ON THE VISCOUS CAMASSA-HOLM EQUATIONS WITH FRACTIONAL DIFFUSION

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Abstract. We study Cauchy problem of a class of viscous Camassa-Holm equations (or Lagrangian averaged Navier-Stokes equations) with fractional diffusion in both smooth bounded domains and in the whole space in two and three dimensions. Order of the fractional diffusion is assumed to be $2s$ with $s \in [n/4, 1)$, which seems to be sharp for the validity of the main results of the paper; here $n = 2, 3$ is the dimension of space. We prove global well-posedness in $C_{[0,+\infty)}(D(A)) \cap L^2_{[0,+\infty),loc}(D(A^{1+s/2}))$ whenever the initial data $u_0 \in D(A)$, where A is the Stokes operator. We also prove that such global solutions gain regularity instantaneously after the initial time. A bound on a higher-order spatial norm is also obtained.

1. Introduction

Hydrodynamic equations with nonlocal effects have attracted a great attention in recent years. While some of the problems are described by nonlocal equations to begin with, many others, especially those concerning interface motion in fluids, are often derived from local equations. See for examples [3, 12, 14, 13, 9, 8, 26] and references therein. As an important type of nonlocality, fractional diffusion arises naturally in many hydrodynamic problems, characterizing nonlocal drift or diffusion [3, 12, 13, 8], or capturing certain thermal and electromagnetic effects [11, 10]. From an analytic point of view, evolution problems with these nonlocal features are of great interest on their own.

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In this paper, we shall study viscous Camassa-Holm equations with fractional diffusion in $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$. Throughout the paper, unless otherwise stated, we shall always assume that

$$\Omega \subset \mathbb{R}^n \text{ is a smooth bounded domain, or } \Omega = \mathbb{R}^n, \text{ with } n = 2, 3, \tag{1}$$

and

$$s \in [n/4, 1). \tag{2}$$

The equations are as follows:

$$\partial_t(1-\alpha^2\Delta)u + u\cdot\nabla(1-\alpha^2\Delta)u - \alpha^2\nabla u^T\cdot\Delta u + \nabla p = -\nu(1-\alpha^2\Delta)A^s u, \tag{3}$$

$$\operatorname{div} u = 0, \quad u|_{t=0} = u_0. \tag{4}$$

Here u denotes a divergence-free fluid velocity field. The constant $\alpha > 0$ characterizes the scale at which fluid motion is averaged, and $\nu > 0$ is the viscosity. $A = \mathcal{P}(-\Delta)$ is the Stokes operator, with $\mathcal{P}$ being the Leray projection operator $\mathcal{P} : L^2(\Omega) \to \{v \in L^2(\Omega) : \operatorname{div} v = 0,\, v\cdot n = 0 \text{ on } \partial\Omega\}$; we always omit the Ω-dependence of A and $\mathcal{P}$. A^s with $s \in [\frac{n}{4}, 1)$ is the spectral fractional Stokes operator which will be defined below, and which is a nonlocal operator in nature. There are alternative (but not necessarily equivalent) ways of defining fractional Stokes operators, but we find the spectral fractional Stokes operator is the easiest to work with for our purpose. The range of s is seen to be sharp from the viewpoint of the energy method (see the proof of Theorem 3.1 below). Initial data for u is specified. When Ω is a smooth bounded domain, we additionally need boundary conditions

$$u = A^s u = 0 \quad \text{on } \partial\Omega. \tag{5}$$

When $s = 1$, equations (3)-(4) are often referred as the classic viscous Camassa-Holm equations, or equivalently the isotropic Lagrangian averaged Navier-Stokes equations (LANS-α) [28]. The inviscid version of the LANS-α equations, or the Lagrangian averaged Euler (LAE-α) equations, were first derived in [24, 23] from a variational formulation, motivated by the fact that the Camassa-Holm equation in one dimension describes geodesic motion on certain diffeomorphism group. An alternative derivation can be found in [21]. Viscosities were later added to the LAE-α equations, giving rise to the LANS-α equations [5, 6, 4]. Its relation to the turbulence theory has been well investigated [16, 17, 22, 30, 7]. Both LAE-α and LANS-α equations can be viewed as closure models when motion at the scales smaller than α is averaged out. Anisotropic generalizations of the LAE-α and LANS-α equations in bounded domains are presented in [29], which takes into account that the covariance tensor of the Lagrangian fluctuation field is not constantly identity matrix throughout the domain and it should evolve with the flow. For a more comprehensive history of the LANS-α equation, we refer the readers to [28] and the references therein. As for results in analysis, a handful of global existence or well-posedness results of the LANS-α equation have been established in periodic boxes [17], in bounded domains and the whole space [15, 2, 1], and on Riemannian manifolds with boundaries [31]; decay of solutions in bounded domains and the whole spaces was also investigated in [2, 1].

Although it is not obvious how fractional diffusion can be physically incorporated into derivations of the Camassa-Holm equations, the specific form of the fractional dissipation in (3) together with the boundary conditions (5) is quite natural from analysis point of view; a similar choice is made in [28]. For simplicity, we only focus on the isotropic fractional LANS-α equations, i.e., the viscous Camassa-Holm equations, although it was suggested that the anisotropic LANS-α equation may be more relavent for bounded domains [29].

Our first result, Theorem 3.1, is the global well-posedness with sharp fractional power s. It may be viewed as a fractional counterpart of classical results by Kieslev-Ladyzenskaya and others for the Navier-Stokes equations [25, 32]. It would also be interesting if one can build rather weak solutions as in [8] for suitable small positive powers s. Next, we show that the global solution gains regularity when $t > 0$, which is stated in Theorem 4.2 and Theorem 4.6. The latter, characterizing the critical case $(n, s) = (2, 1/2)$, is in general not easy to establish, and it may be a starting point for a further regularity theory. Here instead of dealing with commutators associated with nonlocal operators on a bounded domain which could be rather technical, we make use of the fractional semigroups to derive desired estimates. One might need nonlocal commutator estimates when studying higher regularity and boundary regularity. These related issues will be addressed elsewhere.

The rest of the paper is organized as follows. In Section 2, we introduce the spectral fractional Stokes operator and present an equivalent formulation of the equations (3)-(4). Section 3 will be devoted to proving Theorem 3.1 on the global well-posedness result. In Section 4, we prove that the global solution enjoys higher spatial regularity for any positive time. The main results are summarized in Theorem 4.2 for the non-critical case, and in Theorem 4.6 for the critical case, respectively.

2. Preliminaries

We first introduce some notations. Let $\Sigma = \{\phi \in C_0^\infty(\Omega) : \nabla \cdot \phi = 0\}$. As in much literature on mathematical hydrodynamics, let V denote the H^1-completion of Σ; while the L^2-completion of Σ is denoted by H. Define $V^r = H^r(\Omega) \cap V$ for all $r \geq 1$; obviously $V = V^1$.

In the case of $\Omega = \mathbb{R}^n$, $(-\Delta)$ and $\mathcal{P}$ are both Fourier multipliers, and thus they commute. Indeed, for all $f \in \mathscr{S}(\mathbb{R}^n)$, $\widehat{Af}(\xi) = \hat{\mathcal{P}}(\xi)|\xi|^2\hat{f}(\xi)$, where

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\, dx$$

is the Fourier transform of f, and where $\hat{\mathcal{P}}(\xi)$ is the Fourier multiplier associated with $\mathcal{P}$. Hence, A^s can be naturally defined by

$$\widehat{A^s f}(\xi) = \hat{\mathcal{P}}(\xi)|\xi|^{2s}\hat{f}(\xi).$$

Define

$$\|f\|_{D(A^r)(\mathbb{R}^n)} := (\|A^r f\|^2_{L^2(\mathbb{R}^n)} + \|f\|^2_{L^2(\mathbb{R}^n)}\mathbb{1}_{\{r>0\}})^{1/2}.$$

Now consider $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ to be a smooth bounded domain. For the stationary Stokes equation in Ω with zero Dirichlet boundary condition, there exists a sequence of eigenvalues $\{\mu_j\}_{j\in\mathbb{Z}_+} \subset \mathbb{R}_+$ and a sequence of eigenfunctions $\{w_j\}_{j\in\mathbb{Z}_+} \subset L^2(\Omega)$, both depending on Ω, solving

$$Aw_j = \mu_j w_j \text{ in } \Omega, \quad \operatorname{div} w_j = 0, \quad w_j|_{\partial\Omega} = 0,$$

such that $\{\mu_j\}_{j\in\mathbb{Z}_+}$ is non-decreasing in $\mathbb{R}_+$ and $\{w_j\}_{j\in\mathbb{Z}_+}$ forms an orthonormal basis of H. It is known that $w_j \in C^\infty(\Omega) \cap V$ [32]. For all $f \in H$, it has a spectral decomposition

$$f(x) = \sum_{j=1}^\infty f_j w_j(x), \quad f_j = \int_\Omega f(x) w_j(x)\,dx.$$

The infinite sum is understood in the L^2-sense. In fact, $\|f\|_{L^2(\Omega)} = \|\{f_j\}_{j\in\mathbb{Z}_+}\|_{l^2}$. For all $r \in \mathbb{R}$, define

$$D(A^r)(\Omega) = \left\{ f(x) = \sum_{j=1}^\infty f_j w_j(x) : \{\mu_j^r f_j\}_{j\in\mathbb{Z}_+} \in l^2 \right\},$$

with

$$\|f\|_{D(A^r)(\Omega)} := (\|\{\mu_j^r f_j\}_{j\in\mathbb{Z}_+}\|^2_{l^2} + \|\{f_j\}_{j\in\mathbb{Z}_+}\|^2_{l^2}\mathbb{1}_{\{r>0\}})^{1/2}.$$

We shall omit the Ω-dependence in $D(A^r)(\Omega)$ whenever it is convenient. Then for all $f \in D(A^r)$,

$$A^r f(x) := \sum_{j=1}^\infty \mu_j^r f_j w_j(x).$$

Again the infinite sum is understood in the L^2-sense. As a result, $\|f\|_{D(A^r)(\Omega)} = (\|A^r f\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)}\mathbb{1}_{\{r>0\}})^{1/2}$.

Note that when Ω is a smooth bounded domain, the boundary condition (5) is well-defined and it is automatically satisfied in the space $D(A^{1+s/2})$. Indeed, we have the following lemma.

Lemma 2.1. *Suppose $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ is a smooth bounded domain. Then all $D(A^r)$-functions have trace zero if $r > 1/4$.*

Proof. We first show that $D(A) = V^2$. Indeed, for all $f \in D(A)$, there is a sequence of $\{f^n\} \subset C^\infty(\Omega) \cap V$ being finite linear combinations of w_j, such that $f^n \to f$ in the $D(A)$-norm. On one hand, this implies that $\{Af^n\}_{n\in\mathbb{Z}_+}$ forms a Cauchy sequence in $L^2(\Omega)$ and thus $\{f^n\}_{n\in\mathbb{Z}_+}$ is a Cauchy sequence in V^2 thanks to their zero boundary conditions and the regularity theory of the stationary Stokes equation [32, 27]. We assume $f_n \to f_*$ in V^2 for some

$f_* \in V^2$. On the other hand, that $f^n \to f$ in $D(A)$ implies that $f^n \to f$ in $L^2(\Omega)$. Hence, $f = f_* \in V^2$, and $\|f\|_{H^2(\Omega)} \le C\|Af\|_{L^2(\Omega)} \le C\|f\|_{D(A)}$ because the same estimates holds for f^n. This implies that $D(A) \hookrightarrow V^2$. That $V^2 \hookrightarrow D(A)$ is trivial since $\|Af\|_{L^2(\Omega)} \le C\|f\|_{H^2(\Omega)}$.

To this end, for all $f \in D(A)$, we have $f \in L^2$ with $f|_{\partial\Omega} = 0$ and $(-\Delta)f \in L^2$. This gives

$$\|f\|^2_{D(A^{1/2})} = \sum_{j=1}^{\infty}(\mu_j + 1)f_j^2 = \int_\Omega f(1-\Delta)f\,dx = \|f\|^2_{H^1(\Omega)}.$$

This is also true for all $f \in D(A^{1/2})$ since $D(A)$ is dense in $D(A^{1/2})$. Hence, $D(A^{1/2}) = V$.

Since the embeddings $i : D(A^0) \to L^2(\Omega)$ and $i : D(A^{1/2}) \to H_0^1(\Omega)$ are continuous, by interpolation, for $r \in (1/4, 1/2]$, i is continuous from $[D(A^0), D(A^{1/2})]_{2r} = D(A^r)$ to $[L^2(\Omega), H_0^1(\Omega)]_{2r} = H_0^{2r}(\Omega)$ [27]. Functions in $H_0^{2r}(\Omega)$ all have zero trace. For $r > 1/2$, it suffices to note that $D(A^r) \hookrightarrow D(A^{1/2})$. □

Remark 1. When Ω is a smooth bounded domain, in fact, $D(A^r) = V^{2r}$ for all $r \in [1/2, 5/4)$. More generally, by virtue of the interpolation theory [27], $D(A^r) \hookrightarrow V^{2r}$ for all $r \ge 1/2$. When $\Omega = \mathbb{R}^n$, $D(A^r) = V^{2r}$ for all $r \ge 1/2$.

Let $(1-\alpha^2\Delta)^{-1}$ be the inverse of the elliptic operator $(1-\alpha^2\Delta)$ on Ω (with zero Dirichlet boundary condition if Ω is a smooth bounded domain). In the view of $A^s u = 0$ on $\partial\Omega$ if Ω is a smooth bounded domain, it is valid to take $(1-\alpha^2\Delta)^{-1}$ on both sides of (3), and we obtain the following equivalent formulation of the Camassa-Holm equation with fractional diffusion [28]:

$$\partial_t u + \nu A^s u + \mathcal{P}^\alpha[u\cdot\nabla u + \mathcal{U}^\alpha(u,u)] = 0, \tag{6}$$

where with adaptation of notations in [28],

$$\mathcal{U}^\alpha(u_1,u_2) = \alpha^2(1-\alpha^2\Delta)^{-1}\mathrm{div}\,[\nabla u_1\cdot\nabla u_2^T + \nabla u_1\cdot\nabla u_2 - \nabla u_1^T\cdot\nabla u_2], \tag{7}$$

and $\mathcal{P}^\alpha : H_0^1 \cap H^r(\Omega) \to V^r$ $(r \ge 1)$ is the *Stokes projector* [31] uniquely defined by

$$(1-\alpha^2\Delta)\mathcal{P}^\alpha(w) + \nabla p = (1-\alpha^2\Delta)w,$$
$$\mathrm{div}\,\mathcal{P}^\alpha(w) = 0, \quad \mathcal{P}^\alpha(w)|_{\partial\Omega} = 0.$$

For all $r \ge 1$, $\mathcal{P}^\alpha$ is bounded from $H_0^1 \cap H^r(\Omega)$ to V^r [28].

The relation between (3)-(4) and the viscous Camassa-Holm equation can be formally revealed as follows. Assuming sufficient regularity of u, we apply $\mathcal{P}$ to (3) to obtain that

$$\partial_t(1+\alpha^2A)u + \mathcal{P}[u\cdot\nabla(1-\alpha^2\Delta)u - \alpha^2\nabla u^T\cdot\Delta u] = -\nu(1+\alpha^2A)A^s u. \tag{8}$$

Suppose $-\Delta u = Au + \nabla q$ for some q and define

$$v = (1+\alpha^2A)u.$$

Then (8) becomes

$$\partial_t v + \mathcal{P}[u\cdot\nabla v + \alpha^2 u\cdot\nabla\nabla q - \alpha^2\nabla u^T\cdot\Delta u] = -\nu A^s v.$$

Since

$$\begin{aligned}\alpha^2\mathcal{P}[u\cdot\nabla\nabla q - \nabla u^T\cdot\Delta u] &= \alpha^2\mathcal{P}[-\nabla u^T\cdot\nabla q - \nabla u^T\cdot\Delta u]\\ &= \alpha^2\mathcal{P}[\nabla u^T\cdot Au] = \mathcal{P}[\nabla u^T v],\end{aligned}$$

we obtain that

$$\partial_t v + u\cdot\nabla v + \nabla u^T v + \nabla\tilde{p} = -\nu A^s v \tag{9}$$

for some $\tilde{p}$. This recovers the more commonly-used form of the Camassa-Holm equation [2] with fractional diffusions.

3. Global Well-posedness

Our main result on the global well-posedness of the equations (6) and (4) (or equivalently, (3)-(4)), with boundary conditions (5) when Ω is a smooth bounded domain, is as follows.

Theorem 3.1 (Global well-posedness)**.** *Assume (1) and (2), and let $u_0\in D(A)$. Then there exists a unique solution $u\in C_{[0,+\infty)}(D(A))\cap L^2_{[0,+\infty),loc}(D(A^{1+s/2}))$ with $\partial_t u\in L^2_{[0,+\infty),loc}(D(A^{1-s/2}))$ solving (6) with initial condition $u|_{t=0}=u_0$ (and boundary conditions (5) if Ω is a smooth bounded domain). It satisfies*

$$\|u\|_{L^\infty_{[0,+\infty)}(D(A))} + \|A^{1+s/2}u\|_{L^2_{[0,+\infty)}L^2} \le C\|u_0\|_{D(A)}. \tag{10}$$

where $C = C(\alpha,s,n,\nu,\Omega,\|u_0\|_{D(A^{1/2})})$.

As the first step towards the global well-posedness, the following proposition states the local well-posedness result. Note that in the 3-D case, the range of s for the local well-posedness is wider than that in the global well-posedness result.

Proposition 3.1 (Local well-posedness)**.** *Assume (1) and $s\in[\frac{1}{2},1)$, and let $u_0\in D(A)$. Then there exists $T=T(\alpha,\nu,s,n,\Omega,u_0)>0$ and a unique solution $u\in C_{[0,T]}(D(A))\cap L^2_T(D(A^{1+s/2}))$ with $\partial_t u\in L^2_T(D(A^{1-s/2}))$ solving (6) with initial condition $u|_{t=0}=u_0$ (and boundary conditions (5) if Ω is a smooth bounded domain), which satisfies*

$$\|u\|^2_{L^\infty_T(D(A))} + \nu\int_0^T \|A^{1+s/2}u\|^2_{L^2}\,dt \le C\|u_0\|^2_{D(A)}. \tag{11}$$

where C is a universal constant.

Proof. The main ingredient of the proof is the Galerkin approximation. We proceed in two different cases.

Case 1. In this case, assume Ω is a smooth bounded domain.

Step 1. For any $r \geq 0$, let $\mathcal{P}_N$ be the orthogonal projection from V^r to $V_N = \text{span}\{w_1, \cdots, w_N\}$, where w_j's are eigenfunctions of the Stokes operator defined in Section 2. Let u^N solve

$$\partial_t u^N + \nu A^s u^N + \mathcal{P}_N \mathcal{P}^\alpha [u^N \cdot \nabla u^N + \mathcal{U}^\alpha(u^N, u^N)] = 0, \quad u^N|_{t=0} = \mathcal{P}_N u_0. \tag{12}$$

To construct such u^N, assume $u^N = \sum_{j=1}^N a_j(t) w_j$. Then (12) can be written as an ODE system for a_j's, i.e.,

$$\frac{da_j}{dt} + \nu \mu_j^s a_j + \sum_{k,l=1}^N a_k a_l [\langle w_j, w_k \cdot \nabla w_l \rangle + \langle w_j, \mathcal{U}^\alpha(w_k, w_l)\rangle], \quad a_j(0) = \langle w_j, u_0\rangle.$$

Here $\langle \cdot, \cdot \rangle$ denotes the L^2-inner product on Ω. It is not hard to show the inner products all have finite values. Then local existence and uniqueness of u^N follows from the classic ODE theory.

Step 2. We shall derive energy estimates for u^N. It is straightforward to find that

$$\frac{1}{2}\frac{d}{dt}\|u^N\|_{L^2}^2 + \nu \|A^{s/2} u^N\|_{L^2}^2 = -\langle u^N, \mathcal{U}^\alpha(u^N, u^N)\rangle. \tag{13}$$

By the definition of $\mathcal{U}^\alpha$ in (7), for arbitrary $v_1, v_2 \in D(A)$,

$$\begin{aligned}
&\|\mathcal{U}^\alpha(v_1, v_2)\|_{H_0^1(\Omega)} \\
\leq\ & C\|\nabla v_1 \cdot \nabla v_2^T + \nabla v_1 \cdot \nabla v_2 - \nabla v_1^T \cdot \nabla v_2\|_{L^2} \\
\leq\ & C\|v_1\|_{D(A)}\|v_2\|_{D(A)},
\end{aligned}$$

which yields

$$\frac{1}{2}\frac{d}{dt}\|u^N\|_{L^2}^2 + \nu \|A^{s/2} u^N\|_{L^2}^2 \leq C\|u^N\|_{D(A)}^2 \|u^N\|_{L^2}. \tag{14}$$

Next, we derive a higher order estimate. Taking inner product of (12) and $A^2 u^N$,

$$\frac{1}{2}\frac{d}{dt}\|Au^N\|_{L^2}^2 + \nu \|A^{1+s/2} u^N\|_{L^2}^2 = -\langle A^2 u^N, u^N \cdot \nabla u^N\rangle - \langle A^2 u^N, \mathcal{U}^\alpha(u^N, u^N)\rangle. \tag{15}$$

Since u^N is smooth, assuming $-\Delta u^N = Au^N + \nabla p^N$ for some p^N, we derive that

$$\begin{aligned}
&\langle A^2 u^N, u^N \cdot \nabla u^N\rangle \\
=\ & \langle Au^N, (-\Delta)(u^N \cdot \nabla u^N)\rangle \\
=\ & \langle Au^N, u^N \cdot \nabla(-\Delta)u^N - \Delta u^N \cdot \nabla u^N - 2\partial_k u_j^N \partial_{jk} u^N\rangle \\
=\ & \langle Au^N, u^N \cdot \nabla\nabla p^N - \Delta u^N \cdot \nabla u^N - 2\partial_k u_j^N \partial_{jk} u^N\rangle \\
=\ & \langle Au^N, -(\nabla u^N)^T \cdot \nabla p^N - \Delta u^N \cdot \nabla u^N - 2\partial_k u_j^T \partial_{jk} u^N\rangle \\
=\ & \langle Au^N, (\nabla u^N)^T (Au^N + \Delta u^N) - \Delta u^N \cdot \nabla u^N - 2\partial_k u_j^N \partial_{jk} u^N\rangle.
\end{aligned} \tag{16}$$

Combining this with Remark 1 and the assumption $s \in [\frac{1}{2}, 1)$,

$$\begin{aligned} &|\langle A^2u^N, u^N \cdot \nabla u^N\rangle| \\ &\le C\|Au^N\|_{L^{\frac{2n}{n-2s}}}\|\nabla u^N\|_{L^{\frac{n}{s}}}(\|Au^N\|_{L^2} + \|\nabla^2 u^N\|_{L^2}) \\ &\le C\|u^N\|_{D(A^{1+s/2})}\|u^N\|_{D(A^{(1+\frac{n}{2}-s)/2})}\|u^N\|_{D(A)} \\ &\le C\|u^N\|_{D(A^{1+s/2})}\|u^N\|^2_{D(A)}. \end{aligned} \tag{17}$$

In addition, for arbitrary $v_1, v_2 \in D(A^{1+s/2})$,

$$\|\mathcal{P}^\alpha \mathcal{U}^\alpha(v_1, v_2)\|_{D(A)} \le C\|\nabla v_1 \cdot (\nabla v_2)^T + \nabla v_1 \cdot \nabla v_2 - (\nabla v_1)^T \cdot \nabla v_2\|_{H^1}.$$

By Sobolev embedding and Remark 1,

(18)

$$\begin{aligned} &\|\nabla v_1 \cdot (\nabla v_2)^T + \nabla v_1 \cdot \nabla v_2 - (\nabla v_1)^T \cdot \nabla v_2\|_{H^1} \\ &\le C\|\nabla^2 v_1\|_{L^{\frac{2n}{n-2s}}}\|\nabla v_2\|_{L^{\frac{n}{s}}} + C\|\nabla v_1\|_{L^{\frac{n}{s}}}\|\nabla^2 v_2\|_{L^{\frac{2n}{n-2s}}} \\ &\le C\|v_1\|_{D(A^{1+s/2})}\|v_2\|_{D(A^{(1+\frac{n}{2}-s)/2})} + C\|v_1\|_{D(A^{(1+\frac{n}{2}-s)/2})}\|v_2\|_{D(A^{1+s/2})}, \end{aligned}$$

which implies that

$$\|\mathcal{P}^\alpha \mathcal{U}^\alpha(u^N, u^N)\|_{D(A)} \le C\|u^N\|_{D(A)}\|u^N\|_{D(A^{1+s/2})}. \tag{19}$$

Hence,

$$|\langle A^2u^N, \mathcal{U}^\alpha(u^N, u^N)\rangle| \le C\|u^N\|^2_{D(A)}\|u^N\|_{D(A^{1+s/2})}. \tag{20}$$

Note that when Ω is a smooth bounded domain, $\|v\|_{D(A^r)} \le C\|A^r v\|_{L^2}$ for some constant C depending on $r > 0$ and Ω. Combining (14)-(20),

$$\frac{1}{2}\frac{d}{dt}\|u^N\|^2_{D(A)} + \nu\|A^{1+s/2}u^N\|^2_{L^2} \le C\|u^N\|^2_{D(A)}\|A^{1+s/2}u^N\|_{L^2}. \tag{21}$$

By Young's inequality,

$$\frac{d}{dt}\|u^N\|^2_{D(A)} + \nu\|A^{1+s/2}u^N\|^2_{L^2} \le C\nu^{-1}\|u^N\|^4_{D(A)}.$$

Taking time integral yields that

$$(\|u^N\|^2_{L^\infty_T(D(A))} - \|\mathcal{P}_N u_0\|^2_{D(A)}) + \nu\int_0^T \|A^{1+s/2}u^N\|^2_{L^2}\,dt \le C_*T\|u^N\|^4_{L^\infty_T(D(A))}.$$

Here $C_* = C_*(\alpha, s, n, \nu, \Omega)$. By the continuity of u^N in time, if T is taken to be sufficiently small, which only depends on $\|u_0\|_{D(A)}$ and the constant C_* but not on N, we deduce that u^N exists on $[0, T]$ by a continuation argument if needed, and

$$\|u^N\|^2_{L^\infty_T(D(A))} + \nu\int_0^T \|A^{1+s/2}u^N\|^2_{L^2}\,dt \le C\|u_0\|^2_{D(A)}. \tag{22}$$

Here C is a universal constant.

Step 3. Since the bound in (22) is uniform in N, there exists $u \in L^\infty_T(D(A)) \cap L^2_T(D(A^{1+s/2}))$ satisfying (11), such that up to a subsequence, u^N weak-$*$ converges to u in $L^\infty_T(D(A))$ and weakly in $L^2_T(D(A^{1+s/2}))$.

Next we derive an estimate for $\partial_t u^N$. For arbitrary $v_1, v_2 \in D(A^{1+s/2})$,

(23)
$$\|\mathcal{P}^\alpha(v_1 \cdot \nabla v_2)\|_{D(A^{1-s/2})} \leq C\|v_1\|_{D(A)}\|\nabla v_2\|_{H^{2-s}} \leq C\|v_1\|_{D(A)}\|v_2\|_{D(A^{1+s/2})}.$$

Note that in the last inequality, we needed $s \geq 1/2$. Combining this with (12) and (19), we use boundedness of $\mathcal{P}_N$ and $\mathcal{P}^\alpha$ to derive that

$$\begin{aligned}\|\partial_t u^N\|_{D(A^{1-s/2})} &\leq C\|A^s u^N\|_{D(A^{1-s/2})} + C\|u^N\|_{D(A)}\|u^N\|_{D(A^{1+s/2})} \\ &\leq C\|A^{1+s/2}u^N\|_{L^2}(1 + \|u^N\|_{D(A)}).\end{aligned}$$

Thanks to (22), this implies that $\partial_t u^N$ has a uniform-in-N bound in $L^2_T(D(A^{1-s/2}))$. By interpolation and the Aubin-Lions Lemma [32], $u^N \to u$ strongly in $L^p_T(D(A))$ for all $p \in [1,\infty)$. This together with the weak convergence $u^N \rightharpoonup u$ in $L^2_T(D(A^{1+s/2}))$ is then sufficient for passing to the limit $N \to \infty$ in (12), which implies that u is a weak solution. Arguing as above, $\partial_t u \in L^2_T D(A^{1-s/2})$. By a classic argument [32, Lemma 1.2 in Chapter III], u is almost everywhere equal to a continuous function valued in $D(A)$, i.e., $u \in C_{[0,T]}(D(A)) \cap L^2_T(D(A^{1+s/2}))$. Moreover, u satisfies the initial condition in (12) since $\mathcal{P}_N u_0 \to u_0$ strongly in $D(A)$.

Step 4. It remains to show the uniqueness. Suppose there are two solutions u_1 and u_2 for (6) satisfying (11). Define $w = u_1 - u_2$. Then $w \in C_{[0,T]}(D(A)) \cap L^2_T(D(A^{1+s/2}))$ solves

$$\partial_t w + \nu A^s w + \mathcal{P}^\alpha[w \cdot \nabla u_2 + \mathcal{U}^\alpha(w, u_2)] + \mathcal{P}^\alpha[u_1 \cdot \nabla w + \mathcal{U}^\alpha(u_1, w)] = 0$$

with zero initial condition. Similar to (13)-(21), we derive energy estimates for w,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2} + \nu\|A^{s/2}w\|^2_{L^2} \\ &\leq |\langle w, w \cdot \nabla u_2 + \mathcal{U}^\alpha(w, u_2)\rangle| + |\langle w, \mathcal{U}^\alpha(u_1, w)\rangle| \\ &\leq C(\|u_1\|_{D(A)} + \|u_2\|_{D(A)})\|w\|^2_{D(A)},\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|Aw\|^2_{L^2} + \nu\|A^{1+s/2}w\|^2_{L^2} \\ &\leq |\langle A^{1+s/2}w, A^{1-s/2}\mathcal{P}^\alpha(w \cdot \nabla u_2)\rangle| + |\langle Aw, (-\Delta)(u_1 \cdot \nabla w)\rangle| \\ &\quad + |\langle Aw, A\mathcal{P}^\alpha\mathcal{U}^\alpha(w, u_2)\rangle| + |\langle Aw, A\mathcal{P}^\alpha\mathcal{U}^\alpha(u_1, w)\rangle| \\ &\leq C\|A^{1+s/2}w\|_{L^2}\|w\|_{D(A)}\|u_2\|_{D(A^{1+s/2})} \\ &\quad + |\langle Aw, (\nabla u_1)^T(Aw + \Delta w) - \Delta u_1 \cdot \nabla w - 2\partial_k u_{1,j}\partial_{jk}w\rangle| \\ &\quad + C\|w\|_{D(A)}\|w\|_{D(A^{1+s/2})}(\|u_1\|_{D(A^{1+s/2})} + \|u_2\|_{D(A^{1+s/2})}) \\ &\leq C\|A^{1+s/2}w\|_{L^2}\|w\|_{D(A)}(\|A^{1+s/2}u_1\|_{L^2} + \|A^{1+s/2}u_2\|_{L^2}).\end{aligned}$$

Here we note that the derivation in (16) is originally applied to smooth functions, but it also works here for w by an approximation argument. To justify this, (23) will be needed. We omit the details.

Combining these two estimates, by Young's inequality,

$$\begin{aligned}&\frac{d}{dt}\|w\|_{D(A)}^2+\nu\|A^{1+s/2}w\|_{L^2}^2\\ \le\,& C\nu^{-1}\|w\|_{D(A)}^2(\|A^{1+s/2}u_1\|_{L^2}^2+\|A^{1+s/2}u_2\|_{L^2}^2).\end{aligned}$$

By Gronwall's inequality and (11),

$$\|w\|_{L_T^\infty D(A)}^2+\nu\|A^{1+s/2}w\|_{L_T^2L^2}^2\le C(\alpha,s,n,\Omega,\nu,T,\|u_0\|_{D(A)})\|w(0)\|_{D(A)}^2.$$

It follows that $w\equiv 0$ since $w(0)=0$, and thus $u_1\equiv u_2$. This estimate also implies continuous dependence of the solution on the $D(A)$-initial data.

This completes the proof of the local well-posedness in the case of Ω being a smooth bounded domain.

Case 2. Now suppose $\Omega=\mathbb{R}^n$. Let $\eta\in C_0^\infty(\mathbb{R}^n)$ be an even smooth mollifier, such that $\eta\ge 0$ is supported in the unit ball centered at 0, and η has integral 1. Define $\eta_\varepsilon(x)=\varepsilon^{-n}\eta(x/\varepsilon)$. Let u^ε solve

$$\partial_t u^\varepsilon+\nu\eta_\varepsilon*\eta_\varepsilon*A^s u^\varepsilon+\mathcal{P}^\alpha\eta_\varepsilon*[(\eta_\varepsilon*u^\varepsilon)\cdot\nabla(\eta_\varepsilon*u^\varepsilon)+\mathcal{U}^\alpha(\eta_\varepsilon*u^\varepsilon,\eta_\varepsilon*u^\varepsilon)]=0 \tag{24}$$

with initial data $u^\varepsilon|_{t=0}=\eta_\varepsilon*u_0$. We shall view (24) as an ODE $\partial_t u^\varepsilon=F_\varepsilon(u^\varepsilon)$ in $D(A)$, with

$$F_\varepsilon(u^\varepsilon)=-\nu\eta_\varepsilon*\eta_\varepsilon*A^s u^\varepsilon-\mathcal{P}^\alpha\eta_\varepsilon*[(\eta_\varepsilon*u^\varepsilon)\cdot\nabla(\eta_\varepsilon*u^\varepsilon)+\mathcal{U}^\alpha(\eta_\varepsilon*u^\varepsilon,\eta_\varepsilon*u^\varepsilon)].$$

Thanks to (19), (23) and smoothness of η_ε, it is not hard to show that $F_\varepsilon(u^\varepsilon)$ is locally Lipschitz in $u^\varepsilon\in D(A)$. Then local existence and uniqueness of $u^\varepsilon\in C^1_{T_\varepsilon}(D(A))$ follows from ODE theory on Banach spaces.

Then we derive energy estimates for u^ε. Note that in the whole space case, A^s applied to $u^\varepsilon\in D(A)$ is simply a Fourier multiplier, which thus commutes with the mollification by η_ε. We proceed as in the bounded domain case to find that

$$\frac12\frac{d}{dt}\|u^\varepsilon\|_{L^2}^2+\nu\|\eta_\varepsilon*A^{s/2}u^\varepsilon\|_{L^2}\le C\|\eta_\varepsilon*u^\varepsilon\|_{L^2}\|\eta_\varepsilon*u^\varepsilon\|_{D(A)}^2,$$

and

$$\frac12\frac{d}{dt}\|Au^\varepsilon\|_{L^2}^2+\nu\|\eta_\varepsilon*A^{1+s/2}u^\varepsilon\|_{L^2}\le C\|\eta_\varepsilon*u^\varepsilon\|_{D(A)}^2\|\eta_\varepsilon*u^\varepsilon\|_{D(A^{1+s/2})}.$$

We use $\|\eta_\varepsilon*u^\varepsilon\|_{D(A^{1+s/2})}\le\|\eta_\varepsilon*u^\varepsilon\|_{D(A)}+\|\eta_\varepsilon*A^{1+s/2}u^\varepsilon\|_{L^2}$ and Young's inequality to derive that

$$\frac{d}{dt}\|Au^\varepsilon\|_{L^2}^2+\nu\|\eta_\varepsilon*A^{1+s/2}u^\varepsilon\|_{L^2}\le C\|\eta_\varepsilon*u^\varepsilon\|_{D(A)}^3+C\nu^{-1}\|\eta_\varepsilon*u^\varepsilon\|_{D(A)}^4.$$

Combining these estimates, by Young's inequality for convolutions,

$$\frac{d}{dt}\|u^\varepsilon\|_{D(A)}^2+\nu\|\eta_\varepsilon*A^{1+s/2}u^\varepsilon\|_{L^2}\le C\|u^\varepsilon\|_{D(A)}^3+C\nu^{-1}\|u^\varepsilon\|_{D(A)}^4.$$

Taking time integral yields that

$$
\begin{aligned}
&(\|u^\varepsilon\|^2_{L^\infty_T(D(A))} - \|\eta_\varepsilon * u_0\|^2_{D(A)}) + \nu \int_0^T \|\eta_\varepsilon * A^{1+s/2} u^\varepsilon\|^2_{L^2}\, dt \\
&\le C_* T(\|u^\varepsilon\|^3_{L^\infty_T(D(A))} + \nu^{-1}\|u^\varepsilon\|^4_{L^\infty_T(D(A))}).
\end{aligned}
$$

Here $C_* = C_*(\alpha, s, n, \Omega)$. By the continuity of u^ε in time, if T is taken to be sufficiently small, which only depends on $\|u_0\|_{D(A)}$, ν and the constant C_* but not on ε, we deduce that u^ε exists on $[0, T]$ by a continuation argument if needed, and

$$
\|\eta_\varepsilon * u^\varepsilon\|^2_{L^\infty_T(D(A))} + \nu \int_0^T \|\eta_\varepsilon * A^{1+s/2} u^\varepsilon\|^2_{L^2}\, dt \le C\|u_0\|^2_{D(A)}. \tag{25}
$$

Here C is a universal constant.

To this end, since $\eta_\varepsilon * u^\varepsilon$ is uniformly bounded in $L^\infty_T(D(A)) \cap L^2_T(D(A^{1+s/2}))$, there exists $u \in L^\infty_T(D(A)) \cap L^2_T(D(A^{1+s/2}))$ satisfying (11), such that up to a subsequence, $\eta_\varepsilon * u^\varepsilon$ weak-$*$ converges to u in $L^\infty_T(D(A))$ and weakly in $L^2_T(D(A^{1+s/2}))$. We argue as in the bounded domain case that

$$
\|\partial_t u^\varepsilon\|_{D(A^{1-s/2})} \le C\|\eta_\varepsilon * u^\varepsilon\|_{D(A^{1+s/2})}(1 + \|\eta_\varepsilon * u^\varepsilon\|_{D(A)}).
$$

By (25), this implies that $\partial_t(\eta_\varepsilon * u^\varepsilon)$ has a uniform-in-ε bound in $L^2_T(D(A^{1-s/2}))$. By interpolation and the Aubin-Lions Lemma [32], $\eta_\varepsilon * u^\varepsilon \to u$ strongly in $L^p_T(D(A))$ for all $p \in [1, \infty)$. This together with the weak convergence $u^\varepsilon \rightharpoonup u$ in $L^2_T(D(A^{1+s/2}))$ is then sufficient for passing to the limit $\varepsilon \to 0$ in (24) mollified by η_ε, which implies that u is a weak solution. Time continuity of u in $D(A)$ can be justified as before. So are the initial condition and the uniqueness of u.

This completes the proof. □

Now we can prove global well-posedness by combining Proposition 3.1 with a global H^1-energy estimate.

Proof of Theorem 3.1. Take the local solution $u_* \in C_{[0,T]}(D(A)) \cap L^2_T(D(A^{1+s/2}))$ that solves

$$
\partial_t u_* + \nu A^s u_* = -\mathcal{P}^\alpha[u_* \cdot \nabla u_* + \mathcal{U}^\alpha(u_*, u_*)] \text{ in } \Omega \times [0, T], \quad u_*|_{t=0} = u_0. \tag{26}
$$

Take inner product of $(1 - \alpha^2\Delta)u_*$ and (26). It is valid to do so since $u_* \in C_{[0,T]}(D(A)) \cap L^2_T(D(A^{1+s/2}))$ while the right hand side is in $L^2_T D(A^{1-s/2})$, which has been shown in the proof of Proposition 3.1. Taking integration by parts,

(27)
$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|u_*\|^2_{L^2(\Omega)}+\alpha^2\|A^{1/2}u_*\|^2_{L^2(\Omega)})+\nu(\|A^{s/2}u_*\|^2_{L^2(\Omega)}+\alpha^2\|A^{(1+s)/2}u_*\|^2_{L^2(\Omega)})\\
=&\ -\langle(1-\alpha^2\Delta)u_*,[u_*\cdot\nabla u_*+\mathcal{U}^\alpha(u_*,u_*)+(1-\alpha^2\Delta)^{-1}\nabla q]\rangle\\
=&\ -\langle(1-\alpha^2\Delta)u_*,u_*\cdot\nabla u_*\rangle-\langle u_*,\alpha^2\mathrm{div}\,[\nabla u_*\cdot\nabla u_*^T+\nabla u_*\cdot\nabla u_*-\nabla u_*^T\cdot\nabla u_*]\rangle\\
=&\ \alpha^2\langle\Delta u_*,u_*\cdot\nabla u_*\rangle+\alpha^2\langle\partial_j u_*^i,\partial_k u_*^i\partial_k u_*^j+\partial_k u_*^i\partial_j u_*^k-\partial_i u_*^k\partial_j u_*^k\rangle\\
=&\ -\alpha^2\langle\partial_j u_*^i,\partial_j u_*^k\partial_k u_*^i\rangle-\alpha^2\langle\partial_j u_*^i,u_*^k\partial_{kj}u_*^i\rangle+\alpha^2\langle\partial_j u_*^i,\partial_k u_*^i\partial_j u_*^k\rangle=0.
\end{aligned}$$

By a limiting argument, this implies that for all $t\in[0,T]$,

$$\begin{aligned}
&(\|u_*\|^2_{L^2}+\alpha^2\|A^{1/2}u_*\|^2_{L^2})(t)\\
&\quad+2\nu\int_0^t(\|A^{s/2}u_*\|^2_{L^2}+\alpha^2\|A^{(1+s)/2}u_*\|^2_{L^2})(\tau)\,d\tau\\
&\le\|u_0\|^2_{L^2}+\alpha^2\|A^{1/2}u_0\|^2_{L^2}.
\end{aligned}\tag{28}$$

On the other hand, it holds in the scalar distribution sense on $(0,T)$ that [32, Lemma 1.2 in Chapter III]

$$\frac{1}{2}\frac{d}{dt}\|Au_*\|^2_{L^2}+\nu\|A^{1+s/2}u_*\|^2_{L^2}=-\langle A^2u_*,u_*\cdot\nabla u_*+\mathcal{U}^\alpha(u_*,u_*)\rangle.$$

By a limiting argument and the continuity of u_* in $D(A)$, for all $t\in[0,T]$,

$$\begin{aligned}
&\|Au_*\|^2_{L^2}(t)+2\nu\int_0^t\|A^{1+s/2}u_*\|^2_{L^2}\,d\tau\\
&=\|Au_0\|^2_{L^2}-2\int_0^t\langle A^2u_*,u_*\cdot\nabla u_*+\mathcal{U}^\alpha(u_*,u_*)\rangle(\tau)\,d\tau.
\end{aligned}\tag{29}$$

Once again, the derivation in (16) also work for u_* here by an approximation argument. It will be used below to bound the integrand.

To this end, we proceed in two cases.

Case 1. Suppose Ω is a bounded smooth domain. In this case, the norm $\|u_*\|_{D(A^r)}$ is equivalent to the seminorm $\|A^r u_*\|_{L^2}$ for all $r>0$, as all the μ_j's are positive. See Section 2.

Since $s\ge n/4$, by (17),

$$|\langle A^2u_*,u_*\cdot\nabla u_*\rangle|\le C\|u_*\|_{D(A^{1+s/2})}\|u_*\|_{D(A^{(1+s)/2})}\|u_*\|_{D(A)}.\tag{30}$$

Likewise, by (18),

$$\|\nabla u_*\cdot(\nabla u_*)^T+\nabla u_*\cdot\nabla u_*-(\nabla u_*)^T\cdot\nabla u_*\|_{H^1}\le C\|u_*\|_{D(A^{1+s/2})}\|u_*\|_{D(A^{(1+s)/2})},$$

and thus

$$|\langle A^2u_*,\mathcal{U}^\alpha(u_*,u_*)\rangle|\le C\|u_*\|_{D(A)}\|u_*\|_{D(A^{1+s/2})}\|u_*\|_{D(A^{(1+s)/2})}.$$

Combining this with (28), (29) and (30), we obtain that

$$\begin{aligned}
&(\|u_*\|_{D(A)}^2 + \alpha^2\|A^{1/2}u_*\|_{L^2}^2)(t) \\
&\qquad + 2\nu\int_0^t (\|A^{s/2}u_*\|_{D(A)}^2 + \alpha^2\|A^{(1+s)/2}u_*\|_{L^2}^2)(\tau)\,d\tau \\
&\le \|u_0\|_{D(A)}^2 + \alpha^2\|A^{1/2}u_0\|_{L^2}^2 \\
&\qquad + C\int_0^t (\|u_*\|_{D(A)}\|u_*\|_{D(A^{1+s/2})}\|u_*\|_{D(A^{(1+s)/2})})(\tau)\,d\tau.
\end{aligned}$$

Recall that $\|u_*\|_{D(A^r)} \le C\|A^r u_*\|_{L^2}$. By Young's inequality and (28),

$$\begin{aligned}
&\int_0^t (\|u_*\|_{D(A)}\|u_*\|_{D(A^{1+s/2})}\|u_*\|_{D(A^{(1+s)/2})})(\tau)\,d\tau \\
&\le C\int_0^t (\|u_*\|_{D(A)}\|A^{1+s/2}u_*\|_{L^2}\|A^{(1+s)/2}u_*\|_{L^2})(\tau)\,d\tau \\
&\le \nu\int_0^t \|A^{1+s/2}u_*\|_{L^2}^2(\tau)\,d\tau + C\nu^{-1}\int_0^t \|u_*\|_{D(A)}^2(\tau)\|A^{(1+s)/2}u_*\|_{L^2}^2(\tau)\,d\tau.
\end{aligned}$$

Hence,

$$\begin{aligned}
&(\|u_*\|_{D(A)}^2 + \alpha^2\|A^{1/2}u_*\|_{L^2}^2)(t) \\
&\qquad + \nu\int_0^t (\|A^{s/2}u_*\|_{D(A)}^2 + \alpha^2\|A^{(1+s)/2}u_*\|_{L^2}^2)(\tau)\,d\tau \\
&\le \|u_0\|_{D(A)}^2 + \alpha^2\|A^{1/2}u_0\|_{L^2}^2 + C\int_0^t \|u_*\|_{D(A)}^2(\tau)\|A^{(1+s)/2}u_*\|_{L^2}^2(\tau)\,d\tau.
\end{aligned}$$

where $C = C(\alpha, s, n, \Omega, \nu)$. Since $\|A^{(1+s)/2}u_*\|_{L^2} \in L^2([0,T])$ for all $T \in [0,+\infty)$ by (28) with uniform-in-T bound, then a uniform-in-T global bound for $\|u_*\|_{D(A)}$ follows from the Gronwall's inequality. Global well-posedness can be proved by the local well-posedness and a continuation argument, and (10) follows also from the last inequality.

Case 2. Now suppose $\Omega = \mathbb{R}^n$. In this case, we shall slightly change the argument so that the final estimate will be uniform in $T > 0$. We shall take advantage of $A^r u_* = (-\Delta)^r u_*$ in this case.

Again by (17),

$$|\langle A^2 u_*, u_* \cdot \nabla u_*\rangle| \le C\|Au_*\|_{H^s}\|\nabla u_*\|_{H^{\frac{n}{2}-s}}\|u_*\|_{\dot{H}^2},$$

and by (18),

$$|\langle A^2 u_*, \mathcal{U}^\alpha(u_*, u_*)\rangle| \le C\|Au_*\|_{L^2}\|\nabla^2 u_*\|_{H^s}\|\nabla u_*\|_{H^{\frac{n}{2}-s}}.$$

Combining them with (28) and (29), by Young's inequality, we obtain that

$$
\begin{aligned}
&(\|u_*\|_{D(A)}^2+\alpha^2\|A^{1/2}u_*\|_{L^2}^2)(t)\\
&\quad +2\nu\int_0^t(\|A^{s/2}u_*\|_{D(A)}^2+\alpha^2\|A^{(1+s)/2}u_*\|_{L^2}^2)(\tau)\,d\tau\\
\le\ &\|u_0\|_{D(A)}^2+\alpha^2\|A^{1/2}u_0\|_{L^2}^2\\
&+C\int_0^t(\|Au_*\|_{L^2}\|Au_*\|_{D(A^{s/2})}\|A^{1/2}u_*\|_{D(A^{s/2})})(\tau)\,d\tau.
\end{aligned}\tag{31}
$$

By interpolation,

$$\|Au_*\|_{D(A^{s/2})}\le C\|A^{s/2}u_*\|_{D(A)},\quad \|A^{1/2}u_*\|_{D(A^{s/2})}\le C\|A^{s/2}u_*\|_{D(A^{1/2})}.$$

By Young's inequality, (31) then becomes

$$
\begin{aligned}
&(\|u_*\|_{D(A)}^2+\alpha^2\|A^{1/2}u_*\|_{L^2}^2)(t)\\
&\quad +2\nu\int_0^t(\|A^{s/2}u_*\|_{D(A)}^2+\alpha^2\|A^{(1+s)/2}u_*\|_{L^2}^2)(\tau)\,d\tau\\
\le\ &\|u_0\|_{D(A)}^2+\alpha^2\|A^{1/2}u_0\|_{L^2}^2\\
&+C\int_0^t(\|Au_*\|_{L^2}\|A^{s/2}u_*\|_{D(A)}\|A^{s/2}u_*\|_{D(A^{1/2})})(\tau)\,d\tau\\
\le\ &\|u_0\|_{D(A)}^2+\alpha^2\|A^{1/2}u_0\|_{L^2}^2+\nu\int_0^t\|A^{s/2}u_*\|_{D(A)}^2(\tau)\,d\tau\\
&+C\nu^{-1}\int_0^t\|u_*\|_{D(A)}^2(\tau)\|A^{s/2}u_*\|_{D(A^{1/2})}^2(\tau)\,d\tau.
\end{aligned}\tag{32}
$$

By (28), $\|A^{s/2}u_*\|_{L^2_T D(A^{1/2})}\le C(\alpha,\nu,\|u_0\|_{D(A^{1/2})})$. Hence, by the Gronwall's inequality, we obtain a uniform-in-time global bound for $\|u_*\|_{D(A)}$, i.e., for all $t\in[0,T]$,

$$\|u_*\|_{D(A)}(t)\le C(\alpha,s,n,\nu,\|u_0\|_{D(A^{1/2})}).$$

In particular, this bound does not rely on T. Then the global well-posedness and (10) follow as before.

□

4. Improved Regularity of u_*

In this section, we shall show that the global solution u_* gains regularity instantaneously when $t>0$. We proceed in two different cases.

4.1. **Non-critical case:** $s>1/2$. For simplicity, denote $f(u_1,u_2)=-\mathcal{P}^\alpha[u_1\cdot\nabla u_2+\mathcal{U}^\alpha(u_1,u_2)]$. In the proof of Proposition 3.1, we have derived an $D(A^{1-s/2})$-estimate for f (see (18), (19) and (23)). Yet, the following lemma is still useful.

Lemma 4.1. *For all* $r \in (\frac{n}{2}, 2]$,

$$\|f(u_1,u_2)\|_{D(A^{r/2})} \le C\|u_1\|_{D(A)}\|A^{1/2}u_2\|_{D(A^{r/2})},$$

where $C = C(\alpha, r, n, \Omega)$.

Proof. The proof is straightforward. By the boundedness of $\mathcal{P}^\alpha$ and Remark 1,

$$\begin{aligned}
&\|f(u_1,u_2)\|_{D(A^{r/2})}\\
&\le C(\|u_1\cdot\nabla u_2\|_{H^r} + \|\nabla u_1\cdot\nabla u_2^T + \nabla u_1\cdot\nabla u_2 - \nabla u_1^T\cdot\nabla u_2\|_{H^{r-1}})\\
&\le C(\|u_1\|_{H^2}\|\nabla u_2\|_{H^r} + \|\nabla u_1\|_{H^{r-1}}\|\nabla u_2\|_{H^r})\\
&\le C\|u_1\|_{D(A)}(\|A^{1/2}u_2\|_{L^2} + \|A^{(r+1)/2}u_2\|_{L^2}).
\end{aligned} \tag{33}$$

□

Lemma 4.1 roughly shows that the regularity of $f(u_*,u_*)$ is one order lower than that of u_*. Since the backbone equation $\partial_t u_* + \nu A^s u_* = f$ (i.e., (6)) implies that u_* admits regularity $2s$-order higher than f, we immediately obtain improved regularity of u_* when $s > 1/2$ by bootstrapping, which is why we call the case $s > 1/2$ non-critical, and which leads to the following theorem.

Theorem 4.2 (Improved regularity of u_* in the non-critical case)**.** *If* $s > 1/2$*, the global solution* u_* *obtained in Theorem 3.1 satisfies that for all* $r \in [0, s/2]$*, and all* $t > 0$,

$$\|u_*(t)\|_{D(A^{1+r})} \le C(t^{-\frac{r}{s}} + 1)\|u_0\|_{D(A)}, \tag{34}$$

where $C = C(\alpha, s, n, \nu, \Omega, \|u_0\|_{D(A)})$*. In particular, when* $\|u_0\|_{D(A)} \to 0$*,* C *converges to a universal constant depending on* α*,* s*,* n*,* ν *and* Ω.

Proof. Define $r_k = 2 + s + (2s-1)k$ for $k = 1, \cdots, K$, where $K = \left[\frac{1-s}{2s-1}\right]$ is taken in the way that $r_K \le 3$ while $r_{K+1} > 3$. We apply Lemma 4.1 with $r = r_k - 1$ and use interpolation to find that

$$\|f(u_*,u_*)\|_{D(A^{(r_k-1)/2})} \le C(\alpha, s, n, \Omega)\|u_*\|_{D(A)}\|A^{s/2}u_*\|_{D(A^{(r_k-s)/2})}. \tag{35}$$

For any fixed $\varepsilon > 0$, we shall prove regularity and estimate of $u_*(\varepsilon)$. Let $\delta = \varepsilon/(K+2)$ and $t_j = j\delta$ for $j = 0, \cdots, K+2$. It is easy to show that for any $t \ge t_k$, $u_*(t) = e^{-(t-t_k)\nu A^s}u_k + w_k(t)$, where $u_k = u_*(t_k)$, and $w_k(t)$ solves the following Cauchy problem starting from $t = t_k$,

$$\partial_t w_k + \nu A^s w_k = f(u_*,u_*), \quad w_k|_{t=t_k} = 0.$$

It admits an energy estimate, i.e., for all $t \geq 0$,

$$
\begin{aligned}
&\|A^{(r_k-1+s)/2}w_k(t)\|_{L^2}^2 + \int_{t_k}^t \|A^{(r_k-1)/2+s}w_k\|_{L^2}^2 \\
&\leq C\int_{t_k}^t \|A^{(r_k-1)/2}f(u_*,u_*)\|_{L^2}^2\,d\tau \\
&\leq C\int_{t_k}^t \|u_*\|_{D(A)}^2(\|A^{s/2}u_*\|_{L^2}^2 + \|A^{r_k/2}u_*\|_{L^2}^2)\,d\tau.
\end{aligned} \tag{36}
$$

We used (35) in the last inequality. Thanks to (10) in Theorem 3.1 and the global H^1-estimate (28) of u_*, by the definition of r_k,

$$
\begin{aligned}
&\|A^{(r_{k+1}-s)/2}w_k(t)\|_{L^2}^2 + \int_{t_k}^t \|A^{r_{k+1}/2}w_k\|_{L^2}^2 \\
&\leq C\|u_0\|_{D(A)}^2\left(1+\int_{t_k}^t \|A^{r_k/2}u_*\|_{L^2}^2\,d\tau\right),
\end{aligned} \tag{37}
$$

where $C = C(\alpha, s, n, \nu, \Omega, \|u_0\|_{D(A^{1/2})})$. This holds for all $t \geq 0$. For $k = 0$, this together with (10) implies

$$
\|A^{(r_1-s)/2}w_0(t)\|_{L^2}^2 + \int_0^t \|A^{r_1/2}w_0\|_{L^2}^2\,d\tau \leq C\|u_0\|_{D(A)}^2. \tag{38}
$$

For all $k \geq 1$, we write $u_*(\tau) = e^{-(\tau-t_{k-1})\nu A^s}u_{k-1} + w_{k-1}(\tau)$ and derive from (37) that

(39)

$$
\begin{aligned}
&\|A^{(r_{k+1}-s)/2}w_k(t)\|_{L^2}^2 + \int_{t_k}^t \|A^{r_{k+1}/2}w_k\|_{L^2}^2 \\
&\leq C\|u_0\|_{D(A)}^2\int_{t_k}^t \|A^{s/2}e^{-(\tau-t_k)\nu A^s}(A^{(r_k-s)/2}e^{-(t_k-t_{k-1})\nu A^s}u_{k-1})\|_{L^2}^2\,d\tau \\
&\quad + C\|u_0\|_{D(A)}^2\left(1+\int_{t_k}^t \|A^{r_k/2}w_{k-1}\|_{L^2}^2\,d\tau\right) \\
&\leq C\|u_0\|_{D(A)}^2\left(\|A^{(r_k-s)/2}e^{-\delta\nu A^s}u_{k-1}\|_{L^2}^2 + 1 + \int_{t_{k-1}}^t \|A^{r_k/2}w_{k-1}\|_{L^2}^2\,d\tau\right) \\
&\leq C\|u_0\|_{D(A)}^2\left[(\nu\delta)^{-\frac{2s-1}{s}}\|A^{(r_{k-1}-s)/2}u_{k-1}\|_{L^2}^2 + 1 + \int_{t_{k-1}}^t \|A^{r_k/2}w_{k-1}\|_{L^2}^2\,d\tau\right].
\end{aligned}
$$

Here we used energy estimate for semigroup solutions. Since $t \geq 0$ is arbitrary, this implies that

$$
\begin{aligned}
&\|A^{(r_{k+1}-s)/2} w_k(t_{k+1})\|_{L^2}^2 + \int_{t_k}^{\infty} \|A^{r_{k+1}/2} w_k\|_{L^2}^2 \\
&\leq C\|u_0\|_{D(A)}^2 (\nu\delta)^{-\frac{2s-1}{s}} \|A^{(r_{k-1}-s)/2} u_{k-1}\|_{L^2}^2 \\
&\quad + C\|u_0\|_{D(A)}^2 \left[1 + \int_{t_{k-1}}^{\infty} \|A^{r_k/2} w_{k-1}\|_{L^2}^2 \, d\tau\right].
\end{aligned} \tag{40}
$$

Since $u_k = u_*(t_k) = e^{-\delta\nu A^s} u_{k-1} + w_{k-1}(t_k)$ for all $k = 0, \cdots, K+1$,

$$
\begin{aligned}
&\|A^{(r_k-s)/2} u_k\|_{L^2}^2 \\
&\leq C\left(\|A^{(r_k-s)/2} e^{-\delta\nu A^s} u_{k-1}\|_{L^2}^2 + \|A^{(r_k-s)/2} w_{k-1}(t_k)\|_{L^2}^2\right) \\
&\leq C\left[(\nu\delta)^{-\frac{2s-1}{s}} \|A^{(r_{k-1}-s)/2} u_{k-1}\|_{L^2}^2 + \|A^{(r_k-s)/2} w_{k-1}(t_k)\|_{L^2}^2\right].
\end{aligned} \tag{41}
$$

To this end, we claim that for $j = 0, \cdots, K$,

$$
\begin{aligned}
&\|A^{(r_j-s)/2} u_j\|_{L^2}^2 + \|A^{(r_{j+1}-s)/2} w_j(t_{j+1})\|_{L^2}^2 + \int_{t_j}^{\infty} \|A^{r_{j+1}/2} w_j\|_{L^2}^2 \, d\tau \\
&\leq C(\delta^{-\frac{j(2s-1)}{s}} + 1)(\|u_0\|_{D(A)}^2 + 1)^j \|u_0\|_{D(A)}^2,
\end{aligned} \tag{42}
$$

where $C = C(\alpha, s, n, \nu, \Omega, \|u_0\|_{D(A^{1/2})}, j)$. We shall prove this by (40), (41) and induction. Indeed, for $j = 0$, (42) follows from (38) immediately. Now suppose (42) holds for $j \leq k-1$ with some $k \geq 1$. Then for $j = k$, (42) follows immediately from (40) and (41). This justifies the claim.

To this end, we shall perform the last-step improvement to derive an estimate for $u_*(\varepsilon) = e^{-\delta\nu A^s} u_{K+1} + w_{K+1}(t_{K+2})$. Similar to (36) and (39),

$$
\begin{aligned}
&\|A^{1+s/2} w_{K+1}(t_{K+2})\|_{L^2}^2 \\
&\leq C\int_{t_{K+1}}^{\infty} \|Af(u_*, u_*)\|_{L^2}^2 \, d\tau \\
&\leq C\int_{t_{K+1}}^{\infty} \|u_*\|_{D(A)}^2 (\|A^{s/2} u_*\|_{L^2}^2 + \|A^{3/2} u_*\|_{L^2}^2) \, d\tau \\
&\leq C\|u_0\|_{D(A)}^2 \left(1 + \|A^{(3-s)/2} e^{-\delta\nu A^s} u_K\|_{L^2}^2 + \int_{t_{K+1}}^{\infty} \|A^{3/2} w_K\|_{L^2}^2 \, d\tau\right).
\end{aligned}
$$

Note that $r_{K+1} \geq 3$. By (42) with $j = K$,

$$
\begin{aligned}
&\|A^{1+s/2} w_{K+1}(t_{K+2})\|_{L^2}^2 \\
&\leq C\|u_0\|_{D(A)}^2 \left(1 + (\nu\delta)^{-\frac{3-r_K}{s}} \|A^{(r_K-s)/2} u_K\|_{L^2}^2\right) \\
&\quad + C\|u_0\|_{D(A)}^2 \int_{t_{K+1}}^{\infty} \|A^{r_{K+1}/2} w_K\|_{L^2}^2 \, d\tau \\
&\leq C\|u_0\|_{D(A)}^2 (\delta^{-\frac{1-s}{s}} + 1)(\|u_0\|_{D(A)}^2 + 1)^{K+1}.
\end{aligned}
$$

On the other hand, since (41) also holds for $j = K+1$,

$$\begin{aligned}
&\|A^{1+s/2}e^{-\delta\nu A^s}u_{K+1}\|_{L^2}^2\\
&\le C(\nu\delta)^{-\frac{(2+s)-(r_{K+1}-s)}{s}}\|A^{(r_{K+1}-s)/2}u_{K+1}\|_{L^2}^2\\
&\le C(\delta^{-1}+1)(\|u_0\|_{D(A)}^2+1)^K\|u_0\|_{D(A)}^2.
\end{aligned}$$

Combining these two estimates, we obtain that

$$\begin{aligned}
&\|A^{1+s/2}u_*(t_{K+2})\|_{L^2}^2\\
&\le C(\delta^{-1}+1)(\|u_0\|_{D(A)}^2+1)^{K+1}\|u_0\|_{D(A)}^2\\
&\le C(\varepsilon^{-1}+1)(\|u_0\|_{D(A)}^2+1)^{K+1}\|u_0\|_{D(A)}^2,
\end{aligned} \tag{43}$$

where $C = C(\alpha, s, n, \nu, \Omega, \|u_0\|_{D(A^{1/2})}, K)$. Note that K essentially depends on s. By interpolation between (43) and $\|u_*(\varepsilon)\|_{D(A)}^2 \le C\|u_0\|_{D(A)}^2$, we know that for all $r \in [0, s/2]$,

$$\|u_*(\varepsilon)\|_{D(A^{1+r})}^2 \le C(\varepsilon^{-\frac{2r}{s}}+1)\|u_0\|_{D(A)}^2,$$

where $C = C(\alpha, s, n, \nu, \Omega, \|u_0\|_{D(A)})$. In particular, as $\|u_0\|_{D(A)} \to 0$, C converges to a universal constant depending on α, s, n, ν and Ω. Since $\varepsilon > 0$ is arbitrary, (34) is proved. □

4.2. **Critical case:** $s = 1/2$. Now we consider the case $(n, s) = (2, 1/2)$. It is called critical since no easy bootstrapping argument can be applied as before. In what follows, we shall prove, in the fashion of re-constructing the solution, that u_* has local Hölder continuity in time away from $t = 0$ as a function valued in $D(A^{1+s/2})$; while the Hölder norm admits a singularity at $t = 0$ with certain growth rate as $t \to 0^+$. This idea comes from the earlier studies of regularity of L^p-solution of the Navier-Stokes equation and semilinear parabolic equations [18, 20, 19]. To be more precise, we introduce the following definition.

Definition 4.3. Fix $T \in (0, 1]$ and let $w \in C_{[0,T]}(D(A)) \cap L_T^2(D(A^{1+s/2}))$. With $R > 0$ and $\beta \in (0, 1/2)$, we say $w \in B_{R,T}^\beta$ if and only if

(1) $\|w(t)\|_{D(A)} \le R$ for all $t \in [0, T]$;
(2) $\|A^{s/2}w(t)\|_{D(A)} \le Rt^{-1/2}$ for all $t \in (0, T]$;
(3) For all $0 < t \le t+h \le T$,

$$\|w(t+h) - w(t)\|_{D(A)} \le h^\beta t^{-\beta} R, \tag{44}$$

$$\|A^{s/2}(w(t+h) - w(t))\|_{D(A)} \le h^\beta t^{-(\beta+1/2)} R. \tag{45}$$

In fact, homogeneous solutions given by the semigroup $\{e^{-t\nu A^s}\}_{t\ge 0}$ is in this type of sets.

Lemma 4.4. *For all $w_0 \in D(A)$, $w(t) = e^{-t\nu A^s}w_0 \in B_{R,T}^\beta$ with $R = C(\nu, \beta)\|w_0\|_{D(A)}$.*

Proof. It is trivial that $\|w(t)\|_{D(A)} \le \|w_0\|_{D(A)}$ and

$$\begin{aligned}&\|A^{s/2}w(t)\|_{D(A)}\\ &\le \|e^{-t\nu A^s}A^{s/2}w_0\|_{L^2} + \|A^{s/2}e^{-t\nu A^s}Aw_0\|_{L^2}\\ &\le \|A^{s/2}w_0\|_{L^2} + C(\nu)t^{-1/2}\|Aw_0\|_{L^2}\\ &\le C(\nu)t^{-1/2}\|w_0\|_{D(A)}.\end{aligned}$$

In the last inequality, we used the fact that $t \le 1$.

To prove (44), we derive that

$$\begin{aligned}&\|e^{-(t+h)\nu A^s}w_0 - e^{-t\nu A^s}w_0\|_{D(A)}\\ &= \left\|\int_t^{t+h} \nu A^s e^{-\tau\nu A^s}w_0\,d\tau\right\|_{D(A)} \le C\int_t^{t+h}\tau^{-1}\,d\tau\cdot\|w_0\|_{D(A)}\\ &\le C\ln\left(1+\frac{h}{t}\right)\|w_0\|_{D(A)} \le C(\beta)h^\beta t^{-\beta}\|w_0\|_{D(A)}.\end{aligned}$$

Similarly,

$$\begin{aligned}&\|A^{s/2}(e^{-(t+h)\nu A^s}w_0 - e^{-t\nu A^s}w_0)\|_{D(A)}\\ &= \left\|\int_t^{t+h} \nu A^{3s/2} e^{-\tau\nu A^s}w_0\,d\tau\right\|_{D(A)} \le C(\nu)\int_t^{t+h}\tau^{-3/2}\,d\tau\cdot\|w_0\|_{D(A)}\\ &\le C(\nu)t^{-1/2}\left(1-\sqrt{\frac{t}{t+h}}\right)\|w_0\|_{D(A)} \le C(\nu,\beta)\frac{h^\beta}{t^{\beta+1/2}}\|w_0\|_{D(A)}.\end{aligned}$$

In the last inequality, we used the fact that $1-x^{-1/2} \le (x-1)^\beta$ for all $x \ge 1$. Indeed, it is trivial when $x \ge 2$; for $x \in [1,2]$, $1-x^{-1/2} \le 1-x^{-1} \le x-1 \le (x-1)^\beta$. This completes the proof. □

The following lemma is the key to show existence of the solution in the type of sets $B^\beta_{R,T}$.

Lemma 4.5. *Fix $T \in (0,1]$ and $\beta \in (0,1/2)$. For $w_1 \in B^\beta_{R_1,T}$ and $w_2 \in B^\beta_{R_2,T}$, let*

$$v[w_1,w_2](t) = \int_0^t e^{-(t-\tau)\nu A^s} f(w_1,w_2)\,d\tau, \quad t \in [0,T].$$

Then $v[w_1,w_2](t) \in B^\beta_{CR_1R_2,T}$, where $C = C(\alpha,\nu,\Omega,\beta)$.

Proof. It is helpful to first derive some estimates for $f(w_1,w_2)$. By Lemma 4.1 with $r = 2-s$,

$$\|f(w_1,w_2)(t)\|_{D(A^{1-s/2})} \le C\|w_1\|_{D(A)}\|A^{s/2}w_2\|_{D(A)} \le CR_1R_2t^{-1/2}, \tag{46}$$

where $C = C(\alpha, \Omega)$. In addition, for all $t \geq \tau > 0$,

(47)
$$\begin{aligned}
&\|A^{1-s/2}(f(w_1, w_2)(\tau) - f(w_1, w_2)(t))\|_{L^2} \\
&\leq \|f(w_1(\tau) - w_1(t), w_2(\tau))\|_{D(A^{1-s/2})} + \|f(w_1(t), w_2(\tau) - w_2(t))\|_{D(A^{1-s/2})} \\
&\leq C\|w_1(\tau) - w_1(t)\|_{D(A)}\|A^{s/2}w_2(\tau)\|_{D(A)} \\
&\quad + C\|w_1(t)\|_{D(A)}\|A^{s/2}(w_2(\tau) - w_2(t))\|_{D(A)} \\
&\leq C\frac{(t-\tau)^\beta}{\tau^{\beta+1/2}}R_1R_2,
\end{aligned}$$

where $C = C(\alpha, \Omega)$.

For brevity, we write $v[w_1, w_2](t)$ as $v(t)$ in the following.

Step 1. We start from $\|v(t)\|_{D(A)}$. Thanks to (46),

$$\begin{aligned}
\|v(t)\|_{D(A)} &\leq \int_0^t (\|e^{-(t-\tau)\nu A^s}\|_{\mathcal{L}(L^2)} + \|A^{s/2}e^{-(t-\tau)\nu A^s}\|_{\mathcal{L}(L^2)})\|f(w_1, w_2)\|_{D(A^{1-s/2})}\,d\tau \\
&\leq C\int_0^t (1 + (t-\tau)^{-1/2})R_1R_2\tau^{-1/2}\,d\tau \leq CR_1R_2,
\end{aligned}$$

where $C = C(\alpha, \nu, \Omega)$ and $\|\cdot\|_{\mathcal{L}(L^2)}$ denotes the operator norm from $L^2(\Omega)$ to itself. Here we use the fact that $(t-\tau) < T \leq 1$.

Step 2. We make estimate for $\|A^{s/2}v(t)\|_{D(A)}$. Thanks to (46) and (47),

$$\begin{aligned}
&\|A^{s/2}v(t)\|_{D(A)} \\
&\leq \|A^{s/2}v(t)\|_{L^2} + \left\|\int_0^t A^s e^{-(t-\tau)\nu A^s} A^{1-s/2} f(w_1, w_2)(t)\,d\tau\right\|_{L^2} \\
&\quad + \left\|\int_0^t A^s e^{-(t-\tau)\nu A^s}[A^{1-s/2}f(w_1, w_2)(\tau) - A^{1-s/2}f(w_1, w_2)(t)]\,d\tau\right\|_{L^2} \\
&\leq C\|v(t)\|_{D(A)} + \nu^{-1}\left\|\int_0^t \frac{d}{d\eta}\bigg|_{\eta=\tau}\left(e^{-(t-\eta)\nu A^s}\right) A^{1-s/2}f(w_1, w_2)(t)\,d\tau\right\|_{L^2} \\
&\quad + \int_0^t \left\|A^s e^{-(t-\tau)\nu A^s}\right\|_{\mathcal{L}(L^2)} \|A^{1-s/2}(f(w_1, w_2)(\tau) - f(w_1, w_2)(t))\|_{L^2}\,d\tau \\
&\leq CR_1R_2 + \nu^{-1}\|(Id - e^{-t\nu A^s})A^{1-s/2}f(w_1, w_2)(t)\|_{L^2} \\
&\quad + C\nu^{-1}\int_0^t (t-\tau)^{-1}\frac{(t-\tau)^\beta}{\tau^{\beta+1/2}}R_1R_2\,d\tau \\
&\leq Ct^{-1/2}R_1R_2,
\end{aligned}$$

where $C = C(\alpha, \nu, \Omega, \beta)$.

Step 3. We check (44) for v. For all $0 < t \le t+h \le T$, by (46) and (47),

(48)
$$\begin{aligned}
&\|v(t+h)-v(t)\|_{D(A)}\\
&= \left\| \int_0^{t+h} e^{-(t+h-\tau)\nu A^s} f(w_1,w_2)(\tau)\,d\tau - \int_0^t e^{-(t-\tau)\nu A^s} f(w_1,w_2)(\tau)\,d\tau \right\|_{D(A)}\\
&\le \left\| \int_0^{h} e^{-(t+h-\tau)\nu A^s} f(w_1,w_2)(\tau)\,d\tau \right\|_{D(A)}\\
&\quad + \left\| \int_0^{t} e^{-(t-\tau)\nu A^s} (f(w_1,w_2)(\tau+h) - f(w_1,w_2)(\tau))\,d\tau \right\|_{D(A)}\\
&\le \int_0^h \|f(w_1,w_2)(\tau)\|_{L^2}\,d\tau\\
&\quad + \int_0^h \|A^{s/2} e^{-(t+h-\tau)\nu A^s}\|_{\mathcal{L}(L^2)} \|A^{1-s/2} f(w_1,w_2)(\tau)\|_{L^2}\,d\tau\\
&\quad + \int_0^t \|(f(w_1,w_2)(\tau+h) - f(w_1,w_2)(\tau))\|_{L^2}\,d\tau\\
&\quad + \int_0^t \|A^{s/2} e^{-(t-\tau)\nu A^s}\|_{\mathcal{L}(L^2)} \|A^{1-s/2}(f(w_1,w_2)(\tau+h) - f(w_1,w_2)(\tau))\|_{L^2}\,d\tau\\
&\le C\int_0^h (t+h-\tau)^{-1/2} R_1R_2 \tau^{-1/2}\,d\tau + C\int_0^t (t-\tau)^{-1/2} R_1R_2 \frac{h^\beta}{\tau^{\beta+1/2}}\,d\tau\\
&\le CR_1R_2 \int_0^h (t+h-\tau)^{-1/2}\tau^{-1/2}\,d\tau + Ch^\beta t^{-\beta} R_1R_2,
\end{aligned}$$

where $C = C(\alpha,\nu,\Omega,\beta)$. If $t \ge h$,

$$\int_0^h (t+h-\tau)^{-1/2}\tau^{-1/2}\,d\tau \le \int_0^h t^{-1/2}\tau^{-1/2}\,d\tau \le Ch^{1/2}t^{-1/2} \le Ch^\beta t^{-\beta}.$$

Otherwise, if $t < h$,

$$\int_0^h (t+h-\tau)^{-1/2}\tau^{-1/2}\,d\tau \le \int_0^{t+h} (t+h-\tau)^{-1/2}\tau^{-1/2}\,d\tau \le C \le Ch^\beta t^{-\beta}.$$

Combining the above estimates with (48), we find that

$$\|v(t+h)-v(t)\|_{D(A)} \le Ch^\beta t^{-\beta} R_1R_2,$$

which is (44).

Step 4. We check (45) for v. Consider $\|v(t+h)-v(t)\|_{D(A^{1+s/2})}$ with $0 < t \le t+h \le T$. First we assume that $h \le t/2$. It is known that $\|A^{s/2}(v(t+h)-v(t))\|_{D(A)} \le \|A^{s/2}(v(t+h)-v(t))\|_{L^2} + \|A^{1+s/2}(v(t+h)-v(t))\|_{L^2}$. We focus on the second term as the first term can be handled using Step 3.

We calculate that

(49)
$$
\begin{aligned}
&A^{1+s/2}(v(t+h)-v(t))\\
=&\int_0^{t+h} A^s e^{-(t+h-\tau)\nu A^s} A^{1-s/2} f(w_1,w_2)(\tau)\,d\tau\\
&-\int_0^{t} A^s e^{-(t-\tau)\nu A^s} A^{1-s/2} f(w_1,w_2)(\tau)\,d\tau\\
=&\int_0^{t-h} A^s (e^{-h\nu A^s}-Id) e^{-(t-\tau)\nu A^s} A^{1-s/2} (f(w_1,w_2)(\tau)-f(w_1,w_2)(t))\,d\tau\\
&+\int_{t-h}^{t+h} A^s e^{-(t+h-\tau)\nu A^s} A^{1-s/2} (f(w_1,w_2)(\tau)-f(w_1,w_2)(t+h))\,d\tau\\
&-\int_{t-h}^{t} A^s e^{-(t-\tau)\nu A^s} A^{1-s/2} (f(w_1,w_2)(\tau)-f(w_1,w_2)(t))\,d\tau\\
&+\int_0^{t-h} A^s e^{-(t+h-\tau)\nu A^s} A^{1-s/2} f(w_1,w_2)(t)\,d\tau\\
&+\int_{t-h}^{t+h} A^s e^{-(t+h-\tau)\nu A^s} A^{1-s/2} f(w_1,w_2)(t+h)\,d\tau\\
&-\int_0^{t} A^s e^{-(t-\tau)\nu A^s} A^{1-s/2} f(w_1,w_2)(t)\,d\tau\\
=:&\ I_1+I_2+I_3+I_4+I_5+I_6.
\end{aligned}
$$

Take $\beta' = (\beta+\frac{1}{2})/2 \in (\beta, 1/2)$.

$$
\begin{aligned}
\|I_1\|_{L^2} \le \int_0^{t-h} &\|A^{-\beta' s}(e^{-h\nu A^s}-Id)\|_{\mathcal{L}(L^2)} \|A^{(1+\beta')s} e^{-(t-\tau)\nu A^s}\|_{\mathcal{L}(L^2)}\\
&\cdot \|A^{1-s/2}(f(w_1,w_2)(\tau)-f(w_1,w_2)(t))\|_{L^2}\,d\tau.
\end{aligned}
$$

Since $\|A^{-\beta' s}(e^{-h\nu A^s}-Id)\|_{\mathcal{L}(L^2)} \le C(\nu,\beta')h^{\beta'}$ [18] and $h \le t/2$ by assumption,

(50)
$$
\begin{aligned}
\|I_1\|_{L^2} &\le C\int_0^{t-h} h^{\beta'}(t-\tau)^{-(1+\beta')}\frac{(t-\tau)^\beta}{\tau^{\beta+1/2}} R_1R_2\,d\tau\\
&= Ch^{\beta'}R_1R_2 \int_0^{t/2} + \int_{t/2}^{t-h} (t-\tau)^{-(1+\beta'-\beta)}\tau^{-(\beta+1/2)}\,d\tau\\
&\le CR_1R_2h^\beta t^{-(\beta+1/2)},
\end{aligned}
$$

where $C = C(\alpha,\nu,\Omega,\beta)$. In the last step, we used the fact that $h \le t$ and $\beta' \ge \beta$.

By (47),

(51)
$$
\|I_2\|_{L^2} \le C\int_{t-h}^{t+h} (t+h-\tau)^{-1}\cdot\frac{(t+h-\tau)^\beta}{\tau^{-(\beta+1/2)}} R_1R_2 \le CR_1R_2h^\beta t^{\beta+1/2},
$$

and similarly,

$$\|I_3\|_{L^2} \le C \int_{t-h}^{t} (t-\tau)^{-1} \frac{(t-\tau)^{\beta}}{\tau^{\beta+1/2}} R_1 R_2 \le C R_1 R_2 h^{\beta} t^{-(\beta+1/2)}, \tag{52}$$

where $C = C(\alpha, \nu, \Omega, \beta)$.

The rest of the terms in (49) can be handled as follows.

$$\begin{aligned}
&\nu(I_4 + I_5 + I_6) \\
&= (e^{-2h\nu A^s} - e^{-(t+h)\nu A^s}) A^{1-s/2} f(w_1, w_2)(t) \\
&\quad + (Id - e^{-2h\nu A^s}) A^{1-s/2} (f(w_1, w_2)(t+h) - f(w_1, w_2)(t)) \\
&\quad + (Id - e^{-2h\nu A^s}) A^{1-s/2} f(w_1, w_2)(t) \\
&\quad - (Id - e^{-t\nu A^s}) A^{1-s/2} f(w_1, w_2)(t) \\
&= A^{-\beta s}(Id - e^{-h\nu A^s}) A^{\beta s} e^{-t\nu A^s} A^{1-s/2} f(w_1, w_2)(t) \\
&\quad + (Id - e^{-2h\nu A^s}) A^{1-s/2} (f(w_1, w_2)(t+h) - f(w_1, w_2)(t)).
\end{aligned}$$

By virtue of (46) and (47),

$$\begin{aligned}
&\|I_4 + I_5 + I_6\|_{L^2} \\
&\le C\|A^{-\beta s}(Id - e^{-h\nu A^s})\|_{\mathcal{L}(L^2)} \|A^{\beta s} e^{-t\nu A^s}\|_{\mathcal{L}(L^2)} \|A^{1-s/2} f(w_1, w_2)(t)\|_{L^2} \\
&\quad + C\|(Id - e^{-2h\nu A^s})\|_{\mathcal{L}(L^2)} \|A^{1-s/2}(f(w_1, w_2)(t+h) - f(w_1, w_2)(t))\|_{L^2} \\
&\le C h^{\beta} t^{-(\beta+1/2)} R_1 R_2,
\end{aligned} \tag{53}$$

where $C = C(\alpha, \nu, \Omega, \beta)$. Combining (50)-(53), and

$$\|A^{s/2}(v(t+h) - v(t))\|_{L^2} \le C h^{\beta} t^{-\beta} R_1 R_2 \le C h^{\beta} t^{-(\beta+1/2)} R_1 R_2,$$

we establish (45) for v provided that $h \le t/2$.

If $h > t/2$, there exist $N \in \mathbb{N}_+$ and $1 + \kappa \in (\sqrt{3/2}, 3/2]$, such that $t + h = t(1+\kappa)^N$. In fact, it suffices to consider $N = 1, 2, 2^2, \cdots$, and there will be exactly one such N satisfying the above condition; κ will follow from the choice of N. With abuse of notations, let $t_j = t(1+\kappa)^j$ for $j = 0, \cdots, N$. Then by (45) for the case $h \le t/2$,

$$\begin{aligned}
&\|A^{s/2}(v(t_N) - v(t_0))\|_{D(A)} \\
&\le \sum_{j=1}^{N} \|A^{s/2}(v(t_j) - v(t_{j-1}))\|_{D(A)} \\
&\le C R_1 R_2 \sum_{j=1}^{N} (t_j - t_{j-1})^{\beta} t_{j-1}^{-(\beta+1/2)} \\
&\le C R_1 R_2 t^{-1/2} \kappa^{\beta} \sum_{j=1}^{N} (1+\kappa)^{-(j-1)/2} \\
&\le C R_1 R_2 h^{\beta} t^{-(\beta+1/2)},
\end{aligned}$$

where $C = C(\alpha, \nu, \Omega, \beta)$. The last inequality follows from $h \geq t/2$.

This completes the proof. $\square$

With Lemma 4.5, we have the following result in the critical case which re-constructs the solution obtained in Proposition 3.1, yet with refined characterization of its regularity. However, we shall additionally need the initial data to be small.

Proposition 4.1 (Local well-posedness in a refined class)**.** *Assume* (1) *and let* $(n, s) = (2, 1/2)$*. For given* $\beta \in (0, 1/2)$*, there exists an* $\varepsilon = \varepsilon(\alpha, \nu, \Omega, \beta) \in (0, 1]$*, such that if* $u_0 \in D(A)$ *with* $\|u_0\|_{D(A)} \leq \varepsilon$*, then the unique local solution* u *obtained in Proposition 3.1 satisfies* $u \in B^{\beta}_{C\|u_0\|_{D(A)},1}$ *with* $C = C(\alpha, \nu, \Omega, \beta)$*.*

Proof. Instead of proving the regularity of the local solution u directly, the proof uses a fixed-point iteration to re-construct the solution in the refined class.

Let $M = \|u_0\|_{D(A)} < +\infty$. Take $T = 1$ and fix $\beta \in (0, 1/2)$, we denote

$$B := \Big\{u \in C_T(D(A)) \cap L^2_T(D(A^{1+s/2})) : u|_{t=0} = u_0,\ u - e^{-t\nu A^s}u_0 \in B^{\beta}_{M,T}, \\ \|u - e^{-t\nu A^s}u_0\|_{L^\infty_T(D(A))\cap L^2_T(D(A^{1+s/2}))} \leq M\Big\}.$$

B is nonempty (since $e^{-t\nu A^s}u_0 \in B$) and closed in $C_T(D(A))\cap L^2_T(D(A^{1+s/2}))$. By definition, it holds for all $u \in B$ that

$$\|u\|_{L^\infty_T(D(A))\cap L^2_T(D(A^{1+s/2}))} \leq C(\nu)M. \tag{54}$$

Consider the map $Q : u \mapsto Qu := w$, where $u \in B$ and w solves

$$\partial_t w + \nu A^s w = f(u, u), \quad w|_{t=0} = u_0.$$

Since $f(u, u) \in L^2_T(D(A^{1-s/2}))$ thanks to (46), the existence and uniqueness of $w \in L^\infty_T(D(A)) \cap L^2_T(D(A^{1+s/2}))$ can be established easily, e.g., by Galerkin approximation.

We claim that Q is well-defined from B to itself if $M \ll 1$, with smallness of M depending on α, ν, Ω and β. Firstly, $\tilde{w} := Qu - e^{-t\nu A^s}u_0$ solves

$$\partial_t \tilde{w} + \nu A^s \tilde{w} = f(u, u), \quad \tilde{w}|_{t=0} = 0.$$

In the view of energy estimate, (46) and (54),

$$\|\tilde{w}\|_{L^\infty_T(D(A))\cap L^2_T(D(A^{1+s/2}))} \leq C(\nu)\|f(u, u)\|_{L^2_T D(A^{1-s/2})} \leq C_0 M^2, \tag{55}$$

where $C_0 = C_0(\alpha, \nu, \Omega)$. If $M \leq C_0^{-1}$, Qu satisfies

$$\|Qu - e^{-t\nu A^s}u_0\|_{L^\infty_T(D(A))\cap L^2_T(D(A^{1+s/2}))} \leq M.$$

In addition, we may write

$$Qu(t) = e^{-t\nu A^s}u_0 + \int_0^t e^{-(t-\tau)\nu A^s} f(u, u)(\tau)\, d\tau = e^{-t\nu A^s}u_0 + v[u, u](t).$$

Thanks to Lemma 4.5, for all $u \in B$, $Qu - e^{-t\nu A^s}u_0 = v[u,u] \in B^{\beta}_{C_1 M^2, T}$, where $C_1 = C_1(\alpha,\nu,\Omega,\beta)$. Now requiring M to be even smaller if necessary, such that $M \le C_1^{-1}$, we obtain $Qu - e^{-t\nu A^s}u_0 \in B^{\beta}_{M,T}$. This proves the claim.

To this end, with the smallness assumption on M, we define $u^{(0)} = e^{-t\nu A^s}u_0 \in B$, and $u^{(j)} = Qu^{(j-1)} \in B$ for all $j \in \mathbb{N}_+$. We shall prove by induction that for all $j \in \mathbb{N}_+$,

$$\|u^{(j)} - u^{(j-1)}\|_{L^\infty_T(D(A)) \cap L^2_T(D(A^{1+s/2}))} \le (C_2 M)^{j-1} C_0 M^2, \tag{56}$$

$$u^{(j)} - u^{(j-1)} \in B_{(C_3 M)^{j+1}}, \tag{57}$$

for some $C_2 = C(\alpha,\nu,\Omega)$ and $C_3 = C_3(\alpha,\nu,\Omega,\beta)$, while C_0 is defined in (55). Indeed, consider the equation for $\tilde{w}_j = u^{(j)} - u^{(j-1)}$. For all $j \ge 2$,

$$\partial_t \tilde{w}_j + \nu A^s \tilde{w}_j = f(\tilde{w}_{j-1}, u^{(j-1)}) + f(u^{(j-2)}, \tilde{w}_{j-1}), \quad \tilde{w}_j|_{t=0} = 0.$$

Again, by energy estimate, (46) and (54),

$$\begin{aligned} &\|\tilde{w}_j\|_{L^\infty_T(D(A)) \cap L^2_T(D(A^{1+s/2}))} \\ &\le C(\nu)(\|f(\tilde{w}_{j-1}, u^{(j-1)})\|_{L^2_T D(A^{1-s/2})} + \|f(u^{(j-2)}, \tilde{w}_{j-1})\|_{L^2_T D(A^{1-s/2})}) \\ &\le C_2 M \|\tilde{w}_{j-1}\|_{L^\infty_T(D(A)) \cap L^2_T(D(A^{1+s/2}))}, \end{aligned} \tag{58}$$

Now (56) follows immediately from (55) and (58). To show (57), we note that $u^{(1)} - u^{(0)} = v[u^{(0)}, u^{(0)}] \in B^{\beta}_{C_5 (C_4 M)^2, T}$, where C_4 and C_5 are the constants in Lemma 4.4 and Lemma 4.5, respectively. Assuming $4C_5 > 1$, we have

$$u^{(1)} - u^{(0)} = v[u^{(0)}, u^{(0)}] \in B^{\beta}_{(2C_5 C_4 M)^2, T} =: B^{\beta}_{(C_3 M)^2, T}.$$

Now suppose $u^{(j)} - u^{(j-1)} \in B^{\beta}_{(C_3 M)^{j+1}, T}$, by Lemma 4.4 and Lemma 4.5,

$$\begin{aligned} &u^{(j+1)} - u^{(j)} \\ &= v[u^{(j)} - u^{(j-1)}, u^{(j)}] + v[u^{(j-1)}, u^{(j)} - u^{(j-1)}] \\ &\in B^{\beta}_{2C_5 (C_3 M)^{j+1} C_4 M, T} = B^{\beta}_{(C_3 M)^{j+2}, T}. \end{aligned}$$

In the view of (56) and (57), if M is assumed to be even smaller if necessary such that $C_2 M, C_3 M < 1$, then $\{u^{(j)}\}_{j \in \mathbb{N}}$ converges in $L^\infty_T(D(A)) \cap L^2_T(D(A^{1+s/2}))$ and $C^{\beta}_{loc}((0,T]; D(A^{1+s/2}))$ to u_{**}. It is easy to show that u_{**} is a fixed-point of Q, and thus a local solution of (6). By uniqueness result in Proposition 3.1, such u_{**} is unique and $u_{**} = u_*$. Here u_* is the unique global solution from Theorem 3.1. It satisfies the following estimates

$$\|u_* - e^{-t\nu A^s}u_0\|_{L^\infty_T(D(A)) \cap L^2_T(D(A^{1+s/2}))} \le \sum_{j=1}^{\infty} (C_2 M)^{j-1} C_1 M^2 =: C_6 M^2,$$

$$u_* - e^{-t\nu A^s}u_0 \in B^{\beta}_{\sum_{j=1}^{\infty} (C_3 M)^{j+1}, T} =: B^{\beta}_{C_7 M^2, T}.$$

Assuming $M \le 1$, we obtain the desired estimates by estimates on the homogeneous semigroup solution.

This completes the proof. □

Combining Proposition 4.1 with Theorem 3.1 yields the improved regularity of the global solution when $(n, s) = (2, 1/2)$.

Theorem 4.6 (Improved regularity of u_* in the critical case)**.** *Under the assumptions of Proposition 4.1, the unique global solution* $u_* \in C_{[0,+\infty)}(D(A)) \cap L^2_{[0,+\infty),loc}(D(A^{1+s/2}))$ *obtained in Theorem 3.1 satisfies*

(1) $\|A^{s/2}u(t)\|_{D(A)} \leq C\|u_0\|_{D(A)}(1 + t^{-1/2})$ *for all* $t \in (0, +\infty)$*;*

(2) For all $0 < t \leq t + h < +\infty$, $h \in [0, 1]$,

$$\|u(t + h) - u(t)\|_{D(A)} \leq Ch^\beta(1 + t^{-\beta})\|u_0\|_{D(A)},$$

$$\|A^{s/2}(u(t + h) - u(t))\|_{D(A)} \leq Ch^\beta(1 + t^{-(\beta+1/2)})\|u_0\|_{D(A)},$$

where $C = C(\alpha, \nu, \Omega, \beta)$*.*

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