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Spectral radius and Hamiltonian properties of graphs, II

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Abstract

In this paper, we first present spectral conditions for the existence of C_{n-1} in graphs (2-connected graphs) of order n, which are motivated by a conjecture of Erdős. Then we prove spectral conditions for the existence of Hamilton cycles in balanced bipartite graphs. This result presents a spectral analog of Moon-Moser's theorem on Hamilton cycles in balanced bipartite graphs, and extends a previous theorem due to Li and the second author for n sufficiently large. We conclude this paper with two problems on tight spectral conditions for the existence of long cycles of given lengths.

Keywords: spectral radius; Hamiltonicity; minimum degree; long cycle; balanced bipartite graph

Mathematics Subject Classification (2010): 05C50, 15A18, 05C38

1 Introduction

Throughout this paper, we only consider graphs which are simple, finite and undirected. Let $G = (V, E)$ be a graph of order n and size $e(G)$. Let $S \subset V(G)$. We use $G - S$ to denote the subgraph induced by $V(G)\backslash S$. If S consists of only one element, say $S = \{u\}$, then we use $G - u$ instead of $G - \{u\}$. Let $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ be all the eigenvalues of the adjacency matrix $A(G)$ of G. Denote by $\lambda(G) := \lambda_1(G)$ the *spectral radius* of G, $q(G)$ the signless Laplacian spectral radius of G, and $\delta(G)$ the *minimum degree* of G. Let G_1 and G_2 be two graphs. We use $G_1 + G_2$ to denote the *disjoint union* of G_1 and G_2, and $G_1 \vee G_2$ to denote the *join* of G_1 and G_2. Following some notations in [18], for $1 \leq k \leq (n-1)/2$, we define $L_n^k = K_1 \vee (K_k + K_{n-k-1})$ and $N_n^k = K_k \vee (K_{n-2k} + kK_1)$. (Fig 1 illustrates L_n^3 and N_n^3). Note that $L_n^1 = N_n^1$. A graph G is called *Hamiltonian* if it contains a spanning cycle, and is called *pancyclic* if it contains cycles of lengths from 3 to $v(G)$. The *circumference* of a graph refers to the length of a longest cycle in the graph. For terminology and notations not defined here, we refer the reader to Bondy and Murty [6].

Many graph theorists have investigated the relationship between the existence of Hamilton cycles and paths in graphs and the eigenvalues of some associated matrices of graphs, for example,

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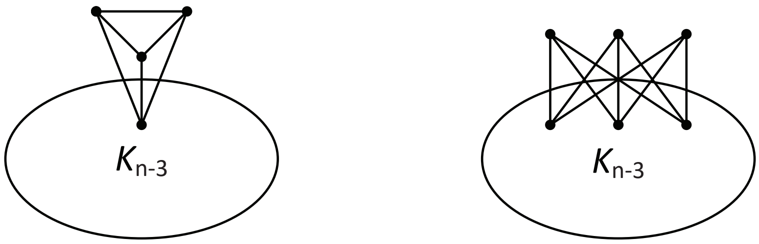

Fig. 1: L_n^3 (left) and N_n^3 (right).

see [7, 12, 29, 22, 18, 21, 23, 19, 20]. Among these results, the following one has received much attention, which is a corollary of a theorem of Ore [25] and of Bondy [5], independently.

Theorem 1.1 (Fiedler and Nikiforov [12]). *Let G be a graph of order n. If $\lambda(G) > n-2$, then G is Hamiltonian unless $G = N_n^1$.*

By introducing the minimum degree of a graph as a new parameter, Li and Ning [18] extended Theorem 1.1 in some sense and obtained spectral analogs of a classical theorem of Erdős [10]. The following theorem is one of the central results in [18]: Let $k \geq 1$ and G be a graph of order $n \geq \max\{6k+5, (k^2+6k+4)/2\}$. If $\delta(G) \geq k$ and $\lambda(G) \geq \lambda(N_n^k)$, then G is Hamiltonian unless $G = N_n^k$.

Since $K_{n-k} \subset N_n^k$ and $K_{n-k} \subset L_n^k$, we have $\lambda(L_n^k) > n-k-1$ and $\lambda(N_n^k) > n-k-1$. Nikiforov [21] further strengthened Li and Ning's theorem for a graph G of order $n \geq k^3 + O(k)$ and $k \geq 2$, by providing a weaker condition that $\lambda(G) \geq n-k-1$. Later, the authors [13] sharpened the result mentioned above for the order of graphs by almost a half.[1]

Since the spectral conditions for C_n are extensively studied, one can naturally consider similar problems for the possible second longest cycle, that is, C_{n-1}. Indeed, our first part is motivated by a conjecture of Erdős that says every graph of order n has a C_{n-1} if its size is at least $\binom{n-2}{2}+4$, which was confirmed by Bondy [2]. The edge number condition above is tight, since one can see the graph $K_1 \vee (K_2 + K_{n-3})$ has $\binom{n-2}{2}+3$ edges but contains no C_{n-1}.

Theorem 1.2. *Let G be a graph of order $n \geq 15$.*
(1) If $\lambda(G) > n-3$, then $C_{n-1} \subseteq G$, unless $G \subseteq K_1 \vee (K_{n-3}+K_2)$ or $G \subseteq \Lambda$ (see Fig 2).
(2) If $q(G) > 2n-6$, then $C_{n-1} \subseteq G$, unless $G \subseteq K_1 \vee (K_{n-3}+K_2)$ or $G \subseteq \Lambda$.

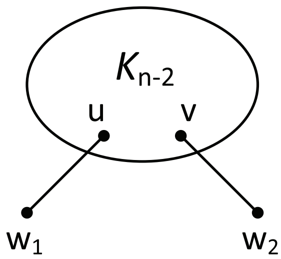

Fig. 2: The graph Λ.

By Lemma 3 in Section 3, Theorem 1.2 implies the following corollary immediately.

[1]For comments on this fact, see Chen and Zhang [8].

Corollary 1. Let G be a graph of order $n \geq 15$.
(1) If $\lambda(G) \geq \lambda(K_1 \vee (K_{n-3} + K_2))$, then $C_{n-1} \subseteq G$ unless $G = K_1 \vee (K_{n-3} + K_2)$.
(2) If $q(G) \geq q(K_1 \vee (K_{n-3} + K_2))$, then $C_{n-1} \subseteq G$ unless $G = K_1 \vee (K_{n-3} + K_2)$.

Considering that the extremal graphs in Corollary 1 contain a cut-vertex, we then consider similar problems among 2-connected graphs. The answers are as follows.

Theorem 1.3. *Let G be a 2-connected graph of order $n \geq 22$.*
(1) If $\lambda(G) > n - 4$, then $C_{n-1} \subseteq G$ unless $G \subseteq K_2 \vee (K_{n-5} + 3K_1)$.
(2) If $q(G) > 2n - 8$, then $C_{n-1} \subseteq G$ unless $G \subseteq K_2 \vee (K_{n-5} + 3K_1)$.

The following corollary follows immediately.

Corollary 2. Let G be a 2-connected graph of order $n \geq 22$.
(1) If $\lambda(G) \geq \lambda(K_2 \vee (K_{n-5} + 3K_1))$, then $C_{n-1} \subseteq G$ unless $G = K_2 \vee (K_{n-5} + 3K_1)$.
(2) If $q(G) \geq q(K_2 \vee (K_{n-5} + 3K_1))$, then $C_{n-1} \subseteq G$ unless $G = K_2 \vee (K_{n-5} + 3K_1)$.

In this paper, we also consider Hamilton cycles in balanced bipartite graphs. Here, a bipartite graph is called *balanced* if its two partite sets X and Y have the same number of vertices. Denote by B_n^k the graph obtained from $K_{n,n}$ by deleting a $K_{k,n-k}$, where $n \geq 2k + 1$. We will improve another theorem of Li and Ning [18] on the spectral condition for Hamilton cycles in balanced bipartite graphs.

Theorem 1.4. *Let $k \geq 1$. Let G be a balanced bipartite graph of order $2n$ with $\delta(G) \geq k$, where $n \geq k^3 + 2k + 4$.*
(1) If G is a proper subgraph of B_n^k, then $\lambda(G) < \sqrt{n(n-k)}$.
(2) If $\lambda(G) \geq \sqrt{n(n-k)}$, then G is Hamiltonian unless $G = B_n^k$.

Since $\lambda(B_n^k) > \sqrt{n(n-k)}$, Theorem 1.4 extends the following theorem in [18] due to Li and the second author (for n sufficiently large).

Theorem 1.5 (Li and Ning [18])**.** *Let $k \geq 1$. Let G be a balanced bipartite graph of order $2n$, where $n \geq (k+1)^2$. If $\delta(G) \geq k$ and $\lambda(G) \geq \lambda(B_n^k)$, then G is Hamiltonian unless $G = B_n^k$.*

We organize this paper as follows. In Section 2, we list necessary preliminaries and prove two structural lemmas. In Section 3, we introduce the Kelmans operation and list several spectral inequalities which will be used in the proof of main theorems. In Section 4, we prove our main results. We conclude this paper in the final section with two problems and some discussions.

2 Structural results

In this section, we list several structural theorems that we rely on.

Theorem 2.1 (Bondy)**.** *(i)* [4] *Let G be a Hamiltonian graph of order n. If $e(G) \geq \frac{n^2}{4} + 1$, then G is pancyclic. (ii)* [3] *Every graph of order $n \geq 4$ and size $e(G) \geq \binom{n-2}{2} + 4$ contains a C_{n-1}.*

The following theorem is a corollary of a theorem of Erdős [10].

Theorem 2.2 (Erdős [10]). *Let G be a 2-connected graph of order $n \geq 13$. If $e(G) \geq \binom{n-2}{2} + 5$, then G is Hamiltonian.*

The next two theorems on Hamiltonian properties of graphs are also useful for us.

Theorem 2.3 (Ore [26]). *Let G be a graph of order n. If $e(G) \geq \binom{n-1}{2}+3$, then G is Hamiltonian-connected.*

Theorem 2.4 (Li and Ning [18]). *Let G be a graph of order $n \geq 6k+5$, where $k \geq 1$. If $\delta(G) \geq k$ and*

$$e(G) > \binom{n-k-1}{2} + (k+1)^2,$$

then G is Hamiltonian unless $G \subseteq L_n^k$ or $G \subseteq N_n^k$.

The following result was proved by Kopylov in [17] (see the last part of [17]), which was originally conjectured by Woodall in [28].

Theorem 2.5 (Kopylov [17]). *Let $n \geq c \geq 5$ and G be a 2-connected graph of order n with circumference less than c. If the minimum degree $\delta(G) \geq k \geq 2$, then*

$$e(G) \leq \max\left\{f(n,k,c), f\left(n, \lfloor \frac{c-1}{2} \rfloor, c\right)\right\},$$

where $f(n,k,c) = \binom{c-k}{2} + k(n-c+k)$.

The following two lemmas, which refine Theorem 2.1(ii), will play the central role in proving Theorems 1.2 and 1.3.

Lemma 1. *Let G be a graph of order $n \geq 15$ and size $e(G) \geq \binom{n-2}{2}$. Then G contains a C_{n-1}, unless $G \subseteq K_1 \vee (K_{n-3} + K_2)$ or $G \subseteq \Lambda$.*

Proof. We prove the lemma by contradiction. Suppose that G contains no C_{n-1} and $G \nsubseteq K_1 \vee (K_{n-3} + K_2)$, and $G \nsubseteq \Lambda$.

Suppose that G is Hamiltonian. Since $e(G) \geq \frac{(n-2)(n-3)}{2} \geq \frac{n^2}{4} + 1$ when $n \geq 10$, by Theorem 2.1(i), G is pancyclic, and thus $C_{n-1} \subseteq G$, a contradiction. Hence G contains no C_n or C_{n-1}. So the circumference of G is less than $n-1$.

Suppose that G is 2-connected. By Theorem 2.5, $e(G) \leq \max\{f(n,2,n-1), f(n, \lfloor \frac{n}{2} \rfloor - 1, n-1)\}$. Since $f(n,2,n-1) = \binom{n-3}{2} + 6$ and $f(n, \lfloor \frac{n}{2} \rfloor - 1, n-1) = \binom{\lceil \frac{n}{2} \rceil}{2} + (\lfloor \frac{n}{2} \rfloor - 1)\lfloor \frac{n}{2} \rfloor$, it is easy to check by WolframAlpha (http://www.wolframalpha.com/) that

$$\max\{f(n,2,n-1), f(n, \lfloor \frac{n}{2} \rfloor - 1, n-1)\} = f(n,2,n-1) = \binom{n-3}{2} + 6$$

when $n \geq 15$. However, we have $e(G) \geq \binom{n-2}{2} > \binom{n-3}{2} + 6$ when $n \geq 10$, a contradiction. So G is not 2-connected.

Suppose that G is disconnected. Let $G = G_1 \cup G_2$, where $G_1 \cap G_2 = \emptyset$ and $v(G_1) \geq v(G_2) \geq 1$. Set $v(G_1) = a \geq \frac{n}{2}$ and $v(G_2) = n - a$. Obviously, $e(G) \leq \binom{a}{2} + \binom{n-a}{2}$, which implies that $\binom{a}{2} + \binom{n-a}{2} \geq \binom{n-2}{2}$. That is, $(a-2)(a+2-n) \geq -1$. Since $a \geq \frac{n}{2} > 7$, we get $a \geq n-2$. Hence $(a,b) = (n-2, 2)$ or $(n-1, 1)$.

If $(a,b)=(n-2,2)$, then $G\subseteq K_{n-2}+K_2\subset K_1\vee(K_{n-3}+K_2)$, a contradiction.

If $(a,b)=(n-1,1)$, then let us consider G_1. Notice that, $e(G_1)=e(G)\geq\binom{n-2}{2}=\binom{v(G_1)-1}{2}>\binom{v(G_1)-2}{2}+4$ when $v(G_1)\geq 11$ $(n\geq 12)$. By Theorem 2.4, G_1 is either Hamiltonian, which contradicts the fact that G contains no C_{n-1}; or $G_1\subseteq K_1\vee(K_{n-3}+K_1)$, which follows $G\subseteq(K_1\vee(K_{n-3}+K_1))+K_1\subset K_1\vee(K_{n-3}+K_2)$, also a contradiction.

Finally, consider the case that G is connected with a cut-vertex, say v. Let $G=G_1\cup G_2$, where $V(G_1)\cap V(G_2)=\{v\}$ and $v(G_1)\geq v(G_2)\geq 2$. Set $v(G_1)=a$ and $v(G_2)=b=n+1-a$, where $\frac{n+1}{2}\leq a\leq n-1$. Thus, we have $\binom{a}{2}+\binom{n+1-a}{2}\geq\binom{n-2}{2}$, which implies $a^2-(n+1)a+3n-3\geq 0$. That is, $(a-3)(a-n+2)\geq -3$. If $a<n-2$, then $(a-3)(a-n+2)\leq 3-a\leq 3-\frac{n+1}{2}<-3$ when $n\geq 12$, a contradiction. Thus, $n-2\leq a\leq n-1$, which implies $(a,b)=(n-2,3)$ or $(n-1,2)$.

If $(a,b)=(n-2,3)$, then $G\subseteq K_1\vee(K_{n-3}+K_2)$, a contradiction.

If $(a,b)=(n-1,2)$, then $e(G_1)=e(G)-1\geq\binom{n-2}{2}-1=\binom{v(G_1)-1}{2}-1$. Suppose that G_1 is 2-connected. Since $e(G_1)\geq\binom{v(G_1)-1}{2}-1\geq\binom{v(G_1)-2}{2}+5$ for $v(G_1)\geq 13$ $(n\geq 14)$, by Theorem 2.2, G_1 is Hamiltonian. Thus, $C_{n-1}\subseteq G_1\subseteq G$, a contradiction. So G_1 contains a cut-vertex. Let $G_1=G_{11}\cup G_{12}$, where $V(G_{11})\cap V(G_{12})=\{w\}$ and $v(G_{11})\geq v(G_{12})\geq 2$. Set $v(G_{11})=s$ and $v(G_{12})=n-s$, where $\frac{n}{2}\leq s\leq n-2$. Thus, we have $\binom{s}{2}+\binom{n-s}{2}\geq\binom{n-2}{2}-1$, which implies $s\geq n-2-\frac{4}{s-2}$. Since $s\geq\frac{n}{2}>7$, we have $s\geq n-2-\frac{4}{5}$. It follows that $s\geq n-2$. So $v(G_{11})=n-2$ and $v(G_{12})=2$. Set $V(G_2)=\{v,x\}$ and $V(G_{12})=\{w,y\}$. If $v=w$, then $G\subseteq K_1\vee(K_{n-3}+2K_1)\subset K_1\vee(K_{n-3}+K_2)$, a contradiction. Thus, $v\neq w$. If $v=y$, then $G\subseteq K_1\vee(K_{n-3}+2K_1)$, a contradiction. So $\{x,v\}\cap\{y,w\}=\emptyset$. In this case, $G\subseteq\Lambda$.

The proof of Lemma 1 is complete. ☐

Lemma 2. *Let G be a 2-connected graph of order $n\geq 22$ and size $e(G)\geq\binom{n-3}{2}-2$. Then G contains a C_{n-1}, unless $G\subseteq K_2\vee(K_{n-5}+3K_1)$.*

Proof. We prove the lemma by contradiction. Suppose G contains no C_{n-1}, and $G\nsubseteq K_2\vee(K_{n-5}+3K_1)$. We shall first prove two claims.

Claim 1. G is not Hamiltonian.

Proof. Suppose that G is Hamiltonian. Notice that we have $e(G)\geq\binom{n-3}{2}-2\geq\frac{n^2}{4}+1$ when $n\geq 14$. By Theorem 2.1(i), G is pancyclic, and thus contains a C_{n-1}, a contradiction. ☐

Claim 2. G contains a 2-cut.

Proof. Suppose that G is 3-connected. Then $\delta(G)\geq 3$. By Theorem 2.5, we have

$$e(G)\leq\max\{f(n,3,n-1),f(n,\lfloor\frac{n}{2}\rfloor-1,n-1)\},$$

where $f(n,k,c)=\binom{c-k}{2}+k(n-c+k)$.

Since $f(n,3,n-1)=\binom{n-4}{2}+12$ and $f(n,\lfloor\frac{n}{2}\rfloor-1,n-1)=\binom{\lceil\frac{n}{2}\rceil}{2}+(\lfloor\frac{n}{2}\rfloor-1)\lfloor\frac{n}{2}\rfloor$, it is easy to check by WolframAlpha (http://www.wolframalpha.com/) that

$$\max\{f(n,3,n-1),f(n,\lfloor\frac{n}{2}\rfloor-1,n-1)\}=f(n,3,n-1)=\binom{n-4}{2}+12$$

when $n\geq 22$. However, we get $e(G)\geq\binom{n-3}{2}-2>\binom{n-4}{2}+12$ when $n\geq 19$, a contradiction.

Now we know G is 2-connected but not 3-connected, so G contains a 2-cut. This proves the claim. $\square$

We choose G_1 and G_2 such that:
(i) $G = G_1 \cup G_2$ with $V(G_1) \cap V(G_2) = \{u, v\}$, where $\{u, v\}$ is a 2-cut;
(ii) $v(G_1) - v(G_2)$ is as large as possible.

In the following, we call $\{G_1, G_2\}$ a *good pair*, if it satisfies both (i) and (ii). Set $v(G_1) = a$ and $v(G_2) = b$. Then $\frac{n+2}{2} \leq a \leq n-1$ and $v(G_2) = n+2-a$. Hence $\binom{a}{2}+\binom{n+2-a}{2}-1 \geq e(G) \geq \binom{n-3}{2}-2$. Therefore, $a^2 - (n+2)a + 5n - 4 \geq 0$, and hence,

$$(a-5)(a-n+3) \geq -11. \tag{1}$$

Since $n \geq 22$, $n-4 \geq \frac{n+2}{2}$. If $a \leq n-4$, then $(a-5)(a-(n-3)) \leq ((n-4)-5)((n-4)-(n-3)) = -(n-9) < -11$ when $n \geq 21$, a contradiction to (1). Thus $a \geq n-3$, which implies that $(a, b) = (n-3, 5)$, or $(a, b) = (n-2, 4)$, or $(a, b) = (n-1, 3)$.

Note that in the case $(a, b) = (n-3, 5)$ or $(a, b) = (n-2, 4)$, there are no vertices of degree two in G, otherwise $\{G_1, G_2\}$ is not a good pair.

Suppose that $(a, b) = (n-3, 5)$. Let $V(G_2)\backslash\{u, v\} = \{w_1, w_2, w_3\}$. We obtain

$$e(G_1) \geq e(G) - e(G_2) \geq \binom{n-3}{2} - 2 - \binom{5}{2} \geq \binom{n-4}{2} + 3 = \binom{v(G_1)-1}{2} + 3$$

when $n \geq 19$. By Theorem 2.3, G_1 is Hamiltonian-connected. Take any (u, v)-Hamilton path in G_1, say P_1. We can see $G_2 - \{u, v\}$ is connected, otherwise there is a vertex of degree 2. If $G_2 - \{u, v\} \cong K_3$, since G is 2-connected, there exists (u, v)-Hamilton path in G_2, say P_2. Then $P_1 \cup P_2$ is a Hamilton cycle in G, a contradiction to Claim 1. If $G_2 - \{u, v\} \cong P_3$, then suppose $G_2 - \{u, v\}$ is the path $w_1w_2w_3$. Since there is no vertex of degree 2 in G, u and v are neighbours of both w_1 and w_3. Then $vP_1uw_1w_2w_3v$ is a Hamilton cycle in G, a contradiction to Claim 1.

Suppose that $(a, b) = (n-2, 4)$. Let $V(G_2)\backslash\{u, v\} = \{w_1, w_2\}$. Since there is no vertex of degree 2 in G, $\{w_1u, w_1v, w_2u, w_2v, w_1w_2\} \subset E(G)$. Let $H := G - w_1$. Then $\delta(H) \geq 2$. We obtain

$$e(H) = e(G) - 3 \geq \binom{n-3}{2} - 5 > \binom{n-4}{2} + 9 = \binom{v(H)-3}{2} + 9$$

for $n \geq 19$. By Theorem 2.4, H is Hamiltonian, unless $H \subseteq L^2_{n-1}$ or $H \subseteq N^2_{n-1}$. If H is Hamiltonian, then $C_{n-1} \subset H \subset G$, a contradiction. Next, we shall show that $H \nsubseteq L^2_{n-1}$. Since $H = G - w_1$ and $N_G(w_1) = \{u, v, w_2\}$, it is easy to prove H is 2-connected. But L^2_{n-1} contains a cut-vertex. Hence $H \nsubseteq L^2_{n-1}$, and it follows $H \subseteq N^2_{n-1}$. Let $\{x_1, x_2\} \subset V(H)$ such that x_1, x_2 are two vertices of degree 2 in N^2_{n-1}. Since there is no vertex of degree 2 in G, x_1 and x_2 must be neighbours of w_1. Since $N_G(w_1) = \{u, v, w_2\}$, $\{x_1, x_2\} \cap \{u, v\} \neq \emptyset$. Hence at least one of u, v has degree 3, say u. Furthermore, the neighbour of u in G other than w_1 and w_2 must not be v, otherwise v is a cut-vertex in G, which contradicts the fact G is 2-connected. Let the neighbour of u in G other than w_1 and w_2 be z. Since

$$e(G - \{u, w_1, w_2\}) = e(G) - 6 \geq \binom{n-3}{2} - 8 \geq \binom{n-4}{2} + 3$$

for $n \geq 15$, we obtain that $G - \{u, w_1, w_2\}$ is Hamiltonian-connected by Theorem 2.3. Take a (z, v)-Hamilton path P in $G - \{u, w_1, w_2\}$. We can see $vPzuw_1w_2v$ is a Hamilton cycle in G, a contradiction to Claim 1.

Finally, consider the case that $(a, b) = (n-1, 3)$. Let $V(G_2) = \{u, v, w\}$ and $H := (G-w) \cup \{uv\}$. Then H is 2-connected. Moreover, $e(H) \geq e(G) - d_G(w) \geq \binom{n-3}{2} - 4$. When $n \geq 18$, we have $e(H) > \binom{v(H)-3}{2} + 9 = \binom{n-4}{2} + 9$. By Theorem 2.4, when $v(H) \geq 17$ (that is, $n \geq 18$), H is Hamiltonian unless $H \subseteq L^2_{n-1}$ or $H \subseteq N^2_{n-1}$. Since H is 2-connected and L^2_{n-1} has a cut-vertex, $H \nsubseteq L^2_{n-2}$. Hence H is Hamiltonian or $H \subseteq N^2_{n-1}$. Assume that H is Hamiltonian. If $uv \in E(G)$, then G contains a C_{n-1}, a contradiction. Thus, $uv \notin E(G)$. If the Hamilton cycle in H, say C, does not pass through the edge uv, then it is also in G, a contradiction. Thus, C passes through uv, and hence there is a (u, v)-Hamilton path in G_1. Together with the path uwv, we can find a Hamilton cycle in G, a contradiction. So $H \subseteq N^2_{n-1} = K_2 \vee (K_{n-5} + 2K_1)$. Let $\{x_1, x_2\} \subseteq V(N^2_{n-1})$ such that each is of degree 2 and let $\{y_1, y_2\}$ be the 2-cut in N^2_{n-1}. Set $V(H) = V(N^2_{n-1})$. Since H is 2-connected, y_1, y_2 are still neighbors of x_1, x_2 in H. So, according to the locations of y_1, y_2, we obtain the following subcases (in the sense of isomorphism): (1) If $\{u, v\} = \{y_1, y_2\}$, then $G \subseteq K_2 \vee (K_{n-5} + 3K_1)$, a contradiction; (2) If $|\{u, v\} \cap \{y_1, y_2\}| = 1$, then G is a subgraph of Γ_1 or Γ_2 (see Fig 3); (3) If $\{u, v\} \cap \{y_1, y_2\} = \emptyset$, then G is a subgraph of a graph in Ψ_1, Ψ_2, or Ψ_3 (see Fig 3). By Proposition 2.1 (whose proof will be presented later), both subcases (2) and (3) contain either a C_{n-1} or a C_n, also a contradiction.

The proof is complete. □

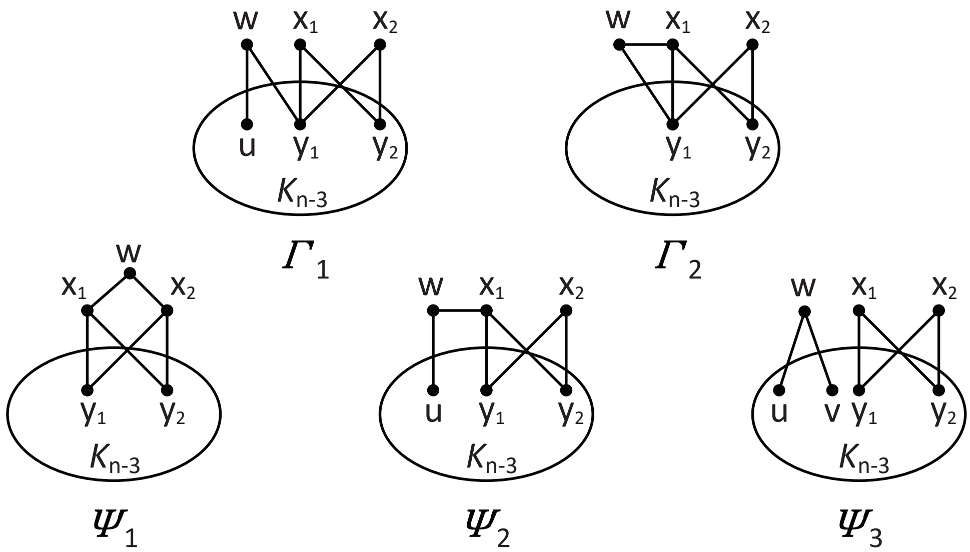

Fig. 3: Γ_1, Γ_2, Ψ_1, Ψ_2 and Ψ_3.

Proposition 2.1. *Let G be a graph of order $n \geq 21$ and size $e(G) \geq \binom{n-3}{2} - 2$. Suppose that the degrees of w, x_1, x_2 in G equal the degrees of those in Γ_i for $i = 1, 2$ and in Ψ_j for $j = 1, 2, 3$, respectively. If G is a spanning subgraph of Ψ_1 or Ψ_2, then G is Hamiltonian. If G is a spanning subgraph of Γ_1, Γ_2 or Ψ_3, then G contains a C_{n-1}.*

Proof. (1) If $G \subseteq \Psi_1$, then let us consider $G - \{w, x_1, x_2\}$. Since $v(G - \{w, x_1, x_2\}) = n - 3$ and

$$e(G - \{w, x_1, x_2\}) = e(G) - 6 \geq \binom{n-3}{2} - 8 \geq \binom{n-4}{2} + 3$$

when $n \geq 15$, we obtain that $G - \{w, x_1, x_2\}$ is Hamiltonian-connected by Theorem 2.3. So there is a (y_1, y_2)-Hamilton path P in $G - \{w, x_1, x_2\}$. Therefore $y_2 P y_1 x_1 w x_2 y_2$ is a Hamilton cycle in G.

(2) If $G \subseteq \Psi_2$, then let us consider $G - \{w, x_1, x_2, y_1\}$. Since $v(G - \{w, x_1, x_2, y_1\}) = n - 4$ and

$$e(G - \{w, x_1, x_2, y_1\}) = e(G) - 4 - d_G(y_1) \geq \binom{n-3}{2} - 2 - 4 - (n-2) \geq \binom{n-5}{2} + 3$$

when $n \geq 16$, we obtain that $G - \{w, x_1, x_2, y_1\}$ is Hamiltonian-connected by Theorem 2.3. So there is a (u, y_2)-Hamilton path P in $G - \{w, x_1, x_2, y_1\}$. Therefore, $y_2 P u w x_1 y_1 x_2 y_2$ is a Hamilton cycle in G.

(3) If $G \subseteq \Gamma_1$, we consider $G - \{w, x_1, x_2, y_1\}$. Since $v(G - \{w, x_1, x_2, y_1\}) = n - 4$ and

$$e(G - \{w, x_1, x_2, y_1\}) = e(G) - 3 - d_G(y_1) \geq \binom{n-3}{2} - 2 - 3 - (n-1) \geq \binom{n-5}{2} + 3$$

when $n \geq 16$, we obtain that $G - \{w, x_1, x_2, y_1\}$ is Hamiltonian-connected by Theorem 2.3. So there is a (u, y_2)-Hamilton path P in $G - \{w, x_1, x_2, y_1\}$. Therefore, $y_2 P u w y_1 x_1 y_2$ is a C_{n-1} in G.

(4) If $G \subseteq \Gamma_2$, we consider $G - \{w, x_1, x_2\}$. Since $v(G - \{w, x_1, x_2\}) = n - 3$ and

$$e(G - \{w, x_1, x_2\}) = e(G) - 6 \geq \binom{n-3}{2} - 8 \geq \binom{n-4}{2} + 3$$

when $n \geq 15$, we obtain that $G - \{w, x_1, x_2\}$ is Hamiltonian-connected by Theorem 2.3. So there is a (y_1, y_2)-Hamilton path P in $G - \{w, x_1, x_2\}$. Therefore, $y_2 P y_1 w x_1 y_2$ is a C_{n-1} in G.

(5) If $G \subseteq \Psi_3$, then we shall prove that there is a C_{n-1} in G. First we claim that $N(v) \cap N(y_1) \neq \emptyset$, otherwise $d_G(v) + d_G(y_1) \leq n$, and it follows that

$$e(G) \leq e(G - \{w, x_1, x_2, v, y_1\}) + d_G(v) + d_G(y_1) + 3 \leq \binom{n-5}{2} + n + 3 < \binom{n-3}{2} - 2$$

when $n > 14$, a contradiction. Hence there exists a vertex $z \in N(v) \cap N(y_1)$. We consider the graph $G - \{w, x_1, x_2, v, y_1, z\}$. Since $v(G - \{w, x_1, x_2, v, y_1, z\}) = n - 6$ and

$$e(G - \{w, x_1, x_2, v, y_1, z\}) \geq e(G) - 6 - 3(n-4) \geq \binom{n-3}{2} - 3n + 4 \geq \binom{n-7}{2} + 3$$

when $n \geq 21$, we obtain that $G - \{w, x_1, x_2, v, y_1, z\}$ is Hamiltonian-connected by Theorem 2.3. So there is a (u, y_2)-Hamilton path P in $G - \{w, x_1, x_2, v, y_1, z\}$. Therefore, $y_2 P u w v z y_1 x_1 y_2$ is a C_{n-1} in G.

The proof is complete. □

The following theorem will be used in the proof of Theorem 1.4.

Theorem 2.6 (Li and Ning [18]). *Let G be a balanced bipartite graph of order $2n$, where $n \geq 2k+1$, $k \geq 1$. If $\delta(G) \geq k$ and $e(G) > n(n-k-1) + (k+1)^2$, then G is Hamiltonian unless $G \subseteq B_n^k$.*

3 Spectral inequalities

First, we introduce the Kelmans operation [16]. Given a graph G and two specified vertices u and v, we construct a new graph $G[u \to v]$ as follows: we delete all edges between u and $S := N(u)\backslash(N(v) \cup \{v\})$ and add all edges between v and S. In notation, $G[u \to v]$ is defined as:

$$V(G[u \to v]) = V(G) \text{ and } E(G[u \to v]) = (E(G)\backslash\{uw : w \in S\}) \cup \{vw : w \in S\}.$$

We call it *the Kelmans operation* (from u to v).

Csikvári [9] proved that the Kelmans operation does not decrease the spectral radius of a graph.

Theorem 3.1 (Csikvári [9]). *Let G be a graph and u, v be two vertices of G. Let $G' = G[u \to v]$. Then $\lambda(G') \geq \lambda(G)$.*

Li and Ning [18] proved a signless spectral radius version.

Theorem 3.2 (Li and Ning [18]). *Let G be a graph and u, v be two vertices of G. Let $G' = G[u \to v]$. Then $q(G') \geq q(G)$.*

Generally speaking, we use these two theorems to determine the extremal graphs, if we already have turned the original problems into similar ones under the condition of number of edges.

The next result helps us to determine the extremal graphs with the help of the two theorems above.

Lemma 3. $\lambda(K_1 \vee (K_{n-3} + K_2)) > \lambda(\Lambda)$ *and* $q(K_1 \vee (K_{n-3} + K_2)) > q(\Lambda)$.

Proof. Let $G' := \Lambda[u \to v]$ (see Fig 2 for the vertices u, v and the graph Λ). Then $G' = K_1 \vee (K_{n-3} + 2K_1)$. By Theorems 3.1 and 3.2, we obtain $\lambda(K_1 \vee (K_{n-3} + 2K_1)) \geq \lambda(\Lambda)$ and $q(K_1 \vee (K_{n-3} + 2K_1)) \geq q(\Lambda)$, respectively. Since $K_1 \vee (K_{n-3} + 2K_1) \subset K_1 \vee (K_{n-3} + K_2)$ and $K_1 \vee (K_{n-3} + K_2)$ is connected, Lemma 3 is proved. □

Finally, we need several inequalities below on spectral radius or signless spectral radius in terms of number of edges and vertices for graphs or bipartite graphs. We mainly use them to get a sufficient condition in terms of number of edges for each problem.

Theorem 3.3 (Nosal [24], Bhattacharya, Friedland, and Peled [1]). *Let G be a bipartite graph. Then $\lambda(G) \leq \sqrt{e(G)}$.*

Theorem 3.4 (Hong [14]). *Let G be a graph of order n. If $\delta(G) \geq 1$, then*

$$\lambda(G) \leq \sqrt{2e(G) - n + 1}.$$

Theorem 3.5 (Hong, Shu, and Fang [15]). *Let G be a connected graph of order n and size $e(G)$. If minimum degree $\delta(G) \geq k \geq 1$, then*

$$\lambda(G) \leq \frac{k - 1 + \sqrt{(k+1)^2 + 4(2e(G) - kn)}}{2}.$$

Theorem 3.6 (Feng and Yu [11]). *Let G be a graph of order n. Then*

$$q(G) \leq \frac{2e(G)}{n-1} + n - 2.$$

4 Proofs

Proof of Theorem 1.2(1). Suppose that $\delta(G)\geq 1$. Then by Theorem 3.4 and the assumption, we obtain

$$\sqrt{2e(G)-n+1}\geq\lambda(G)>n-3,$$

which implies that $2e(G)>(n-2)(n-3)+2$. That is,

$$e(G)\geq\binom{n-2}{2}+2. \tag{2}$$

By Lemma 1, G contains a C_{n-1}, or $G\subseteq K_1\vee(K_{n-3}+K_2)$, or $G\subseteq\Lambda$.

Now we turn to the case that G contains isolated vertices. Since the maximum degree $\triangle(G)\geq\lambda(G)>n-3$, we have $\triangle(G)\geq n-2$, which follows G contains exactly one isolated vertex, say u. Let $H=G-u$. Then $\delta(H)\geq 1$ and $v(H)=n-1$. Again, by Theorem 3.4, we obtain $\sqrt{2e(H)-v(H)+1}=\sqrt{2e(G)-n+2}>n-3$. We get $2e(G)>n^2-5n+7=(n-2)(n-3)+1$. Thus, $e(G)\geq\binom{n-2}{2}+1$. By Lemma 1, G contains a C_{n-1}, or $G\subseteq K_1\vee(K_{n-3}+K_2)$, or $G\subseteq\Lambda$. The proof is complete. □

Proof of Theorem 1.2(2). Recall Theorem 3.6. We obtain $\frac{2e(G)}{n-1}+n-2\geq q(G)>2n-6$, which follows $2e(G)>(n-1)(n-4)$. Since $2e(G)$ and $(n-1)(n-4)$ are both even, we deduce that $2e(G)\geq(n-1)(n-4)+2$, that is, $e(G)\geq\binom{n-2}{2}$. By Lemma 1, G contains a C_{n-1}, or $G\subseteq K_1\vee(K_{n-3}+K_2)$, or $G\subseteq\Lambda$. The proof is complete. □

Proof of Theorem 1.3(1). Since G is 2-connected, we get $\delta(G)\geq 2$. By Theorem 3.5, we have $\lambda(G)\leq\frac{1+\sqrt{9+4(2e(G)-2n)}}{2}$. Hence $\frac{1+\sqrt{9+4(2e(G)-2n)}}{2}>n-4$, which follows that $2e(G)>(n-3)(n-4)+6$. That is, $e(G)>\binom{n-3}{2}+3$. By Lemma 2, G contains a C_{n-1}, or $G\subseteq K_2\vee(K_{n-5}+3K_1)$. The proof is complete. □

Proof of Theorem 1.3(2). By Theorem 3.6, we obtain $\frac{2e(G)}{n-1}+n-2\geq q(G)>2n-8$, which follows $2e(G)>(n-1)(n-6)$. Since $2e(G)$ and $(n-1)(n-6)$ are both even, we infer that $2e(G)\geq(n-1)(n-6)+2$, that is, $e(G)\geq\binom{n-3}{2}-2$. By Lemma 2, G contains a C_{n-1}, or $G\subseteq K_2\vee(K_{n-5}+3K_1)$. The proof is complete. □

Proof of Theorem 1.4(1). Write W for the set of vertices of B_n^k of degree k. Let $X=N(W)$, $Y=N(X)-W$, and $Z=N(Y)-X$ (see Fig 4). Note that $|W|=|X|=k$ and $|Y|=|Z|=n-k$.

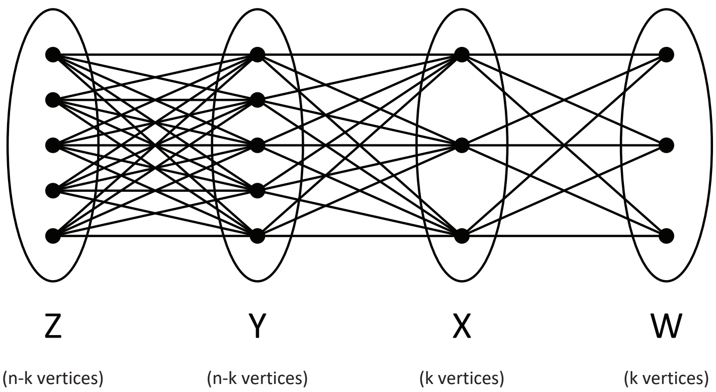

Fig. 4: X, Y, Z, and W in B_n^k.

Claim 1. $\lambda(G) \le \lambda(B_n^k - e)$, where $e = uv$, $u \in Y$, $v \in Z$.

For any proper subgraph G of B_n^k, since $\delta(G) \ge k$, G must contain all the edges incident to W. Thus, either $G \subseteq G_1 := B_n^k - e_1$, where $e_1 = uv \in E(X, Y)$, or $G \subseteq G_2 := B_n^k - e_2$, where $e_2 = uv \in E(Y, Z)$. So $\lambda(G) \le \max\{\lambda(G_1), \lambda(G_2)\}$

Recall $\{u, v\} \subset X \cup Y \cup Z$. By symmetry, there are two cases: (i) $u \in X$, $v \in Y$; (ii) $u \in Y$, $v \in Z$.

Let u' be a vertex in X, v' be a vertex in Y, and w' be a vertex in Z. Then $(B_n^k - u'v')[w' \to u'] = B_n^k - v'w'$. By Theorem 3.1, $\lambda(B_n^k - v'w') \ge \lambda(B_n^k - u'v')$. We have proved Claim 1.

Claim 2. $\lambda(B_n^k - e) < \sqrt{n(n-k)}$, where $e = uv$, $u \in Y$, $v \in Z$.

We prove the claim by contradiction. Let $G' = B_n^k - e$, where $e = uv$, $u \in Y$, $v \in Z$. Set $\lambda := \lambda(G')$. Let $\mathbf{x} = (x_1, \ldots, x_{2n})$ be a positive unit eigenvector to λ. Suppose that

$$\lambda(G') \ge \sqrt{n(n-k)}. \tag{3}$$

Let

$$\begin{aligned} w &:= x_i,\ i \in W, \\ x &:= x_i,\ i \in X, \\ y &:= x_i,\ i \in Y \backslash \{u\}, \\ z &:= x_i,\ i \in Z \backslash \{v\}, \\ s &:= x_u, \\ t &:= x_v. \end{aligned}$$

Note that the $2n$ eigenequations of G' are reduced to the following six types:

$$\lambda w = kx, \tag{4}$$
$$\lambda x = kw + (n-k-1)y + s, \tag{5}$$
$$\lambda y = kx + (n-k-1)z + t, \tag{6}$$
$$\lambda z = (n-k-1)y + s, \tag{7}$$
$$\lambda s = kx + (n-k-1)z, \tag{8}$$
$$\lambda t = (n-k-1)y. \tag{9}$$

From (6) and (8) we have

$$\lambda y - \lambda s = [kx + t + (n-k-1)z] - [kx + (n-k-1)z],$$

that is,

$$t = \lambda(y - s). \tag{10}$$

From (7) and (9) we have

$$\lambda z - \lambda t = (n-k-1)y + s - (n-k-1)y,$$

that is,

$$s = \lambda(z - t). \tag{11}$$

By putting (10) into (11), we obtain

$$s = \lambda\left[z - \lambda(y - s)\right] = \frac{\lambda^2 y - \lambda z}{\lambda^2 - 1}. \tag{12}$$

Hence

$$t = \lambda(y - s) = \lambda\left(y - \frac{\lambda^2 y - \lambda z}{\lambda^2 - 1}\right) = \frac{\lambda^2 z - \lambda y}{\lambda^2 - 1}. \tag{13}$$

By using (12), equation (7) becomes

$$\lambda z = (n - k - 1)y + \frac{\lambda^2 y - \lambda z}{\lambda^2 - 1},$$

from which it follows that

$$z = \frac{\lambda^2 - 1}{\lambda^3}\left(n - k + \frac{1}{\lambda^2 - 1}\right)y. \tag{14}$$

Since $4 < n - k = \lambda(K_{n-k,n-k}) < \lambda < \lambda(K_{n,n}) = n$, we obtain

$$z < \frac{\lambda^2 - 1}{\lambda^3}\left(\lambda + \frac{1}{\lambda^2 - 1}\right)y = \frac{\lambda^3 - \lambda + 1}{\lambda^3} \cdot y < y.$$

Let $f(x) = \frac{x^2-1}{x^3}$. Then $f'(x) = \frac{3-x^2}{x^4} < 0$ when $x > \sqrt{3}$. So $\frac{\lambda^2-1}{\lambda^3}$ decreases when $\lambda > \sqrt{3}$, which follows

$$\begin{aligned} z &= \frac{\lambda^2 - 1}{\lambda^3}(n - k)y + \frac{y}{\lambda^3} \\ &> \frac{n^2 - 1}{n^3}(n - k)y + \frac{y}{n^3} \\ &= \frac{n^3 - kn^2 - n + k + 1}{n^3} \cdot y \\ &> \frac{n^3 - kn^2 - n}{n^3} \cdot y \\ &= \left(1 - \frac{k}{n} - \frac{1}{n^2}\right)y. \end{aligned}$$

Therefore,

$$\left(1 - \frac{k}{n} - \frac{1}{n^2}\right)y < z < y. \tag{15}$$

Note that if we remove all edges between W and X and add the edge uv to G', we obtain the graph $K_{n,n-k} + kK_1$. Let $\mathbf{x}''$ be the restriction of $\mathbf{x}$ to $K_{n,n-k}$. We find that

$$\langle A(K_{n,n-k})\mathbf{x}'', \mathbf{x}''\rangle = \langle A(G')\mathbf{x}, \mathbf{x}\rangle + 2st - 2k^2wx = \lambda + 2(st - k^2wx).$$

Since $\|\mathbf{x}''\| < 1$, $\langle A(K_{n,n-k})\mathbf{x}'', \mathbf{x}''\rangle < \lambda(K_{n,n-k}) = \sqrt{n(n-k)}$, that is,

$$\lambda + 2(st - k^2wx) < \sqrt{n(n-k)}. \tag{16}$$

Recall that $\lambda \geq \sqrt{n(n-k)}$ (see (3)). This assumption, together with (16) yields $st - k^2wx < 0$. Recall that $\lambda w = kx$ (see (4)). We rewrite it by

$$\lambda(st - k^2wx) = \lambda st - k^3x^2 < 0,$$

that is,

$$\lambda st < k^3 x^2. \tag{17}$$

Noting (15), we have

$$\lambda s = \lambda \cdot \frac{\lambda^2 y - \lambda z}{\lambda^2 - 1} > \lambda \cdot \frac{\lambda^2 y - \lambda y}{\lambda^2 - 1} = \frac{\lambda^2}{\lambda + 1} \cdot y > (\lambda - 1)y > (n-k-1)y \geq (k^3 + k + 3)y,$$

and

$$\begin{aligned} t = \frac{\lambda^2 z - \lambda y}{\lambda^2 - 1} &> z - \frac{y}{\lambda - 1} > \left[\left(1 - \frac{k}{n} - \frac{1}{n^2}\right) y - \frac{y}{n-k-1}\right] = \left(1 - \frac{k}{n} - \frac{1}{n^2} - \frac{1}{n-k-1}\right) y \\ &> \left(1 - \frac{k}{n} - \frac{1}{2n} - \frac{3}{2n}\right) y = \left(1 - \frac{k+2}{n}\right) y \geq \left(1 - \frac{k+2}{k^3 + 2k + 4}\right) y. \end{aligned}$$

Thus, we can estimate the left side of the inequality in (17) as follows:

$$\begin{aligned} \lambda st &> (k^3 + k + 3)\left(1 - \frac{k+2}{k^3 + 2k + 4}\right) y^2 \\ &= \frac{k^3(k^3 + 2k + 5) + (k+2)(k+3)}{k^3 + 2k + 4} \cdot y^2 \\ &> k^3 y^2. \end{aligned}$$

Together with (17), we have

$$y^2 < x^2. \tag{18}$$

From (4), (5), (12) and (14), we have

$$\begin{aligned} \left(\lambda - \frac{k^2}{\lambda}\right) x &= (n-k-1)y + s \\ &= (n-k-1)y + \frac{\lambda^2 y - \lambda z}{\lambda^2 - 1} \\ &= (n-k)y + \frac{y - \lambda \cdot \frac{\lambda^2 - 1}{\lambda^3}\left(n - k + \frac{1}{\lambda^2 - 1}\right) y}{\lambda^2 - 1} \\ &= (n-k)y - \frac{n-k-1}{\lambda^2} y \\ &< (n-k)y. \end{aligned}$$

Since

$$n \geq k^3 + 2k + 4 = (k^3 + k) + k + 4 \geq 2\sqrt{k^3 \cdot k} + k + 4 = 2k^2 + k + 4 > \frac{3}{2}k^2 + \frac{1}{2}k + \frac{1}{24},$$

we have

$$n - \frac{k}{2} - \frac{1}{12} = \sqrt{n^2 - kn - \frac{1}{6}\left[n - \left(\frac{3}{2}k^2 + \frac{1}{2}k + \frac{1}{24}\right)\right]} < \sqrt{n(n-k)}.$$

Therefore, we obtain

$$n - k < n - \frac{k}{2} - \frac{1}{12} < \sqrt{n(n-k)} \leq \lambda < \lambda(K_{n,n}) = n. \tag{19}$$

Note that $k^3 + k + 4 > 3k^2$ for all $k \geq 1$, hence $\lambda > n - k \geq k^3 + k + 4 > 3k^2$, and it follows $\frac{k^2}{\lambda} < \frac{1}{3}$. Therefore,

$$x^2 < \left(\frac{n-k}{\lambda - \frac{k^2}{\lambda}}\right)^2 y^2 < \left(\frac{n-k}{n - \frac{k}{2} - \frac{1}{12} - \frac{1}{3}}\right)^2 y^2 < y^2,$$

contradicting (18). Now we have proved Claim 2.

Together with Claims 1 and 2, the proof is complete. □

Proof of Theorem 1.4(2) By the initial condition and Theorem 3.3, $\sqrt{n(n-k)} \leq \lambda(G) \leq \sqrt{e(G)}$. Thus, we obtain

$$e(G) \geq n(n-k) > n(n-k-1) + (k+1)^2$$

when $n \geq (k+1)^2 + 1$. Notice that $k^3 + 2k + 2 \geq (k+1)^2 + 1$ when $k \geq 1$. By Theorem 2.6, G is Hamiltonian or $G \subseteq B_n^k$. By Theorem 1.8, G is Hamiltonian or $G = B_n^k$. The proof is complete. □

5 Concluding remarks

We suggest the following general problems.

Problem 1. Let G be a connected graph of order n. Let s be an integer with $s \geq 1$. Suppose that $\lambda(G) > \lambda(K_1 \vee (K_s + K_{n-s-1}))$, where n is sufficiently large compared to s. Does G contain a C_{n-s+1}?

Problem 2. Let G be a connected graph of order n. Let s be an integer with $s \geq 1$. Suppose that $q(G) > q(K_1 \vee (K_s + K_{n-s-1}))$, where n is sufficiently large compared to s. Does G contain a C_{n-s+1}?

Affirmative answers to these problems will give tight spectral conditions for the existence of cycle C_l, where l is large. One can easily find that Theorems 1.1 and 1.2 give affirmative solutions to Problem 1 for the cases $s = 1$ and $s = 2$, respectively. Theorem 1.2 solves Problem 2 when $s = 2$.

Moreover, we can also consider spectral conditions for consecutive cycles. In this spirit, Theorems 1.1 and 1.2 can be extended as follows, respectively.

Theorem 5.1. *Let G be a graph of order $n \geq 5$. If $\lambda(G) > n-2$, then G is pancyclic unless $G = N_n^1$.*

Theorem 5.2. *Let G be a graph of order $n \geq 15$. If $\lambda(G) > \lambda(K_1 \vee (K_2 + K_{n-3}))$ or $q(G) > q(K_1 \vee (K_2 + K_{n-3}))$, then G contains a cycle C_l for each l such that $3 \leq l \leq n-1$.*

The main ingredient of the proofs comes from a classical theorem proved by Woodall [27].

Theorem 5.3 (Woodall [27]). *Let G be a graph of order $n \geq 2k+3$, where $k \geq 0$ is an integer. If*

$$e(G) \geq \binom{n-k-1}{2} + \binom{k+2}{2} + 1,$$

then G contains a C_l for each l such that $3 \leq l \leq n-k$.

Proof of Theorem 5.1. Since $\Delta(G) \geq \lambda(G) > n-2$, we obtain $\Delta(G) \geq n-1$, which implies that G is connected. By Theorem 3.4, we get $\sqrt{2e(G) - n + 1} > n-2$. We infer that $2e(G) \geq n^2 - 3n + 4 \geq n^2 - 5n + 14$ for $n \geq 5$. This implies that $e(G) \geq \binom{n-2}{2} + \binom{3}{2} + 1$ for $n \geq 5$. By Theorem 5.3, G contains all cycles C_l, where $3 \leq l \leq n-1$. By Theorem 1.1, G contains a

Hamilton cycle or $G = N_n^1$. So G contains all cycles of length from 3 to n. The proof is complete. □

Proof of Theorem 5.2. If $\lambda(G) > \lambda(K_1 \vee (K_2 + K_{n-3}))$, then $\Delta(G) \geq \lambda(G) > n-3$. If $q(G) > q(K_1 \vee (K_2 + K_{n-3}))$, then $2\Delta(G) \geq q(G) > 2(n-3)$. In each case, we deduce $\Delta(G) > (n-3)$, which implies that $\Delta(G) \geq n-2$. It follows that G contains at most one isolated vertex. If G is connected, then by Theorem 3.4, we get $\sqrt{2e(G) - n + 1} > n - 3$. We infer that $2e(G) \geq n^2 - 5n + 8$. If G is not connected, let v be the isolated vertex and $G' := G - v$, then we have $\sqrt{2e(G') - v(G') + 1} > n - 3$, that is $2e(G) \geq n^2 - 5n + 8$. Since $n^2 - 5n + 8 \geq n^2 - 6n + 23$ for $n \geq 15$, this implies that $e(G) \geq \binom{n-3}{2} + \binom{4}{2} + 1$ for $n \geq 15$. By Theorem 5.3, G contains all cycles C_l, where $3 \leq l \leq n-2$. By Corollary 1, G contains a C_{n-1}. The proof is complete. □

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