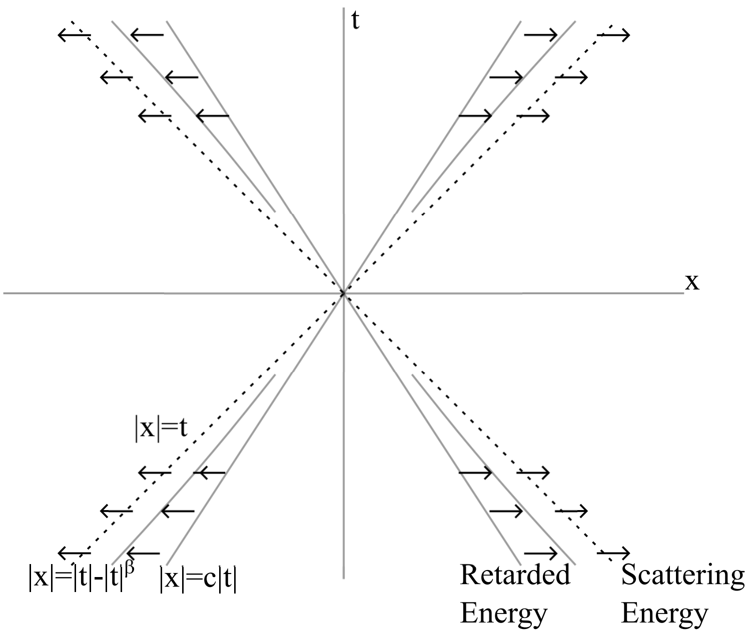

Figure 1: Illustration of travelling energy

Here ε is an sufficiently small positive constant. One can check that $(u_0, 0)$ satisfies all the assumptions on initial data in two main theorems but $u_0 \notin L^{3(p-1)/2}(\mathbb{R}^3)$. The latter implies that $u_0 \notin \dot{H}^{s_p}(\mathbb{R}^3)$ since we have the Sobolev embedding $\dot{H}^{s_p}(\mathbb{R}^3) \hookrightarrow L^{3(p-1)/2}(\mathbb{R}^3)$. As a result we have $(u(\cdot,t), u_t(\cdot,t)) \notin \dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^3)$ for any time t. This is the reason why in Theorem 1.5 we measure the distance of u and free waves by $\dot{H}^1 \times L^2$ norm instead. This is a phenomenon of scattering which has not been covered by previously known results mentioned above. All the solutions discussed in those results come with initial data $(u_0, u_1) \in \dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^3)$.

1.3 The idea

In this subsection we give the main idea of this paper and outline the proof of main theorems. The details can be found in later sections.

Transformation to 1D In order to take full advantage of our radial assumption, we use the following transformation: if u is a radial solution to (CP1), then $w(r,t) = ru(x,t)$, where $|x| = r$, is a solution to one-dimensional wave equation

$$w_{tt} - w_{rr} = -\frac{|w|^{p-1}w}{r^{p-1}}.$$

A basic calculation shows that[1]

$$\begin{aligned} 2\pi &\int_a^b (|w_r(r,t)|^2 + |w_t(r,t)|^2) dr \\ &= 2\pi \left[\int_a^b \left(r^2|u_r(r,t)|^2 + r^2|u_t(r,t)|^2\right) dr + b|u(b,t)|^2 - a|u(a,t)|^2 \right]. \end{aligned} \tag{3}$$

Since for any radial $\dot{H}^1(\mathbb{R}^3)$ function $f(r)$, we have

$$\lim_{r\to 0^+} r|f(r)|^2 = \lim_{r\to+\infty} r|f(r)|^2 = 0.$$

[1]By convention we use notation $u(r,t)$ for the value of a radial function $u(x,t)$ when $|x| = r$

It immediately follows that

$$2\pi \int_0^\infty (|w_r(r,t)|^2 + |w_t(r,t)|^2) dr = \int_{\mathbb{R}^3} \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}|u_t|^2\right) dx. \tag{4}$$

The new solution w also satisfies an energy conservation law

$$E(w, w_t) \doteq 2\pi \int_0^\infty \left(|w_r(r,t)|^2 + |w_t(r,t)|^2 + \frac{2}{p+1} \cdot \frac{|w(r,t)|^{p+1}}{r^{r-1}}\right) dr = E(u, u_t) = \text{Const}.$$

The main tool to understand spatial energy distribution is the energy flux formula for inward and outward energies, whose details are given in Section 3. This idea partially coincides with the channel of energy method. Readers may see Duyckaerts et al. [6], Kenig et al. [16, 17] for more details about the channel of energy method for wave equations.

Inward and Outward Energies Let us first define

Definition 1.7. *Let w be a solution as above. We define the inward and outward energy*

$$E_-(t) = \pi \int_0^\infty \left(|w_r(r,t) + w_t(r,t)|^2 + \frac{2}{p+1} \cdot \frac{|w(r,t)|^{p+1}}{r^{p-1}}\right) dr$$
$$E_+(t) = \pi \int_0^\infty \left(|w_r(r,t) - w_t(r,t)|^2 + \frac{2}{p+1} \cdot \frac{|w(r,t)|^{p+1}}{r^{p-1}}\right) dr$$

We also need to consider their truncated versions

$$E_-(t; r_1, r_2) = \pi \int_{r_1}^{r_2} \left(|w_r(r,t) + w_t(r,t)|^2 + \frac{2}{p+1} \cdot \frac{|w(r,t)|^{p+1}}{r^{p-1}}\right) dr$$
$$E_+(t; r_1, r_2) = \pi \int_{r_1}^{r_2} \left(|w_r(r,t) - w_t(r,t)|^2 + \frac{2}{p+1} \cdot \frac{|w(r,t)|^{p+1}}{r^{p-1}}\right) dr$$

Remark 1.8. *Roughly speaking, the inward energy travels toward the origin as time t increases; the outward energy travels in the opposite direction, as indicated by their names.*

Energy Flux We consider fluxes of inward and outward energies through either characteristic lines $t + r = \text{Const}$, $t - r = \text{Const}$ or the t-axis. This helps to give the following results

(a) Almost all energy is outward energy as $t \to +\infty$.

$$\lim_{t\to+\infty} E_-(t) = 0; \qquad \lim_{t\to+\infty} E_+(t) = E.$$

(b) Almost all energy is inward energy as $t \to -\infty$.

$$\lim_{t\to-\infty} E_+(t) = 0; \qquad \lim_{t\to+\infty} E_-(t) = E.$$

(c) The energy fluxes through characteristic lines are bounded from the above. In particular, the following inequalities hold for any $s, \tau \in \mathbb{R}$.

$$\int_{-\infty}^{s} \frac{|w(s-t,t)|^{p+1}}{(s-t)^{p-1}} dt \lesssim_p E; \qquad \int_{\tau}^{\infty} \frac{|w(t-\tau,t)|^{p+1}}{(t-\tau)^{p-1}} dt \lesssim_p E.$$

Asymptotic Behaviour The characteristic line method gives

$$\frac{\partial}{\partial t}\left[(w_r+w_t)(s-t,t)\right] = -\frac{|w|^{p-1}w(s-t,t)}{(s-t)^{p-1}};$$
$$\frac{\partial}{\partial t}\left[(w_r-w_t)(t-\tau,t)\right] = \frac{|w|^{p-1}w(t-\tau,t)}{(t-\tau)^{p-1}}.$$

Combining these two identities with boundedness of energy fluxes, we have the following convergence holds uniformly in any bounded interval of s or τ.

$$\lim_{t\to-\infty}(w_r+w_t)(s-t,t) = g_-(s); \qquad \lim_{t\to+\infty}(w_r-w_t)(t-\tau,t) = g_+(\tau).$$

We also have the following L^2 convergence for any $s_0,\tau_0\in\mathbb{R}$.

$$\left\|w_r(t-\tau,t)-\frac{1}{2}g_+(\tau)\right\|_{L^2((-\infty,\tau_0])}, \left\|w_t(t-\tau,t)+\frac{1}{2}g_+(\tau)\right\|_{L^2((-\infty,\tau_0])} \to 0, \quad \text{as } t\to+\infty;$$
$$\left\|w_r(s-t,t)-\frac{1}{2}g_-(s)\right\|_{L^2([s_0,\infty))}, \left\|w_t(s-t,t)-\frac{1}{2}g_-(s)\right\|_{L^2([s_0,\infty))} \to 0, \qquad \text{as } t\to-\infty.$$

This gives us the free waves $v^-(x,t)$ and $v^+(x,t)$ in the conclusion part (a) of theorem 1.3.

$$v^-(x,t)=\frac{1}{2|x|}\int_{t-|x|}^{t+|x|} g_-(s)ds; \qquad v^+(x,t)=\frac{1}{2|x|}\int_{t-|x|}^{t+|x|} g_+(\tau)d\tau.$$

We can then prove Part (b) and (c) by considering the energy located in different regions via the energy flux formula. More details are given in Section 4.

Morawetz Estimates Another important ingredient of Theorem 1.5's proof is a more detailed version of the classic Morawetz estimate as given below, which plays an key role in the author's recent work [27] as well. This is a little different from the original inequality given by Perthame and Vega in the work [24], but can be deduced from the original one without difficulty. Please see Subsection 2.3 for more details.

$$\frac{1}{2R}\int_{-\infty}^{+\infty}\int_{|x|<R}\left(\frac{1}{2}|\nabla u|^2+\frac{1}{2}|u_t|^2+\frac{1}{p+1}|u|^{p+1}\right)dxdt+\frac{1}{4R^2}\int_{-\infty}^{+\infty}\int_{|x|=R}|u|^2 d\sigma_R dt$$
$$+\frac{p-3}{2(p+1)R}\int_{-\infty}^{+\infty}\int_{|x|<R}|u|^{p+1}dxdt+\frac{p-1}{2(p+1)}\int_{-\infty}^{+\infty}\int_{|x|>R}\frac{|u|^{p+1}}{|x|}dxdt\leq E.$$

If we pick up the last term in the left hand side and make $R\to 0^+$, we obtain the most frequently used Morawetz estimate:

$$\int_{-\infty}^{+\infty}\int_{\mathbb{R}^3}\frac{|u|^{p+1}}{|x|}dxdt\leq CE.$$

In this work, however, we choose large radius R in the long inequality above and observe an important fact: The first term in the left hand side itself is almost equal to E. This is because for almost all time $t\in[-R,R]$, as long as t is not too closed to R or $-R$, almost all energy concentrates in the ball of radius R, thanks to finite speed of propagation. As a result, we discard all other terms in the left hand and focus on the first term:

$$\int_{-\infty}^{+\infty}\int_{|x|<R}\left(\frac{1}{2}|\nabla u|^2+\frac{1}{2}|u_t|^2+\frac{1}{p+1}|u|^{p+1}\right)dxdt\leq 2RE.$$

We can substitute the right hand side by $\int_{-R}^{R}\int_{\mathbb{R}^3}\left(\frac{1}{2}|\nabla u|^2+\frac{1}{2}|u_t|^2+\frac{1}{p+1}|u|^{p+1}\right)dxdt$, split the left hand side integral into two parts, with $t\in[-R,R]$ and $|t|>R$, respectively, combine the first part integral with the right hand side and finally rewrite the inequality in another form

$$\int_{|t|>R}\int_{|x|<R}\left(\frac{1}{2}|\nabla u|^2+\frac{1}{2}|u_t|^2+\frac{1}{p+1}|u|^{p+1}\right)dxdt$$
$$\leq \int_{-R}^{+R}\int_{|x|>R}\left(\frac{1}{2}|\nabla u|^2+\frac{1}{2}|u_t|^2+\frac{1}{p+1}|u|^{p+1}\right)dxdt.$$

This means that the total contribution of "retarded energy" when $t>|R|$ is always smaller or equal to the total contribution of energy which escapes from the ball of radius R in the time interval $[-R,R]$. This enables us to prove Theorem 1.5 by a contradiction. On one hand, our theory on energy distribution of solutions gives a lower bound of the contribution of "retarded energy", unless the solution scatters. On the other hand, if we assume that the initial data satisfy a suitable decay condition, we can find an upper bound of the escaping energy. Thus we can simply make $R\rightarrow+\infty$ and compare the upper and lower bounds to finish the proof.

1.4 The Structure of This Paper

This paper is organized as follows. In section 2 we collect notations, recall the classic Morawetz estimates and give a few preliminary results. Next in Section 3 we give a general formula of inward and outward energy fluxes. This helps to prove the energy distribution properties of the solutions in Section 4. Finally we prove the scattering of the solution u under an additional decay assumption in the last section.

2 Preliminary Results

2.1 Notations

The $\lesssim$ symbol We use the notation $A\lesssim B$ if there exists a constant c, so that the inequality $A\leq cB$ always holds. In addition, a subscript of the symbol $\lesssim$ indicates that the constant c is determined by the parameter(s) mentioned in the subscript but nothing else. In particular, $\lesssim_1$ means that the constant c is an absolute constant.

Radial functions Let $u(x,t)$ be a spatially radial function. By convention $u(r,t)$ represents the value of $u(x,t)$ when $|x|=r$.

2.2 Uniform Pointwise Estimates

In this subsection we first recall

Lemma 2.1 (Please see Lemma 3.2 of Kenig and Merle's work [20]). *If u is a radial $\dot{H}^1(\mathbb{R}^3)$ function, then*

$$|u(r)|\lesssim_1 r^{-1/2}\|u\|_{\dot{H}^1(\mathbb{R}^3)}.$$

Therefore we always have $|w(r,t)|\lesssim_1 E^{1/2}r^{1/2}$. Because u is not only a $\dot{H}^1(\mathbb{R}^3)$ function but also an $L^{p+1}(\mathbb{R}^3)$ function, this can be further improved if r is large.

Lemma 2.2. *If $w:[0,\infty)\rightarrow\mathbb{R}$ satisfies*

$$2\pi\int_0^\infty\left(|w_r(r)|^2+\frac{2}{p+1}\frac{|w(r)|^{p+1}}{r^{p-1}}\right)dr\leq E,$$

then we have $|w(r)|\lesssim_p E^{2/(p+3)}r^{(p-1)/(p+3)}$.

Proof. First of all, we observe

$$|w(r) - w(r_0)| = \left|\int_{r_0}^{r} w_r(s)ds\right| \leq |r - r_0|^{1/2} \left(\int_{r_0}^{r} |w_r(s)|^2 ds\right)^{1/2} \leq (E/2\pi)^{1/2}|r - r_0|^{1/2}.$$

Thus $w(r)$ converges as $r \to 0^+$. Since the singular integral of $|w(r)|^{p+1}/r^{p-1}$ near zero converges, it is clear that $w(0) = 0$. Plugging $r_0 = 0$ in the estimate above, we re-discover the pointwise estimate $|w(r)| \leq (Er)^{1/2}$. Let us use the notation $S = |w(r_0)|$ for convenience. By the inequality above we have $|w(r) - w(r_0)| \leq S/2$ thus $|w(r)| \geq S/2$ for $r \in [r_0, r_0 + S^2/E]$. The length of the interval here satisfies

$$S^2/E = |w(r_0)|^2/E \leq [(Er_0)^{1/2}]^2/E = r_0.$$

As a result we have

$$E \geq \int_{r_0}^{r_0+S^2/E} \frac{|w(r)|^{p+1}}{r^{p-1}} dr \geq \frac{(S/2)^{p+1}}{(r_0 + S^2/E)^{p-1}} \cdot S^2/E \geq \frac{(S/2)^{p+1}}{(2r_0)^{p-1}} \cdot S^2/E.$$

This means $E \gtrsim_p S^{p+3}/(r_0^{p-1}E)$. Thus we have $S \lesssim_p E^{2/(p+3)} r_0^{(p-1)/(p+3)}$ and finish the proof. □

2.3 Morawetz Estimates

Theorem 2.3 (Please see Perthame and Vega's work [24], we use the 3-dimensional case)**.** *Let u be a solution to (CP1) defined in a time interval $[0, T]$ with a finite energy E. Then we have the following inequality for any $R > 0$. Here σ_R is the regular surface measure of the sphere $|x| = R$.*

$$\frac{1}{2R}\int_0^T\int_{|x|<R}(|\nabla u|^2 + |u_t|^2)dxdt + \frac{1}{2R^2}\int_0^T\int_{|x|=R}|u|^2 d\sigma_R dt + \frac{p-2}{(p+1)R}\int_0^T\int_{|x|<R}|u|^{p+1}dxdt$$
$$+ \frac{p-1}{p+1}\int_0^T\int_{|x|>R}\frac{|u|^{p+1}}{|x|}dxdt + \frac{1}{R^2}\int_{|x|<R}|u(x,T)|^2 dx \leq 2E. \quad (5)$$

Remark 2.4. *The notations p and E represent slightly different constants in the original paper [24] and this current paper. Here we rewrite the inequality in the setting of the current work. The coefficient before the integral $\int_{B(0,R)} |u(T)|^2 dx$ was $\frac{d^2-1}{4R^2}$ (in the 3-dimensional case $\frac{2}{R^2}$) in the original paper. But the author believes that this is a minor typing mistake. It should have been $\frac{d^2-1}{8R^2}$ instead.*

Remark 2.5. *The upper bound of time interval T does not appear in any of the coefficients above. We also have an energy conservation law. As a result, we can substitute the time interval $[0, T]$ by any bounded time interval $[T_1, T_2]$ or even $(-\infty, T]$. If we ignore the final term on the left hand side, we can also use the time interval $(-\infty, \infty)$.*

$$\frac{1}{2R}\int_{-\infty}^{\infty}\int_{|x|<R}(|\nabla u|^2 + |u_t|^2)dxdt + \frac{1}{2R^2}\int_{-\infty}^{\infty}\int_{|x|=R}|u|^2 d\sigma_R dt$$
$$+ \frac{p-2}{(p+1)R}\int_{-\infty}^{\infty}\int_{|x|<R}|u|^{p+1}dxdt + \frac{p-1}{p+1}\int_{-\infty}^{\infty}\int_{|x|>R}\frac{|u|^{p+1}}{|x|}dxdt \leq 2E. \quad (6)$$

This Morawetz estimate plays two different roles in this work. On one hand, it gives a few global integral estimates, which our theory of energy distribution depends on. One of the most popular ones is $\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} \frac{|u(x,t)|^{p+1}}{|x|} dxdt \lesssim E$. On the other hand, this Morawetz estimate also gives direct information on energy distribution, as we mentioned in the introduction part. Let us discuss these aspects one-by-one.

Global Estimates If we pick the second term in the left hand side of (6) and use the radial assumption, we obtain

$$\sup_{r>0} \int_{-\infty}^{+\infty} |u(r,t)|^2 dt \leq \frac{E}{\pi}.$$

We recall $w = ru$, use the Morawetz estimate (6) again and obtain

$$\begin{aligned} &\int_{-\infty}^{+\infty} \int_0^R (|w_r(r,t)|^2 + |w_t(r,t)|^2) drdt \\ &\quad = \int_{-\infty}^{+\infty} \int_0^R r^2 (|u_r(r,t)|^2 + |u_t(r,t)|^2) drdt + \int_{-\infty}^{+\infty} R|u(R,t)|^2 dt \leq \frac{RE}{\pi} \end{aligned}$$

We can also pick the third term in the left hand of (6) and rewrite it in term of w

$$\int_{-\infty}^{+\infty} \int_0^R \frac{|w|^{p+1}}{r^{p-1}} drdt \leq \frac{(p+1)}{2(p-2)\pi} RE.$$

Finally we pick the forth term, make $R \to 0^+$ and rewrite it in term of w

$$\int_{-\infty}^{+\infty} \int_{\mathbb{R}^3} \frac{|u|^{p+1}}{|x|} dxdt \leq \frac{2(p+1)}{p-1} E \Rightarrow \int_{-\infty}^{+\infty} \int_0^{\infty} \frac{|w|^{p+1}}{r^p} drdt \leq \frac{p+1}{2(p-1)\pi} E.$$

In summary, we have (The second line immediately follows the first line)

Corollary 2.6. *Let u be a radial solution to (CP1) with a finite energy E. Then u and $w = ru$ satisfy*

$$\begin{aligned} \int_{-\infty}^{+\infty} \int_0^R \left(|w_r(r,t)|^2 + |w_t(r,t)|^2 + \frac{|w(r,t)|^{p+1}}{r^{p-1}} \right) drdt &\lesssim_p RE; \\ \liminf_{r\to 0^+} \int_{-\infty}^{+\infty} \left(|w_r(r,t)|^2 + |w_t(r,t)|^2 + \frac{|w(r,t)|^{p+1}}{r^{p-1}} \right) dt &\lesssim_p E; \\ \int_{-\infty}^{+\infty} \int_0^{\infty} \frac{|w(r,t)|^{p+1}}{r^p} drdt &\lesssim_p E. \\ \sup_{r>0} \int_{-\infty}^{+\infty} |u(r,t)|^2 dt &\lesssim_p E. \end{aligned}$$

Energy Distribution Information We can combine part of the third term of (6) with the first term, divide both sides by 2 and obtain

$$\begin{aligned} &\frac{1}{2R} \int_{-\infty}^{+\infty} \int_{|x|<R} \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}|u_t|^2 + \frac{1}{p+1}|u|^{p+1} \right) dxdt + \frac{1}{4R^2} \int_{-\infty}^{+\infty} \int_{|x|=R} |u|^2 d\sigma_R dt \\ &\quad + \frac{p-3}{2(p+1)R} \int_{-\infty}^{+\infty} \int_{|x|<R} |u|^{p+1} dxdt + \frac{p-1}{2(p+1)} \int_{-\infty}^{+\infty} \int_{|x|>R} \frac{|u|^{p+1}}{|x|} dxdt \leq E. \end{aligned} \tag{7}$$

This helps to give a decay rate $\int_{-\infty}^{\infty} \int_{|x|>R} \frac{|u(x,t)|^{p+1}}{|x|} dxdt \lesssim R^{-\kappa}$ in the author's recent work [27]. In this work, we only need a weaker version of this inequality. Since every term in the left hand side is nonnegative, we can focus on the first term and obtain

$$\int_{-\infty}^{+\infty} \int_{|x|<R} \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}|u_t|^2 + \frac{1}{p+1}|u|^{p+1} \right) dxdt \leq 2RE.$$

We substitute the right hand side by $\int_{-R}^{R} \int_{\mathbb{R}^3} \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}|u_t|^2 + \frac{1}{p+1}|u|^{p+1} \right) dxdt$, split the integral in the left hand side into two parts, with time $t \in [-R, R]$ and $|t| > R$ respectively, combine the first part with the right hand side and finally obtain

Proposition 2.7. *For any $R > 0$, we have*

$$\int_{|t|>R}\int_{|x|<R}\left(\frac{1}{2}|\nabla u|^2+\frac{1}{2}|u_t|^2+\frac{1}{p+1}|u|^{p+1}\right)dxdt$$
$$\leq \int_{-R}^{+R}\int_{|x|>R}\left(\frac{1}{2}|\nabla u|^2+\frac{1}{2}|u_t|^2+\frac{1}{p+1}|u|^{p+1}\right)dxdt.$$

The following will not be used in the argument of this paper, instead it is a corollary of our theory on energy distribution.

Remark 2.8. *A careful review of Perthame and Vega's calculation shows that for a radial solution u, we actually have an identity for any $R > 0$ and $T_1 < T_2$*

$$\frac{1}{2R}\int_{T_1}^{T_2}\int_{|x|<R}\left(\frac{1}{2}|\nabla u|^2+\frac{1}{2}|u_t|^2+\frac{1}{p+1}|u|^{p+1}\right)dxdt+\frac{1}{4R^2}\int_{T_1}^{T_2}\int_{|x|=R}|u|^2d\sigma_R dt$$
$$+\frac{p-3}{2(p+1)R}\int_{T_1}^{T_2}\int_{|x|<R}|u|^{p+1}dxdt+\frac{p-1}{2(p+1)}\int_{T_1}^{T_2}\int_{|x|>R}\frac{|u|^{p+1}}{|x|}dxdt$$
$$=2\pi\left[\int_0^R\frac{r}{R}w_t(r,T_1)w_r(r,T_1)dr+\int_R^\infty w_t(r,T_1)w_r(r,T_1)dr\right]$$
$$-2\pi\left[\int_0^R\frac{r}{R}w_t(r,T_2)w_r(r,T_2)dr+\int_R^\infty w_t(r,T_2)w_r(r,T_2)dr\right].$$

By the identity $w_t w_r = \frac{1}{4}\left[(w_r+w_t)^2-(w_r-w_t)^2\right]$, Proposition 4.3, Corollary 4.4 and Lemma 4.9, we know that if we make $T_1 \to -\infty$, $T_2 \to +\infty$, then the limit of the right hand side is exactly E. Therefore the inequality (7) *is actually an identity. Please note that the radial assumption is essential because we discard a term in the form of $\int_{T_1}^{T_2}\int_{|x|>R}\mathbf{D}u\cdot\mathbf{D}^2\Phi\cdot\mathbf{D}u\,dxdt$ in the left hand side, where $\mathbf{D}^2\Phi(x)$ is a positive semidefinite matrix whose eigenvalue 0 has a single eigenvector x. This term vanishes only for radial solutions.*

3 Energy Flux for Inward and Outward Energies

In this section we consider the inward and outward energies given in Definition 1.7 and give energy flux formula of them.

3.1 General Energy Flux Formula

We start by introducing the statement of energy flux formula and giving a few remarks. The proof is put in later subsections.

Proposition 3.1 (General Energy Flux)**.** *Let $\Omega \subset \mathbb{R}^+\times\mathbb{R}$ be a closed region in the right half of the $r-t$ space. Its boundary Γ is a simple curve consisting of finite line segments, which are paralleled to either t-axis, r-axis or characteristic lines $t\pm r = 0$, and is oriented counterclockwise. Then we have*

$$\pi\int_\Gamma\left(|w_r+w_t|^2+\frac{2}{p+1}\cdot\frac{|w|^{p+1}}{r^{p-1}}\right)dr+\left(|w_r+w_t|^2-\frac{2}{p+1}\cdot\frac{|w|^{p+1}}{r^{p-1}}\right)dt$$
$$-\frac{2\pi(p-1)}{p+1}\iint_\Omega\frac{|w|^{p+1}}{r^p}drdt=0. \tag{8}$$

$$\pi\int_\Gamma\left(|w_r-w_t|^2+\frac{2}{p+1}\cdot\frac{|w|^{p+1}}{r^{p-1}}\right)dr+\left(-|w_r-w_t|^2+\frac{2}{p+1}\cdot\frac{|w|^{p+1}}{r^{p-1}}\right)dt$$
$$+\frac{2\pi(p-1)}{p+1}\iint_\Omega\frac{|w|^{p+1}}{r^p}drdt=0. \tag{9}$$

Furthermore, there exists a function ξ with $\|\xi\|^2_{L^2(\mathbb{R})} \lesssim_p E$, which is solely determined by u and independent of Ω, so that the identities above also hold for regions Ω with part of its boundary lying on the t-axis. In this case the line integral from the point $(0, t_2)$ downward to $(0, t_1)$ along t-axis is understood as either $-\pi \int_{t_1}^{t_2} |\xi(t)|^2\, dt$ in identity (8) or $\pi \int_{t_1}^{t_2} |\xi(t)|^2\, dt$ in identity (9).

Line integrals We first have a look at what the line integrals look like for different types of boundary line segments. In table 1 we always assume $r_1 < r_2$, $t_1 < t_2$. For example, when we say a line segment $r = r_0$ goes downward, it starts at time t_2 and ends at time t_1.

Table 1: Line integrals in energy flux formula

Boundary type	Inward Energy Case	Outward Energy Case
Horizontally $\rightarrow$	$\int_{r_1}^{r_2} \left(|w_r + w_t|^2 + \frac{2}{p+1} \cdot \frac{|w|^{p+1}}{r^{p-1}}\right) dr$	$\int_{r_1}^{r_2} \left(|w_r - w_t|^2 + \frac{2}{p+1} \cdot \frac{|w|^{p+1}}{r^{p-1}}\right) dr$
Horizontally $\leftarrow$	$-\int_{r_1}^{r_2} \left(|w_r + w_t|^2 + \frac{2}{p+1} \cdot \frac{|w|^{p+1}}{r^{p-1}}\right) dr$	$-\int_{r_1}^{r_2} \left(|w_r - w_t|^2 + \frac{2}{p+1} \cdot \frac{|w|^{p+1}}{r^{p-1}}\right) dr$
$r = r_0 > 0, \uparrow$	$\int_{t_1}^{t_2} \left(|w_r + w_t|^2 - \frac{2}{p+1} \cdot \frac{|w|^{p+1}}{r^{p-1}}\right) dt$	$\int_{t_1}^{t_2} \left(-|w_r - w_t|^2 + \frac{2}{p+1} \cdot \frac{|w|^{p+1}}{r^{p-1}}\right) dt$
$r = r_0 > 0, \downarrow$	$\int_{t_1}^{t_2} \left(-|w_r + w_t|^2 + \frac{2}{p+1} \cdot \frac{|w|^{p+1}}{r^{p-1}}\right) dt$	$\int_{t_1}^{t_2} \left(|w_r - w_t|^2 - \frac{2}{p+1} \cdot \frac{|w|^{p+1}}{r^{p-1}}\right) dt$
$r = 0, \downarrow$	$-\int_{t_1}^{t_2} |\xi(t)|^2 dt$	$+\int_{t_1}^{t_2} |\xi(t)|^2 dt$
$t + r = s, \searrow$	$\frac{4}{p+1} \int_{t_1}^{t_2} \frac{|w(s-t,t)|^{p+1}}{(s-t)^{p-1}} dt$	$2\int_{t_1}^{t_2} |w_r(s-t,t) - w_t(s-t,t)|^2 dt$
$t + r = s, \nwarrow$	$-\frac{4}{p+1} \int_{t_1}^{t_2} \frac{|w(s-t,t)|^{p+1}}{(s-t)^{p-1}} dt$	$-2\int_{t_1}^{t_2} |w_r(s-t,t) - w_t(s-t,t)|^2 dt$
$t - r = \tau, \nearrow$	$2\int_{t_1}^{t_2} |w_r(t-\tau,t) + w_t(t-\tau,t)|^2 dt$	$\frac{4}{p+1} \int_{t_1}^{t_2} \frac{|w(t-\tau,t)|^{p+1}}{(t-\tau)^{p-1}} dt$
$t - r = \tau, \swarrow$	$-2\int_{t_1}^{t_2} |w_r(t-\tau,t) + w_t(t-\tau,t)|^2 dt$	$-\frac{4}{p+1} \int_{t_1}^{t_2} \frac{|w(t-\tau,t)|^{p+1}}{(t-\tau)^{p-1}} dt$

Physical Meaning The physical meaning of the fluxes across characteristic lines, which correspond to light cones in $\mathbb{R}^3$, and other related terms, is given as

- A term in the form of $\int_{t_1}^{t_2} |w_r \pm w_t|^2 dt$ is the amount of energy which moves across characteristic lines due to liner wave propagation.
- A term in the form of $\int_{t_1}^{t_2} \frac{|w|^{p+1}}{r^{p-1}} dt$ is the amount of energy which moves across characteristic lines due to nonlinear effect.
- A term in the form of $\int_{t_1}^{t_2} |\xi(t)|^2 dt$ is the amount of inward energy which moves through the origin and becomes outward energy during the given period of time.
- The double integral in the identities is the amount of inward energy transformed to outward energy by nonlinear effect in the given space-time region.

Notation for Fluxes For convenience we write the energy flux across a characteristic line $t \pm r = \text{Const}$, which corresponds to a light cone in $\mathbb{R}^3$ in the following way.

Definition 3.2 (Notations of Energy fluxes)**.** *Given $s,\tau \in \mathbb{R}$, we define energy fluxes*

$$\begin{aligned}
Q_-^-(s) &= \frac{4\pi}{p+1}\int_{-\infty}^{s} \frac{|w(s-t,t)|^{p+1}}{(s-t)^{p-1}}dt;\\
Q_+^-(s) &= 2\pi\int_{-\infty}^{s} |w_r(s-t,t)-w_t(s-t,t)|^2 dt;\\
Q_-^+(\tau) &= 2\pi\int_{\tau}^{+\infty} |w_r(t-\tau,t)+w_t(t-\tau,t)|^2 dt;\\
Q_+^+(\tau) &= \frac{4\pi}{p+1}\int_{\tau}^{+\infty} \frac{|w(t-\tau,t)|^{p+1}}{(t-\tau)^{p-1}}dt.
\end{aligned}$$

A negative sign upper index means that this is energy flux across the characteristic line $t+r=s$; otherwise this is energy flux across the characteristic line $t-r=\tau$. The lower index indicates whether this is energy flux of inward energy $(-)$ or outward energy $(+)$. We can also consider their truncated version, which is the energy flux across a line segment of a characteristic line.

$$\begin{aligned}
Q_-^-(s;t_1,t_2) &= \frac{4\pi}{p+1}\int_{t_1}^{t_2} \frac{|w(s-t,t)|^{p+1}}{(s-t)^{p-1}}dt, && t_1<t_2\leq s;\\
Q_+^-(s;t_1,t_2) &= 2\pi\int_{t_1}^{t_2} |w_r(s-t,t)-w_t(s-t,t)|^2 dt, && t_1<t_2\leq s;\\
Q_-^+(\tau;t_1,t_2) &= 2\pi\int_{t_1}^{t_2} |w_r(t-\tau,t)+w_t(t-\tau,t)|^2 dt, && \tau\leq t_1<t_2;\\
Q_+^+(\tau;t_1,t_2) &= \frac{4\pi}{p+1}\int_{t_1}^{t_2} \frac{|w(t-\tau,t)|^{p+1}}{(t-\tau)^{p-1}}dt, && \tau\leq t_1<t_2.
\end{aligned}$$

Remark 3.3. *The sums $Q_-^-(s)+Q_+^-(s)$ and $Q_-^+(\tau)+Q_+^+(\tau)$ are exactly full energy fluxes across the lines $t+r=s$ and $t-r=\tau$, respectively. The values of these sums can never exceed the total energy E, because of the already known energy flux formula of full energy. As a result all the energy fluxes defined above are dominated by the full energy E.*

3.2 Variance of $w_r \pm w_t$ along characteristic lines

In this subsection we give a few estimates on the variance of $w_r \pm w_t$ along characteristic lines. They are useful not only in the proof of energy flux formula of inward/outward energy, but also in later sections when we consider the asymptotic behaviour of w as $t\to\pm\infty$. We start by rewriting the equation $w_{tt}-w_{rr} = -\frac{|w|^{p-1}w}{r^{p-1}}$ in the form of

$$(\partial_t-\partial_r)(\partial_t+\partial_r)w = -\frac{|w|^{p-1}w}{r^{p-1}}.$$

Therefore we have

Proposition 3.4. *Let $\tau<t_1<t_2<s$*

$$\begin{aligned}
(w_r+w_t)(s-t_2,t_2)-(w_r+w_t)(s-t_1,t_1) &= -\int_{t_1}^{t_2}\frac{|w|^{p-1}w(s-t,t)}{(s-t)^{p-1}}dt.\\
(w_r-w_t)(t_2-\tau,t_2)-(w_r-w_t)(t_1-\tau,t_1) &= +\int_{t_1}^{t_2}\frac{|w|^{p-1}w(t-\tau,t)}{(t-\tau)^{p-1}}dt.
\end{aligned}$$

Upper bounds of Variance We first give a few upper bounds of the right side integral above. The first one involving energy flux Q is essential to giving the asymptotic behaviour of the solution u as $t\to\infty$. Other bounds are used to show the L^2 convergence of $w_r \pm w_t$ as $r\to 0^+$. For convenience we first give a definition.

Definition 3.5. *Given $s, \tau \in \mathbb{R}$, we define*

$$M^-(s) = \int_{-\infty}^{s} \frac{|w(s-t,t)|^{p+1}}{(s-t)^p} dt; \qquad M^+(\tau) = \int_{\tau}^{\infty} \frac{|w(t-\tau,t)|^{p+1}}{(t-\tau)^p} dt.$$

We claim $M^-, M^+ \in L^1(\mathbb{R})$. Let us take M^+ as an example. We apply the change of variable $\tau = t - r$, recall the classic Morawetz estimate and obtain

$$\int_{-\infty}^{\infty} M^+(\tau) d\tau = \int_{-\infty}^{\infty} \int_{\tau}^{\infty} \frac{|w(t-\tau,t)|^{p+1}}{(t-\tau)^p} dt d\tau = \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{|w(r,t)|^{p+1}}{r^p} dr dt \lesssim_p E.$$

Now we give details about upper bounds of the right hand integrals in Proposition 3.4.

Lemma 3.6. *We have the following estimates for $\tau \leq t_1 < t_2 \leq s$*

$$\int_{t_1}^{t_2} \frac{|w(s-t,t)|^p}{(s-t)^{p-1}} dt \lesssim_p \min\left\{(s-t_2)^{-\frac{p-2}{p+1}} \left(Q_-^-(s;t_1,t_2)\right)^{\frac{p}{p+1}}, (s-t_1)^{\frac{2}{p+1}} M^-(s)^{\frac{p}{p+1}}\right\};$$

$$\int_{t_1}^{t_2} \frac{|w(t-\tau,t)|^p}{(t-\tau)^{p-1}} dt \lesssim_p \min\left\{(t_1-\tau)^{-\frac{p-2}{p+1}} \left(Q_+^+(\tau;t_1,t_2)\right)^{\frac{p}{p+1}}, (t_2-\tau)^{\frac{2}{p+1}} M^+(\tau)^{\frac{p}{p+1}}\right\}.$$

In particular we have ($\tau, s \in \mathbb{R}$)

$$\int_{-\infty}^{s} \frac{|w(s-t,t)|^p}{(s-t)^{p-1}} dt \lesssim_p Q_-^-(s)^{\frac{2}{p+1}} M^-(s)^{\frac{p-2}{p+1}};$$

$$\int_{\tau}^{\infty} \frac{|w(t-\tau,t)|^p}{(t-\tau)^{p-1}} dt \lesssim_p Q_+^+(\tau)^{\frac{2}{p+1}} M^+(\tau)^{\frac{p-2}{p+1}}.$$

Proof. Let us prove the inequalities involving integral along the characteristic line $r + t = s$. The integral along the line $t - r = \tau$ can be dealt with in the same way. The first inequality comes from a straightforward calculation.

$$\begin{aligned}
\int_{t_1}^{t_2} \frac{|w(s-t,t)|^p}{(s-t)^{p-1}} dt &\leq \left(\int_{t_1}^{t_2} \left(\frac{|w(s-t,t)|^p}{(s-t)^{\frac{(p-1)p}{p+1}}}\right)^{\frac{p+1}{p}} dt\right)^{\frac{p}{p+1}} \left(\int_{t_1}^{t_2} \left(\frac{1}{(s-t)^{\frac{p-1}{p+1}}}\right)^{p+1} dt\right)^{\frac{1}{p+1}} \\
&\leq \left(\int_{t_1}^{t_2} \frac{|w(s-t,t)|^{p+1}}{(s-t)^{p-1}} dt\right)^{\frac{p}{p+1}} \left(\int_{t_1}^{t_2} \frac{1}{(s-t)^{p-1}} dt\right)^{\frac{1}{p+1}} \\
&\lesssim_p \left(Q_-^-(s;t_1,t_2)\right)^{\frac{p}{p+1}} \left[(s-t_2)^{2-p} - (s-t_1)^{2-p}\right]^{\frac{1}{p+1}} \\
&\lesssim_p (s-t_2)^{-\frac{p-2}{p+1}} \left(Q_-^-(s;t_1,t_2)\right)^{\frac{p}{p+1}}.
\end{aligned}$$

$$\begin{aligned}
\int_{t_1}^{t_2} \frac{|w(s-t,t)|^p}{(s-t)^{p-1}} dt &\leq \left(\int_{t_1}^{t_2} \left(\frac{|w(s-t,t)|^p}{(s-t)^{\frac{p^2}{p+1}}}\right)^{\frac{p+1}{p}} dt\right)^{\frac{p}{p+1}} \left(\int_{t_1}^{t_2} \left((s-t)^{\frac{1}{p+1}}\right)^{p+1} dt\right)^{\frac{1}{p+1}} \\
&\leq \left(\int_{t_1}^{t_2} \frac{|w(s-t,t)|^{p+1}}{(s-t)^p} dt\right)^{\frac{p}{p+1}} \left(\int_{t_1}^{t_2} (s-t) dt\right)^{\frac{1}{p+1}} \\
&\lesssim_p M^-(s)^{\frac{p}{p+1}} \left[(s-t_1)^2 - (s-t_2)^2\right]^{\frac{1}{p+1}} \\
&\lesssim_p (s-t_1)^{\frac{2}{p+1}} M^-(s)^{\frac{p}{p+1}}.
\end{aligned}$$

In order to prove the last inequality, we split the integral into two parts ($-\infty < T < s$)

$$\int_{-\infty}^{s} \frac{|w(s-t,t)|^p}{(s-t)^{p-1}} dt = \int_{-\infty}^{T} \frac{|w(s-t,t)|^p}{(s-t)^{p-1}} dt + \int_{T}^{s} \frac{|w(s-t,t)|^p}{(s-t)^{p-1}} dt.$$

We then plug in already known upper bounds of these two integrals

$$\int_{\tau}^{\infty} \frac{|w(s-t,t)|^p}{(s-t)^{p-1}} dt \lesssim_p (s-T)^{-\frac{p-2}{p+1}} Q_-^-(s)^{\frac{p}{p+1}} + (s-T)^{\frac{2}{p+1}} M^-(s)^{\frac{p}{p+1}}.$$

Finally we choose the best time $T = s - Q_-^-(s)/M^-(s)$ and finish the proof. □

3.3 Proof of Energy Flux Formula

Now let us give a proof of Proposition 3.1. Without loss of generality let us prove the first identity (8), i.e. the energy flux formula of inward energy. Then we may obtain the energy flux formula of outward energy (9) by either following the same proof as in the inward energy case or writing $E_+ = E - E_-$ then applying the energy flux formula of full energy.

The case away from t-axis If the region Ω is away from the t-axis, then we only need to apply Green's formula on the line integral of the given vector fields and conduct a basic calculation.

$$\begin{aligned}
&\pi \int_{\Gamma} \left(|w_r + w_t|^2 + \frac{2}{p+1} \cdot \frac{|w|^{p+1}}{r^{p-1}} \right) dr + \left(|w_r + w_t|^2 - \frac{2}{p+1} \cdot \frac{|w|^{p+1}}{r^{p-1}} \right) dt \\
=&\pi \iint_{\Omega} \left[\frac{\partial}{\partial r} \left(|w_r + w_t|^2 - \frac{2}{p+1} \cdot \frac{|w|^{p+1}}{r^{p-1}} \right) - \frac{\partial}{\partial t} \left(|w_r + w_t|^2 + \frac{2}{p+1} \cdot \frac{|w|^{p+1}}{r^{p-1}} \right) \right] drdt \\
=&\pi \iint_{\Omega} \left[2(w_r + w_t) \left(w_{rr} - w_{tt} - \frac{|w|^{p-1} w}{r^{p-1}} \right) + \frac{2(p-1)}{p+1} \cdot \frac{|w|^{p+1}}{r^p} \right] drdt \\
=&\frac{2\pi(p-1)}{p+1} \iint_{\Omega} \frac{|w|^{p+1}}{r^p} drdt.
\end{aligned}$$

Here we use the equation $w_{tt} - w_{rr} = -\frac{|w|^{p-1}w}{r^{p-1}}$. In the process of calculation the second derivatives are involved, thus we apply smooth approximation techniques when necessary.

The case with boundary on t-axis Now let us consider the case when part of the boundary is on the t-axis. In this case a limit process $r \to 0^+$ is required. We first apply energy flux formula away from t-axis on the region $\Omega_r = \Omega \cap ([r, \infty) \times \mathbb{R})$ and then make $r \to 0^+$, as shown in figure 2. In order to complete the proof we only need to show that there exists $\xi \in L^2(\mathbb{R})$ so that the following identity holds for all $t_1 < t_2$ and $c_1, c_2 \in \{-1, 0, 1\}$.

$$\lim_{r \to 0^+} \int_{t_1 + c_1 r}^{t_2 + c_2 r} \left(|w_r(r,t) + w_t(r,t)|^2 - \frac{2}{p+1} \cdot \frac{|w(r,t)|^{p+1}}{r^{p-1}} \right) dt = \int_{t_1}^{t_2} |\xi(t)|^2 dt. \tag{10}$$

The integral above consists of two parts, the integrals of $|w_r + w_t|^2$ and $|w|^{p+1}/r^{p-1}$, respectively. We consider their limits one by one. We start with the easier one.

The limit of integral $|w|^{p+1}/r^{p-1}$ We combine the inequality $w(r,t) \lesssim_E r^{1/2}$ with Corollary 2.6 and obtain

$$\int_{-\infty}^{\infty} \frac{|w(r,t)|^{p+1}}{r^{p-1}} dt = \int_{-\infty}^{\infty} \frac{|w(r,t)|^{p-1} r^2 |u(r,t)|^2}{r^{p-1}} dt \lesssim_{p,E} r^{\frac{5-p}{2}} \int_{-\infty}^{\infty} |u(r,t)|^2 dt \lesssim_{p,E} r^{\frac{5-p}{2}}.$$

This vanishes as $r \to 0^+$. Thus we may completely ignore this term in the limit process.

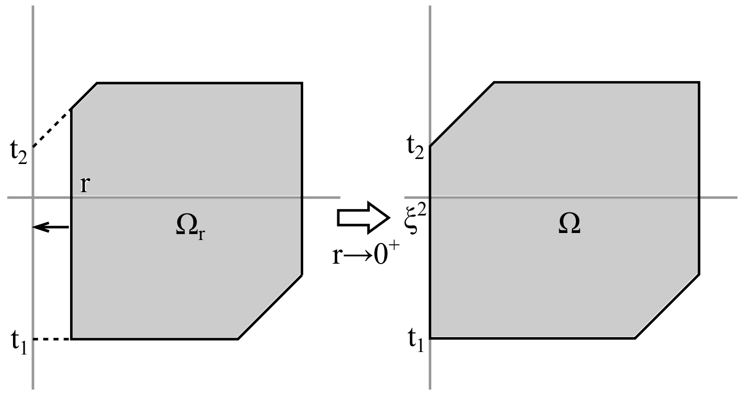

Figure 2: Illustration of the limit process

The limit of integral $|w_r + w_t|^2$ We recall Proposition 3.4 and Lemma 3.6 to obtain the following estimates for $0 < r_1 < r_2$, $s \in \mathbb{R}$. Please note that we also apply change of variable $r = s - t$ here. Thus[2] $r_2 = s - t_1$.

$$\begin{aligned}
&|(w_r + w_t)(r_1, s - r_1) - (w_r + w_t)(r_2, s - r_2)| \lesssim_p r_2^{\frac{2}{p+1}} M^-(s)^{\frac{p}{p+1}};\\
&|(w_r + w_t)(r_1, s - r_1) - (w_r + w_t)(r_2, s - r_2)| \lesssim_p Q_-^-(s)^{\frac{2}{p+1}} M^-(s)^{\frac{p-2}{p+1}}.
\end{aligned}$$

We have already known $M^- \in L^1(\mathbb{R})$ and $Q_-^-(s) \leq E$. As a result we have

$$M^-(s) < +\infty, \text{ a.e. } s \in \mathbb{R}; \qquad\qquad Q_-^-(s)^{\frac{2}{p+1}} M^-(s)^{\frac{p-2}{p+1}} \in L^2_{loc}(\mathbb{R}).$$

Therefore

(I) The first estimate above shows that there exists a measurable function ξ so that

$$\lim_{r\to 0^+} (w_r + w_t)(r, s - r) = \xi(s), \quad \text{for a.e. } s \in \mathbb{R}.$$

(II) By dominated convergence theorem we also have the local L^2 convergence, i.e. We have $\xi \in L^2_{loc}(\mathbb{R})$ and the following limit for any given $s_1 < s_2$

$$\lim_{r\to 0^+} \|(w_r + w_t)(r, \cdot - r) - \xi(\cdot)\|_{L^2([s_1,s_2])} = 0.$$

Thus we also have

$$\begin{aligned}
\lim_{r\to 0^+} \int_{t_1-r}^{t_2-r} |w_r(r,t) + w_t(r,t)|^2 dt &= \lim_{r\to 0^+} \int_{t_1}^{t_2} |w_r(r, s-r) + w_t(r, s-r)|^2 ds\\
&= \int_{t_1}^{t_2} |\xi(t)|^2 dt.
\end{aligned}$$

Completion of the proof Finally we are able to prove (10) by considering lower and upper limits separately. Given any small constant $\varepsilon > 0$, we have

$$\begin{aligned}
&\limsup_{r\to 0^+} \int_{t_1+c_1 r}^{t_2+c_2 r} \left(|w_r(r,t) + w_t(r,t)|^2 - \frac{2}{p+1} \cdot \frac{|w(r,t)|^{p+1}}{r^{p-1}} \right) dt\\
&\qquad \leq \lim_{r\to 0^+} \int_{t_1-r}^{t_2+\varepsilon-r} |w_r(r,t) + w_t(r,t)|^2 dt = \int_{t_1}^{t_2+\varepsilon} |\xi(t)|^2 dt.
\end{aligned}$$

[2]We assume $r_1 < r_2$ and $t_1 < t_2$ here. Thus $s = r_1 + t_2 = r_2 + t_1$.

and

$$\liminf_{r\to 0^+} \int_{t_1+c_1 r}^{t_2+c_2 r} \left(|w_r(r,t)+w_t(r,t)|^2 - \frac{2}{p+1}\cdot\frac{|w(r,t)|^{p+1}}{r^{p-1}}\right) dt$$
$$\geq \lim_{r\to 0^+} \int_{t_1+\varepsilon-r}^{t_2-r} |w_r(r,t)+w_t(r,t)|^2 dt = \int_{t_1+\varepsilon}^{t_2} |\xi(t)|^2 dt.$$

Since ε is an arbitrarily small constant, we may make $\varepsilon \to 0^+$ and obtain the identity (10). Finally we combine the second inequality of Corollary 2.6 with identity (10) to deduce $\|\xi\|^2_{L^2(\mathbb{R})} \lesssim_p E$.

Remark 3.7. *If we follow a similar limit process for the energy flux of outward energy, we will obtain the same L^2 function ξ as in the case of inward energy. There are two different ways to show this.*

- *We may write $E_+ = E - E_-$ and apply the energy flux formula of full energy.*
- *We may apply smooth approximation techniques and use the following fact: If the radial solution $u(x,t)$ is sufficiently smooth near the origin, then a simple calculation shows that $\xi(t) = u(0,t)$. In fact, we have*

$$\lim_{r\to 0^+} |w_r(r,t)+w_t(r,t)|^2 = \lim_{r\to 0^+} |w_r(r,t)-w_t(r,t)|^2 = |u(0,t)|^2 \qquad (11)$$

  This also explains why we prefer $|\xi(t)|^2$ to a single L^1 function in the energy flux formula.

3.4 Energy Flux Formula for Triangle

We can apply our general energy flux formula on a triangle region. This will be frequently used in Section 4.

Proposition 3.8 (Triangle Law)**.** *Given any $s_0 > t_0$, we can define $\Omega = \{(r,t) : t > t_0, r > 0, r+t < s_0\}$ and write*

$$E_-(t_0; 0, s_0 - t_0) = \pi \int_{t_0}^{s_0} |\xi(t)|^2 dt + Q_-^-(s_0; t_0, s_0) + \frac{2\pi(p-1)}{p+1} \iint_\Omega \frac{|w(r,t)|^{p+1}}{r^p} drdt.$$

We can substitute s_0 by $t_0 + r_0$ with $r_0 > 0$ and rewrite this in the form

$$E_-(t_0; 0, r_0) = \pi \int_{t_0}^{t_0+r_0} |\xi(t)|^2 dt + Q_-^-(t_0+r_0; t_0, t_0+r_0) + \frac{2\pi(p-1)}{p+1} \iint_\Omega \frac{|w(r,t)|^{p+1}}{r^p} drdt.$$

4 Energy Distribution of Solutions

4.1 Limits of Inward and Outward Energies

All inward and outward energies are clearly bounded above by the full energy E, since $E = E_- + E_+$. We also know that all the energy fluxes Q's are smaller or equal to E, according to Remark 3.3. Let us give an L^2 bound of ξ:

Lemma 4.1 (L^2 bound of ξ)**.** *Let u be a radial solution to (CP1) with an energy E. Then the function $\xi(t)$ in the energy flux formula satisfies*

$$\pi \int_{-\infty}^{+\infty} |\xi(t)|^2 dt \leq E.$$

Proof. We apply triangle law (Proposition 3.8) on the region $\Omega(s, t_0) = \{(r, t) : r + t \leq s, r > 0, t > t_0\}$ for any $t_0 < s$ and obtain

$$E_-(t_0; 0, s - t_0) = \pi \int_{t_0}^{s} |\xi(t)|^2 dt + Q_-^-(s; t_0, s) + \frac{2(p-1)\pi}{p+1} \int_{\Omega(s,t_0)} \frac{|w(r,t)|^{p+1}}{r^p} drdt,$$

as shown in figure 3. Making $t_0 \to -\infty$ we have

$$\pi \int_{-\infty}^{s} |\xi(t)|^2 dt + Q_-^-(s) + \frac{2(p-1)\pi}{p+1} \iint_{r>0, r+t\leq s} \frac{|w(r,t)|^{p+1}}{r^p} drdt \leq E.$$

Finally we let $s \to +\infty$ in the inequality above to obtain $\pi \int_{-\infty}^{\infty} |\xi(t)|^2 dt \leq E$. ☐

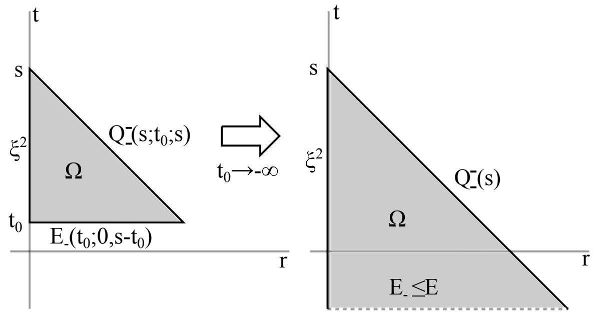

Figure 3: Illustration for proof of Lemma 4.1

Before we consider the monotonicity and asymptotic behaviour of inward and outward energies as $t \to \pm\infty$, let us first give a technical lemma.

Lemma 4.2. *Given any $t_0 \in \mathbb{R}$, we have*

$$\liminf_{r\to+\infty} Q_-^-(t_0 + r, t_0, t_0 + r) = 0.$$

Proof. Fix $r_0 > 0$. We consider the following integral and apply the change of variable $\bar{r} = t_0 + r - t$

$$\begin{aligned}\int_{r_0}^{2r_0} Q_-^-(t_0 + r, t_0, t_0 + r) dr &= \frac{4\pi}{p+1} \int_{r_0}^{2r_0} \int_{t_0}^{t_0+r} \frac{|w(t_0 + r - t, t)|^{p+1}}{(t_0 + r - t)^{p-1}} dtdr \\ &= \frac{4\pi}{p+1} \iint_{\Omega(r_0)} \frac{|w(\bar{r}, t)|^{p+1}}{\bar{r}^{p-1}} d\bar{r}dt \\ &\leq \frac{8\pi r_0}{p+1} \iint_{\Omega(r_0)} \frac{|w(\bar{r}, t)|^{p+1}}{\bar{r}^p} d\bar{r}dt\end{aligned}$$

Here $\Omega(r_0) = \{(\bar{r}, t) : \bar{r} > 0, t > t_0, t_0 + r_0 < t + \bar{r} < t_0 + 2r_0\} \subset [0, 2r_0] \times \mathbb{R}$, as shown in figure 4. Now we are able to apply the mean value theorem to conclude there exists a number $r \in [r_0, 2r_0]$ so that

$$Q_-^-(t_0 + r, t_0, t_0 + r) \leq \frac{8\pi}{p+1} \iint_{\Omega(r_0)} \frac{|w(\bar{r}, t)|^{p+1}}{\bar{r}^p} d\bar{r}dt.$$

If we make $r_0 \to +\infty$ in the argument above and observe the fact

$$\lim_{r_0\to+\infty} \iint_{\Omega(r_0)} \frac{|w(\bar{r}, t)|^{p+1}}{\bar{r}^p} d\bar{r}dt = 0,$$

we obtain the lower limit as desired. ☐

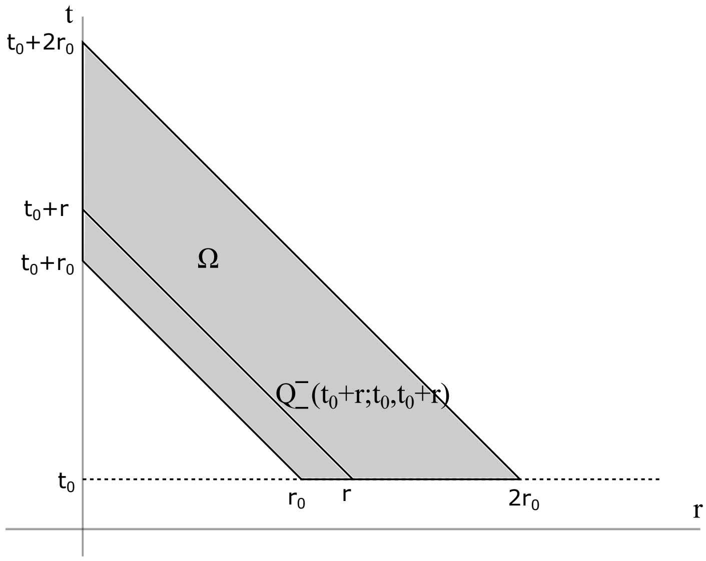

Figure 4: Illustration of integral region

Now we have

Proposition 4.3 (Monotonicity of Inward and Outward Energies)**.** *The inward energy $E_-(t)$ is a decreasing function of t, while the outward energy $E_+(t)$ is an increasing function of t. In addition*

$$\lim_{t\to+\infty} E_-(t) = 0; \qquad\qquad \lim_{t\to-\infty} E_+(t) = 0.$$

Proof. Let us prove the monotonicity and limit of inward energy. The outward energy can be dealt with in the same way. First of all, we apply inward energy flux formula on the region $\Omega = \{(r,t) : r > 0, t_1 < t < t_2, r + t < s\}$ for $s \geq t_2 > t_1$, as shown in the upper half of figure 5.

$$\begin{aligned} &E_-(t_2; 0, s - t_2) - E_-(t_1; 0, s - t_1)\\ &= -\pi \int_{t_1}^{t_2} |\xi(t)|^2 dt - Q_-^-(s; t_1, t_2) - \frac{2\pi(p-1)}{p+1} \iint_\Omega \frac{|w(r,t)|^{p+1}}{r^p} drdt. \end{aligned} \tag{12}$$

Now let us recall the inequality $|w(r,t)| \lesssim_{p,E} r^{(p-1)/(p+3)}$ given by Lemma 2.2. This implies $Q_-^-(s; t_1, t_2) \to 0$ as $s \to \infty$. Therefore we can make $s \to \infty$ in the identity above and obtain

$$E_-(t_2) - E_-(t_1) = -\pi \int_{t_1}^{t_2} |\xi(t)|^2 dt - \frac{2\pi(p-1)}{p+1} \int_{t_1}^{t_2} \int_0^\infty \frac{|w(r,t)|^{p+1}}{r^p} drdt < 0.$$

This gives the monotonicity. Next we apply triangle law with $t_0 \in \mathbb{R}$ and $r_0 > 0$, as shown in the lower part of figure 5

$$\begin{aligned} E_-(t_0; 0, r_0) = \pi \int_{t_0}^{t_0+r_0} |\xi(t)|^2 dt + Q_-^-(t_0 + r_0; t_0, t_0 + r_0)&\\ + \frac{2\pi(p-1)}{p+1} \int_{t_0}^{t_0+r_0} \int_0^{t_0+r_0-t} \frac{|w(r,t)|^{p+1}}{r^p} drdt&. \end{aligned}$$

According to Lemma 4.2, we can take a limit $r_0 \to +\infty$ and write

$$E_-(t_0) = \pi \int_{t_0}^\infty |\xi(t)|^2 dt + \frac{2\pi(p-1)}{p+1} \int_{t_0}^{+\infty} \int_0^{+\infty} \frac{|w(r,t)|^{p+1}}{r^p} drdt. \tag{13}$$

Finally we can make $t_0 \to +\infty$ and finish the proof. $\square$

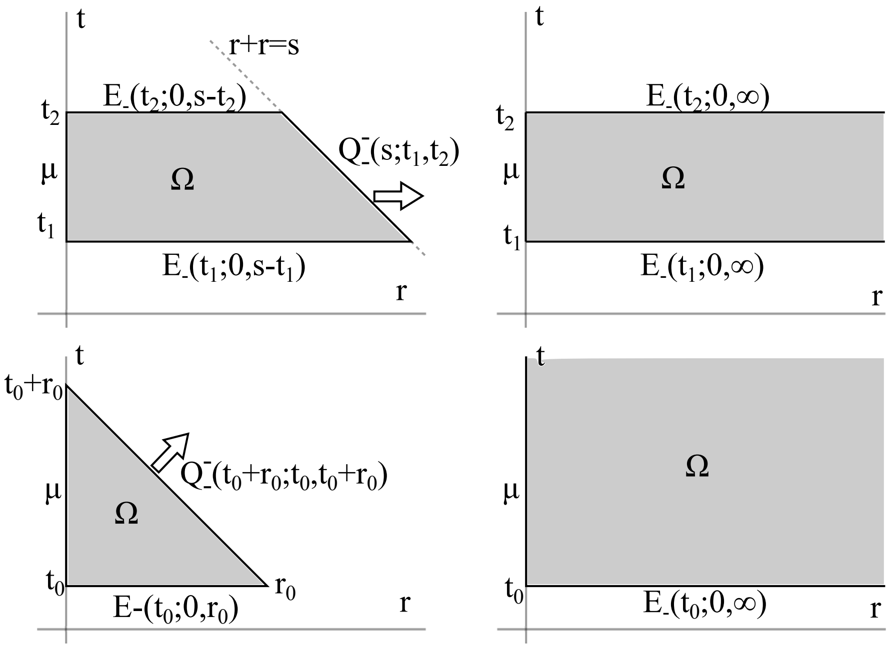

Figure 5: Illustration for proof of Proposition 4.3

Proposition 4.3 immediately gives

Corollary 4.4. *We have the limits*

$$\lim_{t\to-\infty} E_-(t) = E; \qquad \lim_{t\to+\infty} E_+(t) = E.$$

$$\lim_{t\to\pm\infty}\int_0^\infty \frac{|w(r,t)|^{p+1}}{r^{p-1}}dr = 0 \qquad \Leftrightarrow \qquad \lim_{t\to\pm\infty}\int_{\mathbb{R}^3}|u(x,t)|^{p+1}dx = 0.$$

Remark 4.5. *Making $t\to-\infty$ in identity* (13) *we have*

$$\pi\int_{-\infty}^{\infty}|\xi(t)|^2dt + \frac{2\pi(p-1)}{p+1}\int_{-\infty}^{+\infty}\int_0^{+\infty}\frac{|w(r,t)|^{p+1}}{r^p}drdt = E.$$

This means that all the energy eventually changes from inward energy to outward energy by either passing through the origin or nonlinear effect when t moves from $-\infty$ to $+\infty$.

We also have asymptotic behaviour of energy flux

Proposition 4.6. *We have the limits*

$$\lim_{s\to+\infty} Q_-^-(s) = 0; \qquad \lim_{\tau\to-\infty} Q_+^+(\tau) = 0.$$

Proof. Again we only give the proof for inward energy flux. An application of inward energy flux on the parallelogram $\Omega = \{(r,t) : t_0 < t < s, s < r+t < s'\}$ with $t_0 < s < s'$ gives (Please see figure 6)

$$\begin{aligned} &E_-(s;0,s'-s) - E_-(t_0;s-t_0,s'-t_0)\\ &\quad = Q_-^-(s;t_0,s) - Q_-^-(s';t_0,s) - \frac{2\pi(p-1)}{p+1}\int_{t_0}^{s}\int_{s-t}^{s'-t}\frac{|w(r,t)|^{p+1}}{r^p}drdt.\end{aligned}$$

Making $s' \to \infty$ we can discard the term $Q_-^-(s'; t_0, s)$ as in the proof of Proposition 4.3 and rewrite the identity above into

$$E_-(s) + \frac{2\pi(p-1)}{p+1} \int_{t_0}^{s} \int_{s-t}^{\infty} \frac{|w(r,t)|^{p+1}}{r^p} dr dt = E_-(t_0; s - t_0, \infty) + Q_-^-(s; t_0, s).$$

Next we can take a limit as $t_0 \to -\infty$.

$$E_-(s) + \frac{2\pi(p-1)}{p+1} \int_{-\infty}^{s} \int_{s-t}^{\infty} \frac{|w(r,t)|^{p+1}}{r^p} dr dt = \lim_{t \to -\infty} E_-(t; s-t, \infty) + Q_-^-(s).$$

Finally we observe that both terms in left hand converges to zero as $s \to +\infty$ and finish the proof. □

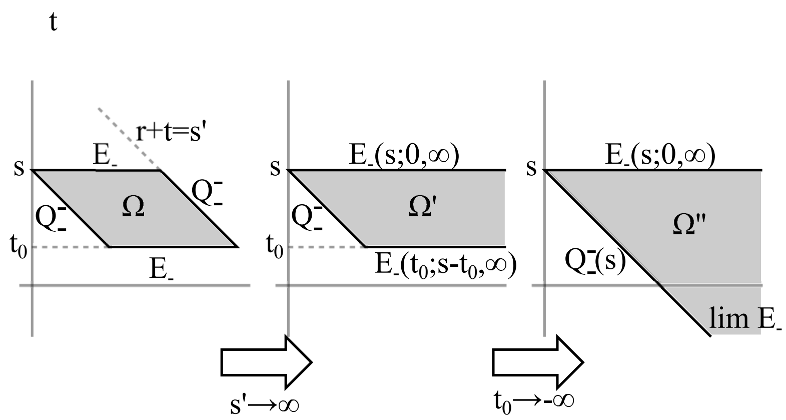

Figure 6: Illustration for proof of Proposition 4.6

4.2 Asymptotic Behaviour of $w_r \pm w_t$

Let us recall that we can write the equation of w in the form of

$$(\partial_t - \partial_r)(\partial_t + \partial_r) w = -\frac{|w|^{p-1} w}{r^{p-1}}.$$

This gives the variance of $w_r \pm w_t$ along characteristic lines $t \pm r = \text{Const}$.

The limit of $w_r \pm w_t$ We combine Proposition 3.4 and Lemma 3.6 to obtain ($\tau < t_1 < t_2 < s$)

$$\begin{aligned} |(w_r + w_t)(s - t_2, t_2) - (w_r + w_t)(s - t_1, t_1)| &\lesssim_p (s - t_2)^{-\frac{p-2}{p+1}} \left(Q_-^-(s; t_1, t_2)\right)^{\frac{p}{p+1}} \\ &\lesssim_{p,E} (s - t_2)^{-\frac{p-2}{p+1}}; \\ |(w_r - w_t)(t_2 - \tau, t_2) - (w_r - w_t)(t_1 - \tau, t_1)| &\lesssim_p (t_1 - \tau)^{-\frac{p-2}{p+1}} \left(Q_+^+(\tau; t_1, t_2)\right)^{\frac{p}{p+1}} \\ &\lesssim_{p,E} (t_1 - \tau)^{-\frac{p-2}{p+1}}. \end{aligned}$$

Here we use the uniform upper bounds $Q_-^-, Q_+^+ \leq E$. This immediately gives the existence of the following pointwise limits as $t \to \pm\infty$

$$\lim_{t \to -\infty} (w_r + w_t)(s - t, t) = g_-(s); \qquad \lim_{t \to +\infty} (w_r - w_t)(t - \tau, t) = g_+(\tau).$$

By Fatou's Lemma, we have $\|g_-\|_{L^2(\mathbb{R})}^2, \|g_+\|_{L^2(\mathbb{R})}^2 \leq E/\pi$. Let us define

$$\tilde{E}_- = \pi \int_{-\infty}^{\infty} |g_-(s)|^2 ds; \qquad \tilde{E}_+ = \pi \int_{-\infty}^{\infty} |g_+(\tau)|^2 d\tau.$$

Energy flux of full energy gives us

$$E(t; s-t, +\infty) \leq E(0; s, \infty), t < 0, s > 0 \quad \Rightarrow \quad \lim_{s \to +\infty} \sup_{t<0} \int_{s-t}^{\infty} |w_r(r,t) + w_t(r,t)|^2 dt = 0;$$
$$E(t; t-\tau, +\infty) \leq E(0; -\tau, \infty), t > 0, \tau < 0 \quad \Rightarrow \quad \lim_{\tau \to -\infty} \sup_{t>0} \int_{t-\tau}^{\infty} |w_r(r,t) - w_t(r,t)|^2 dt = 0.$$

As a result, we have

Proposition 4.7. *There exist functions $g_-(s), g_+(\tau)$ with $\|g_-\|_{L^2(\mathbb{R})}^2, \|g_+\|_{L^2(\mathbb{R})}^2 \leq E/\pi$. so that*

$$|(w_r + w_t)(s-t,t) - g_-(s)| \lesssim_{p,E} (s-t)^{-\frac{p-2}{p+1}}, \qquad t < s;$$
$$|(w_r - w_t)(t-\tau,t) - g_+(\tau)| \lesssim_{p,E} (t-\tau)^{-\frac{p-2}{p+1}}, \qquad t > \tau.$$

Furthermore we have the following L^2 convergence for any $s_0, \tau_0 \in \mathbb{R}$

$$\lim_{t \to -\infty} \|(w_r + w_t)(s-t,t) - g_-(s)\|_{L_s^2([s_0,\infty))} = 0;$$
$$\lim_{t \to +\infty} \|(w_r - w_t)(t-\tau,t) - g_+(\tau)\|_{L_\tau^2((-\infty,\tau_0])} = 0.$$

The scattering target Now we can define

$$V^-(r,t) = \frac{1}{2} \int_{t-r}^{t+r} g_-(s) ds; \qquad V^+(r,t) = \frac{1}{2} \int_{t-r}^{t+r} g_+(\tau) d\tau.$$

One can check $V^{\pm}$ are one-dimensional free wave with $V^{\pm}(0,t) = 0$ and $(V_r^{\pm}, V_t^{\pm}) \in C(\mathbb{R}_t; L^2 \times L^2)$. We can also define

$$v^{\pm}(x,t) = V^{\pm}(|x|,t)/|x|$$

as three-dimensional free waves with energy $\tilde{E}_{\pm}$. By Proposition 4.7 and the fact $\lim_{t \to \pm\infty} E_{\mp}(t) = 0$ (given by Corollary 4.4) we can conduct a simple calculation and conclude for $s_0, \tau_0 \in \mathbb{R}$

$$\lim_{t \to -\infty} \left\| \begin{pmatrix} w_r(\cdot,t) \\ w_t(\cdot,t) \end{pmatrix} - \begin{pmatrix} V_r^-(\cdot,t) \\ V_t^-(\cdot,t) \end{pmatrix} \right\|_{L^2([s_0-t,+\infty))} = 0. \tag{14}$$
$$\lim_{t \to +\infty} \left\| \begin{pmatrix} w_r(\cdot,t) \\ w_t(\cdot,t) \end{pmatrix} - \begin{pmatrix} V_r^+(\cdot,t) \\ V_t^+(\cdot,t) \end{pmatrix} \right\|_{L^2([t-\tau_0,+\infty))} = 0.$$

If we rewrite this in term of the original solution u and $v^{\pm}$ via identity (3), we immediately obtain Part (a) of Theorem 1.3, as long as we can show

$$\lim_{t \to -\infty} (s_0 - t) \left| v^-(s_0 - t, t) - u(s_0 - t, t) \right|^2 = 0;$$
$$\lim_{t \to +\infty} (t - \tau_0) \left| v^+(t - \tau_0, t) - u(t - \tau_0, t) \right|^2 = 0.$$

But these limits are clearly true because

- Proposition 2.2 implies $r|u(r,t)|^2 \lesssim_{p,E} r(r^{-4/(p+3)})^2 = r^{(p-5)/(p+3)}$.

- The definition of $v^{\pm}$ implies the uniform convergence $r|v^{\pm}(r,t)|^2 \rightrightarrows 0$ as $r \to +\infty$.

Next we prove Part (b), i.e. an equivalent condition for scattering.

Proposition 4.8. *If $\tilde{E}_- = E$, then the solution u scatters in the negative time direction*

$$\lim_{t\to-\infty} \left\| \begin{pmatrix} u(\cdot,t) \\ u_t(\cdot,t) \end{pmatrix} - \begin{pmatrix} v^-(\cdot,t) \\ v_t^-(\cdot,t) \end{pmatrix} \right\|_{\dot{H}^1\times L^2(\mathbb{R}^3)} = 0.$$

Proof. Given $\varepsilon > 0$, there exists a number $s_0 \in \mathbb{R}$, so that $\pi \int_{-\infty}^{s_0} |g_-(s)|^2 ds < \varepsilon$. According to Proposition 4.7, for sufficiently large negative time $t < t_1$, we have

$$E(t; s_0 - t, \infty) \geq \pi \int_{s_0-t}^{\infty} |(w_r + w_t)(r,t)|^2 \, dr > E - \varepsilon \Rightarrow E(t; 0, s_0 - t) < \varepsilon. \qquad (15)$$

By the definition of V^- and our assumption on s_0 we also have

$$\pi \int_0^{s_0-t} \left|(V_r^- + V_t^-)(r,t)\right|^2 dr = \pi \int_t^{s_0} |g_-(s)|^2 ds < \varepsilon, \qquad \text{for } t < s_0;$$

$$\pi \int_0^{s_0-t} \left|(V_r^- - V_t^-)(r,t)\right|^2 dr = \pi \int_{2t-s_0}^{t} |g_-(s)|^2 ds \to 0, \qquad \text{as } t \to -\infty.$$

Therefore we have $2\pi \int_0^{s_0-t} \left(|V_r^-|^2 + |V_t^-|^2\right) dr < \varepsilon$ for sufficiently small $t < t_2$. Combining this with (15) we have the following inequality for any $t < \min\{t_1, t_2\}$:

$$2\pi \int_0^{s_0-t} \left(|V_r^- - w_r|^2 + |V_t^- - w_t|^2\right) dr < 4\varepsilon.$$

Combining this with scattering property outside a backward light cone (14), we know for sufficiently large negative t

$$2\pi \int_0^{+\infty} \left(|V_r^- - w_r|^2 + |V_t^- - w_t|^2\right) dr < 4\varepsilon.$$

We can rewrite this in terms of u and v^- by (4) and finish the proof. □

4.3 Location of Remaining Energy

Remaining Energy Proposition 4.8 implies that if we did not have the scattering in the negative time direction, then the difference $E - \tilde{E}_-$ would be positive. Let us try to locate the remaining energy $E - \tilde{E}_-$. Proposition 4.7 tells us that the location is inside the ball $B(\mathbf{0}, s_0 - t)$ for any given s_0 when t is negative and sufficiently large. In other words, this part of energy eventually enters and stays in any backward light cone as $t \to -\infty$. It travels slower than the light. The following lemma, however, shows that its out-going speed is close to the light speed.

Lemma 4.9. *Given $c \in (0,1)$, we have the limit*

$$\lim_{t\to\pm\infty} E_{\pm}(t; 0, c|t|) = 0.$$

Proof. Again we only need to give a proof for inward energy. First of all, we can apply the triangle law on the triangle region $\{(r',t') : t' > t, r' > 0, t' + r' < t + r\}$ with $c|t| < r < \frac{c+1}{2}|t|$.

$$E_-(t; 0, r) = \pi \int_t^{t+r} |\xi(t')|^2 dt' + Q_-^-(t + r; t, t + r) + \frac{2\pi(p-1)}{p+1} \int_t^{t+r} \int_0^{t+r-t'} \frac{|w(r',t')|^{p+1}}{r'^p} dr' dt'.$$

By the upper bound of r we have $E_-(t;0,r) \le P(t) + Q_-^-(t+r;t,t+r)$ where

$$P(t) = \pi \int_t^{\frac{1-c}{2}t} |\xi(t')|^2 dt' + \frac{2\pi(p-1)}{p+1} \int_t^{\frac{1-c}{2}t} \int_0^{\frac{1-c}{2}t-t'} \frac{|w(r',t')|^{p+1}}{r'^p} dr'dt'$$

satisfies $\lim_{t\to-\infty} P(t) = 0$. We can integrate the inequality above for $r \in \left(c|t|, \frac{c+1}{2}|t|\right)$ and obtain

$$\begin{aligned} \frac{1-c}{2}|t| \cdot E_-(t;0,c|t|) &\le \int_{c|t|}^{\frac{c+1}{2}|t|} E_-(t;0,r)dr \\ &\le \frac{1-c}{2}|t|P(t) + \int_{c|t|}^{\frac{c+1}{2}|t|} Q_-^-(t+r;t,t+r)dr. \end{aligned} \tag{16}$$

Following the same argument in Lemma 4.2, as shown in figure 7, we have

$$\begin{aligned} \int_{c|t|}^{\frac{c+1}{2}|t|} Q_-^-(t+r;t,t+r)dr &= \frac{4}{p+1} \iint_{\Omega(t)} \frac{|w(r',t')|^{p+1}}{r'^{p-1}} dr'dt' \\ &\le \frac{2(1+c)|t|}{p+1} \iint_{\Omega(t)} \frac{|w(r',t')|^{p+1}}{r'^p} dr'dt'. \end{aligned}$$

Here $\Omega(t) = \left\{(r',t') : t' > t, r' > 0, (1-c)t < t'+r' < \frac{1-c}{2}t\right\} \subset \{(r',t') : r' \le \frac{c+1}{2}|t|\}$. Plugging this upper bound in (16) and dividing both sides by $\frac{1-c}{2}|t|$, we obtain

$$E_-(t;0,c|t|) \le P(t) + \frac{4(1+c)}{(p+1)(1-c)} \iint_{\Omega(t)} \frac{|w(r',t')|^{p+1}}{r'^p} dr'dt'.$$

Now we can take the limit $t \to -\infty$ on both sides and finish the proof. $\square$

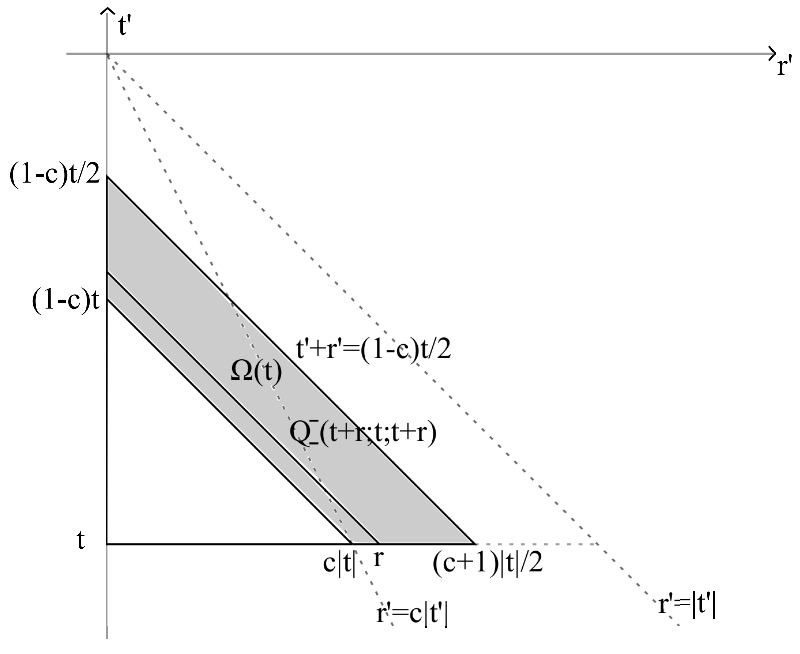

Figure 7: Illustration for proof of Proposition 4.9

This shows that the remaining energy is in the sphere shell $\{x : c|t| < |x| < |t|\}$ as $|t|$ is large. The upper bound of $|x|$ can be further improved, so that we can locate the remaining energy in a region further and further away from the light cone $|x| = |t|$ as t goes to $-\infty$. This is the reason why we give this remaining energy another name "retarded energy".

Lemma 4.10. *Given any $\beta < \frac{2(p-2)}{p+1}$, we have the limit*

$$\lim_{t\to-\infty} E_-(t;|t|-|t|^\beta,+\infty) = \tilde{E}_-.$$

Proof. Our conclusion is a combination of the following two limits. Here we can choose $s_0 = 0$ in identity (18).

$$\lim_{t\to-\infty} E_-(t;|t|-|t|^\beta,|t|) = \pi \int_{-\infty}^0 |g_-(s)|^2 ds; \tag{17}$$

$$\lim_{t\to-\infty} E_-(t;s_0-t,+\infty) = \pi \int_{s_0}^\infty |g_-(s)|^2 ds, \quad s_0 \in \mathbb{R}. \tag{18}$$

We start by proving the first one. Let $I = \left(\int_{-\infty}^0 |g_-(s)|^2 ds\right)^{1/2}$. By Corollary 4.4 we only need to show

$$\lim_{t\to-\infty} \|(w_r+w_t)(r,t)\|_{L_r^2([|t|-|t|^\beta,|t|])} = \lim_{t\to-\infty} \|(w_r+w_t)(s-t,t)\|_{L_s^2([-|t|^\beta,0])} = I.$$

Since we have $\lim_{t\to-\infty} \|g_-\|_{L^2([-|t|^\beta,0])} = I$, it suffices to show

$$\lim_{t\to-\infty} \|(w_r+w_t)(s-t,t) - g_-(s)\|_{L_s^2([-|t|^\beta,0])} = 0.$$

This immediately follows the pointwise estimate given in Proposition 4.7. The same argument as above with the L^2 convergence part of Proposition 4.7 instead proves the second limit (18). □

Retarded energy location Observing the identity

$$E_-(t) = E_-(t;0,c|t|) + E_-(t;c|t|,|t|-|t|^\beta) + E_-(t;|t|-|t|^\beta,\infty),$$

we are able to combine Lemma 4.9, Lemma 4.10, Proposition 4.3 and Corollary 4.4 to prove part (c) of Theorem 1.3, i.e. we have the following limits for any $c \in (0,1)$ and $0 < \beta < \frac{2(p-2)}{p+1}$:

$$\lim_{t\to-\infty} E(t;c|t|,|t|-|t|^\beta) = \lim_{t\to-\infty} E_-(t;c|t|,|t|-|t|^\beta) = E - \tilde{E}_-.$$

Remark 4.11. *A combination of identity* (18) *and Corollary 4.4 gives a limit ($s_0 \in \mathbb{R}$)*

$$\lim_{t\to-\infty} E_-(t;0,s_0-t) = E - \pi \int_{s_0}^\infty |g_-(s)|^2 ds.$$

If we apply the triangle law on the triangle region $\Omega(s,t_0) = \{(r,t) : r > 0, t > t_0, r+t < s\}$, then make $t_0 \to -\infty$ with the limit above in mind and finally consider the limit as $s \to -\infty$, we obtain another expression of the retarded energy

$$E - \tilde{E}_- = \lim_{s\to-\infty} Q_-^-(s).$$

5 Scattering with Additional Decay on Initial Data

In this section we prove Theorem 1.5, i.e. the solution to (CP1) scatters in both time directions if the initial data satisfy additional decay assumptions. Without loss of generality, let us assume $\kappa \in (\kappa_0(p),1)$. Otherwise we can always substitute κ by a slightly smaller $\kappa' \in (\kappa_0(p),1)$. The proof is by a contradiction argument. If the solution failed to scatter in the negative direction, we would have $\tilde{E}_- < E$.

5.1 Additional Contribution by Retarded Energy

According to Part (c) of Theorem 1.3, given any $\beta < \frac{2(p-2)}{p+1}$, there exists a negative time t_1, so that the inequality

$$\int_{|x|<|t|-|t|^\beta} \left(\frac{1}{2}|\nabla u(x,t)|^2 + \frac{1}{2}|u_t(x,t)|^2 + \frac{1}{p+1}|u(x,t)|^{p+1}\right) dx > \frac{E-\tilde{E}_-}{2}.$$

holds for any time $t < t_1$. If R is a large number $R > |t_1|$ and $t \in (-R-R^\beta, -R)$, we have $R < |t| < R + R^\beta \Rightarrow |t|^\beta > R^\beta$. Thus $|t| - |t|^\beta < (R+R^\beta) - R^\beta = R$. This means

$$\int_{|x|<R} \left(\frac{1}{2}|\nabla u(x,t)|^2 + \frac{1}{2}|u_t(x,t)|^2 + \frac{1}{p+1}|u(x,t)|^{p+1}\right) dx > \frac{E-\tilde{E}_-}{2}.$$

As a result we have the following inequality for sufficiently large R:

$$\int_{-R-R^\beta}^{-R}\int_{|x|<R} \left(\frac{1}{2}|\nabla u(x,t)|^2 + \frac{1}{2}|u_t(x,t)|^2 + \frac{1}{p+1}|u(x,t)|^{p+1}\right) dxdt > \frac{E-\tilde{E}_-}{2}\cdot R^\beta. \quad (19)$$

This gives a lower bound of the left hand side of the inequality in Proposition 2.7.

5.2 Upper Bound on Energy Leak

Now let us give an upper bound on the amount of energy escaping the ball $B(0,R) = \{x \in \mathbb{R}^3 : |x| < R\}$ for time $t \in [-R, R]$ under our decay assumption on the energy. In fact we have

Proposition 5.1. *Let u be a solution to (CP1) with a finite energy and satisfy*

$$I = \int_{\mathbb{R}^3} |x|^\kappa \left(\frac{1}{2}|\nabla u_0|^2 + \frac{1}{2}|u_1|^2 + \frac{1}{p+1}|u_0|^{p+1}\right) dx < \infty.$$

Then we have the function

$$I(t) = \int_{|x|>|t|} (|x|-|t|)^\kappa \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}|u_t|^2 + \frac{1}{p+1}|w|^{p+1}\right) dx \le I, \quad t \in \mathbb{R}.$$

Proof. Since the wave equation is time-reversible, it suffices to prove this inequality for $t > 0$. A basic calculation then shows $I'(t) \le 0$ for $t > 0$. Please see [27] for more details. □

Escaping Energy Given any $t \in (-R, R)$, we have

$$\begin{aligned}
&\int_{|x|>R} \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}|u_t|^2 + \frac{1}{p+1}|u|^{p+1}\right) dx \\
&\le (R-|t|)^{-\kappa}\int_{|x|>R} (|x|-|t|)^\kappa \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}|u_t|^2 + \frac{1}{p+1}|w|^{p+1}\right) dx \\
&\le (R-|t|)^{-\kappa} I(t) \le (R-|t|)^{-\kappa} I.
\end{aligned}$$

We integrate t from $-R$ to R and obtain

$$\int_{-R}^{R}\int_{|x|>R} \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}|u_t|^2 + \frac{1}{p+1}|u|^{p+1}\right) dxdt \le \frac{2}{1-\kappa} R^{1-\kappa} I. \quad (20)$$

This is an upper bound of the right hand side of the inequality in Proposition 2.7.

5.3 Completion of the Proof

Plugging both the lower bound (19) and the upper bound (20) in the inequality given by Proposition 2.7, we obtain the following inequality for any given $\beta < \frac{2(p-2)}{p+1}$ and sufficiently large $R > R(u, \beta)$.

$$\frac{E - \tilde{E}_-}{2} \cdot R^\beta \leq \frac{2}{1-\kappa} R^{1-\kappa} I.$$

If $\kappa > \kappa_0(p) = 1 - \frac{2(p-2)}{p+1}$, we can always choose $\beta < \frac{2(p-2)}{p+1}$ so that $\beta > 1 - \kappa$. This immediately gives a contradiction for sufficiently large R's thus finishes the proof.

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