Riesz transform under perturbations via heat kernel regularity

Renjin Jiang & Fanghua Lin

January 9, 2019

Abstract. Let M be a complete non-compact Riemannian manifold. In this paper, we derive sufficient conditions on metric perturbation for stability of L^p-boundedness of the Riesz transform, $p \in (2,\infty)$. We also provide counter-examples regarding in-stability for L^p-boundedness of Riesz transform.

Résumé Soit M une variété riemannienne complète et non compacte. Dans cet article, nous dérivons des conditions suffisantes sur la perturbation métrique pour la stabilité de la L^p bornitude des transformations de Riesz, $p \in (2,\infty)$. De plus, nous fournissons des contre-exemples concernant la stabilité pour la L^p bornitude des transformations de Riesz.

1 Introduction

Let M be a complete, connected and non-compact n-dimensional Riemannian manifold, $n \geq 2$. In this paper, we study the behavior of the Riesz transform under metric perturbations. As a main tool and also a byproduct, we also obtain stability and instability of gradient estimates of harmonic functions and heat kernels under metric perturbation.

Let g_0 and g be two Riemannian metrics on M. Let μ_0, μ, $\mathcal{L}_0$, $\mathcal{L}$, ∇_0, ∇, div_0, div, be the corresponding Riemannian volumes, non-negative Laplace-Beltrami operators, Riemannian gradient operators and divergence operators, generated by g_0 and g, respectively.

Suppose that g_0 and g are comparable on M, i.e., there exist $C \geq 1$ such that for any $x \in M$ and $\mathbf{v} = (v_1, \cdots, v_n) \in T_x(M)$ it holds

$$C^{-1} g^{ij} v_i v_j \leq g_0^{ij} v_i v_j \leq C g^{ij} v_i v_j. \tag{1.1}$$

Then a natural question is: if the Riesz operator $\nabla_0 \mathcal{L}_0^{-1/2}$ is bounded on $L^p(M,\mu_0)$, $p \in (1,\infty)$, is $\nabla \mathcal{L}^{-1/2}$ also bounded on $L^p(M,\mu)$?

2010 *Mathematics Subject Classification*. Primary 58J35; Secondary 58J05; 35B65; 35K05; 42B20.
Key words and phrases: Riesz transform, harmonic functions, heat kernels, perturbation

Note that the case $p = 2$ is trivially true as the Riesz operators $\nabla_0\mathcal{L}_0^{-1/2}$ and $\nabla\mathcal{L}^{-1/2}$ are isometries on $L^2(M,\mu_0)$ and $L^2(M,\mu)$, respectively. For the case $p \in (1,2)$, it was shown by Coulhon and Duong [17] that the Riesz operator is weakly L^1-bounded under a doubling condition and a Gaussian upper bound for the heat kernel. The L^p-boundedness then follows from an interpolating, for all $p \in (1,2)$.

Let $B_0(x,r)$, $B(x,r)$ be open balls induced by the metrics g_0, g respectively, and $V_0(x,r)$ and $V(x,r)$ the volumes $\mu_0(B_0(x,r))$ and $\mu(B(x,r))$ respectively. We say that (M,g_0) satisfies a doubling condition, if there exists $C_D > 0$ such that for any $x \in M$ and for all $r > 0$ that

$$(D) \qquad V_0(x,2r) \le C_D V_0(x,r),$$

and that the heat kernel $P_t(x,y)$ of $e^{-t\mathcal{L}_0}$ satisfies a Gaussian upper bound, if there exists $c, C > 0$ such that

$$(GUB) \qquad P_t(x,y) \le \frac{C}{V(x,\sqrt{t})}\exp\left\{-c\frac{d^2(x,y)}{t}\right\}, \forall\, t > 0 \ \&\ \forall\, x,y \in M.$$

According to [11], under (D), (GUB) is equivalent to a version of the Sobolev-Poincaré inequality that there exists $q > 2$ such that for every ball $B_0(x,r)$ and each $f \in C_0^\infty(B_0(x,r))$,

$$(SI) \qquad \left(\fint_{B_0(x,r)} |f|^q\, d\mu_0\right)^{2/q} \le C_{LS}\left(\fint_{B_0(x,r)} |f|^2\, d\mu_0 + r^2\fint_{B_0(x,r)} |\nabla_0 f|^2\, d\mu_0\right).$$

Above and in what follows, for a measurable set Ω, $\fint_\Omega g\, d\mu_0$ denotes the average of the integrand over it.

It is easy to see that (D) and (SI) are invariant under quasi-isometries. Therefore, if (D) and (GUB) are satisfied on (M,g_0), then they are also satisfied on (M,g), and the Riesz operator $\nabla\mathcal{L}^{-1/2}$ is bounded on $L^p(M,\mu)$ for all $p \in (1,2)$.

The case $p > 2$ is more involved. It was shown in [18] that on a Riemannian manifold (M,g), under (D) together with a scale invariant L^2-Poincaré inequality, $\nabla\mathcal{L}^{-1/2}$ is bounded on $L^p(M,\mu)$, $p \in (2,\infty)$, if and only if, it holds for every ball $B(x,r)$ and every solution to $\mathcal{L}u = 0$ on $B(x,2r)$ that

$$(RH_p) \qquad \left(\fint_{B(x,r)} |\nabla u|^p\, d\mu\right)^{1/p} \le \frac{C}{r}\fint_{B(x,2r)} |u|\, d\mu.$$

See [36] for the case $\mathbb{R}^n$, and [5, 6, 29, 32] for earlier results and also further generalizations. Further, it was shown in [19] that, the local Riesz transform $\nabla(1+\mathcal{L})^{-1/2}$ is bounded on $L^p(M,\mu)$, $p \in (2,\infty)$, if and only if, the above inequality (RH_p) holds for all balls $B(x,r)$ with $r < 1$.

By the perturbation result of Caffarelli and Peral [12], one has a good understanding of the local gradient estimates for elliptic equations on $\mathbb{R}^n$. In particular, for a uniformly elliptic operator $L = -\mathrm{div}_{\mathbb{R}^n} A\nabla_{\mathbb{R}^n}$, if A is uniformly continuous, then [12] implies that any L-harmonic functions satisfies (RH_p) on small balls $B(x,r)$ with $r < 1$ for all $p < \infty$. This gives the L^p-boundedness of the local Riesz transform $\nabla_{\mathbb{R}^n}(1+L)^{-1/2}$ for all $p \in (2,\infty)$.

Then how about the Riesz transform $\nabla_{\mathbb{R}^n} L^{-1/2}$? It was well-known that, for any $p > 2$, there exists a uniformly elliptic operator (Meyer's conic Laplace operator) on $\mathbb{R}^2$ such that the Riesz transform is not bounded on $L^p(\mathbb{R}^2)$; see [7, p.120] and also Section 4. Noting that the conic Laplace operators do not enjoy smoothness at the origin, one may wonder what happens if the coefficients are smooth? We however have the following example.

Proposition 1.1. *For any given $p > 2$ and $n \geq 2$, there exists a $C^\infty(\mathbb{R}^n)$ matrix $A(x)$ satisfying uniformly elliptic condition,*

$$c|\xi|^2 \leq \langle A\xi, \xi\rangle \leq C|\xi|^2, \ \forall\, \xi \in \mathbb{R}^n,$$

and each order of gradients of $A(x)$ is bounded, such that the Riesz operator $\nabla_{\mathbb{R}^n} L^{-1/2}$, $L = -\mathrm{div}_{\mathbb{R}^n} A \nabla_{\mathbb{R}^n}$, is not bounded on $L^p(\mathbb{R}^n)$.

The above proposition implies that apart from the local smoothness (local regularity), some global controls of the perturbation are needed for the stability issue. For some related results we refer to a study of (asymptotically) conic elliptic operators in [33] and other examples motivated by the theory of elliptic homogenization (cf. [4] and [3, Further Remarks]). In fact one can construct (with some extra work) examples of uniformly regular, uniformly elliptic operators with isotropic coefficient matrices $a(x)I_n$ so that the conclusion of the above proposition remains valid.

In what follows, we shall use the Einstein summation convention for repeated indexes, and δ_{ik} the Kronecker delta function. Our main result reads as follows.

Theorem 1.2. *Let g_0, g be two Riemannian metrics on M. Assume that g_0 and g are comparable and there exists $\epsilon > 0$ such that*

$$(GD) \qquad \fint_{B_0(x,r)} |\delta_{ik} - g_0^{ij} g_{jk}|\, d\mu_0 \leq C r^{-\epsilon}, \ \forall\, 1 \leq i, k \leq n, \ \forall x \in M \,\&\, \forall\, r > 1.$$

Suppose that (M, g) satisfies (D) and (GUB). Then if for some $p_0 \in (2, \infty)$, $\nabla_0 \mathcal{L}_0^{-1/2}$ is bounded on $L^p(M, \mu_0)$ and $\nabla(\mathcal{L} + 1)^{-1/2}$ is bounded on $L^p(M, \mu)$ for all $p \in (2, p_0)$, $\nabla \mathcal{L}^{-1/2}$ is bounded on $L^p(M, \mu)$ for all $p \in (2, p_0)$.

Some remarks are in order. First, the above result can not be true if $\epsilon = 0$ as indicated by the Proposition 1.1 though one may replace the algebraic decay condition by a Dini-type condition. Next, if we strengthen the assumption (GUB) to two sides bounds of the heat kernel then we can include the endpoint that $p = p_0$; see Theorem 2.4 below. Moreover, as the L^p-boundedness of the Riesz operator $\nabla \mathcal{L}^{-1/2}$ implies

$$\|\nabla e^{-t\mathcal{L}}\|_{L^p(M,\mu) \to L^p(M,\mu)} \leq \frac{C}{\sqrt{t}}, \ \forall\, t > 0,$$

by [6, 18], one further sees that the gradient estimates for heat kernels and harmonic functions are also stable under such metric perturbations.

Coulhon-Dungey [16] has addressed the stability issue of Riesz transform under perturbations. In [16], no assumptions on the volume growth or the upper bound of the heat kernel were required. However, they assumed the ultra-contractivity, i.e.,

$$\|e^{-t\mathcal{L}}\|_{L^1(M,\mu)\to L^\infty(M,\mu)} \le Ct^{-D/2},\ t\ge 1;$$

and that $\delta_{ik}-g_0^{ij}g_{jk}\in L^q(M,\mu_0)$ for some $q\in[1,\infty)$ instead of (GD); see [16, Theorem 4.1]. In the case of Riemannian manifolds with lower Ricci curvature bound, the ultra-contractivity requires the volume of unit balls is non-collapsing, i.e., $\inf_{x\in M}V(x,1)>0$; see [28, Proposition 3.1]. By [20], there are Riemannian manifolds with non-negative Ricci curvature, on which the volume of unit balls does collapse. Moreover, if (D) and $\inf_{x\in M}V(x,1)>0$ hold, then $\delta_{ik}-g_0^{ij}g_{jk}\in L^q(M,\mu_0)$, $q\in[1,\infty)$, implies (GD).

Recently, Blank, Le Bris and Lions [9, 10] addressed the issue of perturbations related to the elliptic homogenization, their results are rather interesting in comparison with that of [16] for the case that (M_0,h) being Euclidean spaces with a nice periodic metric h. Instead of the ultra-contractivity property as described above, [9, 10] used a continuity argument starting from the estimates established in [2]. We also note that most of conclusions of [9, 10] are also true for systems, while our proofs here and [16] work only for the scalar case.

Our main achievement here is that we find the condition (GD), which works also for the collapsing case. In particular results here cover the case of complete manifolds with non-negative Ricci curvature. In general (GD) allows a much larger class (than L^p $(p<\infty)$) of perturbations. For example, in $\mathbb{R}^n$, a perturbation along a strip $\mathbb{R}^{n-1}\times[0,1]$ satisfies (GD) but is not in L^p for $p<\infty$. For the proof, we shall follow the basic strategy of [16], where the key step is to estimate the difference of the operator norm

$$\|\nabla[(1+t\mathcal{L}_0)^{-1}-(1+t\mathcal{L})^{-1}]\|_{L^p(M,\mu)\to L^p(M,\mu)}\le Ct^{-\alpha-1/2},\ t\ge 1,$$

for some $\alpha>0$. If the ultra-contractivity holds, then such an estimate is relatively easy to establish; see [16, Proposition 2.2]. However, without ultra-contractivity, the proof is more involved. The estimates for the heat kernel and its gradient (cf. [6, 18]), together with (GD) are essential in our proofs.

Let us list several consequences of Theorem 1.2. It is well known that if (M,g) has lower Ricci curvature bound, then the local Riesz transform is L^p-bounded for all $p\in(1,\infty)$; see [6].

Corollary 1.3. *Assume that g, g_0 are two metrics on M, that satisify* (1.1) *and there exists $\epsilon>0$ such that*

$$\fint_{B_0(x,r)}|\delta_{ik}-g_0^{ij}g_{jk}|\,d\mu_0\le Cr^{-\epsilon},\ \forall\,1\le i,k\le n,\ \forall x\in M\ \&\ \forall\, r>1.$$

Suppose that (M,g) has Ricci curvature bounded from below and satisfies (D) and (GUB). Then if $\nabla_0\mathcal{L}_0^{-1/2}$ is bounded on $L^p(M,\mu_0)$ for all $p\in(2,p_0)$, where $p_0\in(2,\infty]$, $\nabla\mathcal{L}^{-1/2}$ is also bounded on $L^p(M,\mu)$ for all $p\in(2,p_0)$.

Note that in particular any compact metric perturbation satisfies (GD).

Corollary 1.4. *Suppose that* (M, g) *and* (M, g_0) *satisfy* (D) *and* (GUB)*, and have Ricci curvature bounded from below. If* g *coincides with* g_0 *outside a compact subset, then for all* $p_0 \in (2, \infty]$, $\nabla_0 \mathcal{L}_0^{-1/2}$ *is bounded on* $L^p(M, \mu_0)$ *for all* $p \in (2, p_0)$*, if and only if,* $\nabla \mathcal{L}^{-1/2}$ *is bounded on* $L^p(M, \mu)$ *for all* $p \in (2, p_0)$.

Carron [13] and Devyver [21] had addressed the question of stability of compact perturbation, under the validity of global Sobolev inequality instead of (D) and (GUB). The global Sobolev inequality in general is a stronger requirement than (D) and (GUB); see [21, Remark 1.1]. On the other hand, the changing of the topology of manifolds is out of reach of this work, which however is allowed in [13, 21], see also [29].

An easy consequence follows for the case of non-negative Ricci curvature.

Corollary 1.5. *Assume that* (M, g_0) *has non-negative Ricci curvature. If* g *coincides with* g_0 *outside a compact subset, then the Riesz transform* $\nabla \mathcal{L}^{-1/2}$ *is bounded on* $L^p(M, \mu)$ *for all* $p \in (1, \infty)$.

Zhang [39] had derived a sufficient condition on the perturbation of a manifold with non-negative Ricci curvature for the stability of Yau's estimate (equivalent to (RH_p) with $p = \infty$, cf. [15, 18, 38]), which implies the boundedness of the Riesz transform for all $p \in (1, \infty)$ by [18, Theorem 1.9]. We did not prove the stability of Yau's estimate, but the advantage of our result is that our condition (GD) is much more explicit and, it is convenient for applications.

Let us mention a few examples that our result can apply. Besides manifolds with non-negative Ricci curvature, conic manifolds (cf. [30, 33]), as well as co-compact covering manifold with polynomial growth deck transformation group (cf. [22]), Lie groups of polynomial growth (cf. [1, 37]) satisfy the doubling condition (D) and (GUB). Indeed the stronger Li-Yau estimate is true (see Theorem 2.4 below). By [26], (GUB) is preserved under gluing operation; see [14, 29] for studies of Riesz transforms in this direction. Therefore, our result applies to these settings if the metric perturbation satisfies (GD). Our result also applies on the Euclidean space $\mathbb{R}^n$ for elliptic operators (including degenerate operators); see Theorem 3.1.

Finally let us state a corollary for the Euclidean case. The balls $B(x, r)$ in the following corollary are induced by the standard Euclidean metric.

Corollary 1.6. *Let* $\mathcal{L}_0 = -\mathrm{div}_{\mathbb{R}^n}(A_0 \nabla_{\mathbb{R}^n})$, $\mathcal{L} = -\mathrm{div}_{\mathbb{R}^n}(A \nabla_{\mathbb{R}^n})$ *be uniformly elliptic operators on* $\mathbb{R}^n$, $n \geq 2$*, with* A_0, A *being uniformly continuous on* $\mathbb{R}^n$*. Suppose that there exists* $\epsilon > 0$ *such that*

$$\fint_{B(y,r)} |A - A_0| \, dx \leq \frac{C}{r^{\epsilon}}, \quad \forall\, y \in \mathbb{R}^n \,\&\, \forall\, r > 1.$$

Then for $p \in (2, \infty)$, $\nabla_{\mathbb{R}^n} \mathcal{L}_0^{-1/2}$ *is bounded on* $L^p(\mathbb{R}^n)$ *if and only if* $\nabla_{\mathbb{R}^n} \mathcal{L}^{-1/2}$ *is bounded on* $L^p(\mathbb{R}^n)$.

The paper is organized as follows. In Section 2, we study the case of manifolds and prove Theorem 1.2 and its corollaries. In Section 3 we discuss the case of degenerate elliptic equations on $\mathbb{R}^n$. In Section 4, we discuss the conic Laplace operators and present the proof of Theorem 1.1 there. In Appendix A, we recall some basic facts regarding the boundedness of functional operators.

2 Metric perturbation on manifolds

In this section, we study the behavior of Riesz transform under metric perturbation on manifolds.

Note that as g, g_0 are comparable on M, the resulting Riemannian volumes μ and μ_0 are also comparable, which implies that for any $p \in [1, \infty]$

$$L^p(M,\mu) = L^p(M,\mu_0),$$

also the boundedness of $\nabla\mathcal{L}_0^{-1/2}$, $\nabla\mathcal{L}^{-1/2}$ on $L^p(M,\mu)$, is equivalent to the boundedness of $\nabla_0\mathcal{L}_0^{-1/2}$, $\nabla_0\mathcal{L}^{-1/2}$ on $L^p(M,\mu_0)$, respectively. In what follows, we shall simply denote by $L^p(M)$ the Lebesgue space $L^p(M,\mu)$ or $L^p(M,\mu_0)$, and denote by $\|\cdot\|_p$, $\|\cdot\|_{p\to p}$ the $L^p(M)$ norm and the operator norm $\|\cdot\|_{L^p(M)\to L^p(M)}$, respectively.

As the consequence of (1.1) also, one sees that the condition (GD) is equivalent to

$$\fint_{B_0(x,r)} |\delta_{ik} - g_0^{ij} g_{jk}|\, d\mu_0 \sim \fint_{B(x,r)} |\delta_{ik} - g_0^{ij} g_{jk}|\, d\mu \le Cr^{-\epsilon},\ \forall\ 1 \le i,k \le n, \forall x \in M \,\&\, \forall\, r > 1.$$

Let us outline the proof of Theorem 1.2, for which we follow the approach in [16]. Note that our main ingredients are Proposition 2.1 and Proposition 2.2 below.

Proof of Theorem 1.2. To show that $\nabla\mathcal{L}^{-1/2}$ is bounded on $L^p(M)$ for $p \in (2, p_0)$, it suffices to show that

$$\|\nabla\mathcal{L}_0^{-1/2} - \nabla\mathcal{L}^{-1/2}\|_{p\to p} \le C. \tag{2.1}$$

We write

$$\begin{aligned}\nabla\mathcal{L}_0^{-1/2} - \nabla\mathcal{L}^{-1/2} &= \frac{1}{\pi}\int_0^1 \nabla[(1+t\mathcal{L}_0)^{-1} - (1+t\mathcal{L})^{-1}]\frac{dt}{\sqrt{t}} \\ &\quad + \frac{1}{\pi}\int_1^\infty \nabla[(1+t\mathcal{L}_0)^{-1} - (1+t\mathcal{L})^{-1}]\frac{dt}{\sqrt{t}}.\end{aligned}$$

By Lemma A.1 and that $\nabla\mathcal{L}_0^{-1/2}$ and $\nabla(1+\mathcal{L})^{-1/2}$ are bounded on $L^p(M)$, one has

$$\left\|\frac{1}{\pi}\int_0^1 \nabla[(1+t\mathcal{L}_0)^{-1} - (1+t\mathcal{L})^{-1}]\frac{dt}{\sqrt{t}}\right\|_{p\to p} \le C. \tag{2.2}$$

For the remaining term, by the following Proposition 2.1 and Proposition 2.2, we see that there exists $\alpha > 0$ such that

$$\|\nabla[(1+t\mathcal{L}_0)^{-1} - (1+t\mathcal{L})^{-1}]\|_{p\to p} \le Ct^{-\alpha}\|\nabla(1+t\mathcal{L})^{-1/2}\|_{p\to p}. \tag{2.3}$$

Note here we may take $\alpha = \min\{\frac{\epsilon(p-1)}{2p}, \frac{\epsilon(p_0-p)}{2p(p_0+p)}\} = \frac{\epsilon(p_0-p)}{2p(p_0+p)}$ if $p_0 < \infty$, and $\alpha = \frac{\epsilon}{2p}$ when $p_0 = \infty$.

By using the boundedness of the local Riesz transform

$$\|\nabla(1+\mathcal{L})^{-1/2}\|_{p\to p}\le C,$$

one obtains for any $t>1$ that

$$\|\nabla(1+t\mathcal{L})^{-1}\|_{p\to p}\le \|\nabla(1+\mathcal{L})^{-1/2}\|_{p\to p}\|(1+\mathcal{L})^{1/2}(1+t\mathcal{L})^{-1}\|_{p\to p}\le C.$$

This together with Lemma A.2 implies that

$$\|\nabla(1+t\mathcal{L})^{-1/2}\|_{p\to p}\le C.$$

Inserting this into (2.3), one finds

$$\|\nabla[(1+t\mathcal{L}_0)^{-1}-(1+t\mathcal{L})^{-1}]\|_{p\to p}\le Ct^{-\alpha},$$

and

$$\|\nabla(1+t\mathcal{L})^{-1}]\|_{p\to p}\le Ct^{-\alpha}+\|\nabla(1+t\mathcal{L}_0)^{-1}\|_{p\to p}\le Ct^{-\alpha\wedge 1/2}. \tag{2.4}$$

Above we used the fact

$$\begin{aligned}\|\nabla(1+t\mathcal{L}_0)^{-1}\|_{p\to p}&\le \|\nabla L_0^{-1/2}\|_{p\to p}\|\mathcal{L}_0^{1/2}(1+t\mathcal{L}_0)^{-1}\|_{p\to p}\\&\le C\left\|\int_0^\infty \mathcal{L}_0^{1/2}e^{-s-st\mathcal{L}_0}\,ds\right\|_{p\to p}\\&\le C\int_0^\infty \frac{e^{-s}}{\sqrt{st}}\,ds\le Ct^{-1/2}.\end{aligned}$$

Now inserting (2.4) into (2.3), we get

$$\|\nabla[(1+t\mathcal{L}_0)^{-1}-(1+t\mathcal{L})^{-1}]\|_{p\to p}\le Ct^{-\alpha-\alpha\wedge 1/2},$$

and

$$\|\nabla(1+t\mathcal{L})^{-1}]\|_{p\to p}\le Ct^{-2\alpha\wedge 1/2}.$$

Repeating this argument finitely many times (depending on α), we arrive at

$$\|\nabla[(1+t\mathcal{L}_0)^{-1}-(1+t\mathcal{L})^{-1}]\|_{p\to p}\le Ct^{-\alpha-1/2},\forall\, t>1, \tag{2.5}$$

and

$$\|\nabla(1+t\mathcal{L})^{-1}]\|_{p\to p}\le Ct^{-1/2}. \tag{2.6}$$

Inserting (2.5) into the term II, we conclude that

$$\left\|\frac{1}{\pi}\int_1^\infty \nabla[(1+t\mathcal{L}_0)^{-1}-(1+t\mathcal{L})^{-1}]\frac{dt}{\sqrt{t}}\right\|_{p\to p}\le C\int_1^\infty t^{-\alpha-1/2}\frac{dt}{\sqrt{t}}\le C. \tag{2.7}$$

Combining the estimates of (2.2) and (2.7), we get

$$\|\nabla\mathcal{L}_0^{-1/2}-\nabla\mathcal{L}^{-1/2}\|_{p\to p}\le C,$$

and hence,

$$\|\nabla\mathcal{L}^{-1/2}\|_{p\to p}\le C,$$

as desired. □

Let us estimate the difference $\nabla[(1+t\mathcal{L}_0)^{-1}-(1+t\mathcal{L})^{-1}]$ for $t>1$ to show (2.3).

Set $|g|=|\det(g_{ij})|$ and $|g_0|=|\det(g_0)_{ij}|$. Let us begin with the formula

$$\begin{aligned}\mathcal{L}_0-\mathcal{L}&=\frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j\right)-\frac{1}{\sqrt{|g_0|}}\partial_i\left(\sqrt{|g_0|}g_0^{ij}\partial_j\right)\\&=\frac{1}{\sqrt{|g|}}\partial_i\left(\left[\sqrt{|g|}g^{ij}-\sqrt{|g_0|}g_0^{ij}\right]\partial_j\right)-\frac{\sqrt{|g|}-\sqrt{|g_0|}}{\sqrt{|g_0|}\sqrt{|g|}}\partial_i\left(\sqrt{|g_0|}g_0^{ij}\partial_j\right),\end{aligned}$$

and for any $t>0$,

$$(1+t\mathcal{L})^{-1}-(1+t\mathcal{L}_0)^{-1}=t(1+t\mathcal{L})^{-1}\left(\mathcal{L}_0-\mathcal{L}\right)(1+t\mathcal{L}_0)^{-1}.$$

Set

$$\begin{aligned}I_t^1&=t\nabla(1+t\mathcal{L})^{-1}\left(\frac{1}{\sqrt{|g|}}\partial_i\left(\left[\sqrt{|g|}g^{ij}-\sqrt{|g_0|}g_0^{ij}\right]\partial_j\right)\right)(1+t\mathcal{L}_0)^{-1}\\&=t\nabla(1+t\mathcal{L})^{-1}\left(\frac{1}{\sqrt{|g|}}\partial_i\left(\left[\sqrt{|g|}(\delta_{ik}-g_0^{ij}g_{jk})g^{kj}\right]\partial_j\right)-\frac{1}{\sqrt{|g|}}\partial_i\left(\left[\sqrt{|g_0|}-\sqrt{|g|}\right]g_0^{ij}\partial_j\right)\right)(1+t\mathcal{L}_0)^{-1}\end{aligned}$$

and

$$\begin{aligned}II_t^2&=t\nabla(1+t\mathcal{L})^{-1}\left(\frac{\sqrt{|g|}-\sqrt{|g_0|}}{\sqrt{|g_0|}\sqrt{|g|}}\partial_i\left(\sqrt{|g_0|}g_0^{ij}\partial_j\right)\right)(1+t\mathcal{L}_0)^{-1}\\&=t\nabla(1+t\mathcal{L})^{-1}\left(1-\sqrt{|g_0|/|g|}\right)\mathcal{L}_0(1+t\mathcal{L}_0)^{-1}.\end{aligned}$$

Note that (1.1) implies that $C^{-1}|g|\le|g_0|\le C|g|$. Moreover, from the assumption

$$\fint_{B_0(x,r)}|\delta_{ik}-g_0^{ij}g_{jk}|\,d\mu_0\le Cr^{-\epsilon},\ \forall\ 1\le i,k\le n,\ \forall x\in M\ \&\ \forall\, r>1,$$

for some $\epsilon>0$, one deduces that for all $x\in M$ and all $r>1$

$$\fint_{B_0(x,r)}|1-\det(g_0^{ij}g_{jk})|\,d\mu_0=\fint_{B_0(x,r)}\left|1-\frac{|g|}{|g_0|}\right|\,d\mu_0\sim\fint_{B(x,r)}\left|1-\frac{|g|}{|g_0|}\right|\,d\mu\le Cr^{-\epsilon}. \tag{2.8}$$

Proposition 2.1. *Assume that (M, g_0) satisfies (D) and (GUB), and that (1.1) holds. Suppose that there exists $\epsilon > 0$ such that*

$$\fint_{B_0(x,r)} |\delta_{ik} - g_0^{ij} g_{jk}|\, d\mu_0 \le C r^{-\epsilon},\ \forall\ 1 \le i, k \le n,\ \forall x \in M\ \&\ \forall\ r > 1.$$

Then for any $p \in (2, \infty)$, there exists $C > 0$ such that for each $t > 1$

$$\|II_t^2\|_{p\to p} \le C t^{-\epsilon(p-1)/2p} \|\nabla(1 + t\mathcal{L})^{-1/2}\|_{p\to p}.$$

Proof. **Step 1.** We claim that it holds for each $t > 1$ that

$$\left\|(1 + t\mathcal{L})^{-1/2}\left(1 - \sqrt{|g_0|/|g|}\right)\right\|_{p\to p} \le C t^{-\epsilon(p-1)/2p}.$$

For any $f \in C_c^\infty(M)$ one has

$$(1 + t\mathcal{L})^{-1/2}\left(1 - \sqrt{|g_0|/|g|}\right) f(x) = C \int_0^\infty e^{-s(1+t\mathcal{L})} \left(1 - |g_0|/|g|\right) f(x) \frac{ds}{\sqrt{s}}.$$

For the first term, by using the fact $|g| \sim |g_0|$ we conclude that

$$\begin{aligned}
\left\|\int_0^{1/t} e^{-s(1+t\mathcal{L})}\left([1 - \sqrt{|g_0|/|g|}]f\right)(x)\frac{ds}{\sqrt{s}}\right\|_p &\le \int_0^{1/t} \left\|e^{-s(1+t\mathcal{L})}\left((1 - \sqrt{|g_0|/|g|})f\right)\right\|_p \frac{ds}{\sqrt{s}}\\
&\le \int_0^{1/t} e^{-s} \left\|(1 - \sqrt{|g_0|/|g|})f\right\|_p \frac{ds}{\sqrt{s}}\\
&\le \frac{C}{\sqrt{t}} \|f\|_p.
\end{aligned}$$

Let $p_t(x, y)$ denote the heat kernel of $e^{-t\mathcal{L}}$. For the remaining estimate, note that the heat kernel of $e^{-st\mathcal{L}}$ satisfies

$$0 < p_{st}(x, y) \le \frac{C}{V(x, \sqrt{st})} e^{-\frac{d(x,y)^2}{cst}},$$

which is a consequence of (M, g_0) satisfying (GUB) and $g \sim g_0$.

Using the Hölder inequality and $|g| \sim |g_0|$ again, one concludes that

$$\begin{aligned}
&\int_M p_{st}(x, y)\left(1 - \sqrt{|g_0|/|g|}(y)\right)|f(y)|\, d\mu(y)\\
&\le \left(\int_M \frac{Ce^{-\frac{d(x,y)^2}{2cst}}}{V(x, \sqrt{st})}|f(y)|^p\, d\mu(y)\right)^{1/p} \left(\int_M \frac{Ce^{-\frac{d(x,y)^2}{2cst}}}{V(x, \sqrt{st})}\left(1 - \sqrt{|g_0|/|g|}(y)\right)^{p'}\, d\mu(y)\right)^{1/p'}\\
&\le \left(\int_M \frac{Ce^{-\frac{d(x,y)^2}{2cst}}}{V(x, \sqrt{st})}|f(y)|^p\, d\mu(y)\right)^{1/p} \left(\sum_{k=1}^\infty \frac{e^{-c2^{2k}}}{V(x, \sqrt{st})}\int_{B(x,2^k\sqrt{st})} \left|1 - \sqrt{|g_0|/|g|}(y)\right|^{p'}\, d\mu(y)\right)^{1/p'}
\end{aligned}$$

$$\leq \left(\int_M \frac{Ce^{-\frac{d(x,y)^2}{2cst}}}{V(x,\sqrt{st})}|f(y)|^p\,d\mu(y)\right)^{1/p}\left(\sum_{k=1}^{\infty}\frac{e^{-c2^k}}{V(x,2^k\sqrt{st})}\int_{B(x,2^k\sqrt{st})}\left|1-\sqrt{|g_0|/|g|}(y)\right|\,d\mu(y)\right)^{1/p'}$$
$$\leq \frac{C}{(st)^{\epsilon/2p'}}\left(\int_M \frac{e^{-\frac{d(x,y)^2}{2cst}}}{V(x,\sqrt{st})}|f(y)|^p\,d\mu(y)\right)^{1/p},$$

where in the last inequality we used the estimate

$$\frac{1}{V(x,2^k\sqrt{st})}\int_{B(x,2^k\sqrt{st})}\left|\frac{\sqrt{|g|}-\sqrt{|g_0|}}{\sqrt{|g|}}\right|\,d\mu = \fint_{B(x,2^k\sqrt{st})}\frac{||g|-|g_0||}{\sqrt{|g|}\sqrt{|g|+|g_0|}}\,d\mu$$
$$\leq C\fint_{B(x,2^k\sqrt{st})}\frac{||g|-|g_0||}{|g_0|}\,d\mu \leq \frac{C}{(2^k\sqrt{st})^{\epsilon}}, \tag{2.9}$$

where the last estimate follows from (2.8). Using this, one deduces that

$$\left\|\int_{1/t}^{\infty} e^{-s(1+t\mathcal{L})}\left(\left|1-\sqrt{|g_0|/|g|}(y)\right|f\right)(x)\frac{ds}{\sqrt{s}}\right\|_p$$
$$\leq C\int_{1/t}^{\infty}\frac{C}{(st)^{\epsilon/2p'}}e^{-s}\left(\int_M\int_M \frac{1}{V(x,\sqrt{st})}e^{-\frac{d(x,y)^2}{2cst}}|f(y)|^p\,d\mu(y)\,d\mu(x)\right)^{1/p}\frac{ds}{\sqrt{s}}$$
$$\leq C\|f\|_p\int_{1/t}^{\infty}\frac{C}{(st)^{\epsilon/2p'}}e^{-s}\frac{ds}{\sqrt{s}}$$
$$\leq Ct^{-\epsilon/2p'}\|f\|_p,$$

where $\epsilon\in(0,1)$, $p'\in(1,2)$. This and the estimate for the first term completes the proof of **Step 1**.

Step 2. Noticing that

$$\|\mathcal{L}_0(1+t\mathcal{L}_0)^{-1}\|_{p\to p} = 1/t\left\|1-(1+t\mathcal{L}_0)^{-1}\right\|_{p\to p}\leq C/t,$$

which together with the first step gives that

$$\|II_t^2\|_{p\to p}\leq Ct^{-\epsilon/2p'}\|\nabla(1+t\mathcal{L})^{-1/2}\|_{p\to p},$$

which completes the proof. □

Recall that $\nabla, \nabla_0, \mathrm{div}, \mathrm{div}_0$ are Riemannian gradients and divergences induced by g, g_0, respectively.

Proposition 2.2. *Assume that* (M,g_0) *satisfies* (D) *and* (GUB)*, and that* (1.1) *holds. Suppose that* $\nabla\mathcal{L}_0^{-1/2}$ *is bounded on* $L^{p_0}(M)$ *for some* $p_0\in(2,\infty)$*, and there exists* $\epsilon>0$ *such that*

$$\fint_{B_0(x,r)}|\delta_{ik}-g_0^{ij}g_{jk}|\,d\mu_0\leq Cr^{-\epsilon},\ \forall\ 1\leq i,k\leq n,\ \forall x\in M\ \&\ \forall\, r>1.$$

Then for each $p\in(2,p_0)$ *there exists* $C>0$ *such that for each* $t>1$

$$\|I_t^1\|_{p\to p}\leq Ct^{-\frac{\epsilon(p_0-p)}{2p(p_0+p)}}\|\nabla(1+t\mathcal{L})^{-1/2}\|_{p\to p}.$$

Proof. Set $A = \{a_{ik}\}_{1\le i,k\le n} = \{g_0^{ij} g_{jk}\}_{1\le i,k\le n}$. For simplicity of notations, we represent I_t^1 in term of Riemannian gradient and divergence as

$$I_t^1 = t\nabla(1+t\mathcal{L})^{-1}\left(\frac{1}{\sqrt{|g|}}\partial_i\left(\left[\sqrt{|g|}(\delta_{ik} - g_0^{il}g_{lk})g^{kj}\right]\partial_j\right) - \frac{1}{\sqrt{|g|}}\partial_i\left(\left[\sqrt{|g_0|} - \sqrt{|g|}\right]g_0^{ij}\partial_j\right)\right)(1+t\mathcal{L}_0)^{-1}$$
$$= t\nabla(1+t\mathcal{L})^{-1}\mathrm{div}\left((I-A)\nabla - \left|\sqrt{|g_0|/|g|}-1\right|\nabla_0\right)(1+t\mathcal{L}_0)^{-1}.$$

Step 1. Noting that (M, g_0) satisfies (D) and (GUB), and that (1.1) holds, (M, g) also satisfies (D) and (GUB). It follows from [17] that $\nabla\mathcal{L}^{-1/2}$ and $\nabla\mathcal{L}_0^{-1/2}$ are bounded on $L^q(M)$ for all $q \in (1, 2)$.

Since $(1+t\mathcal{L})^{-1/2}\mathrm{div}$ is the dual operator of $\nabla(1+t\mathcal{L})^{-1/2}$, and $\nabla\mathcal{L}^{-1/2}$ is bounded on $L^{p'}(M)$, $p' \in (1, 2)$, one has that

$$\left\|(1+t\mathcal{L})^{-1/2}\mathrm{div}\right\|_{p\to p} = \left\|\nabla(1+t\mathcal{L})^{-1/2}\right\|_{p'\to p'} \le C/\sqrt{t},\ \forall\, t > 1.$$

Step 2. We claim that it holds

$$\left\|\left((I-A)\nabla - \left|\sqrt{|g_0|/|g|}-1\right|\nabla_0\right)(1+t\mathcal{L}_0)^{-1}\right\|_{p\to p} \le Ct^{-1/2-\epsilon(p_0-p)/(2p(p_0+p))},\ \forall\, t > 1.$$

For any $f \in C_c^\infty(M)$, let us first show that

$$\left\|((I-A)\nabla)(1+t\mathcal{L}_0)^{-1}f\right\|_p \le Ct^{-1/2-\epsilon(p_0-p)/(2p(p_0+p))}\|f\|_p,$$

the other term can be estimated similarly. We write

$$\left\|((I-A)\nabla)(1+t\mathcal{L}_0)^{-1}f\right\|_p$$
$$\le \int_0^{1/t}\|((I-A)\nabla)e^{-s(1+t\mathcal{L}_0)}f\|_p\,ds + \int_{1/t}^\infty\|((I-A)\nabla)e^{-s(1+t\mathcal{L}_0)}f\|_p\,ds.$$

As

$$\|\nabla\mathcal{L}_0^{-1/2}\|_{p_0\to p_0} \le C,$$

one has

$$\|\nabla\mathcal{L}_0^{-1/2}\|_{p\to p} \le C$$

for any $p \in (2, p_0)$, and hence for any $s > 0$ that

$$(G_p)\qquad \|\nabla e^{-s\mathcal{L}_0}\|_{p\to p} \le Cs^{-1/2}.$$

Therefore by using (G_p) and that A is bounded, one has the estimate

$$\int_0^{1/t}\|((I-A)\nabla)e^{-s(1+t\mathcal{L}_0)}f\|_p\,ds \le C\int_0^{1/t}\|\nabla e^{-s(1+t\mathcal{L}_0)}f\|_p\,ds$$

$$
\begin{aligned}
&\le C\int_0^{1/t}\frac{e^{-s}}{\sqrt{st}}\|f\|_p\,ds\\
&\le Ct^{-1}\|f\|_p. \qquad (2.10)
\end{aligned}
$$

The estimate of the remaining integrand over $(1/t,\infty)$ is more involved. By using the boundedness of $\nabla\mathcal{L}_0^{-1/2}$ on $L^{p_0}(M)$ and [6, Proposition 1.10], we see that, for any $p\in(2,p_0)$ there exist $C,\gamma_p>0$ such that for all $t>0$ and $y\in M$

$$
\int_M |(\nabla_0)_x P_t(x,y)|^p \exp\left\{\gamma_p d^2(x,y)/t\right\}\,d\mu_0(x)\le \frac{C}{t^{p/2}V_0(y,\sqrt{t})^{p-1}}, \qquad (GLY_p)
$$

where $P_t(x,y)$ denotes the heat kernel of $e^{-t\mathcal{L}_0}$. By using (1.1) that $g\sim g_0$, we see that (GLY_p) is equivalent to

$$
\int_M |\nabla_x P_t(x,y)|^p \exp\left\{\gamma_p d^2(x,y)/t\right\}\,d\mu(x)\le \frac{C}{t^{p/2}V_0(y,\sqrt{t})^{p-1}}.
$$

In what follows we shall not distinguish these two estimate.

Let $\gamma\in(0,\gamma_0)$ to be fixed later. By using the Hölder inequality, one sees that

$$
\begin{aligned}
|\nabla e^{-st\mathcal{L}_0}f(x)| &\le C\int_M |\nabla_x P_{st}(x,y)||f(y)|\,d\mu(y)\\
&\le C\left(\int_M |\nabla_x P_{st}(x,y)|^p\exp\left\{\frac{\gamma d^2(x,y)}{st}\right\}V_0(y,\sqrt{t})^{p-1}|f(y)|^p\,d\mu(y)\right)^{1/p}\\
&\quad\times\left(\int_M V_0(y,\sqrt{st})^{-1}\exp\left\{-\frac{c(p)\gamma d^2(x,y)}{st}\right\}\,d\mu(y)\right)^{1/p'}\\
&\le C\left(\int_M |\nabla P_{st}(x,y)|^p\exp\left\{\frac{\gamma d^2(x,y)}{st}\right\}V_0(y,\sqrt{t})^{p-1}|f(y)|^p)\,d\mu(y)\right)^{1/p}. \qquad (2.11)
\end{aligned}
$$

Above in the last inequality we used the doubling condition to conclude that for any $x,y\in M$ and any $r>0$ that

$$
V_0(y,r)^{-1}\le CV_0(x,r+d(x,y))^{-1}\left(\frac{r+d(x,y)}{r}\right)^{\Upsilon}\le CV_0(x,r)^{-1}\left(\frac{r+d(x,y)}{r}\right)^{\Upsilon}
$$

for some $\Upsilon>0$, and therefore

$$
\begin{aligned}
&\int_M V_0(y,\sqrt{st})^{-1}\exp\left\{-\frac{c(p)\gamma d^2(x,y)}{st}\right\}\,d\mu(y)\\
&\le\int_M V_0(x,\sqrt{st})^{-1}\left(\frac{\sqrt{st}+d(x,y)}{\sqrt{st}}\right)^{\Upsilon}\exp\left\{-\frac{c(p)\gamma d^2(x,y)}{st}\right\}\,d\mu(y)\\
&\le\int_M V_0(x,\sqrt{st})^{-1}\exp\left\{-\frac{c(p,\gamma)d^2(x,y)}{st}\right\}\,d\mu(y)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum_{k=1}^{\infty} V_0(x,\ \sqrt{st})^{-1} V_0(x, 2^k \sqrt{st}) \exp\left\{-c(p,\gamma)2^{2k}\right\} \\
&\leq C.
\end{aligned}
$$

Inequality (2.11) gives that

$$
\begin{aligned}
&\left\|(I-A)\nabla e^{-st\mathcal{L}_0} f\right\|_p^p \\
&\leq C \int_M \int_M |(I-A)|^p |\nabla P_{st}(x,y)|^p \exp\left\{\frac{\gamma d^2(x,y)}{st}\right\} V_0(y,\ \sqrt{t})^{p-1} |f(y)|^p \, d\mu(y)\, d\mu(x).
\end{aligned}
$$

Note that $p < p_0$. Letting $\delta = (p_0 - p)/2$, $q = (p+p_0)/2$ and $\gamma \in (0, \gamma_q)$ such that $2(p+\delta)\gamma/p = \gamma_q$, we conclude that

$$
\begin{aligned}
&\int_M |I - A(x)|^p |\nabla_x P_{st}(x,y)|^p \exp\left\{\frac{\gamma d^2(x,y)}{st}\right\} d\mu(x) \\
&\leq \left(\int_M |I - A(x)|^{p(p+\delta)/\delta} \exp\left\{\frac{-(p+\delta)\gamma d^2(x,y)}{\delta st}\right\} d\mu(x)\right)^{\delta/(p+\delta)} \\
&\quad \times \left(\int_M |\nabla_x P_{st}(x,y)|^{p+\delta} \exp\left\{\frac{2(p+\delta)\gamma d^2(x,y)}{pst}\right\} d\mu(x)\right)^{p/(p+\delta)} \\
&\leq \frac{C}{(st)^{p/2} V_0(y,\ \sqrt{st})^{p-p/q}} \left(\int_M |I - A(x)|^{pq/\delta} e^{-q\gamma \frac{d(x,y)^2}{\delta st}} \, d\mu(x)\right)^{\delta/q} \\
&\leq \frac{C}{(st)^{p/2} V_0(y,\ \sqrt{st})^{p-1}} \left(\frac{1}{V_0(y,\ \sqrt{st})} \int_M |I - A(x)|^{pq/\delta} e^{-q\gamma \frac{d(x,y)^2}{\delta st}} \, d\mu(x)\right)^{\delta/q},
\end{aligned}
$$

where by (2.8) one has

$$
\begin{aligned}
&\left(\frac{1}{V_0(y,\ \sqrt{st})} \int_M |I - A(x)|^{pq/\delta} e^{-q\gamma \frac{d(x,y)^2}{\delta st}} \, d\mu(x)\right)^{\delta/q} \\
&\leq \left(\frac{C}{V_0(y,\ \sqrt{st})} \int_M |I - A(x)| e^{-q\gamma \frac{d(x,y)^2}{\delta st}} \, d\mu(x)\right)^{\delta/q} \\
&\leq \left(\sum_{k=1}^{\infty} \frac{e^{-c2^{2k}}}{V_0(y,\ \sqrt{st})} \int_{B(y, 2^k \sqrt{st})} |I - A(x)| \, d\mu(x)\right)^{\delta/q} \\
&\leq C(st)^{-\epsilon\delta/2q}.
\end{aligned}
$$

We can therefore conclude that

$$
\int_M |I - A(x)|^p |\nabla_x P_{st}(x,y)|^p \exp\left\{\frac{\gamma d^2(x,y)}{st}\right\} d\mu(x) \leq \frac{C}{(st)^{p/2+\epsilon\delta/2q} V_0(y,\ \sqrt{st})^{p-1}},
$$

and

$$
\|(I-A)\nabla e^{-st\mathcal{L}_0} f\|_p^p
$$

$$\leq C\int_M\int_M |I-A(x)|^p|\nabla_x P_{st}(x,y)|^p \exp\left\{\frac{\gamma d^2(x,y)}{st}\right\} V_0(y,\sqrt{st})^{p-1}|f(y)|^p\,d\mu(y)\,d\mu(x)$$
$$\leq C\frac{C}{(st)^{p/2+\epsilon\delta/2q}}\|f\|_p^p.$$

We finally get the estimate of the second term by

$$\int_{1/t}^{\infty}\|(I-A)\nabla e^{-s(1+t\mathcal{L}_0)}f\|_p\,ds \leq \int_{1/t}^{\infty}\frac{Ce^{-s}}{(st)^{1/2+\epsilon\delta/(2pq)}}\|f\|_p\,ds \leq Ct^{-1/2-\epsilon\delta/(2pq)}\|f\|_p.$$

This together with (2.10) implies that

$$\left\|(I-A)\nabla(1+t\mathcal{L}_0)^{-1}f\right\|_p \leq Ct^{-1/2-\epsilon\delta/(2pq)}\|f\|_p.$$

By the same proof, one sees that

$$\left\|\left(\left|\sqrt{|g_0|/|g|}-1\right|\nabla_0\right)(1+t\mathcal{L}_0)^{-1}\right\|_{p\to p} \leq Ct^{-1/2-\epsilon\delta/(2pq)}.$$

The above two estimates complete the proof of **Step 2**.

Finally, by combining the estimates from **Step 1** and **Step 2**, we see that

$$\|I_t^1\|_{p\to p} \leq Ct^{-\epsilon(p_0-p)/(2p(p_0+p))}\|\nabla(1+t\mathcal{L})^{-1/2}\|_{p\to p},$$

which completes the proof. □

In Proposition 2.2, there is a loss of integrability, which is somehow nature from the point of view of comparing arguments; see also [12]. If we strengthen the assumption from (GUB) to two side bounds of the heat kernel, then by using the open-ended property of the Riesz transform (cf. [18]) we have the end-point estimate.

We say that the heat kernel satisfies Li-Yau estimate if there exist $C, c > 0$ such that for all $t > 0$ and all $x, y \in M$.

$$(LY)\qquad \frac{C^{-1}}{V(x,\sqrt{t})}\exp\left\{-\frac{d^2(x,y)}{ct}\right\} \leq p_t(x,y) \leq \frac{C}{V(x,\sqrt{t})}\exp\left\{-c\frac{d^2(x,y)}{t}\right\}.$$

By [34, 35, 25], the Li-Yau estimate is equivalent to the fact that (M,g) satisfies (D) and a scale invariant Poincaré inequality (PI), i.e.,

$$(PI)\qquad \fint_{B(x,r)}|f-f_B|\,d\mu \leq Cr\left(\fint_{B(x,r)}|\nabla f|^2\,d\mu\right)^{1/2}.$$

As (D) and (PI) are invariant under quasi-isometries, the Li-Yau estimate is invariant under quasi-isometries.

Proposition 2.3. *Assume that* (M, g_0) *satisfies* (D) *and* (PI)*, and that* (1.1) *holds. Suppose that* $\nabla\mathcal{L}_0^{-1/2}$ *is bounded on* $L^{p_0}(M)$ *for some* $p_0 \in (2,\infty)$*, and there exists* $\epsilon > 0$ *such that*

$$\fint_{B_0(x,r)} |\delta_{ik} - g_0^{ij} g_{jk}|\, d\mu_0 \le C r^{-\epsilon},\ \forall\ 1 \le i,k \le n,\ \forall x \in M \,\&\, \forall\, r > 1.$$

Then there exist $C > 0$ *and* $\alpha > 0$ *such that for each* $t > 1$

$$\|I_t^1\|_{p_0\to p_0} \le C t^{-\alpha} \|\nabla(1+t\mathcal{L})^{-1/2}\|_{p_0\to p_0}.$$

Proof. Note that, under (D) and (PI), the boundedness of the Riesz transform has an open-ended character, cf. [18, Theorem 1.9]. Therefore, there exists $\delta > 0$ such that $\nabla\mathcal{L}_0^{-1/2}$ is bounded on $L^{p_0+\delta}(M)$, which together with [18, Theorem 1.6] implies that

$$(GLY_{p_0+\delta}) \qquad \int_M |\nabla_x P_t(x,y)|^{p+\delta} \exp\left\{\gamma d^2(x,y)/t\right\} d\mu_0(x) \le \frac{C}{t^{(p_0+\delta)/2} V_0(y,\sqrt{t})^{p_0+\delta-1}}.$$

Using $(GLY_{p_0+\delta})$ instead of (GLY_p) in the proof of Proposition 2.2, we see that there exists $\alpha > 0$ such that for each $t > 1$

$$\|I_t^1\|_{p_0\to p_0} \le C t^{-\alpha} \|\nabla(1+t\mathcal{L})^{-1/2}\|_{p_0\to p_0},$$

as desired. □

Theorem 2.4. *Assume that* (M, g_0) *satisfies* (D) *and* (PI)*, and that* (1.1) *holds. Suppose that* $\nabla\mathcal{L}_0^{-1/2}$ *is bounded on* $L^p(M)$ *for some* $p \in (2,\infty)$*, and there exists* $\epsilon > 0$ *such that*

$$\fint_{B_0(x,r)} |\delta_{ik} - g_0^{ij} g_{jk}|\, d\mu_0 \le C r^{-\epsilon},\ \forall\ 1 \le i,k \le n,\ \forall x \in M \,\&\, \forall\, r > 1.$$

Then if $\nabla(1+\mathcal{L})^{-1/2}$ *bounded on* $L^p(M)$*,* $\nabla\mathcal{L}^{-1/2}$ *is bounded on* $L^p(M)$*.*

Proof. The conclusion follows from the same proof of Theorem 1.2, using Proposition 2.3 instead of Proposition 2.2. □

We can now finish the proofs for corollaries of Theorem 1.2.

Proof of Corollary 1.3. Noting that (M, g) has Ricci curvature bounded from below, the local Riesz transform $\nabla(1+\mathcal{L})^{-1/2}$ is bounded on $L^p(M)$ for all $p \in (1,\infty)$; see [6]. The conclusion then follows from Theorem 1.2. □

Proof of Corollary 1.4. If g coincides with g_0 outside a compact subset M_0, then (1.1) holds. By using (D) together with the connectivity of M, one sees there exists $0 < \upsilon \le \Upsilon < \infty$ such that for any $y \in M$ and $0 < r < R < \infty$

$$\frac{1}{C}\left(\frac{R}{r}\right)^{\upsilon} \le \frac{V_0(y,R)}{V_0(y,r)} \le C\left(\frac{R}{r}\right)^{\Upsilon}; \tag{2.12}$$

see [27] for instance.

Note that it holds

$$\fint_{B_0(x,r)} |\delta_{ik} - g_0^{ij} g_{jk}|\, d\mu_0 \le \frac{C}{V_0(x,r)} \int_{M_0 \cap B_0(x,r)} |\delta_{ik} - g_0^{ij} g_{jk}|\, d\mu_0,\ \forall x \in M \,\&\, \forall\, r > 1.$$

Fix $x_0 \in M_0$. If $B_0(x,r) \cap M_0 = \emptyset$, then

$$\fint_{B_0(x,r)} |\delta_{ik} - g_0^{ij} g_{jk}|\, d\mu_0 = 0. \tag{2.13}$$

Otherwise, by using (2.12), one has

$$\begin{aligned} V_0(x_0, \mathrm{diam}(M_0)) &\le C \left(\frac{\mathrm{diam}(M_0)}{r + \mathrm{diam}(M_0)} \right)^{\upsilon} V_0(x_0, r + \mathrm{diam}(M_0)) \\ &\le C \left(\frac{\mathrm{diam}(M_0)}{r + \mathrm{diam}(M_0)} \right)^{\upsilon} V_0(x, 2r + 2\mathrm{diam}(M_0)) \\ &\le C \left(\frac{\mathrm{diam}(M_0)}{r + \mathrm{diam}(M_0)} \right)^{\upsilon} \left(\frac{r + \mathrm{diam}(M_0)}{r} \right)^{\Upsilon} V_0(x, r), \end{aligned}$$

and therefore,

$$\begin{aligned} &\fint_{B_0(x,r)} |\delta_{ik} - g_0^{ij} g_{jk}|\, d\mu_0 \\ &\le C \left(\frac{\mathrm{diam}(M_0)}{r + \mathrm{diam}(M_0)} \right)^{\upsilon} \left(\frac{r + \mathrm{diam}(M_0)}{r} \right)^{\Upsilon} \frac{1}{V_0(x_0, \mathrm{diam}(M_0))} \int_{M_0} |\delta_{ik} - g_0^{ij} g_{jk}|\, d\mu_0. \end{aligned}$$

This together with (2.13) implies that for all $x \in M$ and all $r > 1$ it holds

$$\fint_{B_0(x,r)} |\delta_{ik} - g_0^{ij} g_{jk}|\, d\mu_0 \le \frac{C}{r^{\upsilon}},$$

where C depends on M_0.

By applying Theorem 1.2, together with the fact that (M, g) and (M, g_0) have lower Ricci curvature bounds, we see $\nabla_0 \mathcal{L}_0^{-1/2}$ is bounded on $L^p(M)$ for all $p \in (2, p_0)$, if and only if, $\nabla \mathcal{L}^{-1/2}$ is bounded on $L^p(M)$ for all $p \in (2, p_0)$. □

Proof of Corollary 1.5. Note that since (M, g_0) has non-negative Ricci curvature, $\nabla_0 \mathcal{L}_0^{-1/2}$ is bounded on $L^p(M)$ for all $p \in (1, \infty)$; see [6]. This together with Corollary 1.4 implies that $\nabla \mathcal{L}^{-1/2}$ is bounded on $L^p(M)$ for all $p \in (2, \infty)$.

Moreover, since (D) and (GUB) hold on (M, g_0) as a consequence of non-negative Ricci curvature (cf. [31]), (D) and (GUB) hold on (M, g). By [17] we see that $\nabla \mathcal{L}^{-1/2}$ is bounded on $L^p(M)$ for all $p \in (1, 2)$. □

3 Degenerate elliptic equations

In this section, we deal with degenerate elliptic equations on Euclidean spaces. Let $A_2(\mathbb{R}^n)$ denote the collection of A_2-Muckenhoupt weights, and $QC(\mathbb{R}^n)$ denote the collection of all quasi-conformal weights, i.e., $w \in QC(\mathbb{R}^n)$ if there exists a quasi-conformal mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $w = |J_f|^{1-2/n}$, where J_f denotes the determinant of the gradient matrix Df; see [24].

For $w, w_0 \in A_2(\mathbb{R}^n) \cup QC(\mathbb{R}^n)$, denote by

$$V(x,r) = \int_{B(x,r)} w\,dy, \quad V_0(x,r) = \int_{B(x,r)} w_0\,dy, \ \ \forall\, x \in \mathbb{R}^n \,\&\, r > 0.$$

For $w, w_0 \in A_2(\mathbb{R}^n) \cup QC(\mathbb{R}^n)$, the volumes V, V_0 satisfy the doubling condition, and there are scale-invariant Poincaré inequality (PI) on the spaces $(\mathbb{R}^n, w\,dx)$ and $(\mathbb{R}^n, w_0\,dx)$; see [24].

We will assume that $C^{-1}w \le w_0 \le Cw$ in what follows. As a consequence of the assumption, it holds that $C^{-1}V(x,r) \le V_0(x,r) \le CV(x,r)$ for any $x \in \mathbb{R}^n$ and $r > 0$, and $L^p(w) = L^p(w_0)$ for any $p > 0$. In what follows, we will not distinguish $L^p(w)$ and $L^p(w_0)$, and denote by $\|\cdot\|_{p\to p}$ the operator norm $\|\cdot\|_{L^p(w)\to L^p(w)}$.

In this section, we use ∇, div to denote the gradient operator and divergence operator on $\mathbb{R}^n$. We use the notation $B(x,r)$ for open ball under usual Euclidean metric of $\mathbb{R}^n$. The proof of results in this section is similar to that of Theorem 1.2, thus we only sketch their proofs.

Theorem 3.1. *Let A, A_0 be $n \times n$ matrixes that satisfy uniformly elliptic conditions, and $w_0, w \in A_2(\mathbb{R}^n) \cup QC(\mathbb{R}^n)$ with*

$$C^{-1}w(x) \le w_0(x) \le Cw(x), \ a.e.\ x \in \mathbb{R}^n.$$

Suppose there exists $\epsilon > 0$ such that

$$\frac{1}{V_0(y,r)} \int_{B(y,r)} \left(|A - A_0| + \frac{|w_0 - w|}{w_0} \right) w_0\,dx \le \frac{C}{r^\epsilon}, \quad \forall\, y \in \mathbb{R}^n \,\&\, r > 1.$$

Let $\mathcal{L} = -\frac{1}{w}\mathrm{div}(wA\nabla)$ and $\mathcal{L}_0 = -\frac{1}{w_0}\mathrm{div}(w_0A_0\nabla)$. Then if $\nabla\mathcal{L}_0^{-1/2}$ and $\nabla(1+\mathcal{L})^{-1/2}$ are bounded on $L^p(w)$ for some $p \in (2,\infty)$, $\nabla\mathcal{L}^{-1/2}$ is bounded on $L^p(w)$.

Proof. Let us begin with the formula that for any $t > 0$

$$\nabla(1+t\mathcal{L}_0)^{-1} - \nabla(1+t\mathcal{L})^{-1} = t\nabla(1+t\mathcal{L})^{-1}(\mathcal{L}_0 - \mathcal{L})(1+t\mathcal{L}_0)^{-1},$$

and

$$\begin{aligned}\nabla\mathcal{L}_0^{-1/2} - \nabla\mathcal{L}^{-1/2} &= \frac{1}{\pi}\int_0^1 \nabla[(1+t\mathcal{L}_0)^{-1} - (1+t\mathcal{L})^{-1}]\frac{dt}{\sqrt{t}} \\ &+ \frac{1}{\pi}\int_1^\infty \nabla[(1+t\mathcal{L}_0)^{-1} - (1+t\mathcal{L})^{-1}]\frac{dt}{\sqrt{t}}.\end{aligned}$$

Step 1. By Lemma A.1 and our assumptions that $\nabla\mathcal{L}_0^{-1/2}$ and $\nabla(1+\mathcal{L})^{-1/2}$ are bounded on $L^p(w_0)$, we have

$$\left\|\int_0^1 \nabla[(1+t\mathcal{L}_0)^{-1}-(1+t\mathcal{L})^{-1}]\frac{dt}{\sqrt{t}}\right\|_{p\to p}\le C.$$

Step 2. For the remaining term, by the following Proposition 3.2 and Proposition 3.3, we see that there exists $\alpha>0$ such that for any $t>1$

$$\|\nabla[(1+t\mathcal{L}_0)^{-1}-(1+t\mathcal{L})^{-1}]\|_{p\to p}\le Ct^{-\alpha}\|\nabla(1+t\mathcal{L})^{-1/2}\|_{p\to p}. \tag{3.1}$$

With this, repeating the same proof after (2.3) in the proof of Theorem 1.2, we conclude that $\nabla\mathcal{L}^{-1/2}$ is bounded on $L^p(w)$. □

Let us prove (3.1) in the following two propositions. Note that

$$\mathcal{L}_0-\mathcal{L}=\frac{1}{w}\mathrm{div}\,[((w-w_0)A-w_0(A_0-A))\nabla]-\frac{w-w_0}{w_0w}\mathrm{div}(w_0A_0\nabla),$$

and set

$$I_t^1=t\nabla(1+t\mathcal{L})^{-1}\left(\frac{1}{w}\mathrm{div}\left([(w-w_0)A-w_0(A_0-A)]\,\nabla(1+t\mathcal{L}_0)^{-1}\right)\right),$$

and

$$II_t^2=t\nabla(1+t\mathcal{L})^{-1}\left(\frac{w-w_0}{w}\mathcal{L}_0(1+t\mathcal{L}_0)^{-1}\right).$$

Then we have the following representation

$$\int_1^\infty \nabla[(1+t\mathcal{L}_0)^{-1}-(1+t\mathcal{L})^{-1}]\frac{dt}{\sqrt{t}}=\int_1^\infty(I_t^1+II_t^2)\frac{dt}{\sqrt{t}}.$$

Proposition 3.2. *Let A,A_0 be $n\times n$ matrixes that satisfy uniformly elliptic conditions, and $w_0,w\in A_2(\mathbb{R}^n)\cup QC(\mathbb{R}^n)$ with*

$$C^{-1}w(x)\le w_0(x)\le Cw(x),\ a.e.\ x\in\mathbb{R}^n.$$

Suppose there exists $\epsilon>0$ such that

$$\frac{1}{V_0(y,r)}\int_{B(y,r)}\left(\frac{|w_0-w|}{w_0}\right)w_0\,dx\le\frac{C}{r^\epsilon},\quad \forall\, y\in\mathbb{R}^n\ \&\ r>1.$$

Then for each $p\in(2,\infty)$, there exists $C>0$ such that for each $t>1$ it holds

$$\|II_t^2\|_{p\to p}\le Ct^{-\epsilon(p-1)/2p}\|\nabla(1+t\mathcal{L})^{-1/2}\|_{p\to p}.$$

Proof. Note that (D) and (PI) hold since $w_0, w \in A_2(\mathbb{R}^n) \cup QC(\mathbb{R}^n)$, the heat kernels $p_t^{\mathcal{L}}(x,y)$, $p_t^{\mathcal{L}_0}(x,y)$ of $e^{-t\mathcal{L}}$, $e^{-t\mathcal{L}_0}$ satisfy (LY) estimates by [34, 35], i.e.,

$$p_t^{\mathcal{L}}(x,y),\ p_t^{\mathcal{L}_0}(x,y) \sim \frac{1}{V(x,\sqrt{t})} e^{-\frac{d(x,y)^2}{ct}}.$$

Using this, and the assumption that

$$\frac{1}{V_0(y,r)} \int_{B(y,r)} \frac{|w_0 - w|}{w_0} w_0\, dx \le \frac{C}{r^\epsilon}, \quad \forall\, y \in \mathbb{R}^n\ \&\ r > 1,$$

we follow the proof of Proposition 2.1 to see that

$$\left\| (1 + t\mathcal{L})^{-1/2} \frac{w - w_0}{w} \right\|_{p \to p} \le C t^{-\epsilon(p-1)/2p}.$$

This, together with the estimate

$$\|\mathcal{L}_0 (1 + t\mathcal{L}_0)^{-1}\|_{p\to p} \le C/t$$

for all $t > 1$, implies

$$\|II_t^2\|_{p\to p} \le C t^{-\epsilon(p-1)/2p} \|\nabla (1 + t\mathcal{L})^{-1/2}\|_{p\to p}.$$

□

Proposition 3.3. *Let A, A_0 be $n \times n$ matrixes that satisfy uniformly elliptic conditions, and $w_0, w \in A_2(\mathbb{R}^n) \cup QC(\mathbb{R}^n)$ with*

$$C^{-1} w(x) \le w_0(x) \le C w(x),\ a.e.\ x \in \mathbb{R}^n.$$

Suppose there exists $\epsilon > 0$ such that

$$\frac{1}{V_0(y,r)} \int_{B(y,r)} \left(|A - A_0| + \frac{|w_0 - w|}{w_0} \right) w_0\, dx \le \frac{C}{r^\epsilon}, \quad \forall\, y \in \mathbb{R}^n\ \&\ r > 1.$$

Let $\mathcal{L} = -\frac{1}{w}\mathrm{div}(wA\nabla)$ and $\mathcal{L}_0 = -\frac{1}{w_0}\mathrm{div}(w_0 A_0 \nabla)$. Then if $\nabla \mathcal{L}_0^{-1/2}$ is bounded $L^p(w)$ for some $p \in (2,\infty)$, there exist C, $\alpha > 0$ such that for any $t > 1$

$$\|I_t^1\|_{p\to p} \le C t^{-\alpha} \|\nabla (1 + t\mathcal{L})^{-1/2}\|_{p\to p}.$$

Proof. Recall that

$$I_t^1 = t\nabla (1 + t\mathcal{L})^{-1} \left(\frac{1}{w} \mathrm{div} w \left[\left(\left(1 - \frac{w_0}{w}\right) A - \frac{w_0}{w}(A_0 - A) \right) \nabla (1 + t\mathcal{L}_0)^{-1} \right] \right),$$

Step 1. Since $(1 + t\mathcal{L})^{-1/2} \frac{1}{w}\mathrm{div} w$ is the dual operator of $\nabla(1 + t\mathcal{L})^{-1/2}$, and $\nabla \mathcal{L}^{-1/2}$ is bounded on $L^{p'}(w)$, one has that

$$\left\| (1 + t\mathcal{L})^{-1/2} \frac{1}{w} \mathrm{div} w \right\|_{p\to p} = \left\| \nabla (1 + t\mathcal{L})^{-1/2} \right\|_{p'\to p'} \le C/\sqrt{t}.$$

Step 2. As $w_0, w \in A_2(\mathbb{R}^n) \cup QC(\mathbb{R}^n)$, (D) and (PI) hold. By [18, Theorem 1.6 & Theorem 1.9], we see that there exists $p_0 > p$ such that $\nabla\mathcal{L}_0^{-1/2}$ is bounded on $L^{p_0}(w)$ and

$$(GLY_{p_0}) \qquad \int_{\mathbb{R}^n} |\nabla_x p_t^{\mathcal{L}_0}(x,y)|^{p_0} \exp\left\{\gamma d^2(x,y)/t\right\} w_0(x)\,dx \le \frac{C}{t^{p_0/2} V_0(y, \sqrt{t})^{p_0-1}},$$

Using this together with the assumption

$$\frac{1}{V_0(y,r)} \int_{B(y,r)} \left(|A - A_0| + \frac{|w_0 - w|}{w_0}\right) w_0\,dx \le \frac{C}{r^\epsilon}, \quad \forall\, y \in \mathbb{R}^n \,\&\, r > 1,$$

we follow the proof of Proposition 2.2 and Proposition 2.3 to conclude that there exists $\alpha > 0$ such that

$$\left\|\left(\left(1 - \frac{w_0}{w}\right)A - \frac{w_0}{w}(A_0 - A)\right)\nabla(1 + t\mathcal{L}_0)^{-1}\right\|_{p\to p} \le Ct^{-1/2-\alpha}.$$

The above two steps give the desired estimates. □

Finally by using Theorem 3.1 and the result of [12] we can finish the proof of Corollary 1.6. Note that L^p-boundedness of the Riesz transform on $L^p(\mathbb{R}^n)$ for $p \in (1,2)$ is always true if the operator is uniformly elliptic; see [17].

Proof of Corollary 1.6. By [12], if the matrix A is uniformly continuous on $\mathbb{R}^n$, then every solution to $\mathcal{L}u = 0$ on $B(x,r)$, $r < 1$, satisfies

$$\left(\fint_{B(x,r/2)} |\nabla u|^q\,dy\right)^{1/q} \le \frac{C}{r} \fint_{B(x,r)} |u|\,dy$$

for any $q < \infty$. This implies the local Riesz operator $\nabla(1 + \mathcal{L})^{-1/2}$ is L^q-bounded for any $q < \infty$; see [6, 19]. The same holds for $\mathcal{L}_0$. The conclusion then follows from Theorem 3.1. □

For the homogenized elliptic operator $\mathcal{L}_0 = -\mathrm{div}A\nabla$ (cf. [2]), it was known by [2] that $\nabla\mathcal{L}_0^{-1/2}$ is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1,\infty)$. Therefore, if $\mathcal{L} = -\mathrm{div}B\nabla$ with B uniformly continuous and satisfying

$$\fint_{B(y,r)} |A - B|\,dx \le \frac{C}{r^\epsilon}, \quad \forall\, y \in \mathbb{R}^n \,\&\, \forall\, r > 1,$$

for some $\epsilon > 0$, then Corollary 1.6 implies that $\nabla\mathcal{L}^{-1/2}$ is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1,\infty)$.

4 Examples

In this section, we discuss the counter-examples regarding unboundedness of the Riesz operator.

4.1 Conic Laplace operator

Let us start from the Meyer's example; see [7], and also [33] for general asymptotically conic elliptic operators.

On the plane. Consider $\mathcal{L} = -\mathrm{div}A\nabla$, where

$$A(x) = I + \frac{\beta(\beta+2)}{|x|^2}\begin{pmatrix} x_2^2 & -x_1x_2 \\ -x_1x_2 & x_1^2 \end{pmatrix},$$

where $\beta \in (-1,\infty)$. Then $f(x) = |x|^\beta x_1$ is a (weak-)solution to $\mathcal{L}f = 0$ on $\mathbb{R}^2$. In particular, for $\beta \in (-1,0)$, $\nabla f(x)$ is not locally L^p integrable around the origin for any $p \geq 2/|\beta|$. This implies the Riesz operator $\nabla\mathcal{L}^{-1/2}$ can not be bounded on $L^p(\mathbb{R}^2)$ for any $p \geq 2/|\beta|$; see [36, 18].

To see the geometric meaning of $\mathcal{L}$, let us rewrite $\mathcal{L}$ in the polar coordinates. First let $\lambda = \beta(\beta+2)+1$, and write

$$A(x) = I + \frac{\beta(\beta+2)}{|x|^2}\begin{pmatrix} x_2^2 & -x_1x_2 \\ -x_1x_2 & x_1^2 \end{pmatrix} = \frac{1}{|x|^2}\begin{pmatrix} x_1^2 & x_1x_2 \\ x_1x_2 & x_2^2 \end{pmatrix} + \lambda I + \frac{\lambda}{|x|^2}\begin{pmatrix} -x_1^2 & -x_1x_2 \\ -x_1x_2 & -x_2^2 \end{pmatrix}.$$

Set

$$A_1(x) = \frac{1}{|x|^2}\begin{pmatrix} x_1^2 & x_1x_2 \\ x_1x_2 & x_2^2 \end{pmatrix},$$

and $A_2(x) = A(x) - A_1(x)$. Then in the polar coordinates, the operator $\mathcal{L}_1 = -\mathrm{div}A_1\nabla$ has the representation

$$\mathcal{L}_1 f = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right),$$

and $\mathcal{L}_2 = -\mathrm{div}A_2\nabla$ can be represented as

$$\mathcal{L}_2 f = -\frac{\lambda}{r^2}\frac{\partial^2 f}{\partial\theta^2}.$$

This means

$$\mathcal{L}f = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) - \frac{\lambda}{r^2}\frac{\partial^2 f}{\partial\theta^2}.$$

It is easy to see that $f(x) = f(r,\theta) = r^{1+\beta}\cos\theta$ satisfies $\mathcal{L}f = 0$ on $\mathbb{R}^2$ (in the weak sense).

On $\mathbb{R}^N$, $N \geq 3$. It is straight to generalize the above operator to higher dimension as

$$\mathcal{L}f = -\frac{1}{r^{N-1}}\frac{\partial}{\partial r}\left(r^{N-1}\frac{\partial f}{\partial r}\right) - \frac{\lambda}{r^2}\Delta_{\mathbb{S}^{N-1}}f,$$

where $\Delta_{\mathbb{S}^{N-1}}$ is the spherical Laplacian operator, and $\lambda > 0$. In the Euclidean coordinates, the operator has the form $\mathcal{L}f = -\mathrm{div}A\nabla f$, where

$$A(x) = \frac{1}{|x|^2}A_N + \lambda I - \frac{\lambda}{|x|^2}A_N,$$

where $A_N = \{x_i x_j\}_{1\le i,j\le N}$ and $\lambda \in (0,\infty)$.

Then functions $f_i(x) = |x|^\beta x_i$, where $\beta > -1$ satisfying

$$\beta = \sqrt{\frac{N^2}{4} + \lambda(N-1) - N + 1} - \frac{N}{2} = \sqrt{\left(\frac{N}{2}-1\right)^2 + \lambda(N-1)},$$

$1 \le i \le N$, satisfy $\mathcal{L} f_i = 0$ on $\mathbb{R}^N$.

In particular, if $\lambda \in (0,1)$, then $\beta \in (-1,0)$ and the gradient ∇f_i does not belong to $L^p_{\rm loc}(\mathbb{R}^N)$ for any

$$p \ge \frac{N}{|\beta|} = N\left(\frac{N}{2} - \sqrt{\left(\frac{N}{2}-1\right)^2 + \lambda(N-1)}\right)^{-1}.$$

This implies that the Riesz transform $\nabla\mathcal{L}^{-1/2}$ can not be bounded on $L^p(\mathbb{R}^N)$ for $p \ge N/|\beta|$; see [36, 18].

Viewing $\lambda(N-1)$ as the lowest non-zero eigenvalue of the operator $\lambda\Delta_{\mathbb{S}^{N-1}}$, this range coincides with Lin [33] of the conical elliptic operators, and also Li [30] on general conic manifolds.

As $\beta \in (-1,0)$, the above generalization of conic Laplacian operator to higher dimensions gives counter-example of failure of the boundedness of the Riesz transform for any $p > N$.

For the counter-example regarding the case $p \in (2,N]$, let us consider the operator $\mathcal{L} = -{\rm div} A\nabla$ given by the matrix

$$A(x) = I + \frac{\beta(\beta+2)}{x_1^2 + x_2^2}\begin{pmatrix} x_2^2 & -x_1x_2 & 0\cdots & 0\\ -x_1x_2 & x_1^2 & 0\cdots & 0\\ 0 & 0 & 0\cdots & 0\\ \cdots & & & \\ 0 & 0 & 0\cdots & 0\end{pmatrix},$$

where $\beta \in (-1,0)$. The functions $f_1(x) = (x_1^2+x_2^2)^{\beta/2}x_1$, $f_2(x) = (x_1^2+x_2^2)^{\beta/2}x_2$ are weak solutions to $\mathcal{L}u = 0$ on $\mathbb{R}^N$. However, the gradients ∇f_1, ∇f_2 do not belong to $L^p_{\rm loc}(\mathbb{R}^N)$ for any

$$p \ge \frac{2}{|\beta|}.$$

This shows the corresponding Riesz transform can not bounded on $L^p(\mathbb{R}^N)$ for any $p \ge 2/|\beta|$.

4.2 Uniformly elliptic operator with smooth coefficients

The previous examples show that on $\mathbb{R}^N$, $N \ge 2$, for any $p > 2$, there are uniformly elliptic operators such that the Riesz transform is not L^p-bounded. However, these operators are not smooth at the origin. We next provide a uniformly elliptic operator with smooth coefficients such that the Riesz transform is not L^p-bounded, for any given $p > 2$. The idea of the construction comes from the study of asymptotically conic elliptic operators in [33] and the homogenization theory (cf. [3, 4]).

Theorem 4.1. *For any given matrix $A(x)$ with measurable coefficients satisfying uniformly elliptic conditions, there exists a smooth matrix $B(x)$ satisfying uniformly elliptic conditions, whose derivatives of any order are bounded, and a sequence $\{r_k\}_{k\in\mathbb{N}}$, $r_k \to \infty$, such that $B(r_k x) \to A(x)$ a.e. on $\mathbb{R}^n$.*

Proof. Let $0 \le \psi \in C^\infty(\mathbb{R}^n)$ be a mollifier, i.e. $\operatorname{supp}\psi \subset B(0,1)$ and $\int_{\mathbb{R}^n} \psi\, dx = 1$.

Choose an increasing sequence $\{r_k\}_{k\in\mathbb{N}}$, $1 < r_k \to \infty$, such that

$$r_k < r_k^2 < 2r_k^2 < \sqrt{r_{k+1}} - 1$$

for any $k \in \mathbb{N}$.

Set

$$\tilde{B}(x) := \begin{cases} A(x/r_k), & \text{if } x \in \bigcup_{k\in\mathbb{N}} B(0, 2r_k^2) \setminus B(0, \sqrt{r_k} - 1), \\ I_{n\times n}, & \text{other } x. \end{cases}$$

Then $\tilde{B}$ satisfies the uniformly elliptic conditions. Let

$$B(x) := \tilde{B} * \psi(x).$$

Then B is a C^∞ matrix satisfying uniform ellipticity and any order of its gradients is bounded.

Note that for $x \in B(0, r_k) \setminus B(0, 1/\sqrt{r_k})$, for all $m > k$ it holds that

$$r_m x \in B(0, r_k r_m) \setminus B(r_m/\sqrt{r_k}) \subset B(0, r_m^2) \setminus B(\sqrt{r_m}).$$

By the choice of B, we see that it holds for all $m > k$ that

$$B(r_m x) = \int_{\mathbb{R}^n} A(y/r_m)\psi(r_m x - y)\, dy = r_m^n \int_{\mathbb{R}^n} A(y)\psi(r_m(x - y))\, dy.$$

Letting $m \to \infty$ yields that for a.e. $x \in B(0, r_k) \setminus B(0, 1/\sqrt{r_k})$ it holds

$$\lim_{m\to\infty} B(r_m x) = A(x).$$

Therefore, we see that for a.e. $x \in \mathbb{R}^n$, it holds

$$\lim_{m\to\infty} B(r_m x) = A(x).$$

□

We can now finish the proof of Proposition 1.1.

Proof of Proposition 1.1. From previous subsection, for any $p > 2$ on $\mathbb{R}^n$, $n \ge 2$, we may find a uniformly elliptic operator $\mathcal{L} = -\mathrm{div}A\nabla$ such that there exists a solution u to $\mathcal{L}u = 0$ on $\mathbb{R}^n$ and ∇u is not L^p integrable near the origin.

By Theorem 4.1, we may find a smooth matrix $B(x)$ satisfying the uniform ellipticity and a positive sequence $\{r_k\}_{k\in\mathbb{N}}$, $r_k \to \infty$, such that $B(r_k x) \to A(x)$ a.e. on $\mathbb{R}^n$.

We claim that for $\mathcal{L}_0 = -\mathrm{div}B\nabla$ the Riesz transform $\nabla\mathcal{L}_0^{-1/2}$ is not bounded on $L^p(\mathbb{R}^n)$.

Let us argue by contradiction. Assume that $\nabla\mathcal{L}_0^{-1/2}$ is bounded on $L^p(\mathbb{R}^n)$. Then by [36, 18], one sees that there exists $C>0$ such that for any ball $B(x,2r)$ and any $\mathcal{L}_0$-harmonic function v on $B(x,2r)$, it holds

$$(RH_p)\qquad \left(\fint_{B(x,r)}|\nabla v|^p\,dy\right)^{1/p}\le \frac{C}{r}\fint_{B(x,2r)}|v|\,dy.$$

For any $k\in\mathbb{N}$ consider

$$\begin{cases}-\mathrm{div}B(r_k\cdot)\nabla v_k=0, & \text{on } B(0,1),\\ \qquad v_k=u, & \text{on } \partial B(0,1).\end{cases}$$

Then there exists a subsequence, still denoted by $\{v_k\}_{k\in\mathbb{N}}$, such that v_k converges weakly to $\tilde u$ in $W^{1,2}(B(0,1))$. Moreover, by using the G-convergence (cf. [8]), there exists a limit operator $\tilde{\mathcal{L}}$, such that $\tilde{\mathcal{L}}$ is a uniformly elliptic operator and $\tilde{\mathcal{L}}\tilde u=0$. By the construction of B, one can infer that $\tilde{\mathcal{L}}=\mathcal{L}$ and $\tilde u=u$. Moreover, the boundary condition implies that

$$\|v_k\|_{W^{1,2}(B(0,1))}\le C\|u\|_{W^{1,2}(B(0,1))}.$$

Note that $v_k(x/r_k)\in W^{1,2}(B(0,r_k))$ satisfies $-\mathrm{div}B\nabla v_k(\cdot/r_k)=0$. By using (RH_p), one has

$$\left(\fint_{B(0,r_k/2)}r_k^{-p}|\nabla v_k(y/r_k)|^p\,dy\right)^{1/p}\le\frac{C}{r_k}\fint_{B(0,r_k)}|v_k(y/r_k)|\,dy\le\frac{C}{r_k}\fint_{B(0,1)}|v_k(x)|\,dx,$$

and hence,

$$\left(\fint_{B(0,1/2)}|\nabla v_k(x)|^p\,dx\right)^{1/p}\le C\fint_{B(0,1)}|v_k(x)|\,dx\le C\|u\|_{W^{1,2}(B(0,1))}.$$

By applying Poincaré inequality, one has $v_k\in L^p(B(0,1/2))$ with

$$\|v_k\|_{L^p(B(0,1/2))}\le C\|u\|_{W^{1,2}(B(0,1))}.$$

These imply that $\{v_k\}_{k\in\mathbb{N}}$ is a bounded sequence in $W^{1,p}(B(0,1/2))$, and there exists a subsequence $\{v_{k_j}\}_{j\in\mathbb{N}}$ such that v_{k_j} converges weakly to u_0 in $W^{1,p}(B(0,1/2))$.

This further implies $u=u_0\in W^{1,p}(B(0,1/2))$. This contradicts with the fact ∇u is not L^p-integrable around the origin. Therefore, our claim holds, i.e., the Riesz transform $\nabla\mathcal{L}_0^{-1/2}$ is not bounded on $L^p(\mathbb{R}^n)$, where $\mathcal{L}_0=-\mathrm{div}B\nabla$. □

A Appendix

In this appendix, we provide some basic fact of functional calculus. These estimates are more or less well-known (see the proofs in [16] for instance), we include them for completeness. Note that (D) or (GUB) is not required in this appendix. Let X be a locally compact, separable, metrisable, and connected space equipped with a Borel measure μ that is finite on compact sets and strictly positive on non-empty open sets. Consider a strongly local and regular Dirichlet form $\mathscr{E}$ with the domain $\mathscr{D}$ on $L^2(X,\mu)$; see [23, 18] for precise definitions. Such a form can be written as

$$\mathscr{E}(f,g) = \int_X d\Gamma(f,g) = \int_X \langle \nabla f, \nabla g \rangle \, d\mu$$

for all $f, g \in \mathscr{D}$.

Corresponding to such a Dirichlet form $\mathscr{E}$, there exists an operator denoted by $\mathcal{L}$, acting on a dense domain $\mathscr{D}(\mathcal{L})$ in $L^2(X,\mu)$, $\mathscr{D}(\mathcal{L}) \subset \mathscr{D}$, such that for all $f \in \mathscr{D}(\mathcal{L})$ and each $g \in \mathscr{D}$,

$$\int_X f(x)\mathcal{L}g(x)\, d\mu(x) = \mathscr{E}(f,g).$$

The heat semigroup further satisfies for any $q \in [1,\infty]$ and $t > 0$ that

$$\|e^{-t\mathcal{L}}\|_{q\to q} \le 1,$$

where we denote the operator norm $\|\cdot\|_{L^p(X,\mu)\to L^p(X,\mu)}$ by $\|\cdot\|_{p\to p}$. This implies for any $s,t>0$ that

$$\|(s+t\mathcal{L})^{-1}\|_{q\to q} \le \int_0^\infty \|e^{-r(s+t\mathcal{L})}\|_{q\to q}\, dr \le \int_0^\infty e^{-rs}\, dr \le \frac{1}{s}. \tag{A.1}$$

and

$$\|(s+t\mathcal{L})^{-1/2}\|_{q\to q} \le C\int_0^\infty \|e^{-r(s+t\mathcal{L})}\|_{q\to q}\frac{dr}{\sqrt{r}} \le C\int_0^\infty e^{-rs}\frac{dr}{\sqrt{r}} \le \frac{C}{\sqrt{s}}. \tag{A.2}$$

Lemma A.1. *Let $(X,\mu,\mathscr{E})$ be a Dirichlet metric measure space. If the local Riesz transform $|\nabla(1+\mathcal{L})^{-1/2}|$ is bounded on $L^p(X,\mu)$ for some $p\in(2,\infty)$, then the operator*

$$\int_0^1 |\nabla(1+t\mathcal{L})^{-1}|\frac{dt}{\sqrt{t}}$$

is bounded on $L^p(X,\mu)$.

Proof. Note that

$$\begin{aligned}\pi(1+\mathcal{L})^{-1/2} - \int_0^1 (1+t\mathcal{L})^{-1}\frac{dt}{\sqrt{t}} &= \int_0^1 [(1+t+t\mathcal{L})^{-1} - (1+t\mathcal{L})^{-1}]\frac{dt}{\sqrt{t}} + \int_1^\infty (1+t+t\mathcal{L})^{-1}\frac{dt}{\sqrt{t}}\\ &= -\int_0^1 -t(1+t+t\mathcal{L})^{-1}(1+t\mathcal{L})^{-1}\frac{dt}{\sqrt{t}} + \int_1^\infty (1+t+t\mathcal{L})^{-1}\frac{dt}{\sqrt{t}}.\end{aligned}$$

From this and using (A.1), (A.2) and the condition that $|\nabla(1+\mathcal{L})^{-1/2}|$ is bounded on $L^p(X,\mu)$, we deduce that

$$\int_0^1 \left\|-t|\nabla(1+t+t\mathcal{L})^{-1}|(1+t\mathcal{L})^{-1}\right\|_{p\to p}\frac{dt}{\sqrt{t}}$$

$$
\begin{aligned}
&\le \int_0^1 \left\| -t|\nabla(1+\mathcal{L})^{-1/2}(1+\mathcal{L})^{1/2}(1+t+t\mathcal{L})^{-1}|(1+t\mathcal{L})^{-1}\right\|_{p\to p} \frac{dt}{\sqrt{t}}\\
&\le \int_0^1 C\sqrt{t}\left\|(t+t\mathcal{L})^{1/2}(1+t+t\mathcal{L})^{-1}(1+t\mathcal{L})^{-1}\right\|_{p\to p} \frac{dt}{\sqrt{t}}\\
&\le \int_0^1 C\left\|(1+t+t\mathcal{L})^{-1/2}(1+t\mathcal{L})^{-1}\right\|_{p\to p}\, dt\\
&\le C,
\end{aligned}
$$

and

$$
\begin{aligned}
\int_1^\infty \|\nabla(1+t+t\mathcal{L})^{-1}\|_{p\to p}\frac{dt}{\sqrt{t}} &\le \int_1^\infty \|\nabla(1+\mathcal{L})^{-1/2}(1+\mathcal{L})^{1/2}(1+t+t\mathcal{L})^{-1}\|_{p\to p}\frac{dt}{\sqrt{t}}\\
&\le \int_1^\infty C\|(t+t\mathcal{L})^{1/2}(1+t+t\mathcal{L})^{-1}\|_{p\to p}\frac{dt}{t}\\
&\le \int_1^\infty C\|(1+t+t\mathcal{L})^{-1/2}\|_{p\to p}\frac{dt}{t}\\
&\le \int_1^\infty C\frac{dt}{t\sqrt{(1+t)}} \le C.
\end{aligned}
$$

The above two estimates imply that

$$
\left\|\int_0^1 |\nabla(1+t\mathcal{L})^{-1}|\frac{dt}{\sqrt{t}}\right\|_{p\to p} \le C + C\|\nabla(1+\mathcal{L})^{-1/2}\|_{p\to p} \le C.
$$

□

Lemma A.2. *Let $(X,\mu,\mathscr{E})$ be a Dirichlet metric measure space. Suppose that for some $p\in(2,\infty)$, $|\nabla(1+\mathcal{L})^{-1/2}|$ is bounded on $L^p(X,\mu)$. Assume for some $\nu\in[0,1/2)$, it holds for any $t>1$ that*

$$
\|\nabla(1+t\mathcal{L})^{-1}\|_{p\to p} \le Ct^{-\nu}. \tag{A.3}
$$

Then for any $t>1$ it holds

$$
\|\nabla(1+t\mathcal{L})^{-1/2}\|_{p\to p} \le Ct^{-\nu}.
$$

Proof. The boundedness of $|\nabla(1+\mathcal{L})^{-1/2}|$ on $L^p(X,\mu)$ implies that for any $r>0$

$$
\||\nabla e^{-r\mathcal{L}}|\|_{p\to p} \le \|\nabla(1+\mathcal{L})^{-1/2}\|_{p\to p}\|(r+r\mathcal{L})^{1/2}e^{-r(1+\mathcal{L})}\|\|_{p\to p}\frac{e^r}{\sqrt{r}} \le Cr^{-1/2}e^r. \tag{A.4}
$$

Write

$$
|\nabla(t+t\mathcal{L})^{-1/2} - \nabla(1+t\mathcal{L})^{-1/2}| \le \frac{1}{\pi}\int_0^\infty [e^{-s}-e^{-st}]|\nabla e^{-st\mathcal{L}}|\frac{ds}{\sqrt{s}}
$$

$$= \int_0^{1/t} \cdots + \int_{1/t}^{\infty} \cdots =: I + II.$$

For the term I, by using (A.4) and the fact $|e^{-st} - e^{-s}| \le Cst$ for any $t > 1$ and $s \le 1/t$, one has

$$\|I\|_{p\to p} \le C \int_0^{1/t} Cst \|\nabla e^{-st\mathcal{L}}\|_{p\to p} \frac{ds}{\sqrt{s}} \le \int_0^{1/t} \frac{Cste^{st}}{\sqrt{st}} \frac{ds}{\sqrt{s}} \le \frac{C}{\sqrt{t}}.$$

For the term II, one has via (A.3) that

$$\begin{aligned}
\|II\|_{p\to p} &\le \int_{1/t}^{\infty} (e^{-s} - e^{-st}) \|\nabla e^{-st\mathcal{L}}\|_{p\to p} \frac{ds}{\sqrt{s}} \\
&\le \int_{1/t}^{\infty} e^{-s} \|\nabla (1 + st\mathcal{L})^{-1}\|_{p\to p} \|(1 + st\mathcal{L}) e^{-st\mathcal{L}}\|_{p\to p} \frac{ds}{\sqrt{s}} \\
&\le \int_{1/t}^{\infty} e^{-s} (st)^{-\nu} \frac{ds}{\sqrt{s}} \le Ct^{-\nu},
\end{aligned}$$

where $\nu \in [0, 1/2)$. The estimates of I and II imply that

$$\begin{aligned}
\|\nabla (1 + t\mathcal{L})^{-1/2}\|_{p\to p} &\le \|\nabla (t + t\mathcal{L})^{-1/2}\|_{p\to p} + \|\nabla [(t + t\mathcal{L})^{-1/2} - (1 + t\mathcal{L})^{-1/2}]\|_{p\to p} \\
&\le Ct^{-1/2} + Ct^{-\nu} \le Ct^{-\nu}.
\end{aligned}$$

□

Acknowledgments

R. Jiang was partially supported by NNSF of China (11671039, 11771043), F.H. Lin was in part supported by the National Science Foundation Grant DMS-1501000.

References

[1] G. Alexopoulos, An application of homogenization theory to harmonic analysis: Harnack inequalities and Riesz transforms on Lie groups of polynomial growth, *Canad. J. Math.* 44 (1992), 691-727. [5]

[2] M. Avellaneda, F.H. Lin, L^p bounds on singular integrals in homogenization, *Comm. Pure Appl. Math.* 44 (1991), 897-910. [4, 20]

[3] M. Avellaneda, F.H. Lin, Homogenization of elliptic problems with L^p boundary data, *Appl. Math. Optim.* 15 (1987), 93-107. [3, 22]

[4] M. Avellaneda, F.H. Lin, Compactness methods in the theory of homogenization, *Comm. Pure Appl. Math.* 40 (1987), 803-847. [3, 22]

[5] P. Auscher, T. Coulhon, Riesz transform on manifolds and Poincaré inequalities, *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, (5) 4 (2005), 531-555. [2]

[6] P. Auscher, T. Coulhon, X.T. Duong, S. Hofmann, Riesz transform on manifolds and heat kernel regularity, *Ann. Sci. École Norm. Sup.*, (4) 37 (2004), 911-957. [2, 3, 4, 12, 15, 16, 20]

[7] P. Auscher, P. Tchamitchian, Square root problem for divergence operators and related topics, *Astérisque* No. 249 (1998), viii+172 pp. [3, 21]

[8] A. Bensoussan, J.L. Lions, G. Papanicolaou, *Asymptotic analysis for periodic structures*, AMS Chelsea Publishing, Providence, RI, 2011. xii+398 pp. [24]

[9] X. Blanc, C. Le Bris, P.L. Lions, On correctors for linear elliptic homogenization in the presence of local defects, arXiv:1801.10335, to appear in *Communications in Partial Differential Equations*. [4]

[10] X. Blanc, C. Le Bris, P.L. Lions, On correctors for linear elliptic homogenization in the presence of local defects: the case of advection-diffusion, arXiv:1801.10330, to appear *Journal de Mathématiques Pures et Appliquées*. [4]

[11] S. Boutayeb, T. Coulhon, A. Sikora, A new approach to pointwise heat kernel upper bounds on doubling metric measure spaces, *Adv. Math.* 270 (2015), 302-374. [2]

[12] L. Caffarelli, I. Peral, On $W^{1,p}$ estimates for elliptic equations in divergence form, *Comm. Pure Appl. Math.*, 51 (1998), 1-21. [2, 14, 20]

[13] G. Carron, Riesz transforms on connected sums, Festival Yves Colin de Verdière, *Ann. Inst. Fourier (Grenoble)* 57 (2007), 2329-2343. [5]

[14] G. Carron, T. Coulhon, A. Hassell, Riesz transform and L^p-cohomology for manifolds with Euclidean ends, *Duke Math. J.* 133 (2006), 59-93. [5]

[15] S.Y. Cheng, S.T. Yau, Differential equations on Riemannian manifolds and their geometric applications, *Comm. Pure Appl. Math.*, 28 (3) (1975) 333-354. [5]

[16] T. Coulhon, N. Dungey, Riesz transform and perturbation, *J. Geom. Anal.* 17 (2007), 213-226. [4, 6, 24]

[17] T. Coulhon, X.T. Duong, Riesz transforms for $1 \le p \le 2$, *Trans. Amer. Math. Soc.*, 351 (1999), 1151-1169. [2, 11, 16, 20]

[18] T. Coulhon, R. Jiang, P. Koskela, A. Sikora, Gradient estimates for heat kernels and harmonic functions, arXiv:1703.02152. [2, 3, 4, 5, 14, 15, 20, 21, 22, 24, 25]

[19] T. Coulhon, R. Jiang, P. Koskela, A. Sikora, Local gradient estimates for heat kernels and harmonic functions, preprint. [2, 20]

[20] C.B. Croke, H. Karcher, Volumes of small balls on open manifolds: lower bounds and examples, *Trans. Am. Math. Soc.* 309 (1988) 753-762. [4]

[21] B. Devyver, A perturbation result for the Riesz transform, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) 14 (2015), 937-964. [5]

[22] N. Dungey, Heat kernel estimates and Riesz transforms on some Riemannian covering manifolds, *Math. Z.*, 247 (2004), 765-794. [5]

[23] M. Fukushima, Y. Oshima, M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, de Gruyter Studies in Mathematics, Vol. 19, Walter de Gruyter & Co., Berlin, 1994. [25]

[24] E. B. Fabes, C. E. Kenig, R. P. Serapioni, The local regularity of solutions of degenerate elliptic equations, *Comm. Partial Differential Equations*, 7 (1982), 77-116. [17]

[25] A. Grigor'yan, The heat equation on noncompact Riemannian manifolds, (Russian) *Mat. Sb.*, 182 (1991), 55-87; translation in Math. USSR-Sb. 72 (1992), 47-77. [14]

[26] A. Grigor'yan, L. Saloff-Coste, Heat kernel on manifolds with ends, *Ann. Inst. Fourier (Grenoble)* 59 (2009), 1917-1997. [5]

[27] J. Heinonen, P. Koskela, N. Shanmugalingam, J. Tyson, *Sobolev spaces on metric measure spaces. An approach based on upper gradients*. New Mathematical Monographs, 27. Cambridge University Press, Cambridge, 2015. xii+434 pp. [16]

[28] R. Jiang, The Li-Yau inequality and heat kernels on metric measure spaces, *J. Math. Pures Appl.*, (9) 104 (2015), 29-57. [4]

[29] R. Jiang, Riesz transform via heat kernel and harmonic functions on non-compact manifolds, arXiv:1710.00518. [2, 5]

[30] H.Q. Li, La transformation de Riesz sur les variétés coniques, *J. Funct. Anal.*, 168 (1999), 145-238. [5, 22]

[31] P. Li, S.T. Yau, On the parabolic kernel of the Schrödinger operator, *Acta Math.*, 156 (1986), 153-201. [16]

[32] X.D. Li, Riesz transforms for symmetric diffusion operators on complete Riemannian manifolds, *Rev. Mat. Iberoam.* 22 (2006), 591-648 [2]

[33] F.H. Lin, *Asymptotically conic elliptic operators and Liouville type theorems*, Geometric analysis and the calculus of variations, 217-238, Int. Press, Cambridge, MA, 1996. [3, 5, 21, 22]

[34] L. Saloff-Coste, A note on Poincaré, Sobolev, and Harnack inequalities, *I.M.R.N.* (2) (1992) 27-38. [14, 19]

[35] L. Saloff-Coste, Uniformly elliptic operators on Riemannian manifolds, *J. Differential Geom.*, 36 (1992), 417-450. [14, 19]

[36] Z.W. Shen, Bounds of Riesz transforms on L^p spaces for second order elliptic operators, *Ann. Inst. Fourier (Grenoble)*, 55 (2005), 173-197. [2, 21, 22, 24]

[37] N.T. Varopoulos, Analysis on Lie groups, *J. Funct. Anal.* 76 (1988), 346-410. [5]

[38] S.T. Yau, Harmonic functions on complete Riemannian manifolds, *Comm. Pure Appl. Math.*, 28 (1975), 201-228. [5]

[39] Q.S. Zhang, Stability of the Cheng-Yau gradient estimate, *Pacific J. Math.*, 225 (2006), 379-398. [5]

Renjin Jiang
Center for Applied Mathematics, Tianjin University, Tianjin 300072, China
rejiang@tju.edu.cn

Fanghua Lin
Courant Institute of Mathematical Sciences, 251 Mercer Street, New York, NY 10012, USA
linf@cims.nyu.edu