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Constructing Strebel differentials via Belyi maps on the Riemann sphere

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Abstract In this manuscript, by using Belyi maps and dessin d'enfants, we construct some concrete examples of Strebel differentials with four double poles of residues $1, 1, 1, 1$ on the Riemann sphere. We also prove that they have either two double zeroes or four simple zeroes. In particular, we show that they have two double zeroes if and only if their poles are coaxial, under which we find their explicit expressions. On the other hand, for those differentials with four non-coaxial poles and whose metric ribbon graphs have edges of rational lengths, we characterize them optimally in terms of Belyi maps in the sense that the Belyi maps used here have minimal degree, and work out the explicit expressions of the five simplest ones among them. As an application, we could give some explicit cone spherical metrics on the Riemann sphere.

Keywords Strebel differential · Metric ribbon graph · Dessin d'enfant · Belyi map · Cone spherical metric

Mathematics Subject Classification (2010) Primary 30F30 · Secondary 14H57

1 Introduction

Let X be a compact Riemann surface, and let Ω_X denote its cotangent bundle. Then a (meromorphic) *quadratic differential* q is a (meromorphic) global

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section of the line bundle $\Omega_X^{\otimes 2}$. Let $\mathrm{Crit}(q)$ denote the set of zeroes and poles of q. Then q is holomorphic and nowhere vanishing on $X \setminus \mathrm{Crit}(q)$. The restriction of q on $X \setminus \mathrm{Crit}(q)$ defines a conformal flat metric, which is called *q-metric*. With respect to this metric, a curve γ is called a *horizontal geodesic* if $q > 0$ along γ. More precisely, in a coordinate chart $\{U, z\}$ of $X \setminus \mathrm{Crit}(q)$, if q has form $f_U(z)dz^2$, then the corresponding q-metric is $|f_U(z)|dzd\bar{z}$ and the horizontal geodesic $\gamma(t)$ satisfies $f_U(\gamma(t))\gamma'(t)^2 > 0$. A maximal horizontal geodesic is called a *horizontal trajectory*. In general, a horizontal trajectory of q may be a closed curve (*closed*), or bounded by points in $\mathrm{Crit}(q)$ (*critical*), or neither (*recurrent*).

A quadratic differential q with at most double poles is called *Jenkins-Strebel* if the union of all non-closed horizontal trajectories and $\mathrm{Crit}(q)$ is compact and of measure zero, or equivalently, q has no recurrent trajectories. Such differentials are first investigated by Jenkins [4] to solve an extremal problem. Later, K. Strebel proves many astonishing results about these quadratic differentials in his famous book [11]. One of the most important existence theorems is about a special class of Jenkins-Strebel differentials, which are called *Strebel differentials* (see Sect. 2 for more details). Arbarello and Cornalba [1] give a directly proof of the existence and uniqueness of Strebel differentials. However, except these general pure existence theorems, it seems that seldom have people talked about how to construct Strebel differentials, not to mention explicit expressions of such differentials.

In contrast to few explicit constructions of Strebel differentials[12], there are many applications of Strebel differentials in mathematics, such as studying Teichmüller theory and the moduli space of pointed compact Riemann surfaces[6,7]. Furthermore, Kontsevich[5,13] uses the cell decomposition induced by Strebel differentials to prove Witten's conjecture.

A cone spherical metric is a conformal metric on a compact Riemann surface with constant Gaussian curvature +1 and isolated conical singularities. In [10], we have shown that all periods of Strebel differentials are real. By using this fact, we give a canonical construction of cone spherical metrics by Strebel differentials. In more detail, suppose $a_1, a_2, \cdots, a_n$ are the residues of a Strebel differential q at $p_1, p_2, \cdots, p_n$ and $m_1, m_2, \cdots, m_l$ are the multiplicities of the zeroes $z_1, z_2, \cdots, z_l$ of q. Then the corresponding cone spherical metric represents the divisor

$$D = \sum_{i=1}^{n}(a_i - 1)p_i + \sum_{j=1}^{l}\frac{m_j}{2}z_j,$$

which is equivalent to that the metric has cone angle $2\pi a_i$ at p_i and cone angle $\pi(m_j+2)$ at z_j, respectively. Hence in order to obtain some explicit cone spherical metrics, we only need to construct some concrete Strebel differentials.

Note that if q is a Strebel differential on $\mathbb{P}^1$ and $f\colon X \to \mathbb{P}^1$ is a branched covering such that the critical values of f are in $\{\text{critical trajectories of } q\} \cup \mathrm{Crit}(q)$, then $f^*(q)$ is also a Strebel differential on X. Therefore, it is significant to obtain some explicit examples of Strebel differentials on $\mathbb{P}^1$. On the other hand, Mulase and Penkava give a construction of a Riemann surface X and a Strebel differential by a metric ribbon graph in [8]. In particular, if the metric ribbon graph has rational ratios of the lengths, then the corresponding Strebel differential is a pullback by some Belyi map $f: X \to \mathbb{P}^1$ of the differential

$$q_0 = -\frac{1}{4\pi^2}\frac{dz^2}{z(z-1)^2}.$$

However, even if $X = \mathbb{P}^1$, we could *not* obtain the expressions of Strebel differentials by following their process. The purpose of this manuscript is to present an improvement of that result on $\mathbb{P}^1$ for a special case. That is, we will give the explicit expressions of Belyi maps and show that the Belyi maps have minimal degrees.

We focus in this manuscript on the construction of Strebel differentials with 4 double poles and residue vector $(1,1,1,1)$ on $\mathbb{P}^1$. Let q be a meromorphic quadratic differential on the Riemann sphere with 4 double poles at $0, 1, \lambda, \infty$ and residue vector $(1,1,1,1)$. Then we can express q as

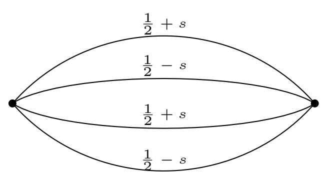
$$q = q_{\lambda,\mu} = -\frac{dz^2}{4\pi^2}\left(\frac{1}{z^2} + \frac{1}{(z-1)^2} + \frac{1}{(z-\lambda)^2} + \frac{\mu - 2z}{z(z-1)(z-\lambda)}\right), \quad (1)$$

where $\mu \in \mathbb{C}$ is a free complex parameter.

Theorem 1 *Suppose that q is a Strebel differential with form (1) on $\mathbb{P}^1$. Then q has either two double zeroes or four simple zeroes. Moreover, the Strebel differential q has two double zeroes if and only if $\lambda \in \mathbb{R}\backslash\{0,1\}$ and*

$$\mu = \mu(\lambda) = \begin{cases} 2\lambda + 2, & \lambda < 0; \\ 2 - 2\lambda, & 0 < \lambda < 1; \\ 2\lambda - 2, & \lambda > 1. \end{cases}$$

In this case, the metric ribbon graph of $q_{\lambda,\mu(\lambda)}$ for any $\lambda \in \mathbb{R}\backslash\{0,1\}$ can be realized by some $\lambda_0 \in [\frac{1}{2}, 1)$ (see Fig. 1).



where $s = \frac{\arcsin(2\lambda_0 - 1)}{\pi}$.

Fig. 1: *The corresponding metric ribbon graphs if $\lambda_0 \in [\frac{1}{2}, 1)$.*

As a consequence, suppose the residues of the Strebel differential q are all equal. Then q has 4 simple zeroes if and only if $\lambda \in \mathbb{C} \setminus \mathbb{R}$, i.e., the double poles of q are non-coaxial. Unfortunately, we have not yet obtained all the explicit expressions of q in this general case. Denote by $q \sim q'$ if two differentials q and q' coincide up to a non-zero complex constant multiple. Then we have

Theorem 2 *Let q be a Strebel differential on $\mathbb{P}^1$ with four simple zeroes and residue vector $(1,1,1,1)$. Then the metric ribbon graph of q coincides with the graph in Figure 2.*

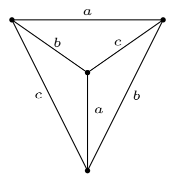


Fig. 2: *Metric ribbon graphs with residues $(1,1,1,1)$ and 4 simple zeroes.*

where $a,b,c>0$ and $a+b+c=1$. Suppose that $a,b,c\in\mathbb{Q}_{>0}$ and $a+b+c=1$. Then there exists a Belyi map f such that $f^*(q_0)\sim q$ and f could be decomposed to be $f=g\circ x^2$ with another Belyi map g. Furthermore, if d is the minimal positive integer such that da, db and dc are all integers, then

$$\min\{\deg f \mid f \textit{ is a Belyi map and } f^*(q_0)\sim q\} = \begin{cases} 2d & \textit{if } 2\mid d,\\ 4d & \textit{if } 2\nmid d,\end{cases}$$

and we obtain the explicit expressions of Belyi maps and Strebel differentials when (a,b,c) equals one of the following five triples:

$$\left(\frac{1}{2},\frac{1}{4},\frac{1}{4}\right),\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right),\left(\frac{1}{3},\frac{1}{6},\frac{1}{2}\right),\left(\frac{1}{3},\frac{1}{2},\frac{1}{6}\right),\left(\frac{2}{3},\frac{1}{6},\frac{1}{6}\right),$$

which exhaust all the possibilities such that the Belyi maps have minimal degrees equal to either 8 or 12.

In [9], Mulase and Penkava conjectured that *if there exists a Strebel differential q on X such that all lengths of critical trajectories of q are algebraic but not rational under q-metric, then the pointed Riemann surface $\big(X,(p_1,p_2,\cdots,p_n)\big)$ could not be defined over $\overline{\mathbb{Q}}$*. The examples of Strebel differentials we construct provide more evidence for this conjecture. As an application on cone spherical metrics, we have

Corollary 1 *The moduli space of cone spherical metrics with four conical singularities of angles $3\pi,3\pi,3\pi,3\pi$ on $\mathbb{P}^1$ has a subspace homeomorphic to the quotient space of the triangle region $\{(a,b,c)\in\mathbb{R}^3 \mid a,b,c>0,\ a+b+c=1\}$ by the group $\mathbb{Z}/3\mathbb{Z}$ generated by the cyclic transformation $(a,b,c)\mapsto(b,c,a)$.*

The organization of this manuscript is as follows. In Section 2, for the convenience of readers, we recall in detail the existence theorem of Strebel differentials and the correspondence between Strebel differentials and metric ribbon graphs. As an application, we give the proof of Corollary 1. The proof of Theorem 1 occupies the whole of Section 3. We prove Theorem 2 in Section 4.

2 Preliminaries

In this section, we will recall some basic results about Strebel differentials, such as the existence theorem/definition given by K. Strebel, Harer's one-to-one correspondence between pointed compact Riemann surfaces with Strebel differentials and metric ribbon graphs. For more details, one can see [8].

Theorem 3 ([11, Theorem 23.5]) *Let X be a compact Riemann surface of genus g with n marked points $p_1, p_2, \cdots, p_n$, and $a_1, \cdots, a_n \in \mathbb{R}_{>0}$. If $2-2g-n<0$, then there exists a unique quadratic differential $q \in H^0\big(X, \Omega_X^{\otimes 2}(2p_1 + 2p_2 + \cdots + 2p_n)\big)$ such that*

1. *p_i is a double pole with residue a_i of q for $i = 1, 2, \cdots, n$.*
2. *The union of all non-closed trajectories is a set of measure zero.*
3. *Every closed trajectory is a circle around some p_i.*

Then the quadratic differential q is called a *Strebel differential*, and $(a_1, \cdots, a_n)$ is called the *residue vector* of q. In [11], K. Strebel also proved that the closure of any recurrent trajectory is a subset of X of positive measure. Hence, the second condition in Theorem 3 is equivalent to say q has no recurrent trajectories. If $\{U, z\}$ is a local coordinate around p_i with $z(p_i) = 0$, then the local expression of q on U is

$$\left(-\frac{a_i^2}{z^2} + \frac{b_i}{z} + h_i(z)\right)\frac{dz^2}{4\pi^2},$$

where $h_i(z)$ is a holomorphic function on U. By the third condition, we know that, for each p_i, the union of all closed trajectories around p_i is an open punctured disc and p_i is the center of the disc.

At a zero of multiplicity m of q, there are $m+2$ half critical trajectories emanating from the zero. Moreover, the *critical graph* constituted by critical trajectories and the zeroes of q is connected. Each edge of the critical graph has a length measured by the q-metric. Hence we obtain a connected metric graph Γ drawn on X, which is called a *metric ribbon graph*. Moreover, the cell decomposition of X induced by Γ has n discs. The number n is called the *number of boundary components* of Γ. Note that, at any vertex v of Γ, the orientation of X induces a cyclic ordering of the half edges incident to v. As an example, a metric ribbon graph on $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ is essentially the same thing as a planar metric graph with natural cyclic ordering at each vertex.

Given a metric ribbon graph Γ, Mulase and Penkava [8, Theorem 5.1] proved that there exists a Riemann surface X and a Strebel differential q on it such that Γ coincides with the critical graph of q. Therefore, there exists a correspondence between Riemann surfaces with Strebel differentials and metric ribbon graphs. Furthermore, Harer [3] proved that this correspondence is actually an orbifold isomorphism

$$\begin{aligned} \mathfrak{M}_{g,n} \times \mathbb{R}_{>0}^n \to \coprod_{\Gamma} \frac{\mathbb{R}_{>0}^{e(\Gamma)}}{\mathrm{Aut}_\partial(\Gamma)}, \\ (X, (p_1, p_2, \cdots, p_n)) \times (a_1, a_2, \cdots, a_n) \mapsto \Gamma. \end{aligned} \tag{2}$$

where $\mathfrak{M}_{g,n}$ is the moduli space of compact Riemann surfaces of genus g with n marked ordered points, Γ runs over all ribbon graphs with degree of each vertex ≥ 3 and with n boundary components, $e(\Gamma)$ is the number of edges of Γ, and $\mathrm{Aut}_\partial(\Gamma)$ is the automorphism group of the ribbon graph Γ. Then by combining the correspondence (2) and Theorem 2, we could give the proof of Corollary 1.

Proof (Proof of Corollary 1) Let S denote the set of all Strebel differentials on $\mathbb{P}^1$ with expressions (1) and 4 simple zeroes. For any $q \in S$, by the construction in [10], we could obtain a cone spherical metric representing the divisor $D = \sum_{j=1}^{4} \frac{1}{2} z_j$, i.e., the metric has singular angles $3\pi, 3\pi, 3\pi, 3\pi$ at z_1, z_2, z_3, z_4. Suppose Strebel differentials $q_1, q_2 \in S$ have the same zero points. Then $q_1 = q_2$ by the expression (1). Hence, we always obtain different spherical metrics from distinct Strebel differentials in S. By the correspondence (2), There exists a bijective correspondence between S and $\{(a,b,c) \in \mathbb{R}^3_{>0} \mid a+b+c = 1\}/\mathrm{Aut}_\partial(\Gamma)$, where Γ is the underlying ribbon graph in Fig. 2. Note that the automorphism group of Γ is $\mathbb{Z}/3\mathbb{Z}$(see [8, Definition 1.8]). Hence, we are done. □

It is well known that compact Riemann surfaces are essentially the same things as nonsingular complex algebraic curves. There is an extraordinarily beautiful theory which was launched by Grothendieck to determine when an algebraic curve is defined over the field $\overline{\mathbb{Q}}$ of algebraic numbers.

Theorem 4 ([2, Theorem 3.1]) *Let X be a compact Riemann surface. Then X is defined over $\overline{\mathbb{Q}}$ if and only if there exists a non-constant holomorphic map $f\colon X \to \mathbb{P}^1$ with at most three critical values.*

By taking a Möbius transformation, we could assume that the critical values of f are contained in the set $\{0, 1, \infty\}$. Such a map f is called a *Belyi map* on X. A *dessin d'enfant* (or *child's drawing*) $\mathcal{D}$ on X is the inverse image of the line segment $[0,1] \subset \mathbb{P}^1$ by a Belyi map $f\colon X \to \mathbb{P}^1$. If we assign black colour to the vertices in $f^{-1}(0)$ and white to those in $f^{-1}(1)$, then $\mathcal{D}$ is a bicoloured graph embedded in X and vertices connected by an edge have different colours. In fact, $\mathcal{D}$ is a connected bicoloured graph and there exists a one-to-one correspondence between Belyi maps and dessins d'enfants on X(see [2, Chapter 4]).

3 Strebel differentials with two double zeroes

In this section, we give all expressions of Strebel differentials whose zero partitions are $4 = 2 + 2$(i.e. 2 double zeroes) and residue vectors $(1, 1, 1, 1)$.

The strategy of our construction is to study the holomorphic map f from $\mathbb{P}^1$ to $\mathbb{P}^1$ of degree 4 such that f^*q_0 has 4 double poles. Firstly, we prove that the zero partition of f^*q_0 can only be $2 + 2$ or $1 + 1 + 1 + 1$(i.e. 4 simple zeroes). Secondly, we give the expression of Strebel differential for $\lambda = \frac{1}{2}$ by writing down f with only 2 critical values. Thirdly, through the research of the branched covering f with 3 critical values and the critical graph of f^*q_0, we work out all expressions of Strebel differentials if $\lambda \in (\frac{1}{2}, 1)$. Then by considering Möbius transformations $z \mapsto 1-z$ and $z \mapsto \frac{1}{z}$, we obtain all expressions of Strebel differentials for $\lambda \in \mathbb{R}\backslash\{0,1\}$. At last, we show that these differentials are all the Strebel differentials with residue vector $(1,1,1,1)$ and 2 double zeroes by investigating the corresponding metric ribbon graphs.

Suppose that q_1 and q_2 have exactly 4 double poles at (p_1, p_2, p_3, p_4) with the same residue vector and no other poles. Then they have the same leading coefficient (i.e. the coefficient of $\frac{dz^2}{z^2}$) at p_i. Hence $q_1 - q_2$ has at most simple poles at $p_i (i = 1,2,3,4)$ and no other poles on $\mathbb{P}^1$. On the other hand, we know that $\dim_{\mathbb{C}} H^0\big(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^{\otimes 2}(p_1 + \cdots + p_4)\big) = \dim_{\mathbb{C}} H^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}}^1\big) = 1$. Therefore the meromorphic quadratic differentials which have exactly 4 double poles at $(p_1, \cdots, p_4) = (0, 1, \lambda, \infty)$ with residue vector $(1,1,1,1)$ have the form of

$$\begin{aligned} q &= -\frac{dz^2}{4\pi^2}\left(\frac{1}{z^2} + \frac{1}{(z-1)^2} + \frac{1}{(z-\lambda)^2} + \frac{\mu - 2z}{z(z-1)(z-\lambda)}\right) \\ &= -\frac{dz^2}{4\pi^2}\frac{z^4 + (\mu - 2(\lambda+1))z^3 + (2(\lambda^2+\lambda+1) - \mu(\lambda+1))z^2 + (\lambda\mu - 2\lambda(\lambda+1))z + \lambda^2}{z^2(z-1)^2(z-\lambda)^2}, \end{aligned}$$

where $\mu \in \mathbb{C}$ is a parameter.

For the zero partition of the quadratic differential q with residue vector $(1,1,1,1)$ on the Riemann sphere, we have the following property:

Lemma 1 *Let q be a quadratic differential on $\mathbb{P}^1$ with 4 double poles and residue vector $(1,1,1,1)$. Then the zero partition of q is either $4 = 2+2$ or $4 = 1+1+1+1$, i.e. q has 2 double zeroes or 4 simple zeroes.*

Proof In order to investigate the multiplicities of zeroes of q, we only need to consider the numerator of the expression of q. The discriminant of the polynomial

$$z^4 + (\mu - 2(\lambda+1))z^3 + (2(\lambda^2+\lambda+1) - \mu(\lambda+1))z^2 + (\lambda\mu - 2\lambda(\lambda+1))z + \lambda^2 \tag{3}$$

is

$$\lambda^2(\lambda-1)^2(\mu - 2 + 2\lambda)^2(\mu - 2 - 2\lambda)^2(\mu + 2 - 2\lambda)^2.$$

Hence, q has a multiple zero if and only if $\mu = 2(1-\lambda)$, $2(1+\lambda)$ or $2(\lambda - 1)$. For these three cases, the corresponding expressions of (3) are $(z^2 - 2\lambda z + \lambda)^2$, $(z^2 - \lambda)^2$ or $(z^2 - 2z + \lambda)^2$ respectively. As a result, q has either 4 simple zeroes or 2 double zeroes. □

Therefore, if q is a Strebel differential on the Riemann sphere with 4 double poles and residue vector $(1,1,1,1)$, then the zero partition of q can only be $4 = 2+2$ or $4 = 1+1+1+1$. For these two partitions, we could determine their ribbon graphs:

Lemma 2 *Suppose q is a Strebel differential with residue vector $(1,1,1,1)$. If the zero partition of q is $4 = 2+2$, then its ribbon graph looks like Fig. 3.*

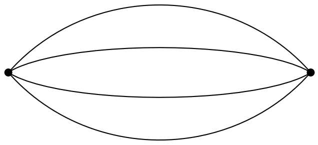


Fig. 3: Ribbon graph for the partition $4 = 2 + 2$.

If the zero partition of q *is* $4 = 1+1+1+1$*, then its ribbon graph is shown in Fig. 4.*

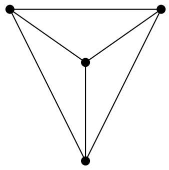


Fig. 4: Ribbon graph for the partition $4 = 1 + 1 + 1 + 1$.

Proof For the first case, let Γ be the metric ribbon graph of q. Suppose there exists a loop l in Γ. Then the Riemann sphere is divided into 2 regions by l. Let D be the one of the 2 regions containing no vertex of Γ. Since the total length of the boundary of each boundary component of Γ is 1, we conclude that there is no other loop in the interior of D, which means that D is a boundary component of Γ and the length of l is 1. Then the length of boundaries of the other boundary component besides D touched by l is greater than 1, contradiction! Hence, Γ is a planar graph with 2 vertices and no loop. The only possible graph is Fig. 3 since the degree of each vertex is 4.

For the second case, we also show that there is no loop in Γ. Otherwise, note that we can assume D contains at most one vertex of Γ. If there is no vertex in D, the argument is the same as the first case. Suppose there is a vertex v in D. Then there is a small loop in D incident to v with the length of 1, contradiction! Then Γ can only be the graph in Fig. 4 and the graph in Fig. 5.

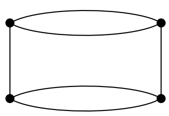


Fig. 5: A fake ribbon graph.

However, for Fig. 5, it can not be a metric ribbon graph such that the corresponding Strebel differential has residue vector $(1, 1, 1, 1)$. □

Now, we give a simple meromorphic quadratic differential which plays the role of a building block in the following construction (see [8], Example 4.4).

Example 1 Consider the following meromorphic quadratic differential on $\mathbb{P}^1$

$$q_0' = \frac{1}{4\pi^2}\frac{dz^2}{z(1-z)}.$$

It has simple poles at 0 and 1, and a double pole at ∞. By solving differential equations, we know that the line segment $[0, 1]$ is a critical horizontal trajectory of length $1/2$. The space $\mathbb{P}^1$ minus $[0, 1]$ and ∞ is covered by a collection of closed horizontal trajectories which are confocal ellipses

$$z = a\cos\theta + 1/2 + \sqrt{-1}\, b\sin\theta,$$

where a and b are positive constants that satisfy $a^2 = b^2 + 1/4$. The length of each closed horizontal trajectory is 1 (denoted by dotted curves in Fig. 6).

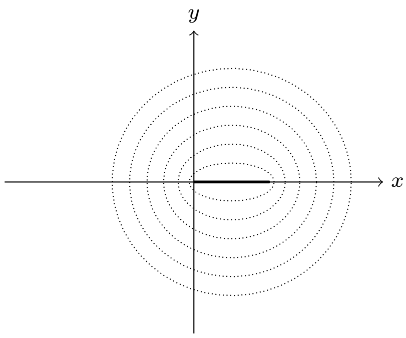


Fig. 6: Horizontal trajectories of q_0'.

Let $\phi(z) = \frac{z}{z-1}$ and $q_0 = \phi^*(q_0') = -\frac{1}{4\pi^2}\frac{dz^2}{z(z-1)^2}$. Then q_0 has simple poles at $0, \infty$, and a double pole at 1 with residue 1. Note that for any compact Riemann surface X and a holomorphic map $f\colon X \to \mathbb{P}^1$, $f^*(q_0)$ has only finite critical trajectories and no recurrent trajectories since f is proper. From now on, we fix the compact Riemann surface $X = \mathbb{P}^1$ and denote by x the coordinate of the domain space $\mathbb{P}^1$ and z that of the target space $\mathbb{P}^1$.

By Theorem 3, we know that, for any given $\lambda \in \mathbb{C}\backslash\{0, 1\}$, the value of μ in (1) is unique if q is Strebel. In order to obtain the value $\mu(\lambda)$ for some λ, let us consider $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ be a branched covering such that $f^*(q_0)$ has 4 double poles with residue vector $(1, 1, 1, 1)$ and no simple poles on $\mathbb{P}^1$, then

– $\deg f = 4$;

– 1 is not a critical value of f;
– the local ramification degrees > 1 over 0 and ∞.

By the Riemann-Hurwitz formula, the total branching order $\nu(f) = 6$. We consider the following two cases:

Case 1 (**The expression of the Strebel differential for $\lambda = \frac{1}{2}$**)
If the local ramification degrees over 0 and ∞ are both 4, we can assume that $f(0) = 0, f(\infty) = \infty$ and f has the form of $cx^4,\ c \in \mathbb{C}^*$. For simplicity, let $f(x) = x^4$, then

$$f^*(q_0) = -\frac{dx^2}{4\pi^2}\frac{16x^2}{(x^2+1)^2(x^2-1)^2},$$

and its critical graph is shown in Fig. 7.

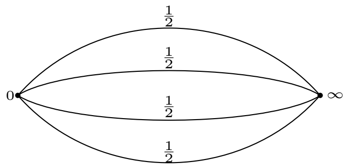


Fig. 7: Metric critical graph of $f^*(q_0)$.

Since $f^*(q_0)$ has only 4 critical trajectories and its critical graph is connected, it is a Strebel differential on $\mathbb{P}^1$. Consider the Möbius transformation $\varphi(x) = \frac{1-(1-i)x}{-1+(1+i)x}$, then

$$\varphi^* f^* q_0 = -\frac{dx^2}{4\pi^2}\frac{x^4 - 2x^3 + 2x^2 - x + 1/4}{x^2(x-1)^2(x-1/2)^2},$$

which is a Strebel differential with four double poles at $(0, 1, 1/2, \infty)$ and residue vector $(1, 1, 1, 1)$.

Case 2 (**The expressions of Strebel differentials for $\lambda \in (\frac{1}{2}, 1)$**)
If the local ramification degrees over 0 and ∞ are $(2, 2)$ and 4 respectively, we can assume $f(x) = \frac{1}{c^2}x^2(x-1)^2$ with $c \in \mathbb{C}^*$. Then the double poles of f^*q_0 are located at

$$\frac{1+\sqrt{1+4c}}{2}, \frac{1-\sqrt{1+4c}}{2}, \frac{1+\sqrt{1-4c}}{2}, \frac{1-\sqrt{1-4c}}{2}.$$

Taking a Möbius transformation

$$\varphi(x) = \frac{x - \frac{1+\sqrt{1+4c}}{2}}{x - \frac{1+\sqrt{1-4c}}{2}} \cdot \frac{\frac{1-\sqrt{1+4c}}{2} - \frac{1+\sqrt{1-4c}}{2}}{\frac{1-\sqrt{1+4c}}{2} - \frac{1+\sqrt{1+4c}}{2}},$$

we have

$$\varphi\left(\frac{1-\sqrt{1-4c}}{2}\right)=\frac{1+\sqrt{1-16c^2}}{2\sqrt{1-16c^2}},$$

i.e. the Möbius transformation φ sends the location of four double poles to $(0,1,\infty,\lambda)$.

$$\lambda=\frac{1+\sqrt{1-16c^2}}{2\sqrt{1-16c^2}},$$
$$c^2=\frac{\lambda(\lambda-1)}{4(2\lambda-1)^2}.$$

A routine computation gives rise to $f'(x)=\frac{2}{c^2}x(x-1)(2x-1)$. Thus the ramification points of f are $0,1,\frac{1}{2},\infty$ and

$$f(0)=f(1)=0,$$
$$f(\infty)=\infty,$$
$$f\left(\frac{1}{2}\right)=\frac{1}{16c^2}.$$

Hence, for any $c^2\neq\frac{1}{16}$, f^*q_0 has 4 double poles with residue vector $(1,1,1,1)$ and 2 double zeroes at $\frac{1}{2},\infty$. If $f(\frac{1}{2})=\frac{1}{16c^2}\in(-\infty,0)$, the horizontal trajectories of f^*q_0 are the graph in Fig. 8. (In the rest of this manuscript, we always denote by dotted curves the closed horizontal trajectories and solid curves the critical horizontal trajectories. The same type dots are mapped to the same point by f.)

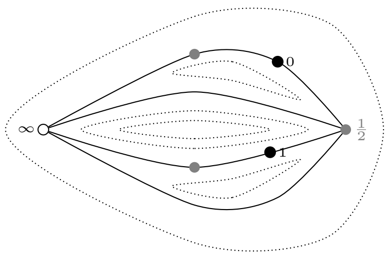


Fig. 8: Horizontal trajectories of Strebel differentials with 2 double zeroes

Therefore, f^*q_0 is a Strebel differential if $c^2<0$. In what follows, we assume that $c^2<0$, i.e. $\lambda\in(\frac{1}{2},1)$, then

$$f^*q_0=-\frac{dx^2}{4\pi^2}\frac{4c^2(2x-1)^2}{(x^2-x-c)^2(x^2-x+c)^2}.$$

The inverse transformation of $\varphi(x)$ is

$$\begin{aligned}\varphi^{-1}(x) &= \frac{2(1+\sqrt{1-4c})x - \left(1+\sqrt{1+4c}+\sqrt{1-4c}+\sqrt{\frac{1-4c}{1+4c}}\right)}{2x-(1+\sqrt{\frac{1-4c}{1+4c}})}\cdot\frac{1}{2} \\ &= \frac{1+\sqrt{1-4c}}{2} - \frac{\frac{4c}{\sqrt{1+4c}}}{2x-\left(1+\sqrt{\frac{1-4c}{1+4c}}\right)}.\end{aligned}$$

By a direct calculation, we have

$$(\varphi^{-1})^* f^* q_0 = -\frac{dx^2}{4\pi^2}\frac{\left(x-\frac{1}{2}(1+\sqrt{\frac{1-4c}{1+4c}})\right)^2\left(x-\frac{1}{2}(1+\sqrt{\frac{1+4c}{1-4c}})\right)^2}{x^2(x-1)^2\left(x-\frac{1+\sqrt{1-16c^2}}{2\sqrt{1-16c^2}}\right)^2}.$$

Therefore

$$\begin{aligned}&\lambda = \frac{1+\sqrt{1-16c^2}}{2\sqrt{1-16c^2}} \in (1/2,1),\\ &\mu(\lambda) = 2-2\lambda.\end{aligned}$$

Proof (Proof of Theorem 1) By Cases 1 and 2, we know all expressions of Strebel differentials when $\lambda \in [\frac{1}{2},1)$. In order to obtain all the expressions of Strebel differentials for $\mathbb{R}\backslash\{0,1\}$, we consider the Möbius transformation $x \mapsto 1-x$, then q becomes to

$$-\frac{dx^2}{4\pi^2}\left(\frac{1}{x^2}+\frac{1}{(x-1)^2}+\frac{1}{(x-(1-\lambda))^2}+\frac{2-\mu(\lambda)-2x}{x(x-1)(x-(1-\lambda))}\right).$$

Hence $\mu(1-\lambda) = 2-\mu(\lambda)$. By considering $x \mapsto 1/x$, we get $\mu(1/\lambda) = \mu(\lambda)/\lambda$. To sum up all results, the quadratic differential

$$q = -\frac{dx^2}{4\pi^2}\left(\frac{1}{x^2}+\frac{1}{(x-1)^2}+\frac{1}{(x-\lambda)^2}+\frac{\mu-2x}{x(x-1)(x-\lambda)}\right)$$

is a Strebel differential if μ and λ satisfy the relation shown by Fig. 9.

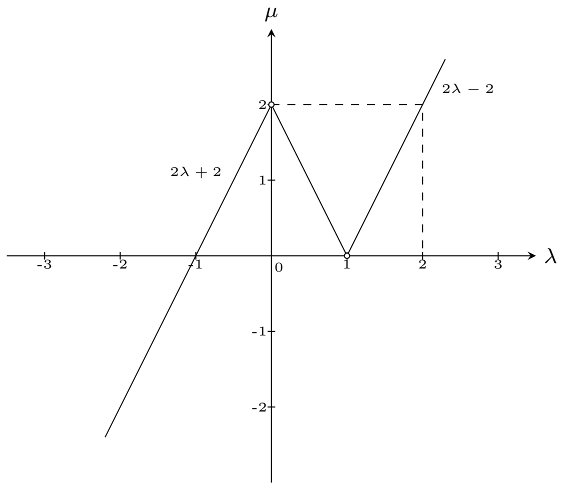


Fig. 9: The relation between μ and λ when q is Strebel.

Now consider $\lambda \in [\frac{1}{2}, 1)$, then $\mu(\lambda) = 2 - 2\lambda$ and the length of the arc from 0 (or 1) to $\frac{1}{2}$ in Fig. 8 is

$$\begin{aligned}\left|\int_{f(0)}^{f(\frac{1}{2})} \sqrt{q_0}\right| &= \left|\int_0^{\frac{1}{16c^2}} \sqrt{q_0}\right| \\ &= \left|\int_0^{\frac{1}{1-16c^2}} \sqrt{q_0'}\right| \\ &= \frac{1}{2\pi}\int_0^{(2\lambda-1)^2} \frac{dz}{\sqrt{z(1-z)}} \\ &= \frac{1}{\pi}\int_0^{\arcsin(2\lambda-1)} d\theta \quad (z = \sin^2\theta) \\ &= \frac{\arcsin(2\lambda-1)}{\pi},\end{aligned}$$

which means that we obtain all the metric ribbon graphs as in Fig. 8. Since the metric ribbon graphs of Strebel differentials with 2 double zeroes and residue vector $(1,1,1,1)$ are exhausted by Fig. 8, we complete our proof by Lemma 2 and Harer's correspondence. □

4 Strebel differentials with four simple zeroes

Let q be a Strebel differential with residue vector $(1,1,1,1)$ and 4 simple zeroes on the Riemann sphere. Then its ribbon graph Γ has 4 boundary components and 4 vertices of degrees 3. The graph Γ can only be Fig. 4 by Lemma 2.

In the rest of this section, we give the proof of the remaining part of Theorem 2. In Proposition 1, we prove that f factors through another Belyi map of degree $\frac{1}{2}\deg f$, and then show that f has minimal degree in Proposition 2. We also give 5 explicit examples by our construction at the end of this section.

Proposition 1 *Let $f(x)\colon \mathbb{P}^1 \to \mathbb{P}^1$ be a Belyi map satisfying*

- f^*q_0 *is a Strebel differential;*
- f^*q_0 *has exactly 4 simple zeroes;*
- f^*q_0 *has exactly 4 double poles with the same residues.*

Then there exists a Belyi map $g(x)$ so that $f = g \circ x^2$.

Proof By the conditions of f^*q_0, we know that the skeleton of the ribbon graph of f^*q_0 is Fig. 4. Since the residues of f^*q_0 are equal to each other, i.e., the local ramification degrees over 1 of f are the same to each other. As a result, $\deg f = 4d$ for some positive integer $d \geq 2$. Note that the ribbon graph of f^*q_0 has exactly 4 vertices of degree 3. Hence there are exactly 4 points of local ramification degree 3 in $f^{-1}(0) \cup f^{-1}(\infty)$, and the other points have local ramification degree 2. Then the branching type over $0, \infty, 1$ of f has only two possible cases for $d \geq 3$ (one case for $d = 2$)

- ♦ $(3^4, 2^{2d-6})$, 2^{2d}, d^4;
- ♦ $(3^2, 2^{2d-3})$, $(3^2, 2^{2d-3})$, d^4.

For the first case[1], the vertices of degree 3 in the dessin (we modify the definition of dessin to the inverse image of segment $[-\infty, 0]$ in this argument) of f have the same colour since they are all contained in $f^{-1}(0)$. We can draw the dessin as in Fig. 10 if the points in the interior of edges are omitted.

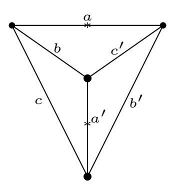


Fig. 10: Dessins for the first case.

In order to guarantee the residues of f^*q_0 are equal to each other, the edges $a(b, c)$ and $a'(b', c',$ respectively) must be with the same coloured points(black

[1] The data means that the local ramification degrees over $0, \infty$ and 1 are $(3,3,3,3,\underbrace{2,\cdots,2}_{2d-6}), (\underbrace{2,\cdots,2}_{2d})$ and (d,d,d,d) respectively. The meaning of the second one is similar.

or white). Assume that the colour of the middle point on the edge a is $*$. Then the dessin of f is a pullback by x^2 of the dessin shown in Fig. 11.

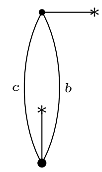


Fig. 11: The initial dessin to pull back.

For the second case, the proof is similar and Example 4 is an explicit construction. □

Proposition 2 *Suppose q is the Strebel differential corresponding to the metric ribbon graph as in Fig. 2. Let*

$$F_q = \{f\colon \mathbb{P}^1 \to \mathbb{P}^1 \mid f^*q_0 = c \cdot q \text{ for some nonzero complex number } c\}.$$

For any given 3 positive rational numbers a, b, c satisfying $a+b+c = 1$, let d be the minimal positive integer so that $da, db, dc \in \mathbb{Z}$. Then $GCD(da, db, dc) = 1$ and

- *if $2 \mid d$, then $\min_{f\in F_q} \deg f = 2d$;*
- *if $2 \nmid d$, then $\min_{f\in F_q} \deg f = 4d$.*

Proof If $\mathrm{GCD}(da, db, dc) = k > 1$, then $k|(da + db + dc) = d$. Let $d' = \frac{d}{k}$, we have $d'a, d'b, d'c \in \mathbb{Z}$. Contradiction!

For the first case, there must be two odd numbers in (da, db, dc) since $\mathrm{GCD}(da, db, dc) = 1$ and $da + db + dc = d(even)$. Without loss of generality, we assume db and dc are odd and da is even. We can draw a dessin as in Fig. 12.

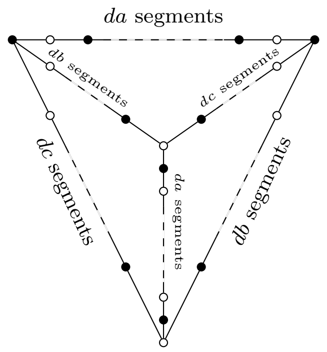


Fig. 12: The minimal dessin if d is even.

The degree of the corresponding Belyi map f is $2d$ and $f \in F_q$. Suppose there exists $g \in F_q$ such that $\deg g = 2d' < 2d$, then each edge of dessin associated to g has $d'a(d'b$ or $d'c)$ segments i.e. $d'a, d'b, d'c \in \mathbb{Z}$. Which has a contradiction with the minimality of d.

For the second one, the possible parity of (da, db, dc) is (even, even, odd) or (odd, odd, odd). Similarly, consider the dessin d'enfant as in Fig. 13.

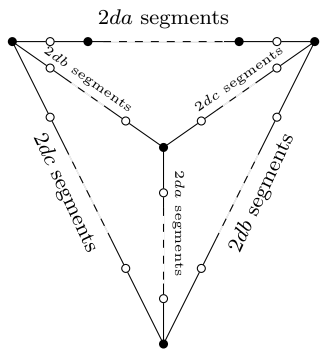


Fig. 13: The minimal dessin if d is odd.

Then the Belyi map f associated to this dessin has degree $4d$ and $f \in F_q$. Since there does not exist bicolour triangle such that the parity of the number of segments on 3 edges is (even, even, odd) or (odd, odd, odd), $\deg f$ is minimal. □

At the very end of this section, we give some examples by our own method.

Example 2 Consider the dessin d'enfant in Fig. 14 (which can be viewed as a metric ribbon graph with $(a, b, c) = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$),

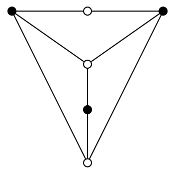


Fig. 14: A dessin corresponds to a Belyi map of degree 8.

The corresponding Belyi map is

$$f(x) = -\frac{1}{2^{12}} \frac{(x-1)^2(9x^2+14x+9)^3}{x^3(x+1)^2},$$

and the local ramification degrees over $0, 1, \infty$ are $(2,3,3), (2,2,2,2)$ and $(2,3,3)$ respectively. The points in $f^{-1}(1)$ satisfy the following equation

$$27x^4 + 36x^3 + 2x^2 + 36x + 27 = 0,$$

whose roots are

$$x_1 = -\frac{1}{3}\left(\frac{4}{\sqrt{3}} + 1 + 2\sqrt{\frac{2}{3}(\sqrt{3}-1)}\right), \quad x_2 = -\frac{1}{3}\left(\frac{4}{\sqrt{3}} + 1 - 2\sqrt{\frac{2}{3}(\sqrt{3}-1)}\right),$$
$$x_3 = \frac{1}{3}\left(\frac{4}{\sqrt{3}} - 1 + 2i\sqrt{\frac{2}{3}(\sqrt{3}+1)}\right), \quad x_4 = \frac{1}{3}\left(\frac{4}{\sqrt{3}} - 1 - 2i\sqrt{\frac{2}{3}(\sqrt{3}+1)}\right).$$

The pullback of q_0 by f is

$$f^*q_0 = +\frac{dx^2}{4\pi^2} \frac{4096x(9x^2+14x+9)}{(27x^4+36x^3+2x^2+36x+27)^2}.$$

Then

$$q_1 = +\frac{dx^2}{4\pi^2} \frac{1024x(9x^2+14x+9)}{(27x^4+36x^3+2x^2+36x+27)^2}$$

is a Strebel differential with 4 simple zeroes and 4 double poles with residue vector $(1,1,1,1)$. Consider the Möbius transformation $x \mapsto \frac{x-x_2}{x-x_3} \cdot \frac{x_1-x_3}{x_1-x_2}$, then q_1 becomes to

$$-\frac{dx^2}{4\pi^2} \cdot \frac{x^4 - (2+\sqrt{2}i)x^3 - (\frac{1}{2} - \frac{3\sqrt{2}}{2}i)x^2 + (\frac{3}{2} - 3\sqrt{2}i)x - \frac{23}{8} + \frac{5\sqrt{2}}{4}i}{x^2(x-1)^2(x-(\frac{1}{2}+\frac{5\sqrt{2}}{4}i))^2}.$$

Hence, $\lambda = \frac{1}{2} + \frac{5\sqrt{2}}{4}i$ and $\mu(\lambda) = 1 + \frac{3\sqrt{2}}{2}i$.

Example 3 Let us consider a Belyi map $f(x)$ corresponding to the dessin in Fig. 15

$$\frac{f(x)}{1+f(x)} = -\frac{64x^3(x^3-1)^3}{(8x^3+1)^3}.$$

The local ramification degrees of $f(x)$ over $1, 0, \infty$ are $(3,3,3,3), (3,3,3,3)$ and $(2,2,2,2,2,2)$ respectively.

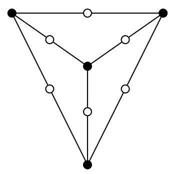


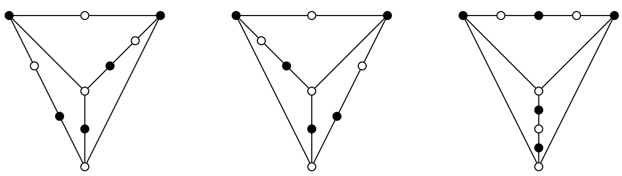
Fig. 15: A dessin corresponds to a Belyi map of degree 12.

Similarly, we could get a Strebel differential

$$-\frac{dx^2}{4\pi^2} \cdot \frac{x^4 - (2 - \frac{2}{\sqrt{3}}i)x^3 + (1-\sqrt{3}i)x^2 + \frac{4i}{\sqrt{3}}x - (\frac{1}{2} + \frac{\sqrt{3}}{2}i)}{x^2(x-1)^2(x-(\frac{1}{2}-\frac{\sqrt{3}}{2}i))^2}.$$

Hence $\mu(\frac{1}{2} - \frac{\sqrt{3}}{2}i) = 1 - \frac{\sqrt{3}}{3}i$. In fact, we have a simpler way to figure out $\mu(\frac{1}{2} - \frac{\sqrt{3}}{2}i)$. Note that $1 - \lambda = 1/\lambda$ if $\lambda = \frac{1}{2} - \frac{\sqrt{3}}{2}i$. Then $2 - \mu(\lambda) = \frac{\mu(\lambda)}{\lambda}$, which implies $\mu(\lambda) = 1 - \frac{\sqrt{3}}{3}i$.

Example 4 If $\deg f = 12$ and $\frac{1}{3^2} \cdot f^* q_0$ has 4 simple zeroes and 4 double poles with residue vector $(1,1,1,1)$, the local ramification degrees over $0, \infty, 1$ can also be $(3^2, 2^3), (3^2, 2^3), 3^4$ respectively. The only possible dessins d'enfants of $\frac{f}{f-1}$ are shown in Fig. 16.



Γ_1 Γ_2 Γ_3

Fig. 16: The possible ribbon graphs of $f^* q_0$.

They can also be viewed as metric ribbon graphs with $(a,b,c) = (\frac{1}{3}, \frac{1}{6}, \frac{1}{2})$, $(\frac{1}{3}, \frac{1}{2}, \frac{1}{6})$, $(\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$ respectively. In order to write down an explicit Belyi map with the above branching type, by Proposition 1, we only need to construct a Belyi map $g(x)$ of degree 6 with branching type $\big((1,2,3),(1,2,3),(3,3)\big)$. Up to a scalar factor, $g(x)$ has the form of

$$g(x) = \frac{x^3(x-a_1)(x-a_2)^2}{(x-a_3)(x-a_4)^2},$$

and $f(x) = g(\frac{a_3x+a_1}{x+1}) \circ x^2 = g \circ \frac{a_3x^2+a_1}{x^2+1}$ up to scale. The derivative of $g(x)$ is

$$g'(x) = \frac{x^2(x-a_2)}{(x-a_3)^2(x-a_4)^3} \cdot h(x),$$

where

$$\begin{aligned} h(x) =& 3x^4 - (2a_1+a_2+4a_3+5a_4)x^3 + (3a_1a_3+2a_2a_3+4a_1a_4+3a_2a_4+6a_3a_4)x^2 \\ &- (a_1a_2a_3+2a_1a_2a_4+5a_1a_3a_4+4a_2a_3a_4)x + 3a_1a_2a_3a_4. \end{aligned}$$

Since $h(x)$ has 2 double zeroes, we may assume that $h(x) = 3(x-1)^2(x-a_5)^2$ and $g(1) = g(a_5)$. By comparing the coefficients of $h(x)$, we find that a_5 satisfies the following equation

$$t^6 + 6t^5 + 15t^4 + 36t^3 + 15t^2 + 6t + 1 = 0,$$

and we have

$$\begin{aligned} a_1 &= -\frac{1}{2}(1+10a_5+35a_5^2+15a_5^3+6a_5^4+a_5^5), \\ a_2 &= \frac{1}{4}(7+16a_5+35a_5^2+15a_5^3+6a_5^4+a_5^5), \\ a_3 &= \frac{1}{16}(21+45a_5+166a_5^2+70a_5^3+29a_5^4+5a_5^5), \\ a_4 &= \frac{1}{20}(3a_5-61a_5^2-25a_5^3-11a_5^4-2a_5^5). \end{aligned}$$

Claim: The polynomial $P(t) = t^6+6t^5+15t^4+36t^3+15t^2+6t+1 \in \mathbb{Q}[t]$ **is irreducible.**

Proof Note that $P(t) = t^6+6t^5+15t^4+36t^3+15t^2+6t+1 = (t+1)^6+16t^3$. Hence

$$\begin{aligned} P(t) &= (t+1)^6 = (t^2+1)(t^4+1) \text{ in } \mathbb{F}_2[t], \\ P(t) &= t^6+1 = (t^2+1)^3 = (t^2+1)(t^4+2t^2+1) \text{ in } \mathbb{F}_3[t]. \end{aligned}$$

Suppose that $P(t)$ is reducible in $\mathbb{Z}[t]$. Since t^2+1 is irreducible in $\mathbb{F}_3[t]$, we have a polynomial factorization $P(t) = P_1(t)P_2(t)$ in $\mathbb{Z}[t]$, where $\deg P_1(t) = 2, \deg P_2(t) = 4$. Hence, we can assume that

$$\begin{aligned} P_1(t) &= t^2 + 6c_0t + 1, \\ P_2(t) &= t^4 + 6d_0t^3 + (2+6d_1)t^2 + 6d_2t + 1. \end{aligned}$$

The coefficients of t and t^5 of $P_1(t)P_2(t)$ are $6(c_0+d_2)$ and $6(c_0+d_0)$ respectively, which imply that $d_0=d_2$ and $c_0+d_0=1$. The coefficient of t^2 is $3+6d_1+36c_0d_0=15$, therefore $c_0d_0=0$ and $d_1=2$. The only two possible factors are

$$\begin{aligned}
&\text{case 1}\\
&\qquad P_1(t)=t^2+6t+1\\
&\qquad P_2(t)=t^4+14t^2+1\\
&\text{case 2}\\
&\qquad P_1(t)=t^2+1\\
&\qquad P_2(t)=t^4+6t^3+14t^2+6t+1
\end{aligned}$$

However, both are impossible to satisfy $P(t)=P_1(t)P_2(t)$. By Gauss lemma, we know that $P(t)$ in irreducible in $\mathbb{Q}[t]$. $\square$

As a consequence of this claim, we know that $a_1, a_2, a_3, a_4, a_5, 0, 1$ are pairwise distinct. In order to obtain the concrete expression of $f(x)$, we only need to solve the equation $P(t)=0$. Luckily, $P(t)$ is solvable by radicals.

$$(t+1)^6=-16t^3 \Longrightarrow (t+1)^2=e^{\frac{i\pi}{3}+\frac{2i\pi}{3}\cdot k}2^{\frac{4}{3}}t(k=0,1,2).$$

We get the roots:

$$\begin{aligned}
&\frac{1}{2}\left(-2+2^{1/3}+2^{1/3}\sqrt{3}i\pm\sqrt{-4+(2-2^{1/3}-2^{1/3}\sqrt{3}i)^2}\right),\\
&-1-2^{1/3}\pm 2^{1/3}\sqrt{1+2^{2/3}},\\
&\frac{1}{2}\left(-2+2^{1/3}-2^{1/3}\sqrt{3}i\pm\sqrt{-4+(2-2^{1/3}+2^{1/3}\sqrt{3}i)^2}\right).
\end{aligned}$$

By the construction we know that if (a_1,a_2,a_3,a_4,a_5) is a solution of $g(x)$, then $(\frac{a_1}{a_5},\frac{a_2}{a_5},\frac{a_3}{a_5},\frac{a_4}{a_5},\frac{1}{a_5})$ is also a solution of $g(x)$ and these two Belyi maps are equivalent under Möbius transformations. Hence, we only need to consider

$$\begin{aligned}
t_0&=\frac{1}{2}\left(-2+2^{1/3}+2^{1/3}\sqrt{3}i-\sqrt{-4+(2-2^{1/3}-2^{1/3}\sqrt{3}i)^2}\right),\\
t_1&=-1-2^{1/3}-2^{1/3}\sqrt{1+2^{2/3}},\\
t_2&=\frac{1}{2}\left(-2+2^{1/3}-2^{1/3}\sqrt{3}i-\sqrt{-4+(2-2^{1/3}+2^{1/3}\sqrt{3}i)^2}\right).
\end{aligned}$$

As before, we can figure out exact values of λ and μ. For example, the expression of λ corresponding to t_1 is

$$\frac{\left(2i+2^{11/12}\sqrt{\frac{5\times 2^{1/6}\sqrt{2+2^{1/3}}(2+2^{2/3})+2\left(11+7\times 2^{1/3}+7\times 2^{2/3}+5\sqrt{2(2+2^{1/3})}\right)}{2^{1/6}(-18+6\times 2^{1/3}+5\times 2^{2/3})+\sqrt{2+2^{1/3}}(-10-2\times 2^{1/3}+9\times 2^{2/3})}}\right)^2}{\left(-2i+2^{11/12}\sqrt{\frac{5\times 2^{1/6}\sqrt{2+2^{1/3}}(2+2^{2/3})+2\left(11+7\times 2^{1/3}+7\times 2^{2/3}+5\sqrt{2(2+2^{1/3})}\right)}{2^{1/6}(-18+6\times 2^{1/3}+5\times 2^{2/3})+\sqrt{2+2^{1/3}}(-10-2\times 2^{1/3}+9\times 2^{2/3})}}\right)^2},$$

which is too complicated. Here we give the approximate values of λ and μ corresponding to each t_k $(k = 0, 1, 2)$:

$$\lambda_0 = 1.3157 - 1.5429i, \qquad \mu_0 = 1.6586 - 1.87049i;$$
$$\lambda_1 = 0.9726 + 0.2324i, \qquad \mu_1 = 0.3689 + 0.04346i;$$
$$\lambda_2 = 1.3157 + 1.5429i, \qquad \mu_2 = 1.6586 + 1.87049i.$$

Now we want to give the correspondence between λ_i and the ribbon graphs in Fig. 16. At first, we note that the branching type of $g(x)$ is $\big((1,2,3),(1,2,3),(3,3)\big)$. Up to colour exchange and isomorphism there are 3 possible dessins of $g(x)$ (see Fig. 17).

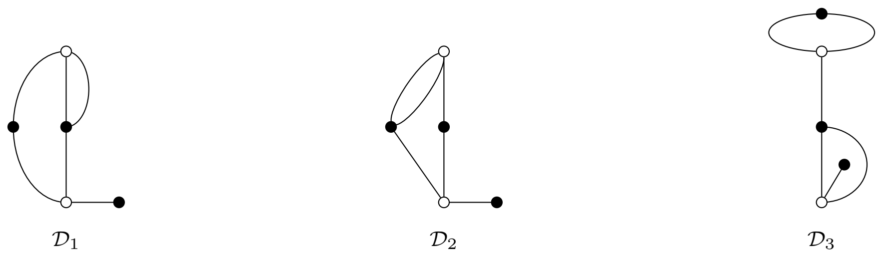


Fig. 17: The possible dessins of $g(x)$.

By definition, the ribbon graph of f^*q_0 is the inverse image of $[-\infty, 0]$(i.e. the negative real axis) by $f(x)$ and the dessin of $g(x)$ is the inverse image of segment $[0, 1]$ by g. In order to obtain the ribbon graphs corresponding to dessins of $g(x)$. We need to construct "dual graph" of these dessins. For example, the gray graph in Fig. 18 is the dual dessin corresponding to $\mathcal{D}_2$.

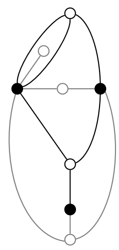


Fig. 18: Construction a new dessin from the old one $\mathcal{D}_2$.

Hence, the ribbon graph corresponding to $\mathcal{D}_2$ can be constructed as in Fig. 19.

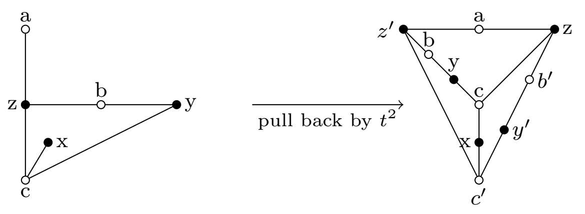


Fig. 19: The ribbon graph associated to dessin $\mathcal{D}_2$.

By a similar procedure, we could construct ribbon graphs corresponding to the other two dessins $\mathcal{D}_1$ and $\mathcal{D}_3$. In summary, we have

$$\mathcal{D}_1 \leftrightarrow \Gamma_1$$
$$\mathcal{D}_2 \leftrightarrow \Gamma_2$$
$$\mathcal{D}_3 \leftrightarrow \Gamma_3.$$

We first observe that the ribbon graph Γ_1 is the image of the ribbon graph Γ_2 by the complex conjugation $z \mapsto \overline{z}$, an orientation reversing homeomorphism. On the other hand, the equation $(t+1)^2 = -2^{\frac{4}{3}}t$ is fixed by complex conjugation. Hence, the corresponding ribbon graph of (λ_1, μ_1) is Γ_3. By directly computing the inverse image of the segment $[0,1]$ by g, we know that the corresponding dessin of (λ_0, μ_0) is $\mathcal{D}_1$. Therefore

$$(\lambda_0, \mu_0) \leftrightarrow \Gamma_1$$
$$(\lambda_1, \mu_1) \leftrightarrow \Gamma_3$$
$$(\lambda_2, \mu_2) \leftrightarrow \Gamma_2.$$

By the above examples and Propositions 1 and 2, we complete the proof of Theorem 2.

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