Identifying Constant Curvature Manifolds, Einstein Manifolds, and Ricci Parallel Manifolds *

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Abstract

We establish variational formulas for Ricci upper and lower bounds, as well as a derivative formula for the Ricci curvature. Combining these with derivative and Hessian formulas of the heat semigroup developed from stochastic analysis, we identify constant curvature manifolds, Einstein manifolds and Ricci parallel manifolds by using analytic formulas and semigroup inequalities. Moreover, explicit Hessian estimates are derived for the heat semigroup on Einstein and Ricci parallel manifolds.

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1 Introduction

Let $(M, \mathbf{g})$ be a d-dimensional complete Riemannian manifold. Let $TM = \cup_{x\in M} T_x M$ be the bundle of tangent vectors of M, where for every $x \in M$, $T_x M$ is the tangent space at point x. We will denote $\langle u, v\rangle = \mathbf{g}(u, v)$ and $|u| = \sqrt{\langle u, u\rangle}$ for $u, v \in T_x M, x \in M$. Let $\mathcal{R}$, $\mathcal{R}ic$ and $\mathcal{S}ec$ denote the Riemannian curvature tensor, Ricci curvature and sectional curvature respectively.

When M has constant Ricci curvature, i.e. $\mathcal{R}ic = K\mathbf{g}$ (simply denote by $\mathcal{R}ic = K$) for some constant K, the metric is a vacuum solution of Einstein field equations (in particular

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for $d = 4$ in general relativity), and M is called an Einstein manifold. In differential geometry, a basic problem is to characterize topological and geometry properties of Einstein manifolds. For instance, according to Thorpe [16] and Hitchin [17], if a four-dimensional compact manifold admits an Einstein metric, then

$$\chi(M) \geq \frac{3}{2}|\tau(M)|,$$

where χ is the Euler characterization and τ is the signature, and the equality holds if and only if M is a flat torus, a Calabi-Yau manifold, or a quotient thereof. Recently, Brandle [6] showed that Einstein manifolds with positive isotropy curvature are space forms, while with nonnegative isotropy curvature are locally symmetric, and an ongoing investigation is to classify Einstein manifolds of both positive and negative sectional curvatures.

Since the curvature tensor is determined by the sectional curvature, M is called a constant curvature manifold if $\mathcal{S}ec = k$ for some constant $k \in \mathbb{R}$. Complete simply connected constant curvature manifolds are called space forms, which are classified by hyperbolic space (negative constant sectional curvature), Euclidean space (zero sectional curvature) and unit sphere (positive sectional curvature) respectively. All other connected complete constant curvature manifolds are quotients of space forms by some group of isometries. Constant curvature manifolds are Einstein but not vice versa.

A slightly larger class of manifolds are Ricci parallel manifolds, where the Ricci curvature is constant under parallel transports; that is, $\nabla \mathcal{R}ic = 0$ where ∇ is the Levi-Civita connection. An Einstein manifold is Ricci parallel but the inverse is not true. When the manifold is simply connected and indecomposable (i.e. does not split as non-trivial Riemannian products), these two properties are equivalent, see e.g. [11, Chapter XI].

In this paper, we aim to identify the above three classes of manifolds by using integral formulas with respect to the volume measure and derivative inequalities of the heat semigroup. To state the main results, we introduce some notations where most are standard in the literature.

For $f, g \in C^2(M)$ and $x \in M$, consider the Hilbert-Schmidt inner product of the Hessian tensors Hess_f and Hess_g:

$$\langle \mathrm{Hess}_f, \mathrm{Hess}_g \rangle_{HS}(x) = \sum_{i,j=1}^{d} \mathrm{Hess}_f(\Phi^i, \Phi^j)\mathrm{Hess}_g(\Phi^i, \Phi^j),$$

where $\Phi = (\Phi^i)_{1 \leq i \leq d} \in O_x(M)$, the space of orthonormal bases of T_xM. Then the Hilbert-Schmidt norm of Hess_f reads

$$\|\mathrm{Hess}_f\|_{HS} = \sqrt{\langle \mathrm{Hess}_f, \mathrm{Hess}_f \rangle_{HS}}\,.$$

For a symmetric 2-tensor T and a constant K, we write $\mathcal{T} \geq K$ if

$$\mathcal{T}(u,u) \geq K|u|^2, \quad u \in TM.$$

Similarly, $\mathcal{T} \leq K$ means $\mathcal{T}(u,u) \leq K|u|^2, u \in TM$. Let $\mathcal{T}^\# : TM \to TM$ be defined by

$$\langle \mathcal{T}^\#(u), v \rangle = \mathcal{T}(u,v), \quad u, v \in T_xM, x \in M.$$

Then $\mathcal{T}^\#$ is a symmetric map, i.e. $\langle \mathcal{T}^\# u, v\rangle = \langle \mathcal{T}^\# v, u\rangle$ for $u, v \in T_xM, x\in M$. Let

$$\|\mathcal{T}\|(x) = \sup\big\{|\mathcal{T}^\#(u)|:\ u\in T_xM, |u|\le 1\big\},\quad x\in M.$$

A C^1 map $Q: TM\to TM$ with $QT_xM\subset T_xM$ for $x\in M$ is called constant, if $\nabla(Qv)(x)=0$ holds for any $x\in M$ and vector field v with $\nabla v(x)=0$. So, M is Ricci parallel if only if $\mathcal{R}ic^\#$ is a constant map.

For any symmetric 2-tensor $\mathcal{T}$, define

$$(\mathcal{R}\mathcal{T})(v_1,v_2) := \mathrm{tr}\big\langle \mathcal{R}(\cdot, v_2)v_1, \mathcal{T}^\#(\cdot)\big\rangle = \sum_{i=1}^d \big\langle \mathcal{R}(\Phi^i, v_2)v_1, \mathcal{T}^\#(\Phi^i)\big\rangle, \tag{1.1}$$

where $v_1, v_2\in T_xM, x\in M, \Phi=(\Phi^i)_{1\le i\le d}\in O_x(M)$. Since T is symmetric, so is $\mathcal{R}T$. Let

$$\|\mathcal{R}\|(x) = \sup\big\{\|\mathcal{R}\mathcal{T}\|(x):\ \mathcal{T} \text{ is a symmetric 2-tensor},\ \|\mathcal{T}\|(x)\le 1\big\},$$
$$\|\mathcal{R}\|_\infty = \sup_{x\in M}\|\mathcal{R}\|(x).$$

For a smooth tensor $\mathcal{T}$, consider the Bochner Laplacian

$$\Delta\mathcal{T} := \mathrm{tr}\big(\nabla.\nabla.\mathcal{T}\big).$$

Then $\frac{1}{2}\Delta$ generates a contraction semigroup $P_t = \mathrm{e}^{\frac{t}{2}\Delta}$ in the L^2 space of tensors, see [13, Theorems 2.4 and 3.7] for details. In Subsection 3.1, we will prove a probabilistic formula of $P_t\mathcal{T}$, which is a smooth tensor when $\mathcal{T}$ is smooth with compact support. Precisely, for any $x\in M$ and $\Phi\in O_x(M)$, let $\Phi_t(x)$ be the horizontal Brownian motion starting at Φ, and let $X_t(x) := \pi\Phi_t(x)$ be the Brownian motion starting at x, see (3.3) and (3.5) below for details. Then $\|_t = \Phi_t(x)\Phi^{-1}: T_xM\to T_{X_t(x)}M$ is called the stochastic parallel transport along the Brownian path. Both $X_t(x)$ and $\|_t$ do not depend on the choice of the initial value $\Phi\in O_x(M)$. When the manifold is stochastically complete (i.e. the Brownian motion is non-explosive), for a bounded n-tensor $\mathcal{T}$ we have

$$(P_t\mathcal{T})(v_1,\cdots,v_n) = \mathbb{E}\big[\mathcal{T}(\|_t\, v_1,\cdots,\|_t\, v_n)\big],\quad v_1,\cdots,v_n\in T_xM.$$

In the following, we will take this regular (rather than L^2) version of the heat semigroup P_t.

Finally, For $v\in T_xM$, let $W_t(v)\in T_{X_t(x)}M$ solve the following covariant differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t(x)^{-1}W_t(v) = -\frac{1}{2}\Phi_t^{-1}(x)\mathcal{R}ic^\#(W_t(v)),\quad W_0(v)=v. \tag{1.2}$$

W_t is called the damped stochastic parallel transport. When $\mathcal{R}ic\ge K$ for some constant $K\in\mathbb{R}$, we have $|W_t(v)|\le \mathrm{e}^{-\frac{K}{2}t}|v|,\ t\ge 0, v\in T_xM$.

In Section 2 and Sections 6-8, we will present a number of identifications of constant curvature manifolds, Einstein manifolds, and Ricci parallel manifolds. In particular, the following assertions are direct consequences of Theorems 2.1, 6.1, 7.1 and 8.1 below.

(A) Constant curvature. Let $k \in \mathbb{R}$. Each of the following assertions is equivalent to $\mathcal{S}ec = k$:

(A_1) For any $t \geq 0$ and $f \in C_0^\infty(M)$, $\mathrm{Hess}_{P_t f} = \mathrm{e}^{-dkt} P_t \mathrm{Hess}_f + \frac{1}{d}(1 - \mathrm{e}^{-dkt})(P_t \Delta f)\mathbf{g}$.

(A_2) For any $f \in C_0^\infty(M)$, $\mathrm{Hess}_{\Delta f} - \Delta \mathrm{Hess}_f = 2k\big\{(\Delta f)\mathbf{g} - d\, \mathrm{Hess}_f\big\}$.

(A_3) For any $f \in C_0^\infty(M)$,

$$\frac{1}{2}\Delta \|\mathrm{Hess}_f\|_{HS}^2 - \langle \mathrm{Hess}_{\Delta f}, \mathrm{Hess}_f\rangle_{HS} - \|\nabla \mathrm{Hess}_f\|_{HS}^2 = 2k\big(d\|\mathrm{Hess}_f\|_{HS}^2 - (\Delta f)^2\big),$$

where $\|\nabla \mathrm{Hess}_f\|_{HS}^2 := \sum_{i=1}^d \|\nabla_{\Phi^i} \mathrm{Hess}_f\|_{HS}^2$, $\Phi = (\Phi^i)_{1 \leq i \leq d} \in O_x(M)$.

(A_4) For any $x \in M, u \in T_x M$ and $f \in C_0^\infty(M)$ with $\mathrm{Hess}_f(x) = u \otimes u$ (i.e. $\mathrm{Hess}_f(v_1, v_2) = \langle u, v_1\rangle\langle u, v_2\rangle, v_1, v_2 \in T_x M$),

$$\big(\mathrm{Hess}_{\Delta f} - \Delta \mathrm{Hess}_f\big)(v,v) = 2k\big(|u|^2|v|^2 - \langle u, v\rangle^2\big), \quad v \in T_x M.$$

According to the Bochner-Weitzenböck formula, for any constant $K \in \mathbb{R}$, $\mathcal{R}ic = K$ is equivalent to each of the following formulas:

$$\begin{aligned}
&\nabla P_t f = \mathrm{e}^{-\frac{t}{2}K} P_t \nabla f, \quad f \in C_0^\infty(M), \quad t \geq 0,\\
&\Delta \nabla f - \nabla \Delta f = K \nabla f, \quad f \in C_0^\infty(M),\\
&\frac{1}{2}\Delta |\nabla f|^2 - \langle \nabla \Delta f, \nabla f\rangle - \|\mathrm{Hess}_f\|_{HS}^2 = K|\nabla f|^2, \quad f \in C_0^\infty(M).
\end{aligned}$$

So, (A_1)-(A_3) can be regarded as the corresponding formulas for $\mathcal{S}ec = k$. Below, we present some other identifications of Einstein manifolds.

(B) Einstein manifolds. Let μ be the volume measure, and denote $\mu(f) = \int_M f \mathrm{d}\mu$ for $f \in L^1(\mu)$. M is Einstein if and only if

$$\begin{aligned}
&\mu\big(\langle \nabla f, \nabla g\rangle\big)\mu\big((\Delta f)^2 - \|\mathrm{Hess}_f\|_{HS}^2\big)\\
&= \mu\big(|\nabla f|^2\big)\mu\big((\Delta f)(\Delta g) - \langle \mathrm{Hess}_f, \mathrm{Hess}_g\rangle_{HS}\big), \quad f, g \in C_0^\infty(M).
\end{aligned} \tag{1.3}$$

Moreover, for any $K \in \mathbb{R}$, $\mathcal{R}ic = K$ is equivalent to each of the following assertions:

(B_1) $\|\mathcal{R}\|_\infty < \infty$, and for any $t \geq 0, f \in C_0^\infty(M)$,

$$\mathrm{Hess}_{P_t f} = \mathrm{e}^{-Kt} P_t \mathrm{Hess}_f + \int_0^t \mathrm{e}^{-Ks} P_s(\mathcal{R}\mathrm{Hess}_{P_{t-s} f})\mathrm{d}s.$$

(B_2) For any $f \in C_0^\infty(M)$,

$$\frac{1}{2}\big\{\mathrm{Hess}_{\Delta f} - \Delta \mathrm{Hess}_f\big\} = (\mathcal{R}\mathrm{Hess}_f) - K\mathrm{Hess}_f.$$

(B_3) For any $f \in C_0^\infty(M)$,

$$\mu\big((\Delta f)^2 - \|\mathrm{Hess}_f\|_{HS}^2\big) = K\mu\big(|\nabla f|^2\big).$$

(B_4) For any $f, g \in C_0^\infty(M)$,

$$\mu\big((\Delta f)(\Delta g) - \langle \mathrm{Hess}_f, \mathrm{Hess}_g\rangle_{HS}\big) = K\mu\big(\langle \nabla f, \nabla g\rangle\big).$$

(B_5) There exists $h : [0,\infty) \times M \to [0,\infty)$ with $\lim_{t\to 0} h(t,\cdot) = 0$ such that

$$\big|P_t|\nabla f|^2 - \mathrm{e}^{Kt}|\nabla P_t f|^2\big| \le h(t,\cdot)\big(\|\mathrm{Hess}_{P_t f}\|_{HS}^2 + P_t\|\mathrm{Hess}_f\|_{HS}^2\big), \quad t \ge 0, f \in C_0^\infty(M).$$

(C) Ricci Parallel manifolds. M is Ricci parallel if and only if

$$\int_M \big\{(\Delta f)^2 - \|\mathrm{Hess}_f\|_{HS}^2 - \langle Q\nabla f, \nabla f\rangle\big\}\mathrm{d}\mu = 0, \quad f \in C_0^\infty(M)$$

holds for some constant symmetric linear map $Q : TM \to TM$, and in this case $\mathcal{R}ic^\# = Q$. They are also equivalent to each of the following statements:

(C_1) There exists a function $h : [0,\infty) \times M \to [0,\infty)$ with $\lim_{t\downarrow 0} t^{-\frac{1}{2}} h(t,\cdot) = 0$ such that

$$\|\mathrm{Hess}_{P_t f} - P_t \mathrm{Hess}_f\| \le h(t,\cdot)\big(P_t\|\mathrm{Hess}_f\| + \|\mathrm{Hess}_{P_t f}\|\big), \quad t \ge 0, f \in C_b^2(M).$$

(C_2) $\|\mathcal{R}\|_\infty < \infty$, and for any $x \in M, t \ge 0$, $f \in C_0(M)$ and $v_1, v_2 \in T_x M$,

$$\mathrm{Hess}_{P_t f}(v_1, v_2) - \mathbb{E}\big[\mathrm{Hess}_f(W_t(v_1), W_t(v_2))\big] = \mathbb{E}\int_0^t \big(\mathcal{R}\mathrm{Hess}_{P_{t-s} f}\big)\big(W_s(v_1), W_s(v_2)\big)\mathrm{d}s.$$

(C_3) For any $f \in C_0^\infty(M)$ and $x \in M$,

$$\big(\mathrm{Hess}_{\Delta f} - \Delta \mathrm{Hess}_f\big)(v_1, v_2) = 2\big(\mathcal{R}\mathrm{Hess}_f\big)(v_1, v_2) - 2\mathcal{R}ic(v_1, \mathrm{Hess}_f^\#(v_2)), \quad v_1, v_2 \in T_x M.$$

(C_4) For any $x \in M$ and $f \in C_0^\infty(M)$ with $\mathrm{Hess}_f(x) = 0$,

$$(\Delta \mathrm{Hess}_f)(v_1, v_2) = \mathrm{Hess}_{\Delta f}(v_1, v_2), \quad v_1, v_2 \in T_x M.$$

Since a symmetric 2-tensor is determined by its diagonal, one may take $v_1 = v_2$ in (C_2)-(C_4).

To prove these results, we establish the following variational and derivative formulas for $\mathcal{R}ic$, see Remark 2.1 and Theorem 5.1 below.

(D) Formulas of $\mathcal{R}ic$. Let $\overline{\mathcal{R}ic}$ and $\underline{\mathcal{R}ic}$ be the exact upper and exact lower bounds of $\mathcal{R}ic$. Then

$$\underline{\mathcal{R}ic} = \inf\Big\{\mu\big((\Delta f)^2 - \|\mathrm{Hess}_f\|_{HS}^2\big):\ f \in C_0^\infty(M), \mu(|\nabla f|^2) = 1\Big\},$$
$$\overline{\mathcal{R}ic} = \sup\Big\{\mu\big((\Delta f)^2 - \|\mathrm{Hess}_f\|_{HS}^2\big):\ f \in C_0^\infty(M), \mu(|\nabla f|^2) = 1\Big\}.$$

Moreover, let $x \in M$, $v_1, v_2 \in T_xM$. For any $f \in C_b^4(M)$ with $\nabla f(x) = v_1, \mathrm{Hess}_f(x) = 0$,

$$(\nabla_{v_2}\mathcal{R}ic)(v_1, v_1) = 2\lim_{t\downarrow 0}\frac{(P_t\mathrm{Hess}_f - \mathrm{Hess}_{P_tf})(v_1, v_2)}{t} = \big(\Delta\mathrm{Hess}_f - \mathrm{Hess}_{\Delta f}\big)(v_1, v_2).$$

The remainder of the paper is organized as follows. In Section 2, we present variational formulas of Bakry-Emery-Ricci upper and lower bounds, as well as integral characterizations of Einstein and Ricci parallel manifolds. In Section 3, we recall derivative and Hessian formulas of P_t developed from stochastic analysis. Using these formulas we estimate the Hessian of P_t in Section 4, and establish a formula for $\nabla\mathcal{R}ic$ in Section 5. Finally, by applying results presented in Sections 3-5, we identify constant curvature manifolds, Einstein manifolds, and Ricci parallel manifolds in Sections 6-8 respectively.

2 Characterizations of Bakry-Emery-Ricci curvature

Let $V \in C^2(M)$, $\mu_V(\mathrm{d}x) = \mathrm{e}^{V(x)}\mu(\mathrm{d}x)$, and $L_V = \Delta + \nabla V$. Then L_V is symmetric in $L^2(\mu_V)$. Consider the Bakry-Emery-Ricci curvature

$$\mathcal{R}ic_V := \mathcal{R}ic - \mathrm{Hess}_V.$$

Definition 2.1. The manifold M is called V-Einstein if $\mathcal{R}ic_V = K$ for some constant $K \in \mathbb{R}$, while it is called $\mathcal{R}ic_V$ parallel if $\nabla\mathcal{R}ic_V = 0$ (i.e. $\mathcal{R}ic_V^\# : TM \to TM$ is a constant map).

By the integral formula of Bochner-Weitzenböck, we have

$$\int_M \big\{(L_Vf)^2 - \|\mathrm{Hess}_f\|_{HS}^2 - \mathcal{R}ic_V(\nabla f, \nabla f)\big\}\mathrm{d}\mu_V = 0, \quad f \in C_0^\infty(M). \tag{2.1}$$

According to Theorem 2.1 below, this formula identifies the curvature $\mathcal{R}ic_V$, and provides sharp upper and lower bounds of $\mathcal{R}ic_V$, as well as integral characterizations of V-Einstein and $\mathcal{R}ic_V$ parallel manifolds.

Theorem 2.1. *Let $K \in \mathbb{R}$ be a constant, and let $Q : TM \to TM$ be a symmetric continuous linear map.*

(1) $\mathcal{R}ic_V^\# = Q$ *if and only if*

$$\int_M \big\{(L_Vf)^2 - \|\mathrm{Hess}_f\|_{HS}^2 - \langle Q\nabla f, \nabla f\rangle\big\}\mathrm{d}\mu_V = 0, \quad f \in C_0^\infty(M). \tag{2.2}$$

Consequently, M is $\mathcal{R}ic_V$ parallel if and only if (2.2) *holds for some symmetric constant linear map $Q : TM \to TM$.*

(2) *For any* $V \in C^2(M)$ *and* $K \in \mathbb{R}$, $\mathcal{R}ic_V \ge K$ *if and only if*

$$\int_M \Big\{(L_V f)^2 - \|\mathrm{Hess}_f\|_{HS}^2\Big\}\mathrm{d}\mu_V \ge K \int_M |\nabla f|^2 \mathrm{d}\mu_V, \quad f \in C_0^\infty(M); \tag{2.3}$$

while $\mathcal{R}ic_V \le K$ *if and only if*

$$\int_M \Big\{(L_V f)^2 - \|\mathrm{Hess}_f\|_{HS}^2\Big\}\mathrm{d}\mu_V \le K \int_M |\nabla f|^2 \mathrm{d}\mu_V, \quad f \in C_0^\infty(M). \tag{2.4}$$

(3) M *is* V*-Einstein if and only if*

$$\begin{aligned} &\mu_V\big(|\nabla f|^2\big)\cdot \mu_V\big((L_V f)(L_V g) - \langle \mathrm{Hess}_f, \mathrm{Hess}_g\rangle_{HS}\big)\\ &= \mu_V\big(\langle \nabla f, \nabla g\rangle\big)\cdot \mu_V\big((L_V f)^2 - \|\mathrm{Hess}_f\|_{HS}^2\big), \quad f, g \in C_0^\infty(M). \end{aligned} \tag{2.5}$$

Moreover, for any constant $K \in \mathbb{R}$, $\mathcal{R}ic_V = K$ *is equivalent to each of*

$$\int_M \big\{(L_V f)^2 - \|\mathrm{Hess}_f\|_{HS}^2 - K|\nabla f|^2\big\}\mathrm{d}\mu_V = 0, \quad f \in C_0^\infty(M), \tag{2.6}$$

and

$$\int_M \big\{(L_V f)(L_V g) - \langle \mathrm{Hess}_f, \mathrm{Hess}_g\rangle_{HS} - K\langle \nabla f, \nabla g\rangle\big\}\mathrm{d}\mu_V = 0, \quad f, g \in C_0^\infty(M). \tag{2.7}$$

Remark 2.1. Theorem 2.1(2) provides the following variational formulas of the upper and lower bounds of $\mathcal{R}ic_V$. Let

$$\overline{\mathcal{R}ic_V} = \sup\{\mathcal{R}ic_V(u,u) : \ u \in TM, |u| = 1\}, \quad \underline{\mathcal{R}ic_V} = \inf\{\mathcal{R}ic_V(u,u) : \ u \in TM, |u| = 1\}.$$

We have

$$\begin{aligned} \underline{\mathcal{R}ic_V} &= \inf\Big\{\mu_V\big((L_V f)^2 - \|\mathrm{Hess}_f\|_{HS}^2\big) : \ f \in C_0^\infty(M), \mu_V(|\nabla f|^2) = 1\Big\},\\ \overline{\mathcal{R}ic_V} &= \sup\Big\{\mu_V\big((L_V f)^2 - \|\mathrm{Hess}_f\|_{HS}^2\big) : \ f \in C_0^\infty(M), \mu_V(|\nabla f|^2) = 1\Big\}. \end{aligned}$$

To prove Theorem 2.1, we need the following lemma.

Lemma 2.2. *For a continuous symmetric linear map* $Q : TM \to TM$*, if*

$$\int_M \langle Q\nabla f, \nabla f\rangle \mathrm{d}\mu_V \ge 0, \quad f \in C_0^\infty(M), \tag{2.8}$$

then $Q \ge 0$*; that is,* $\langle Qu, u\rangle \ge 0$ *for* $u \in TM$.

Proof. Using $\mathrm{e}^V Q$ replacing Q, we may and do assume that $V = 0$ so that $\mu_V = \mu$ is the volume measure.

(a) We first consider $M = \mathbb{R}^d$ for which we have $Q = (q_{ij})_{1\le i,j\le d}$ with $q_{ij} = q_{ji}$ for some continuous functions q_{ij} on $\mathbb{R}^d$. Thus,

$$\langle Qu, v\rangle = \sum_{i,j=1}^d q_{ij}u_iv_j, \quad u, v \in \mathbb{R}^d.$$

Without loss of generality, we only prove that $Q(0) \ge 0$. Using the eigenbasis of $Q(0)$, we may and do assume that $Q(0) = \mathrm{diag}\{q_1, \cdots, q_d\}$. It suffices to prove $q_l \ge 0$ for $1 \le l \le d$. For any $f \in C_0^\infty(\mathbb{R}^d)$, let

$$f_n(x) = f \circ \phi_n(x), \quad (\phi_n(x))_i := nx_i \text{ if } i \ne l, \ (\phi_n(x))_l := n^2x_l, \quad n \ge 1.$$

By (2.8),

$$\begin{aligned}
0 &\le \sum_{i,j=1}^d \int_{\mathbb{R}^d} q_{ij}(x)(\partial_i f_n)(x)\partial_j f_n(x)\mathrm{d}x\\
&= \int_{\mathbb{R}^d} \Big(n^4 q_{ll}(x)(\partial_l f)^2 \circ \phi_n(x) + 2n^3 \sum_{j\ne l} q_{jl}(x)\{(\partial_l f)(\partial_j f)\} \circ \phi_n(x)\\
&\qquad + n^2 \sum_{i,j\ne l} q_{ij}(x)\{(\partial_i f)(\partial_j f)\} \circ \phi_n(x)\Big)\mathrm{d}x\\
&= \int_{\mathbb{R}^d} \Big(n^{3-d} q_{ll} \circ \phi_n^{-1}(x)(\partial_l f)^2(x) + 2n^{2-d} \sum_{j\ne l} q_{jl} \circ \phi_n^{-1}(x)\{(\partial_l f)(\partial_j f)\}(x)\\
&\qquad + n^{1-d} \sum_{i,j\ne l} q_{ij} \circ \phi_n^{-1}(x)\{(\partial_i f)(\partial_j f)\}(x)\Big)\mathrm{d}x,
\end{aligned}$$

where the last step is due to the integral transform $x \mapsto \phi_n^{-1}(x)$. Since $\phi_n^{-1}(x) \to 0$ as $n \to \infty$, multiplying both sides by n^{d-3} and letting $n \to \infty$ we arrive at

$$\int_{\mathbb{R}^d} q_l(\partial_l f)^2(x)\mathrm{d}x \ge 0, \quad f \in C_0^\infty(\mathbb{R}^d).$$

Thus, $q_l \ge 0$ as wanted.

(b) In general, for $x_0 \in M$, we take a neighborhood $O(x_0)$ of x_0 such that it is diffeomorphic to $\mathbb{R}^d$ with x_0 corresponding to $0 \in \mathbb{R}^d$. Let $\psi : O(x_0) \to \mathbb{R}^d$ with $\psi(x_0) = 0$ be a diffeomorphism. Then under the local charts induced by ψ,

$$\langle Q\nabla(f \circ \psi), \nabla(f \circ \psi)\rangle\mathrm{d}\mu = \sum_{i,j=1}^d q_{ij}(x)(\partial_i f)(x)(\partial_j f)(x)\mathrm{d}x, \quad f \in C_0^\infty(\mathbb{R}^d)$$

holds for some symmetric matrix-valued continuous functional $(q_{ij})_{1\le i,j\le d}$. Therefore, by step **(a)**, (2.8) implies $\sum_{i,j=1}^d q_{ij}(x)u_iu_j \ge 0, x, u \in \mathbb{R}^d$. In particular, $Q(x_0) \ge 0$. □

Proof of Theorem 2.1. (1) By the Bochner-Weitzenböck formula, we have

$$\frac{1}{2}L_V|\nabla f|^2 - \langle \nabla f, \nabla L_V f\rangle = \mathcal{R}ic_V(\nabla f, \nabla f) + \|\mathrm{Hess}_f\|_{HS}^2, \quad f \in C_0^\infty(M). \tag{2.9}$$

So, $\mathcal{R}ic_V^{\#} = Q$ implies

$$\frac{1}{2}L_V|\nabla f|^2 - \langle \nabla f, \nabla L_V f\rangle = \langle Q\nabla f, \nabla f\rangle + \|\mathrm{Hess}_f\|_{HS}^2.$$

Integrating both sides with respect to μ_V proves (2.2).

On the other hand, integrating both of (2.9) with respect to μ_V, we obtain (2.1). This together with (2.2) implies

$$\int_{\mathbb{R}^d} \big\langle \mathcal{R}ic_V^{\#}(\nabla f) - Q\nabla f, \nabla f\big\rangle \, \mathrm{d}\mu_V = 0, \quad f \in C_0^\infty(M). \tag{2.10}$$

Therefore, by Lemma 2.2 for $\mathcal{R}ic_V^{\#} - Q$ replacing Q, we prove $\mathcal{R}ic_V^{\#} = Q$.

(2) By integrating both sides of (2.9) with respect to μ_V, we see that $\mathcal{R}ic_V^{\#} \geq K$ implies (2.3). On the other hand, applying Lemma 2.2 to $Q = \mathcal{R}ic_V^{\#} - K$, if (2.3) holds then $\mathcal{R}ic_V^{\#} \geq K$. Similarly, we can prove the equivalence of $\mathcal{R}ic_V^{\#} \leq K$ and (2.4). Therefore, Theorem 2.1(2) holds.

(3) By assertion (2), $\mathcal{R}ic_V = K$ is equivalent (2.6). It remains to prove that (2.5) is equivalent to the Einstein property, since this together with the equivalence of $\mathcal{R}ic_V = K$ and (2.6) implies the equivalence of (2.6) and (2.7).

If M is V-Einstein, there exists a constant $K \in \mathbb{R}$ such that $\mathcal{R}ic_V = K$. Let $f, g \in C_0^\infty(M)$ with $\mu_V(|\nabla f|^2) > 0$, let $f_s = f + sg$. Then there exists $s_0 > 0$ such that $\mu_V(|\nabla f_s|^2) > 0$ for $s \in [0, s_0]$. By (2.6), we have

$$h(s) := \frac{\mu_V((L_V f_s)^2 - \|\mathrm{Hess}_{f_s}\|_{HS}^2)}{\mu_V(|\nabla f_s|^2)} = K, \quad s \in [0, s_0].$$

So,

$$\begin{aligned} &\frac{\mu_V(|\nabla f|^2)\mu_V((L_V f)(\Delta g) - \langle \mathrm{Hess}_f, \mathrm{Hess}_g\rangle) - \mu_V(\langle \nabla f, \nabla g\rangle)\mu_V((L_V f)^2 - \|\mathrm{Hess}_f\|_{HS}^2)}{\mu_V(|\nabla f|^2)^2} \\ &= \frac{1}{2}h'(0) = 0. \end{aligned}$$

Therefore, (2.5) holds.

On the other hand, for $f \in C_0^\infty(M)$ with $\mu_V(|\nabla f|^2) > 0$, let

$$K = \frac{\mu_V((L_V f)^2 - \|\mathrm{Hess}_f\|_{HS}^2)}{\mu_V(|\nabla f|^2)} \in \mathbb{R}.$$

By the equivalence of $\mathcal{R}ic_V^{\#} = K$ and (2.6), it suffices to prove

$$\mu_V\big((L_V g)^2 - \|\mathrm{Hess}_g\|_{HS}^2\big) = K\mu_V(|\nabla g|^2), \quad g \in C_0^\infty(M). \tag{2.11}$$

By the definition of K, this formula holds when g is a linear combination of f and 1. So, we assume that f, g and 1 are linear independent. In this case,

$$g_s := (1-s)f + sg, \quad s \in [0,1]$$

satisfies $\mu_V(|\nabla g_s|^2) > 0$. Let

$$K(s) := \frac{\mu_V((L_V g_s)^2 - \|\mathrm{Hess}_{g_s}\|_{HS}^2)}{\mu_V(|\nabla g_s|^2)}, \quad s \in [0,1].$$

Then (2.5) implies

$$\begin{aligned} K'(s) = \frac{2}{\mu_V(|\nabla g_s|^2)^2} \Big\{ & \mu_V(|\nabla g_s|^2) \mu_V\big((L_V g_s) L_V(g-f) - \langle \mathrm{Hess}_{g_s}, \mathrm{Hess}_{g-f} \rangle_{HS}\big) \\ & - \mu_V(\langle \nabla g_s, \nabla(g-f) \rangle) \mu_V\big((L_V g_s)^2 - \|\mathrm{Hess}_{g_s}\|_{HS}^2\big) \Big\} = 0, \quad s \in [0,1]. \end{aligned}$$

Therefore,

$$\frac{\mu_V((L_V g)^2 - \|\mathrm{Hess}_g\|_{HS}^2)}{\mu_V(|\nabla g|^2)} = K(1) = K(0) = K,$$

that is, (2.11) holds as desired. □

3 Derivative and Hessian formulas of P_t

In this section, by using the (horizontal) Brownian motion, we first formulate the heat semigroup P_t acting on tensors, then recall the derivative and Hessian formulas of P_t on functions.

3.1 Brownian motion and heat semigroup on tensors

Consider the projection operator from the orthonormal frame bundle $O(M)$ onto M:

$$\pi : O(M) \to M; \quad \pi\Phi = x \text{ if } \Phi \in O_x(M).$$

Then for any $a \in \mathbb{R}^d$ and $\Phi = (\Phi^i)_{1 \le i \le d} \in O(M)$,

$$\Phi a := \sum_{i=1}^d a_i \Phi^i \in T_{\pi\Phi} M;$$

and for any $v \in T_{\pi\Phi} M$,

$$\Phi^{-1} v := \sum_{i=1}^d \langle v, \Phi^i \rangle e_i \in \mathbb{R}^d,$$

where $\{e_i\}_{1 \le i \le d}$ is the canonical orthonormal basis of $\mathbb{R}^d$.

For any $a \in \mathbb{R}^d$ and $\Phi \in O(M)$, let $\Phi(s)$ be the parallel transport of Φ along the geodesic $s \mapsto \exp[s\Phi a], s \geq 0$. We have

$$H_a(\Phi) := \frac{\mathrm{d}}{\mathrm{d}s}\Phi(s)\big|_{s=0} \in T_\Phi O(M),$$

which is a horizontal vector field on $O(M)$. Let $H_i = H_{e_i}$. Then $(H_i)_{1\leq i\leq d}$ forms the canonical orthonormal basis for the space of horizontal vector fields:

$$H_a = \sum_{i=1}^d a_i H_i, \quad a = (a_i)_{1\leq i\leq d} \in \mathbb{R}^d.$$

We call

$$\Delta_{O(M)} := \sum_{i=1}^d H_i^2 \tag{3.1}$$

the horizontal Laplacian on $O(M)$. For any smooth n-tensor $\mathcal{T}$,

$$\mathcal{T}^{O(M)}(\Phi) := \big(\mathcal{T}(\Phi^{i_1}, \cdots, \Phi^{i_n})\big)_{1\leq i_1,\cdots,i_n\leq d} \in \otimes^n\mathbb{R}^d, \quad \Phi = (\Phi^i)_{1\leq i\leq d} \in O(M)$$

gives rise to a smooth $\otimes^n\mathbb{R}^d$-valued function on $O(M)$. Let Δ be the Bochner Laplacian Δ. For any $x \in M$, $v_1, \cdots, v_n \in T_xM$ and $\Phi \in O_x(M)$, we have

$$\begin{aligned} &(\Delta\mathcal{T})(v_1, \cdots, v_n) = \{\Delta_{O(M)}\mathcal{T}^{O(M)}(\Phi)\}(\Phi^{-1}v_1, \cdot, \Phi^{-1}v_n), \\ &(\nabla_v\mathcal{T})(v_1, \cdots, v_n) = (\nabla_{H_{\Phi^{-1}v}}\mathcal{T}^{O(M)})(\Phi^{-1}v_1, \cdots, \Phi^{-1}v_n), \quad v \in T_xM. \end{aligned} \tag{3.2}$$

Now, consider the following SDE on $O(M)$:

$$\mathrm{d}\Phi_t = H(\Phi_t) \circ \mathrm{d}B_t := \sum_{i=1}^d H_i(\Phi_t) \circ \mathrm{d}B_t^i, \tag{3.3}$$

where $B_t := (B_t^i)_{1\leq i\leq d}$ is the d-dimensional Brownian motion on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with natural filtration $\mathscr{F}_t^B := \sigma(B_s : s \leq t)$ (by convention, we always take the completion of a σ field). The solution is called the Horizonal Brownian motion on $O(M)$. Let ζ be the life time of the solution. When $\zeta = \infty$ (i.e. the solution is non-explosive) we call the manifold M stochastically complete. It is the case when

$$\mathcal{R}ic \geq -c(1 + \rho^2) \tag{3.4}$$

for some constant $c > 0$, where ρ is the Riemannian distance to some fixed point, see the proof of Proposition 3.1 blow. See also [8] and references within for the stochastic completeness under weaker conditions.

Let $X_t := \pi\Phi_t$ for $t \in [0, \zeta)$. Then $(X_t)_{t\in[0,\zeta)}$ solves the SDE

$$\mathrm{d}X_t = \Phi_t(X_t) \circ \mathrm{d}B_t, \tag{3.5}$$

and is called the Brownian motion on M. For any $x \in M$, let $X_t(x)$ be the solution of (3.5) with $X_0 = x$. Note that $X_t(x)$ does not depend on the choice of the initial value $\Phi_0 \in O_x(M)$ for (3.3), and for fixed initial value $\Phi_0 \in O_x(M)$, both $\Phi_t(X_t)$ and B_t are measurable with respect to $\mathscr{F}_t^X := \sigma(X_s : s \le t)$. Therefore, $\mathscr{F}_t^B = \mathscr{F}_t^X, t \ge 0$. When the initial point x is clearly given in the context, we will simply denote $X_t(x)$ by X_t.

Let

$$\|_t := \Phi_t \Phi_0^{-1} : T_{X_0}M \to T_{X_t}M$$

be the stochastic parallel transport along X_t, which also does not depend on the choice of Φ_0. For any smooth n-tensor $\mathcal{T}$ with compact support, let

$$(P_t\mathcal{T})(v_1, \cdots, v_n) = \mathbb{E}\big[1_{\{t<\zeta\}}\mathcal{T}(\|_t\, v_1, \cdots, \|_t\, v_n)\big], \quad v_1, \cdots, v_n \in T_xM. \tag{3.6}$$

As is well known in the function (i.e. 0-tensor) setting, by (3.1), the first formula in (3.2), (3.3) and Itô's formula we have the forward/backward Kolmogorov equations

$$\partial_t P_t\mathcal{T} = \frac{1}{2}\Delta P_t\mathcal{T} = \frac{1}{2}P_t\Delta\mathcal{T}, \quad \mathcal{T} \in C_0^\infty. \tag{3.7}$$

For later use, we present the following exponential estimate of the Brownian motion under condition (3.4).

Proposition 3.1. *Assume* (3.4). *Then there exist constants* $r, c(r) > 0$ *such that*

$$\mathbb{E}\exp[\mathrm{e}^{-rt}\rho^2(X_t)] \le \exp\Big[\rho^2(x) + \frac{c(r)(1-\mathrm{e}^{-rt})}{r}\Big], \quad t \ge 0, x \in M. \tag{3.8}$$

Consequently, if

$$\lim_{\rho\to\infty} \frac{\log \|\nabla\mathcal{R}ic\|}{\rho^2} = 0, \tag{3.9}$$

then for any $\varepsilon \in (0,1)$ *and* $p \ge 1$, *there exists a positive function* $C_{\varepsilon,p} \in C([0,\infty))$ *such that*

$$\mathbb{E}\|\nabla\mathcal{R}ic\|^p(X_t(x)) \le \mathrm{e}^{\varepsilon\rho^2(x) + C_{\varepsilon,p}(t)}, \quad t \ge 0, x \in M. \tag{3.10}$$

Proof. Let ρ be the Riemannian distance to a fixed point $o \in M$. By the Laplacian comparison theorem, (3.4) implies

$$\Delta\rho \le c_1(\rho + \rho^{-1})$$

outside $\{o\} \cup \mathrm{cut}(o)$ for some constant $c_1 > 0$, where $\mathrm{cut}(o)$ is the cut-locus of o. By Itô's formula of $\rho(X_t)$ given in [10], this gives

$$\mathrm{d}\rho(X_t)^2 \le c_2\{1 + \rho(X_t)^2\}\mathrm{d}t + 2\rho(X_t)\mathrm{d}b_t$$

for some constant $c_2 > 0$ and an one-dimensional Brownian motion b_t. By Itô's formula, for any $r > 2 + c_2$ there exists a constant $c(r) > 0$ such that

$$\mathrm{d}\exp[\mathrm{e}^{-rt}\rho^2(X_t)] \le \exp[\mathrm{e}^{-rt}\rho^2(X_t)]\Big(\{(c_2 + c_2\rho^2(X_t))\mathrm{e}^{-rt} - r\mathrm{e}^{-rt}\rho^2(X_t)$$

$$+ 2\mathrm{e}^{-2rt}\rho^2(X_t)\big\}\mathrm{d}t + 2\mathrm{e}^{-rt}\rho(X_t)\mathrm{d}b_t\Big)$$
$$\leq \exp[\mathrm{e}^{-rt}\rho^2(X_t)]\big\{c(r)\mathrm{e}^{-rt}\mathrm{d}t + 2\mathrm{e}^{-rt}\rho(X_t)\mathrm{d}b_t\big\}.$$

Therefore, (3.8) holds.

On the other hand, by (3.9) we may find a positive function $c_{p/\varepsilon} \in C([0,\infty))$ such that

$$\|\nabla\mathcal{R}ic^{\#}\|^{p/\varepsilon} \leq \exp\big[\mathrm{e}^{-t}\rho^2 + c_{p/\varepsilon}(t)\big], \quad t \geq 0.$$

Combining this with (3.8) for $r = 1$, we obtain

$$\mathbb{E}\|\nabla\mathcal{R}ic\|^p(X_t(x)) \leq \big(\mathbb{E}\|\nabla\mathcal{R}ic\|^{p/\varepsilon}(X_t)\big)^{\varepsilon}$$
$$\leq \big(\mathrm{e}^{\rho^2(x)+c(1)+c_{\varepsilon/p}(t)}\big)^{\varepsilon} \leq \mathrm{e}^{\varepsilon\rho^2(x)+\varepsilon c(1)+\varepsilon c_{p/\varepsilon}(t)}.$$

Therefore, (3.10) holds for $C_{\varepsilon,p}(t) := \varepsilon[c(1) + c_{p/\varepsilon}(t)]$. □

3.2 Derivative formula of P_t

In this subsection, we assume

$$\mathcal{R}ic \geq -h(\rho) \ \text{ for some positive } h \in C([0,\infty)) \text{ with } \lim_{r\to\infty}\frac{h(r)}{r^2} = 0. \tag{3.11}$$

Let $W_t : T_xM \to T_{X_t}M$ be defined in (1.2). By (3.11) and Proposition 3.1, we have

$$\mathbb{E}\sup_{s\in[0,t]}\|W_s\|^p < \infty, \quad t > 0. \tag{3.12}$$

We have (see e.g. [7, 8])

$$\nabla_v P_t f(x) = \mathbb{E}\langle\nabla f(X_t(x)), W_t(v)\rangle, \quad t \geq 0, f \in C_b^1(M). \tag{3.13}$$

Following the idea of [7, 14], it is standard to establish the Bismut type formula using (3.13). By $\partial_t P_t f = \frac{1}{2}\Delta P_t f$, (3.5) and Itô's formula, we have

$$\mathrm{d}P_{t-s}f(X_s) = \langle\nabla P_{t-s}f(X_s), \Phi_s\mathrm{d}B_s\rangle,$$

so that

$$f(X_t(x)) = P_t f(x) + \int_0^t \langle\nabla P_{t-s}f(X_s(x)), \Phi_s(x)\mathrm{d}B_s\rangle. \tag{3.14}$$

In particular, $P_{t-s}f(X_s)$ is a martingale. Next, by (3.13) and the Markov property, we have

$$\big\langle\nabla P_{t-s}f(X_s), W_s(v)\big\rangle = \mathbb{E}\big(\big\langle\nabla f(X_t), W_t(v)\big\rangle\big|\mathscr{F}_s^B\big), \quad s \in [0,t],$$

which is again a martingale. Indeed, according to e.g. [19, (2.2.7)], we have

$$\mathrm{d}\langle\nabla P_{t-s}f(X_s), W_s(v)\rangle = \mathrm{Hess}_{P_{t-s}f}(\Phi_s\mathrm{d}B_s, W_s(v)), \quad s \in [0,t]. \tag{3.15}$$

So, for any adapted process $h \in C^1([0,t])$ such that $h_0 = 0, h_t = 1$, (3.13), (3.14) and (3.15) imply

$$\begin{aligned}
&\mathbb{E}\bigg[f(X_t(x))\int_0^t \dot h_s\langle W_s(v), \Phi_s(x)\mathrm{d}B_s\rangle\bigg] = \mathbb{E}\int_0^t \dot h_s\langle W_s(v), \nabla P_{t-s}f(X_s(x))\rangle\mathrm{d}s\\
&= \mathbb{E}\bigg[h_s\langle W_s(v), \nabla P_{t-s}f(X_s(x))\rangle\big|_0^t - \int_0^t h_s\mathrm{d}\langle W_s(v), \nabla P_{t-s}f(X_s(x))\rangle\bigg]\\
&= \mathbb{E}\langle W_t(v), \nabla f(X_t(x))\rangle = \nabla_v P_t f(x).
\end{aligned}$$

Therefore,

$$\nabla_v P_t f(x) = \mathbb{E}\bigg[f(X_t(x))\int_0^t \dot h_s\langle W_s(v), \Phi_s(x)\mathrm{d}B_s\rangle\bigg],\quad t>0, x\in M, f\in C_b^1(M). \tag{3.16}$$

This type formula is named after J.-M. Bismut, K.D. Elworthy and X.-M. Li because of their pioneering work [5] and [7]. The present version is due to [14] and has been applied in [2, 15] to derive gradient estimates using local curvature conditions.

3.3 Hessian formula of P_t

To calculate Hess_{P_tf}, we introduce the following doubled damped parallel transport $W_t(v_1, v_2)$ for $v_1, v_2 \in T_xM$:

$$\begin{aligned}
\Phi_t(x)^{-1}W_t^{(2)}(v_1,v_2) =& \frac{1}{2}\int_0^t \Phi_s^{-1}\big\{(\tilde\nabla\mathcal{R}ic^{\#})(W_s(v_2))W_s(v_1) - \mathcal{R}ic^{\#}(W_s^{(2)}(v_1,v_2))\big\}\mathrm{d}s\\
&+ \int_0^t \Phi_s(x)^{-1}\mathcal{R}(\Phi_s(x)\mathrm{d}B_s, W_s(v_2))W_s(v_1),\quad t\ge 0,
\end{aligned}\tag{3.17}$$

where the cycle derivative $\tilde\nabla\mathcal{R}ic^{\#}$ is defined by

$$\langle(\tilde\nabla\mathcal{R}ic^{\#})(u_2)u_1, u_3\rangle := (\nabla_{v_3}\mathcal{R}ic)(u_1,u_2) - (\nabla_{u_1}\mathcal{R}ic)(u_2,u_3) - (\nabla_{u_2}\mathcal{R}ic)(u_1,u_3) \tag{3.18}$$

for $u_1, u_2, u_3 \in T_yM, y\in M$. According to Proposition 3.1 and (3.12), conditions (3.9) and (3.11) imply

$$\mathbb{E}\sup_{s\in[0,t]}\|W_s^{(2)}\|^p < \infty,\quad p\ge 1, t>0.$$

Proposition 3.2 ([1, 12])**.** *Assume* (3.9) *and* (3.11)*. Then for any* $f\in C_b^2(M)$ *and* $v_1, v_2 \in T_xM$*,*

$$\mathrm{Hess}_{P_tf}(v_1,v_2) = \mathbb{E}\big\{\mathrm{Hess}_f(W_t(v_1), W_t(v_2)) + \langle\nabla f(X_t(x)), W_t^{(2)}(v_1,v_2)\rangle\big\}. \tag{3.19}$$

Proof. Let $v_2(s)$ be the parallel transport of v_2 along the geodesic $s\mapsto \exp[sv_1], s\ge 0$. According to (3.10), we define the following covariant derivative of W_t:

$$W_t^{(2)}(v_1,v_2) := \nabla_{v_1}W_t(v_2) = \frac{\mathrm{d}}{\mathrm{d}s}W_t(v_s(s))\big|_{s=0}.$$

By (3.13),

$$\mathrm{Hess}_{P_tf}(v_1,v_2) = \mathbb{E}\big\{\mathrm{Hess}_f(W_t(v_1),W_t(v_2)) + \langle \nabla f(X_t(x)), W_t^{(2)}(v_1,v_2)\rangle\big\}.$$

It remains to prove that $W_t^{(2)}$ satisfies (3.17). Since in the present setting $\mathscr{F}_t^X = \mathscr{F}_t^B$, this follows from formula (7) in [1], see also (3.1) in [12]. □

In the same spirit of deducing the Bismut type derivative formula (3.16) from (3.13), Bismut type Hessian formulas of P_tf have been presented in [1, 7, 12] by using (3.19). In Section 4 we will use the following local version of Hessian formula, which follows from [1, Theorem 2.1] and [1, Proof of Theorem 3.1] for e.g. $D_1 = B(x,1), D_2 = B(x,2)$, where $B(x,r)$ is the open geodesic ball at x with radius r.

Proposition 3.3 ([1])**.** *Let M be a complete noncompact Riemannian manifold. Let*

$$\tau_i(x) = \inf\{t\ge 0 : X_t(x)\in \partial B(x,i)\},\quad i=1,2.$$

There exists a positive function $C\in C(M)$ such that for any $x\in M$, $v_1,v_2\in T_xM$ with $|v_1,|v_2|\le 1$, and $f\in \mathscr{B}_b(M)$,

$$\mathrm{Hess}_{P_tf}(v_1,v_2) = \mathbb{E}\big[P_{t-t\wedge\tau_1(x)}f(X_{t\wedge\tau_1(x)}(x))M_t + P_{t-t\wedge\tau_2(x)}f(X_{t\wedge\tau_2(x)}(x))N_t\big],\quad t>0. \tag{3.20}$$

holds for some adapted continuous processes $(M_t,N_t)_{t\ge0}$ determined by $(X_t(x))_{0\le t\le\tau_2(x)}$ such that

$$\mathbb{E}\big[|N_t|+|M_t|\big]\le \frac{C(x)}{t\wedge 1},\quad t>0. \tag{3.21}$$

4 Hessian estimates and applications

In this section, we first present Hessian estimates of P_t for Einstein and Ricci parallel manifolds, then apply these results to describe the lower and upper bounds of the Ricci curvature.

Recall that for any $x\in M$ and $f\in C^2(M)$,

$$\|\mathrm{Hess}_f\|(x) := \sup\{|\mathrm{Hess}_f(u,v)| :\ u,v\in T_xM, |u|,|v|\le 1\},$$
$$\|\mathrm{Hess}_f\|_{HS}^2(x) := \sum_{i,j=1}^d \mathrm{Hess}_f(\Phi^i,\Phi^j)^2,\quad \Phi=(\Phi^i)_{1\le i\le d}\in O_x(M).$$

Theorem 4.1. *Let M be a Ricci parallel manifold with $\|\mathcal{R}\|_\infty<\infty$. Then for any $x\in M, t\ge 0$, $f\in C_0(M)$ and $v_1,v_2\in T_xM$,*

$$\begin{aligned}&\mathrm{Hess}_{P_tf}(v_1,v_2) - \mathbb{E}\big[\mathrm{Hess}_f(W_t(v_1),W_t(v_2))\big]\\ &= \mathbb{E}\int_0^t \big(\mathcal{R}\mathrm{Hess}_{P_{t-s}f}\big)\big(W_s(v_1),W_s(v_2)\big)\mathrm{d}s,\end{aligned} \tag{4.1}$$

where $\mathcal{R}\mathrm{Hess}_{P_{t-s}f}$ is defined in (1.1) *for $T=\mathrm{Hess}_{P_{t-s}f}$. Consequently:*

(1) *If $\mathcal{R}ic \geq K$, then for any $f \in C_b^2(M)$,*

$$\|\mathrm{Hess}_{P_t f}\| \leq \mathrm{e}^{(\|\mathcal{R}\|_\infty - K)t} P_t \|\mathrm{Hess}_f\|, \quad t \geq 0. \tag{4.2}$$

(2) *If $\mathcal{R}ic = K$, then*

$$\|\mathrm{Hess}_{P_t f}\|_{HS}^2 \leq \mathrm{e}^{2(\|\mathcal{R}\|_\infty - K)t} P_t \|\mathrm{Hess}_f\|_{HS}^2, \quad t \geq 0. \tag{4.3}$$

Proof. We fix $t > 0$ and $f \in C_b^2(M)$. Let $\mathbf{d}$ be the exterior differential. By e.g. [19, (2.2.6)] we have

$$\mathrm{d}(\mathbf{d}P_{t-s}f)(X_s) = \nabla_{\Phi_s \mathrm{d}B_s}(\mathbf{d}P_{t-s}f)(X_s) + \frac{1}{2}\mathcal{R}ic(\cdot, \nabla P_{t-s}f(X_s))\mathrm{d}s, \quad s \in [0,t]. \tag{4.4}$$

Equivalently,

$$\Phi_t^{-1}\nabla f(X_t) = \Phi_0^{-1}\nabla P_t f + \frac{1}{2}\int_0^t \Phi_s^{-1}\mathcal{R}ic^{\#}(\nabla P_{t-s}f(X_s))\mathrm{d}s + \int_0^t \mathrm{Hess}^{\#}_{P_{t-s}f}(\Phi_s \mathrm{d}B_s). \tag{4.5}$$

On the other hand, since $\tilde{\nabla}\mathcal{R}ic^{\#} = 0$, (3.17) becomes

$$\Phi_t^{-1}W_t^{(2)}(v_1, v_2) = \int_0^t \Phi_s^{-1}\mathcal{R}(\Phi_s \mathrm{d}B_s, W_s(v_2))W_s(v_1) - \frac{1}{2}\int_0^t \Phi_s^{-1}\mathcal{R}ic^{\#}(W_s^{(2)}(v_1, v_2))\mathrm{d}s.$$

Combining this with (**??**), we obtain

$$\mathbb{E}\langle \nabla f(X_t), W_t^{(2)}(v_1, v_2)\rangle = \mathbb{E}\int_0^t \mathrm{tr}\Big\{\mathrm{Hess}_{P_{t-s}f}(\cdot, \mathcal{R}(\cdot, W_s(v_2))W_s(v_s))\Big\}. \tag{4.6}$$

Plugging (4.6) into (3.19) gives

$$\begin{aligned}
&\mathrm{Hess}_{P_t f}(v_1, v_2) - \mathbb{E}\big[\mathrm{Hess}_f(W_t(v_1), W_t(v_2))\big] \\
&= \mathbb{E}\int_0^t \mathrm{tr}\big(\big\langle \mathcal{R}(\cdot, W_s(v_2))W_s(v_1), \mathrm{Hess}^{\#}_{P_{t-s}f}(\cdot)\big\rangle\big)\mathrm{d}s \\
&= \mathbb{E}\int_0^t \big(\mathcal{R}\mathrm{Hess}_{P_{t-s}f}\big)(W_s(v_1), W_s(v_2))\mathrm{d}s.
\end{aligned}$$

Therefore, (4.1) holds.

Below we prove (4.2) and (4.3) for Ricci parallel and Einstein manifolds respectively.

(a) (4.1) and $\mathcal{R}ic \geq K$ imply (4.2). If $\mathcal{R}ic \geq K$, then (1.2) implies

$$|W_t(v)| \leq \mathrm{e}^{-\frac{1}{2}Kt}|v|.$$

So, according to (4.1), for any $s > 0$ we have

$$\|\mathrm{Hess}_{P_s f}\| \leq \mathrm{e}^{-Ks}P_s\|\mathrm{Hess}_f\| + \|\mathcal{R}\|_\infty \int_0^s \mathrm{e}^{-Kr}P_r\|\mathrm{Hess}_{P_{s-r}f}\|\mathrm{d}r.$$

Letting

$$\phi(s) = \mathrm{e}^{-K(t-s)} P_{t-s} \|\mathrm{Hess}_{P_s f}\|, \quad s \in [0, t],$$

we obtain

$$\begin{aligned}\phi(s) &\le \mathrm{e}^{-K(t-s)} P_{t-s} \bigg(\mathrm{e}^{-sK} P_s \|\mathrm{Hess}_f\| + \|\mathcal{R}\|_\infty \int_0^s \mathrm{e}^{-rK} P_r \|\mathrm{Hess}_{P_{s-r} f}\| \mathrm{d}r \bigg) \\ &\le \mathrm{e}^{-Kt} P_t \|\mathrm{Hess}_f\| + \|\mathcal{R}\|_\infty \int_0^t \mathrm{e}^{-K(t+r-s)} P_{t+r-s} \|\mathrm{Hess}_{P_{s-r} f}\| \mathrm{d}r.\end{aligned}$$

Using the change of variable $\theta = s - r$, we arrive at

$$\phi(s) \le \phi(0) + \|\mathcal{R}\|_\infty \int_0^s \phi(\theta) \mathrm{d}\theta, \quad s \in [0, t].$$

By Gronwall's lemma, this implies

$$\phi(t) \le \phi(0) \mathrm{e}^{\|\mathcal{R}\|_\infty t},$$

which is equivalent to (4.2).

(b) Let $\mathcal{R}ic = K$. Then (1.2) implies $W_s(v) = \mathrm{e}^{-\frac{K}{2}s} v$. So, for $x \in M, v \in T_x M$ and $\Phi_0 \in O_x(M)$, (4.1) implies

$$\begin{aligned}\mathrm{Hess}_{P_t f}(v, \Phi_0^k) = &\, \mathrm{e}^{-Kt} \mathbb{E}\big[\mathrm{Hess}_f(\|_t v, \Phi_t^k)\big] \\ &+ \mathbb{E} \int_0^t \mathrm{e}^{-sK} \big(\mathcal{R} \mathrm{Hess}_{P_{t-s} f}\big)(\Phi_s^k, \|_s v) \mathrm{d}s.\end{aligned} \tag{4.7}$$

Let

$$\phi_k(s) = \mathrm{e}^{-K(t-s)}\, \mathbb{E} \|\mathrm{Hess}^{\#}_{P_s f}(\Phi_{t-s}^k)\|, \quad s \in [0, t], 1 \le k \le d.$$

By (4.7) and the Markov property, for any $0 \le s_2 < s_1 \le t$ we have

$$\begin{aligned}&\mathrm{Hess}_{P_{s_1} f}(\|_{t-s_1} v, \Phi_{t-s_1}^k) = \mathrm{e}^{-K(s_1 - s_2)}\, \mathbb{E}\big(\mathrm{Hess}_{P_{s_2} f}(\|_{t-s_2} v, \Phi_{t-s_2}^k) \big| \mathscr{F}_{t-s_1}^x\big) \\ &+ \int_0^{s_1 - s_2} \mathrm{e}^{-rK} \mathbb{E}\Big(\big(\mathcal{R} \mathrm{Hess}_{P_{s_1 - r} f}\big)(\Phi_{t-s_1+r}^k, \|_{t-s_1+r} v) \Big| \mathscr{F}_{t-s_1}^x \Big) \mathrm{d}r.\end{aligned}$$

So,

$$\begin{aligned}&I_{k,v}(s_1, s_2) \\ &:= \mathbb{E}\Big| \mathrm{e}^{-(t-s_1)K} \mathrm{Hess}_{P_{s_1} f}(\|_{t-s_1} v, \Phi_{t-s_1}^k) - \mathrm{e}^{-K(t-s_2)} \mathbb{E}\big(\mathrm{Hess}_{P_{s_2} f}(\|_{t-s_2} v, \Phi_{t-s_2}^k) \big| \mathscr{F}_{t-s_1}^x\big) \Big| \\ &\le |v| \mathrm{e}^{-(t-s_1)K} \|\mathcal{R}\|_\infty \mathbb{E} \int_0^{s_1 - s_2} \mathrm{e}^{-rK} |\mathrm{Hess}^{\#}_{P_{s_1 - r} f}(\Phi_{t-s_1+r}^k)| \mathrm{d}r \\ &= |v| \|\mathcal{R}\|_\infty \int_{s_2}^{s_1} \mathrm{e}^{-K(t-\theta)} \mathbb{E} |\mathrm{Hess}^{\#}_{P_\theta f}(\Phi_{t-\theta}^k)| \mathrm{d}\theta \\ &= |v| \|\mathcal{R}\|_\infty \int_{s_1}^{s_2} \phi_k(\theta) \mathrm{d}\theta,\end{aligned}$$

where we have used the change of variable $\theta = s_1 - r$. Then

$$\begin{aligned}\phi_k(s_1) - \phi_k(s_2) &\le \sup_{|v|\le 1} I_{k,v}(s_1,s_2)\\ &\le \|\mathcal{R}\|_\infty \int_{s_1}^{s_2} \phi_k(\theta)\mathrm{d}\theta, \quad 0\le s_2\le s_1\le t.\end{aligned}$$

By Gronwall's lemma, this implies

$$|\mathrm{Hess}^{\#}_{P_tf}(\Phi_0^k)| = \phi_k(t) \le \mathrm{e}^{\|\mathcal{R}\|_\infty t}\phi_k(0) = \mathrm{e}^{(\|\mathcal{R}\|_\infty - K)t}\mathbb{E}|\mathrm{Hess}^{\#}_f(\Phi_t^k)|, \quad 1\le k\le d.$$

Therefore,

$$\begin{aligned}\|\mathrm{Hess}_{P_tf}\|_{HS}^2 &= \sum_{k=1}^d |\mathrm{Hess}^{\#}_{P_tf}(\Phi_0^k)|^2 \le \mathrm{e}^{2(\|\mathcal{R}\|_\infty - K)t}\sum_{k=1}^d \left(\mathbb{E}|\mathrm{Hess}^{\#}_f(\Phi_t^k)|\right)^2\\ &\le \mathrm{e}^{2(\|\mathcal{R}\|_\infty - K)t} P_t \|\mathrm{Hess}_f\|_{HS}^2.\end{aligned}$$

□

Next, we apply the above results to characterize the lower and upper bounds of $\mathcal{R}ic$ for Ricci parallel manifolds.

Theorem 4.2. *Let M be a Ricci parallel manifold. Then for any constant $K\in\mathbb{R}$, the following statements are equivalent each other:*

(1) $\mathcal{R}ic \ge K$.

(2) *For any $f\in C_0^\infty(M)$ and $t\ge 0$,*

$$\frac{\mathrm{e}^{Kt} - \mathrm{e}^{2(K-\|\mathcal{R}\|_\infty)t}}{2\|\mathcal{R}\|_\infty - K}\|\mathrm{Hess}_{P_tf}\|^2 \le P_t|\nabla f|^2 - \mathrm{e}^{Kt}|\nabla P_tf|^2.$$

(3) *For any $f\in C_0^\infty(M)$ and $t\ge 0$,*

$$\|\mathrm{Hess}_{P_tf}\|^2\int_0^t \frac{\mathrm{e}^{Ks} - \mathrm{e}^{2(K-\|\mathcal{R}\|_\infty)s}}{2\|\mathcal{R}\|_\infty - K}\mathrm{d}s \le P_tf^2 - (P_tf)^2 - \frac{\mathrm{e}^{Kt}-1}{K}|\nabla P_tf|^2.$$

(4) *For any $f\in C_0^\infty(M)$ and $t\ge 0$,*

$$\begin{aligned}&P_tf^2 - (P_tf)^2 - \frac{1-\mathrm{e}^{-Kt}}{K}P_t|\nabla f|^2\\ &\le -\|\mathrm{Hess}_{P_tf}\|^2\,\mathrm{e}^{2(K-\|\mathcal{R}\|_\infty)t}\int_0^t \frac{\mathrm{e}^{2(\|\mathcal{R}\|_\infty - K)s} - \mathrm{e}^{-Ks}}{2\|\mathcal{R}\|_\infty - K}\mathrm{d}s.\end{aligned}$$

Proof. **(a)** (1) $\Rightarrow$ (2). Let $t>0$ and $f\in C_b^2(M)$. By (3.5) and Itô's formula, we have

$$\begin{aligned}\text{(4.8)}\qquad \mathrm{d}|\nabla P_{t-s}f|^2(X_s) &= \Big(\frac{1}{2}\Delta|\nabla P_{t-s}f|^2(X_s) - \langle \nabla P_{t-s}f, \nabla\Delta P_{t-s}f\rangle(X_s)\Big)\mathrm{d}s\\ &\quad + 2\langle \nabla|\nabla P_{t-s}f|^2(X_s), \Phi_s\mathrm{d}B_s\rangle,\quad s\in[0,t].\end{aligned}$$

By the Bochner-Weitzenböck formula and $\mathcal{R}ic\geq K$, we obtain

$$\begin{aligned}&\frac{1}{2}\Delta|\nabla P_{t-s}f|^2(X_s) - \langle \nabla P_{t-s}f, \nabla\Delta P_{t-s}f\rangle(X_s)\\ &= \mathcal{R}ic(\nabla P_{t-s}f, \nabla P_{t-s}f)(X_s) + \|\mathrm{Hess}_{P_{t-s}f}\|_{HS}^2(X_s)\\ &\geq K|\nabla P_{t-s}f|^2(X_s) + \|\mathrm{Hess}_{P_{t-s}f}\|_{HS}^2(X_s).\end{aligned}$$

Then (4.8) implies

$$\mathrm{d}|\nabla P_{t-s}f|^2(X_s) \geq \big(K|\nabla P_{t-s}f|^2 + \|\mathrm{Hess}_{P_{t-s}f}\|_{HS}^2\big)(X_s)\mathrm{d}s + 2\langle \nabla|\nabla P_{t-s}f|^2(X_s), \Phi_s\mathrm{d}B_s\rangle$$

for $s\in[0,t]$. Combining this with (4.2), we arrive at

$$\begin{aligned}&P_t|\nabla f|^2 - \mathrm{e}^{Kt}|\nabla P_t f|^2 \geq \int_0^t \mathrm{e}^{K(t-s)}P_s\|\mathrm{Hess}_{P_{t-s}f}\|_{HS}^2\mathrm{d}s\\ &\geq \int_0^t \mathrm{e}^{K(t-s)}\mathrm{e}^{-2(\|\mathcal{R}\|_\infty-K)s}\|\mathrm{Hess}_{P_t f}\|^2\mathrm{d}s\\ &= \frac{\mathrm{e}^{Kt}-\mathrm{e}^{2(K-\|\mathcal{R}\|_\infty)t}}{2\|\mathcal{R}\|_\infty - K}\|\mathrm{Hess}_{P_t f}\|^2.\end{aligned}$$

(b) (2) implies (3) and (4). By (3.5) and Itô's formula, we have

$$\mathrm{d}(P_{t-s}f)^2(X_s) = |\nabla P_{t-s}f|^2(X_s)\mathrm{d}s + \langle \nabla|P_{t-s}f|^2(X_s), \Phi_s\mathrm{d}B_s\rangle,\quad s\in[0,t].$$

So,

$$\text{(4.9)}\qquad P_t f^2 - (P_t f)^2 = \int_0^t P_s|\nabla P_{t-s}f|^2\mathrm{d}s.$$

Combining this with (2) and (4.2), we obtain

$$\begin{aligned}&P_t f^2 - (P_t f)^2\\ &\geq \int_0^t \Big\{|\nabla P_t f|^2\mathrm{e}^{Ks} + \frac{\mathrm{e}^{Ks}-\mathrm{e}^{2(K-\|\mathcal{R}\|_\infty)s}}{2\|\mathcal{R}\|_\infty - K}\|\mathrm{Hess}_{P_t f}\|^2\Big\}\mathrm{d}s.\end{aligned}$$

Then (3) is proved.

Similarly, (4.2) and (2) imply

$$\begin{aligned}&\mathrm{e}^{-Ks}P_t|\nabla f|^2 - P_{t-s}|\nabla P_s f|^2\\ &\geq \frac{1-\mathrm{e}^{(K-2\|\mathcal{R}\|_\infty)s}}{2\|\mathcal{R}\|_\infty - K}P_{t-s}\|\mathrm{Hess}_{P_s f}\|^2\end{aligned}$$

$$\geq e^{2(K-\|\mathcal{R}\|_\infty)t}\|\mathrm{Hess}_{P_tf}\|^2\cdot\frac{e^{2(\|\mathcal{R}\|_\infty-K)s}-e^{-Ks}}{2\|\mathcal{R}\|_\infty-K},$$

which together with (4.9) gives (4).

(c) Each of (3) and (4) implies (1). For $v\in T_xM$ with $|v|=1$, take $f\in C_0^\infty(M)$ such that

$$\nabla f(x)=v,\quad \mathrm{Hess}_f(x)=0.$$

We have

$$\lim_{t\downarrow 0}\|\mathrm{Hess}_{P_tf}\|^2\int_0^t\frac{e^{Ks}-e^{2(K-\|\mathcal{R}\|_\infty)s}}{2\|\mathcal{R}\|_\infty-K}\mathrm{d}s=0. \tag{4.10}$$

On the other hand, by the Bochner-Weitzenböck formula we have (see [19, Theorem 2.2.4]),

$$\frac{1}{2}\mathcal{R}ic(v,v)=\lim_{t\downarrow 0}\frac{P_t|\nabla f|^p(x)-|\nabla P_tf|^p(x)}{pt},\quad p>0. \tag{4.11}$$

Combining this with (3), (4.9) and (4.10), we obtain

$$\begin{aligned}0&\leq 2\lim_{t\downarrow 0}\frac{P_tf^2-(P_tf)^2-\frac{e^{Kt}-1}{K}|\nabla P_tf|^2}{t^2}\\&=\lim_{t\downarrow 0}\frac{2}{t^2}\int_0^t\left\{P_s|\nabla P_{t-s}f|^2-e^{Ks}|\nabla P_tf|^2\right\}(x)\mathrm{d}s=\mathcal{R}ic(v,v)-K,\end{aligned}$$

Therefore, (3) implies (1). Similarly, (4) also implies (1). □

The following result provides corresponding characterizations for the Ricci upper bound.

Theorem 4.3. *Let M be a Ricci parallel manifold. Then for any constant $K\in\mathbb{R}$, the following are equivalent each other:*

(1) $\mathcal{R}ic\leq K$.

(2) *For any $f\in C_0^\infty(M)$ and $t\geq 0$,*

$$\frac{e^{(2\|\mathcal{R}\|_\infty-K)t}-1}{2\|\mathcal{R}\|_\infty-K}dP_t\|\mathrm{Hess}_f\|^2\geq P_t|\nabla f|^2-e^{Kt}|\nabla P_tf|^2.$$

(3) *For any $f\in C_0^\infty(M)$ and $t\geq 0$,*

$$\begin{aligned}&dP_t\|\mathrm{Hess}_f\|^2\int_0^t\frac{e^{(2\|\mathcal{R}\|_\infty-K)t-Ks}-e^{2(\|\mathcal{R}\|_\infty-K)s}}{2\|\mathcal{R}\|_\infty-K}\mathrm{d}s\\&\geq P_tf^2-(P_tf)^2-\frac{e^{Kt}-1}{K}|\nabla P_tf|^2.\end{aligned}$$

(4) *For any $f\in C_0^\infty(M)$ and $t\geq 0$,*

$$P_tf^2-(P_tf)^2-\frac{1-e^{-Kt}}{K}P_t|\nabla f|^2\quad\geq -dP_t\|\mathrm{Hess}_f\|^2\int_0^t\frac{e^{2(\|\mathcal{R}\|_\infty-K)s}-e^{-Ks}}{2\|\mathcal{R}\|_\infty-K}\mathrm{d}s.$$

Proof. By using

$$\|\text{Hess}_f\|_{HS}^2 \le d\|\text{Hess}_f\|^2, \tag{4.12}$$

the proof is completely similar to that of Theorem 4.2. For instance, below we only show the proof of (1) implying (2).

By $\mathcal{R}ic \le K$ and (4.8), we have

$$\mathrm{d}|\nabla P_{t-s}f|^2(X_s) \le \big(K|\nabla P_{t-s}f|^2 + \|\text{Hess}_{P_{t-s}f}\|_{HS}^2\big)(X_s)\mathrm{d}s + 2\langle\nabla|\nabla P_{t-s}f|^2(X_s), \Phi_s\mathrm{d}B_s\rangle$$

for $s \in [0,t]$. Combining this with (4.2) and (4.12), we arrive at

$$\begin{aligned} P_t|\nabla f|^2 - \mathrm{e}^{Kt}|\nabla P_t f|^2 &\le d\int_0^t \mathrm{e}^{K(t-s)}P_s\|\text{Hess}_{P_{t-s}f}\|^2\mathrm{d}s \\ &\le d\int_0^t \mathrm{e}^{K(t-s)}\mathrm{e}^{2(\|\mathcal{R}\|_\infty - K)(t-s)}P_t\|\text{Hess}_f\|^2\mathrm{d}s \\ &= \frac{\mathrm{e}^{(2\|\mathcal{R}\|_\infty - K)t} - 1}{2\|\mathcal{R}\|_\infty - K}P_t\|\text{Hess}_f\|^2. \end{aligned}$$

Then (1) implies (2). We therefore omit other proofs. □

5 Formula of $\nabla\mathcal{R}ic$

Theorem 5.1. *For any $x \in M$, $v_1, v_2 \in T_xM$ and $f \in C_b^4(M)$ with $\nabla f(x) = v_1, \text{Hess}_f(x) = 0$, there holds*

$$(\nabla_{v_2}\mathcal{R}ic)(v_1, v_1) = 2\lim_{t\downarrow 0}\frac{(P_t\text{Hess}_f - \text{Hess}_{P_tf})(v_1, v_2)}{t} = \big(\Delta\text{Hess}_f - \text{Hess}_{\Delta f}\big)(v_1, v_2). \tag{5.1}$$

Consequently, M is Ricci parallel if and only if $\Delta\text{Hess}_f = \text{Hess}_{\Delta f}$ holds at any point $x \in M$ and $f \in C_0^\infty(M)$ with $\text{Hess}_f(x) = 0$.

When $f \in C_0^\infty(M)$, the second equation in (5.1) follows from (3.7). By a standard approximation argument, this equation holds for all $f \in C_b^2(M)$. So, it suffices to prove the first equation or the formula (5.2) below for $x \in M$ and $f \in C_0^\infty(M)$ with $\text{Hess}_f(x) = 0$. Here, we prove both of them by using analytic and probabilitstic arguments respectively, since each proof has its own interest.

Analytic Proof. For any $x \in M$ and $f \in C_0^\infty(M)$ with $\text{Hess}_f(x) = 0$, we intend to prove

$$(\nabla_v\mathcal{R}ic)(\nabla f, \nabla f) = (\Delta\text{Hess}_f)(\nabla f, v) - \text{Hess}_{\Delta f}(\nabla f, v), \quad v \in T_xM. \tag{5.2}$$

According to the Bochner-Weitzenböck formula, we have

$$\mathcal{R}ic^\#(\nabla f) = \Delta\nabla f - \nabla\Delta f, \tag{5.3}$$

where $\Delta\nabla f := \Phi(\Delta_{O(M)}\nabla_f^{O(M)})(\Phi)$ is independent of $\Phi \in O(M)$. Consequently,

$$\mathcal{R}ic(\nabla f, \nabla f) = \frac{1}{2}\Delta|\nabla f|^2 - \langle \nabla\Delta f, \nabla f\rangle - \|\mathrm{Hess}_f\|_{HS}^2. \tag{5.4}$$

Since $\mathrm{Hess}_f(x) = 0$, (5.4) and (5.3) imply that at point x,

$$\begin{aligned}
&(\nabla\mathcal{R}ic)(\nabla f, \nabla f) = \nabla\{\mathcal{R}ic(\nabla f, \nabla f)\}\\
&= \frac{1}{2}\nabla\Delta|\nabla f|^2 - \mathrm{Hess}_{\Delta f}^{\#}(\nabla f) - \mathrm{Hess}_f^{\#}(\nabla\Delta f) - 2\|\mathrm{Hess}_f\|_{HS}\nabla\|\mathrm{Hess}_f\|\\
&= \frac{1}{2}\Delta\{\nabla|\nabla f|^2\} - \frac{1}{2}\mathcal{R}ic^{\#}(\nabla|\nabla f|^2) - \mathrm{Hess}_{\Delta f}^{\#}(\nabla f)\\
&= \Delta\{\mathrm{Hess}_f^{\#}(\nabla f)\} - \mathrm{Hess}_{\Delta f}^{\#}(\nabla f)\\
&= (\Delta\mathrm{Hess}_f)^{\#}(\nabla f) + \mathrm{Hess}_f^{\#}(\Delta\nabla f) + 2\mathrm{tr}\big\{(\nabla.\mathrm{Hess}_f^{\#})(\mathrm{Hess}_f^{\#}(\cdot))\big\} - \mathrm{Hess}_{\Delta f}^{\#}(\nabla f)\\
&= (\Delta\mathrm{Hess}_f)^{\#}(\nabla f) - \mathrm{Hess}_{\Delta f}^{\#}(\nabla f).
\end{aligned}$$

Therefore, (5.2) holds. □

Probabilistic Proof. We first consider bounded $\nabla\mathcal{R}ic$ and $\mathcal{R}$, then extend to the general case by using Proposition 3.3.

(a) Assume that $\|\mathcal{R}\|_\infty + \|\nabla\mathcal{R}ic\|_\infty < \infty$. Let $x \in M$ and $v_1, v_2 \in T_xM$. We take $f \in C_b^4(M)$ such that $\nabla f(x) = v_1$ and $\mathrm{Hess}_f(x) = 0$. Below we only consider functions taking value at point x. Since $\mathrm{Hess}_f(x) = 0$, there exists a constant $c > 0$ such that

$$P_t\|\mathrm{Hess}_f\|_{HS}^2(x) = \frac{1}{2}\int_0^t P_s\Delta\|\mathrm{Hess}_f\|_{HS}^2(x)\mathrm{d}s \le ct, \quad t \ge 0. \tag{5.5}$$

Then there exists a constant $c_1 > 0$ such that

$$P_s\|\mathrm{Hess}_f\| \le \sqrt{P_s\|\mathrm{Hess}_f\|^2} \le c_1\sqrt{s}, \quad s \in [0,1]. \tag{5.6}$$

Since $\nabla f(x) = v_1$ and $\nabla P_s f(x)$ is smooth in s, this together with (4.5) yields

$$\mathbb{E}|\nabla f(X_s) - \|_s v_1| \le c_2 s, \quad s \in [0,1] \tag{5.7}$$

for some constant $c_2 > 0$. Moreover, by (1.2) there exists a constant $c_3 > 0$ such that

$$|W_s(v_i) - \|_s v_i| \le c_3 s, \quad s \in [0,1], i = 1, 2. \tag{5.8}$$

Combining (5.6)-(5.8) with (3.10), (3.17) and (3.18), for small $t > 0$ we arrive at

$$\begin{aligned}
\mathbb{E}\langle\nabla f(X_t), W_t^{(2)}(v_1, v_2)\rangle =& \frac{1}{2}\mathbb{E}\int_0^t \big\langle(\tilde{\nabla}\mathcal{R}ic^{\#})(W_s(v_2))W_s(v_1), \nabla f(X_s)\big\rangle\mathrm{d}s\\
&+ \sum_{i=1}^d \mathbb{E}\int_0^t \mathrm{Hess}_{P_{t-s}f}\big(\Phi_s^i, \mathcal{R}(\Phi_s^i, W_s(v_2))W_s(v_1)\big)\mathrm{d}s\\
=& \,\mathrm{o}(t) + \frac{1}{2}\langle(\tilde{\nabla}\mathcal{R}ic^{\#})(v_2)v_1, v_1\rangle t = \mathrm{o}(t) - \frac{1}{2}(\nabla_{v_2}\mathcal{R}ic)(v_1, v_1)t,
\end{aligned} \tag{5.9}$$

where the last step follows from (3.18).

On the other hand, by (3.19), (5.6) and (5.8), there exist a constant $c_4 > 0$ such that for small $t > 0$,

$$\begin{aligned}&\mathbb{E}\langle\nabla f(X_t), W_t^{(2)}(v_1, v_2)\rangle = \mathrm{Hess}_{P_t f}(v_1, v_2) - \mathbb{E}\mathrm{Hess}_f(W_t(v_1), W_t(v_2))\\&= \mathrm{Hess}_{P_t f}(v_1, v_2) - P_t\mathrm{Hess}_f(v_1, v_2) + \mathrm{O}(t)P_t\|\mathrm{Hess}_f\|\\&= \mathrm{Hess}_{P_t f}(v_1, v_2) - P_t\mathrm{Hess}_f(v_1, v_2) + \mathrm{o}(t).\end{aligned}$$

Combining this with (5.9) we derive the desired the first equation in (5.1).

(b) In general, let M be a complete non-compact Riemannian manifold. For fixed $x \in M$ we take $D = B(x, 4)$. Let $f \in C^\infty(\bar{D})$ with $f|_{\partial D} = 0, f|_{B(x,3)} = 1$, $|\nabla f| = 1$ on ∂D and $f > 0$ in D. Then $(D, f^{-2}g)$ is a complete Riemannian manifold. We use superscript D to denote quantities on this manifold, for instance, $\mathcal{R}^D$ is the Riemannian tensor on $(D, f^{-2}g)$. Then both $\mathcal{R}^D$ and $\nabla^D\mathcal{R}ic^D$ are bounded. So, by (a), for P_t^D the heat semigroup on D,

$$(\nabla_{v_2}\mathcal{R}ic)(v_1, v_1) = \lim_{t\downarrow 0}\frac{(P_t^D\mathrm{Hess}_f^D - \mathrm{Hess}^D_{P_t^D f})(v_1, v_2)}{t}. \tag{5.10}$$

Since $f = 1$ in $B(x, 3)$, we may construct the horizontal Brownian motion $\Phi_t^D(y)$ on D with $\Phi_0^D(y) = \Phi_0 \in O_y(M)$ such that

$$\Phi_t^D(y) = \Phi_t(y), \quad t \le \tau_3(y), \quad X_t^D(y) = X_t(y), t \le \tau_3(y),$$

where

$$\tau_3(y) = \inf\{t \ge 0 : X_t(y) \in \partial B(x, 3)\}.$$

Noting that

$$P_t f(y) = \mathbb{E}[f(X_t(y))1_{\{t<\zeta\}}], \quad P_t^D f(y) = \mathbb{E}[f(X_t^D(y))], \quad f \in \mathscr{B}_b(M),$$

where ζ is the life time of X_t, we obtain

$$|P_t f(y) - P_t^D(y)| \le \|f\|_\infty\mathbb{P}(\tau_3(y) \le t), \quad t \ge 0, f \in \mathscr{B}_b(M).$$

Combining this with [3, Lemma 2.3], we may find constants $c_1, c_2 > 0$ such that

$$|P_t f(y) - P_t^D f(y)| \le c_1\|f\|_\infty \mathrm{e}^{-c_2/t}, \quad t \in (0, 1], y \in B(x, 2). \tag{5.11}$$

Consequently,

$$|P_t\mathrm{Hess}_f(v_1, v_2) - P_t^D\mathrm{Hess}_f(v_1, v_2)| \le c_1\|\mathrm{Hess}_f\|_\infty\mathrm{e}^{-c_2/t}, \quad t \in (0, 1]. \tag{5.12}$$

Moreover, since $X_t = X_t^D \in B(x, 2)$ before time τ_2, Proposition 3.3 and (5.11) imply that at point x,

$$\begin{aligned}&|\mathrm{Hess}_{P_t f}(v_1, v_2) - \mathrm{Hess}^D_{P_t^D f}(v_1, v_2)|\\&\le \mathbb{E}\big[(|P_{t-t\wedge\tau_1}f(X_{t\wedge\tau_1}) - P^D_{t-t\wedge\tau_1}f(X^D_{t\wedge\tau_1})| \cdot |M_t|\big]\\&\quad + \mathbb{E}\big[|P_{t-t\wedge\tau_2}f(X_{t\wedge\tau_2}) - P^D_{t-t\wedge\tau_2}f(X^D_{t\wedge\tau_2})|)|N_t|\big]\\&\le c_1\mathrm{e}^{-c_2/t}\|f\|_\infty\frac{C(x)}{t}, \quad t \in (0, 1].\end{aligned}$$

Combining this with (5.10) and (5.12), we prove the first equation in (5.1). □

6 Identification of constant curvature

Theorem 6.1. *Let $k \in \mathbb{R}$. Then each of the following assertions is equivalent to $\mathcal{S}ec = k$:*

(1) *For any $t \geq 0$ and $f \in C_0^\infty(M)$,*

$$\text{Hess}_{P_t f} = \mathrm{e}^{-dkt} P_t \text{Hess}_f + \frac{1}{d}(1 - \mathrm{e}^{-dkt})(P_t \Delta f)\mathbf{g}. \tag{6.1}$$

(2) *For any $f \in C_0^\infty(M)$,*

$$\text{Hess}_{\Delta f} - \Delta \text{Hess}_f = 2k(\Delta f)\mathbf{g} - 2dk\text{Hess}_f. \tag{6.2}$$

(3) *For any $x \in M, u \in T_x M$ and $f \in C_0^\infty(M)$ with $\text{Hess}_f(x) = u \otimes u$ (i.e. $\text{Hess}_f(v_1, v_2) = \langle u, v_1 \rangle \langle u, v_2 \rangle, v_1, v_2 \in T_x M$),*

$$\big(\text{Hess}_{\Delta f} - \Delta \text{Hess}_f\big)(v, v) = 2k\big(|u|^2 |v|^2 - \langle u, v \rangle^2\big), \quad v \in T_x M. \tag{6.3}$$

(4) *For any $f \in C_0^\infty(M)$,*

$$\frac{1}{2}\Delta \|\text{Hess}_f\|_{HS}^2 - \langle \text{Hess}_{\Delta f}, \text{Hess}_f \rangle_{HS} - \|\nabla \text{Hess}_f\|_{HS}^2 = 2k\big(d\|\text{Hess}_f\|_{HS}^2 - (\Delta f)^2\big). \tag{6.4}$$

To prove this result, we need the following lemma where $\mathcal{R}T$ is defined in (1.1).

Lemma 6.2. *If the sectional curvature $\mathcal{S}ec = k$ for some constant k, then for any symmetric 2-tensor T,*

$$\mathcal{R}T = k\text{tr}(T)\mathbf{g} - kT.$$

Proof. Let $\tilde{T} = k\text{tr}(T)\mathbf{g} - kT$. Since both $\mathcal{R}T$ and $\tilde{T}$ are symmetric, it suffices to prove

$$(\mathcal{R}T)(v, v) = \tilde{T}(v, v), \quad v \in T_x, |v| = 1, x \in M. \tag{6.5}$$

Let $v \in T_x M$ with $|v| = 1$. By the symmetry of T, there exists $\Phi = (\Phi^i)_{1 \leq i \leq d} \in O_x(M)$ such that

$$T^\#(\Phi^i) = \lambda_i \Phi^i, \quad 1 \leq i \leq d$$

holds for some constants $\lambda_i, 1 \leq i \leq d$. Then $\mathcal{S}ec = k$ implies

$$\begin{aligned}(\mathcal{R}T)(v, v) &= \sum_{i=1}^d \big\langle \mathcal{R}(\Phi^i, v)v, T^\#(\Phi^i) \big\rangle = \sum_{i=1}^d \lambda_i \big\langle \mathcal{R}(\Phi^i, v)v, \Phi^i \big\rangle \\ &= k \sum_{i=1}^d \lambda_i \big(1 - \langle \Phi^i, v \rangle^2\big) = k\text{tr}(T) - kT(v, v) = \tilde{T}(v, v).\end{aligned}$$

Therefore, (6.5) holds. □

Proof of Theorem 6.1. Obviously, (3) follows from (2). Next, by (3.7) and taking derivative of (6.1) with respect to t at $t=0$, we obtain (6.2). So, (1) implies (2). Moreover, by chain rule we have

$$\frac{1}{2}\Delta\|\mathrm{Hess}_f\|_{HS}^2 = \langle \Delta\mathrm{Hess}_f, \mathrm{Hess}_f\rangle_{HS} + \|\nabla\mathrm{Hess}_f\|_{HS}^2. \tag{6.6}$$

Then (4) follows from (2) and the identity $\langle(\delta f)\mathbf{g}, \mathrm{Hess}_f\rangle_{HS} = (\Delta f)^2$. To complete the proof, below we prove "$\mathcal{S}ec = k \Rightarrow$ (1)", "(3) $\Rightarrow \mathcal{S}ec = k$ " and "(4) $\Rightarrow \mathcal{S}ec = k$ " respectively.

(a) $\mathcal{S}ec = k \Rightarrow$ (1). Let $\mathcal{S}ec = k$. Then $\mathcal{R}ic = (d-1)k$. By (1.2), (4.1), we have $W_s(v) = \mathrm{e}^{-\frac{K}{2}s} \, \|_s \, v, \ s\geq 0, v\in TM$, and for any $x\in M, v_1, v_2\in T_xM$,

$$\mathrm{Hess}_{P_tf}(v_1,v_2) = \mathrm{e}^{-Kt}P_t\mathrm{Hess}_f(v_1,v_2) + \int_0^t \mathrm{e}^{-Ks}P_s(\mathcal{R}\mathrm{Hess}_{P_{t-s}f})(v_1,v_2)\mathrm{d}s.$$

Noting that $\Delta P_{t-s}f = P_{t-s}\Delta f$ and $\langle \|_s \, v_1, \|_s \, v_2\rangle = \langle v_1, v_2\rangle$, this together with Lemma 6.2 gives

$$\begin{aligned}
&\mathrm{Hess}_{P_tf}(v_1,v_2) - \mathrm{e}^{-Kt}P_t\mathrm{Hess}_f(v_1,v_2)\\
&= \int_0^t \mathrm{e}^{-Ks}\Big(P_s\big\{k\langle v_1,v_2\rangle P_{t-s}\Delta f\big\} - kP_s(\mathrm{Hess}_{P_{t-s}f})(v_1,v_2)\Big)\mathrm{d}s, \quad t\geq 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{Hess}_{P_tf}(v_1,v_2) &= \mathrm{e}^{-(K+k)t}P_t\mathrm{Hess}_f(v_1,v_2) + k\langle v_1,v_2\rangle\int_0^t \mathrm{e}^{-(K+k)s}P_t\Delta f\mathrm{d}s\\
&= \mathrm{e}^{-dkt}P_t\mathrm{Hess}_f(v_1,v_2) + \frac{1-\mathrm{e}^{-dkt}}{d}(P_t\Delta f)\langle v_1,v_2\rangle.
\end{aligned}$$

So, (6.1) holds.

(b) (3) $\Rightarrow \mathcal{S}ec = k$. By taking $u=0$, (3) implies that for any $f\in C_0^\infty(M)$ with $\mathrm{Hess}_f(x) = 0$,

$$(\Delta\mathrm{Hess}_f - \mathrm{Hess}_{\Delta f})(v,v) = 0, \quad v\in T_xM.$$

By the symmetry of $\Delta\mathrm{Hess}_f - \mathrm{Hess}_{\Delta f}$, this is equivalent to

$$(\Delta\mathrm{Hess}_f - \mathrm{Hess}_{\Delta f})(v_1,v_2) = 0, \quad v_1,v_2\in T_xM.$$

So, for any $v_1,v_2\in T_xM$, by taking $f\in C_0^\infty(M)$ such that $\nabla f(x) = v_1$ and $\mathrm{Hess}_f(x) = 0$, we deduce from Theorem 5.1 that

$$(\nabla_{v_2}\mathcal{R}ic)(v_1,v_1) = \big(\Delta\mathrm{Hess}_f - \mathrm{Hess}_{\Delta f}\big)(v_1,v_2) = 0.$$

Thus, M is Ricci parallel. By Theorem 4.1, (4.1) holds. Due to (1.2) and Itô's formula, by taking derivative of (4.1) with respect to t at $t=0$, we obtain

$$\begin{aligned}
&\frac{1}{2}\mathrm{Hess}_{\Delta f}(v_1,v_2)\\
&= \frac{1}{2}(\Delta\mathrm{Hess}_f)(v_1,v_2) - \mathcal{R}ic(v_1,\mathrm{Hess}_f^{\#}(v_2)) + \mathrm{tr}\langle\mathcal{R}(\cdot,v_2)v_1,\mathrm{Hess}_f^{\#}(\cdot)\rangle, \quad f\in C_0^\infty(M).
\end{aligned} \tag{6.7}$$

Now, letting $u, v \in T_xM$ with $|u| = |v| = 1$ and $\langle u, v\rangle = 0$, and combining (6.7) with (6.3) for $v_1 = v_2 = v$, and $f \in C_0^\infty(M)$ with $\mathrm{Hess}_f(x) = u \otimes u$, we arrive at

$$\begin{aligned} k &= k(|u|^2|v|^2 - \langle u, v\rangle^2) = \frac{1}{2}(\mathrm{Hess}_{\Delta f} - \Delta \mathrm{Hess}_f)(v, v) \\ &= -\mathcal{R}ic(v, \mathrm{Hess}_f^{\#}(v)) + \mathrm{tr}\langle \mathcal{R}(\cdot, v)v, \mathrm{Hess}_f^{\#}(\cdot)\rangle = \mathcal{S}ec(u, v). \end{aligned}$$

Therefore, $\mathcal{S}ec = k$.

(c) (4) $\Rightarrow \mathcal{S}ec = k$. By (6.6), (6.4) is equivalent to

$$\frac{1}{2}\big\langle \Delta \mathrm{Hess}_f - \mathrm{Hess}_{\Delta f}, \mathrm{Hess}_f\big\rangle_{HS} = k\big(d\|\mathrm{Hess}_f\|_{HS}^2 - (\Delta f)^2\big). \tag{6.8}$$

We first prove that (6.8) implies $\nabla \mathcal{R}ic = 0$. Let $f \in C_0^\infty(M)$ with $\mathrm{Hess}_f(x) = 0$. For any $u \in T_xM$, let $h \in C_0^\infty(M)$ with $\mathrm{Hess}_h(x) = u \otimes u$. Applying (6.8) to $f_s := f + sh, s \geq 0$, we obtain at point x that

$$\begin{aligned} &\frac{s}{2}(\Delta \mathrm{Hess}_f - \mathrm{Hess}_{\Delta f})(u, u) + \frac{s^2}{2}(\Delta \mathrm{Hess}_h - \mathrm{Hess}_{\Delta h})(u, u) \\ &= k\big(ds^2\|\mathrm{Hess}_h\|_{HS}^2 - s^2(\Delta h)^2\big), \quad s > 0. \end{aligned}$$

Multiplying by s^{-1} and letting $s \to 0$, we arrive at $(\Delta \mathrm{Hess}_f - \mathrm{Hess}_{\Delta f})(u, u) = 0$. As shown above, this implies $\nabla \mathcal{R}ic = 0$.

Next, we prove $\mathcal{S}ec = k$. Since $\nabla \mathcal{R}ic = 0$, (6.7) holds. For $x \in T_xM$ and $u, v \in T_xM$ with $|u| = |v| = 1$ and $\langle u, v\rangle = 0$, take $f \in C_0^\infty(M)$ such that

$$\mathrm{Hess}_f(x) = u \otimes v + v \otimes u.$$

Then at point x,

$$\mathrm{Hess}_f^{\#}(\cdot) = \langle u, \cdot\rangle v + \langle v, \cdot\rangle u.$$

So, by (6.7) and (6.8) we obtain

$$\begin{aligned} 2kd &= k\big(d\|\mathrm{Hess}_f\|_{HS}^2 - (\Delta f)^2\big)(x) \\ &= \frac{1}{2}\langle \Delta \mathrm{Hess}_f - \mathrm{Hess}_{\Delta f}, \mathrm{Hess}_f\rangle_{HS}(x) = (\Delta \mathrm{Hess}_f - \mathrm{Hess}_{\Delta f})(u, v) \\ &= 2\mathcal{R}ic(u, \mathrm{Hess}_f^{\#}(v)) - 2\mathrm{tr}\langle \mathcal{R}(\cdot, v)u, \mathrm{Hess}_f^{\#}(\cdot)\rangle = 2\mathcal{R}ic(u, u) + 2\mathcal{S}ec(u, v). \end{aligned} \tag{6.9}$$

Letting $\{v_1\}_{1\leq i\leq d-1}$ be orthonormal and orthogonal to u, replacing v by v_i and sum over i leads to

$$2kd(d-1) = 2(d-1)\mathcal{R}ic(u, u) + 2\mathcal{R}ic(u, u) = 2d\mathcal{R}ic(u, u).$$

Thus, $\mathcal{R}ic(u, u) = (d-1)k$, and (6.9) implies $\mathcal{S}ec(u, v) = k$.

□

7 Identifications of Einstein manifolds

Theorem 7.1. *For any constant $K \in \mathbb{R}$, the following statements are equivalent each other:*

(1) *M is an Einstein manifold with $\mathcal{R}ic = K$.*

(2) *$\|\mathcal{R}\|_\infty < \infty$, and for any $x \in M, t \geq 0$, $f \in C_0(M)$ and $v_1, v_2 \in T_xM$,*

$$\text{Hess}_{P_tf}(v_1, v_2) - e^{-Kt} P_t \text{Hess}_f(v_1, v_2) = \int_0^t e^{-Ks} P_s\big(\mathcal{R}\text{Hess}_{P_{t-s}f}\big)(v_1, v_2) ds. \tag{7.1}$$

(3) *$\|\mathcal{R}\|_\infty < \infty$, and*

$$\begin{aligned}&\frac{e^{Kt} - e^{2(K-\|\mathcal{R}\|_\infty)t}}{2\|\mathcal{R}\|_\infty - K} \|\text{Hess}_{P_tf}\|_{HS}^2 \leq P_t|\nabla f|^2 - e^{Kt}|\nabla P_t f|^2 \\ &\leq \frac{e^{(2\|\mathcal{R}\|_\infty - K)t} - 1}{2(d-1)\|\mathcal{R}\|_\infty - K} P_t \|\text{Hess}_f\|_{HS}^2, \quad f \in C_b^2(M), t \geq 0.\end{aligned}$$

(4) *There exists $h : [0, \infty) \times M \to [0, \infty)$ with $\lim_{t\to 0} h(t, \cdot) = 0$ such that*

$$\big|P_t|\nabla f|^2 - e^{Kt}|\nabla P_t f|^2\big| \leq h(t, \cdot)\big(\|\text{Hess}_{P_tf}\|_{HS}^2 + P_t\|\text{Hess}_f\|_{HS}^2\big), \quad t \geq 0, f \in C_0^\infty(M).$$

(5) *$\|\mathcal{R}\|_\infty < \infty$, and*

$$\begin{aligned}&\|\text{Hess}_{P_tf}\|_{HS}^2 \int_0^t \frac{e^{Ks} - e^{2(K-\|\mathcal{R}\|_\infty)s}}{2\|\mathcal{R}\|_\infty - K} ds \leq P_t f^2 - (P_t f)^2 - \frac{e^{Kt} - 1}{K} |\nabla P_t f|^2 \\ &\leq \big(P_t\|\text{Hess}_f\|_{HS}^2\big) \int_0^t \frac{e^{(2\|\mathcal{R}\|_\infty - K)t - Ks} - e^{2(\|\mathcal{R}\|_\infty - K)s}}{2\|\mathcal{R}\|_\infty - K} ds, \quad f \in C_b^2(M), t \geq 0.\end{aligned}$$

(6) *$\|\mathcal{R}\|_\infty < \infty$, and*

$$\begin{aligned}&-\big(P_t\|\text{Hess}_f\|_{HS}^2\big) \int_0^t \frac{e^{2(\|\mathcal{R}\|_\infty - K)s} - e^{-Ks})}{2\|\mathcal{R}\|_\infty - K} ds \leq P_t f^2 - (P_t f)^2 - \frac{1 - e^{-Kt}}{K} P_t|\nabla f|^2 \\ &\leq -\|\text{Hess}_{P_tf}\|_{HS}^2 e^{2(K-\|\mathcal{R}\|_\infty)t} \int_0^t \frac{e^{2(\|\mathcal{R}\|_\infty - K)s} - e^{-Ks}}{2\|\mathcal{R}\|_\infty - K} ds, \quad f \in C_b^2(M), t \geq 0.\end{aligned}$$

(7) *There exists $\tilde{h} : [0, \infty) \times M \to [0, \infty)$ with $\lim_{t\to 0} t^{-1}\tilde{h}(t, \cdot) = 0$ such that*

$$\begin{aligned}&\min\left\{\left|P_t f^2 - (P_t f)^2 - \frac{e^{Kt} - 1}{K}|\nabla P_t f|^2\right|, \left|P_t f^2 - (P_t f)^2 - \frac{1 - e^{-Kt}}{K} P_t|\nabla f|^2\right|\right\} \\ &\leq \tilde{h}(t, \cdot)\big(\|\text{Hess}_{P_tf}\|_{HS}^2 + P_t\|\text{Hess}_f\|_{HS}^2\big), \quad t \geq 0, f \in C_0^\infty(M).\end{aligned}$$

(8) *For any $f \in C_0^\infty(M)$,*

$$\frac{1}{2}\big\{\text{Hess}_{\Delta f} - \Delta \text{Hess}_f\big\} = (\mathcal{R}\text{Hess}_f) - K\text{Hess}_f.$$

Proof. Obviously, (3) implies (4), each of (5) and (6) implies (7). According to Theorem 4.1 for $\mathcal{R}ic = K$, (1) implies (2). Moreover, by taking derivative of (7.1) with respect to t at $t = 0$, we obtain (8). So, it suffices to prove that (1) implies (3); (3) implies (5) and (6); and each of (4), (7) and (8) implies (1).

(a) (1) $\Rightarrow$ (3). By the Bochner-Weitzenböck formula and using $\mathcal{R}ic = K$, we obtain

$$\begin{aligned}&\frac{1}{2}\Delta|\nabla P_{t-s}f|^2(X_s) - \langle \nabla P_{t-s}f, \nabla \Delta P_{t-s}f\rangle(X_s)\\&= \mathcal{R}ic(\nabla P_{t-s}f, \nabla P_{t-s}f)(X_s) + \|\mathrm{Hess}_{P_{t-s}f}\|_{HS}^2(X_s)\\&= K|\nabla P_{t-s}f|^2(X_s) + \|\mathrm{Hess}_{P_{t-s}f}\|_{HS}^2(X_s).\end{aligned}$$

Then (4.8) implies

$$\mathrm{d}|\nabla P_{t-s}f|^2(X_s) = \big(K|\nabla P_{t-s}f|^2 + \|\mathrm{Hess}_{P_{t-s}f}\|_{HS}^2\big)(X_s)\mathrm{d}s + 2\langle \nabla|\nabla P_{t-s}f|^2(X_s), \Phi_s \mathrm{d}B_s\rangle$$

for $s \in [0,t]$. Thus,

$$P_t|\nabla f|^2 - \mathrm{e}^{Kt}P_t|\nabla f|^2 = \int_0^t \mathrm{e}^{K(t-s)}P_s\|\mathrm{Hess}_{P_{t-s}f}\|_{HS}^2\mathrm{d}s. \tag{7.2}$$

Since by (4.3)

$$\|\mathrm{Hess}_{P_{t-s}f}\|_{HS}^2 \le \mathrm{e}^{2(\|\mathcal{R}\|_\infty - K)(t-s)}P_{t-s}\|\mathrm{Hess}_f\|_{HS}^2,$$

it follows from (7.2) that

$$\begin{aligned}P_t|\nabla f|^2 - \mathrm{e}^{Kt}P_t|\nabla f|^2 &\le \int_0^t \mathrm{e}^{K(t-s)+2(\|\mathcal{R}\|_\infty - K)(t-s)}P_t\|\mathrm{Hess}_f\|_{HS}^2\mathrm{d}s\\&= \frac{\mathrm{e}^{(2\|\mathcal{R}\|_\infty - K)t}-1}{2\|\mathcal{R}\|_\infty - K}P_t\|\mathrm{Hess}_f\|^2.\end{aligned}$$

So, the second inequality in (3) holds. Similarly, (4.3) implies

$$P_s\|\mathrm{Hess}_{P_{t-s}f}\|_{HS}^2 \ge \mathrm{e}^{-2(\|\mathcal{R}\|_\infty - K)s}\|\mathrm{Hess}_{P_sP_{t-s}f}\|_{HS}^2 = \mathrm{e}^{-2(\|\mathcal{R}\|_\infty - K)s}\|\mathrm{Hess}_{P_tf}\|_{HS}^2,$$

the first inequality in (3) also follows from (7.2).

(b) (3) $\Rightarrow$ (5) and (6). By (3) and (4.3) we have

$$\begin{aligned}&\frac{\mathrm{e}^{Ks} - \mathrm{e}^{2(K-\|\mathcal{R}\|_\infty)s}}{2\|\mathcal{R}\|_\infty - K}\|\mathrm{Hess}_{P_tf}\|_{HS}^2 \le P_s|\nabla P_{t-s}f|^2 - \mathrm{e}^{Ks}|\nabla P_tf|^2\\&\le \frac{\mathrm{e}^{(2\|\mathcal{R}\|_\infty - K)s}-1}{2\|\mathcal{R}\|_\infty - K}P_s\|\mathrm{Hess}_{P_{t-s}f}\|_{HS}^2\\&\le \frac{\mathrm{e}^{((2\|\mathcal{R}\|_\infty - K)t - K(t-s)} - \mathrm{e}^{2(\|\mathcal{R}\|_\infty - K)(t-s)}}{2\|\mathcal{R}\|_\infty - K}P_t\|\mathrm{Hess}_f\|_{HS}^2.\end{aligned}$$

This together with (4.9) ensures (5).

Similarly, (4.3) and (3) imply

$$\begin{aligned}&\frac{\mathrm{e}^{-Ks}(\mathrm{e}^{(2\|\mathcal{R}\|_\infty - K)s}-1)}{2\|\mathcal{R}\|_\infty - K}P_t\|\mathrm{Hess}_f\|_{HS}^2 \ge \mathrm{e}^{-Ks}P_t|\nabla f|^2 - P_{t-s}|\nabla P_sf|^2\\&\ge \frac{1-\mathrm{e}^{(K-2\|\mathcal{R}\|_\infty)s}}{2\|\mathcal{R}\|_\infty - K}P_{t-s}\|\mathrm{Hess}_{P_sf}\|_{HS}^2\\&\ge \mathrm{e}^{2(K-\|\mathcal{R}\|_\infty)t}\|\mathrm{Hess}_{P_tf}\|_{HS}^2 \cdot \frac{\mathrm{e}^{2(\|\mathcal{R}\|_\infty - K)s} - \mathrm{e}^{-Ks}}{2\|\mathcal{R}\|_\infty - K},\end{aligned}$$

which together with (4.9) gives (6).

(c) (4) $\Rightarrow$ (1). For $v \in T_xM$ with $|v| = 1$, take $f \in C_0^\infty(M)$ such that

$$\nabla f(x) = v, \quad \mathrm{Hess}_f(x) = 0.$$

Then (4.11) holds. Moreover, since $\mathrm{Hess}_{P_tf}(x)$ is smooth in t, $\mathrm{Hess}_f(x) = 0$ implies

$$\|\mathrm{Hess}_{P_tf}(x)\| \le c(x)t, \quad t \in [0,1]$$

for some constant $c(x) > 0$. Combining this with (4.11), (5.5) and (4), we obtain

$$\begin{aligned} 0 &= \lim_{t\downarrow 0} \frac{P_t|\nabla f|^2(x) - \mathrm{e}^{Kt}|\nabla P_tf|^2(x)}{t} \\ &= \lim_{t\downarrow 0} \left(\mathcal{R}ic(v,v) + \frac{1-\mathrm{e}^{Kt}}{t} P_t|\nabla f|^2(x) \right) = \mathcal{R}ic(v,v) - K. \end{aligned}$$

Therefore, $\mathcal{R}ic(v,v) = K|v|^2$ holds for all $v \in TM$. By the symmetry of $\mathcal{R}ic$ and g, this is equivalent to $\mathcal{R}ic = K$.

(d) (7) $\Rightarrow$ (1). Let v and f be in **(c)**. In the spirit of (4.11) and using (4.9), we have

$$\begin{aligned} &2\lim_{t\downarrow 0} \frac{P_tf^2 - (P_tf)^2 - \frac{\mathrm{e}^{Kt}-1}{K}|\nabla P_tf|^2}{t^2} \\ &= \lim_{t\downarrow 0} \frac{2}{t^2} \int_0^t \big\{P_s|\nabla P_{t-s}f|^2 - \mathrm{e}^{Ks}|\nabla P_tf|^2\big\}(x)\mathrm{d}s = \mathcal{R}ic(v,v) - K, \end{aligned}$$

and

$$\begin{aligned} &2\lim_{t\downarrow 0} \frac{P_tf^2 - (P_tf)^2 - \frac{1-\mathrm{e}^{-Kt}}{K}P_t|\nabla f|^2}{t^2} \\ &= \lim_{t\downarrow 0} \frac{2}{t^2} \int_0^t \big\{P_s|\nabla P_{t-s}f|^2 - \mathrm{e}^{-Ks}P_t|\nabla f|^2\big\}(x)\mathrm{d}s = K - \mathcal{R}ic(v,v). \end{aligned}$$

Thus, multiplying the inequality in (7) by t^{-2} and letting $t \to 0$, we prove $\mathcal{R}ic(v,v) - K = 0$. That is, (1) holds.

(e) (8) $\Rightarrow$ (1). For any $v_1, v_2 \in T_xM$, take $f \in C_0^\infty(M)$ such that $\nabla f(x) = v_1$, $\mathrm{Hess}_f(x) = 0$. According to Theorem 5.1, (8) implies

$$(\nabla_{v_2}\mathcal{R}ic)(v_1, v_1) = 0.$$

So, M is Ricci parallel, and as shown in the proof of Theorem 4.1 that (6.7) holds. Taking $v_1 = v_2 = v$ for $v \in T_xM$ with $|v| = 1$, and letting $f \in C_0^\infty(M)$ such that $\mathrm{Hess}_f(x) = v \otimes v$, (6.7) implies

$$\frac{1}{2}\big\{\mathrm{Hess}_{\Delta f} - \Delta \mathrm{Hess}_f\big\}(v,v) = -\mathcal{R}ic(v,v) + \mathrm{tr}\langle \mathcal{R}(\cdot, v)v, \mathrm{Hess}_f^\#(\cdot)\rangle = -\mathcal{R}ic(v,v).$$

Combining this with (8) we obtain

$$-\mathcal{R}ic(v,v) = -K\mathrm{Hess}_f(v,v) = -K.$$

So, (1) holds.

$\square$

8 Identifications of Ricci Parallel manifolds

Theorem 8.1. *The following assertions are equivalent each other:*

(1) M *is a Ricci parallel manifold.*

(2) $\|\mathcal{R}\|_\infty < \infty$, *and* (4.1) *holds for any* $x \in M, t \geq 0,\ f \in C_0(M), v_1, v_2 \in T_xM$.

(3) $\|\mathcal{R}\|_\infty < \infty$, *and for any constant* $K \in \mathbb{R}$ *with* $\mathcal{R}ic \geq K$, $t \geq 0, f \in C_b^2(M)$,

$$\|\mathrm{Hess}_{P_tf} - P_t\mathrm{Hess}_f\| \leq \left(\frac{\|\mathcal{R}ic\|_\infty(1-\mathrm{e}^{-Kt})}{K} + \mathrm{e}^{(\|\mathcal{R}\|_\infty - K)t} - \mathrm{e}^{-Kt}\right)P_t\|\mathrm{Hess}_f\|. \tag{8.1}$$

(4) *There exists a function* $h : [0,\infty)\times M \to [0,\infty)$ *with* $\lim_{t\downarrow 0} t^{-\frac{1}{2}}h(t,\cdot) = 0$ *such that*

$$\|\mathrm{Hess}_{P_tf} - P_t\mathrm{Hess}_f\| \leq h(t,\cdot)\big(P_t\|\mathrm{Hess}_f\| + \|\mathrm{Hess}_{P_tf}\|\big), \quad t \geq 0, f \in C_0^\infty(M). \tag{8.2}$$

(5) *For any* $f \in C_0^\infty(M)$ *and* $x \in M$,

$$\big(\mathrm{Hess}_{\Delta f} - \Delta\mathrm{Hess}_f\big)(v_1,v_2) = 2\big(\mathcal{R}\mathrm{Hess}_f\big)(v_1,v_2) - 2\mathcal{R}ic(v_1, \mathrm{Hess}_f^{\#}(v_2)), \quad v_1, v_2 \in T_xM.$$

(6) *For any* $x \in M$ *and* $f \in C_0^\infty(M)$ *with* $\mathrm{Hess}_f(x) = 0$,

$$(\Delta\mathrm{Hess}_f)(v_1,v_2) = \mathrm{Hess}_{\Delta f}(v_1,v_2), \quad v_1, v_2 \in T_xM.$$

Proof. The equivalence of (1) and (6) follows from Theorem 5.1, (1) implying (2) is included in Theorem 4.1, (5) follows from (2) by taking derivative of (4.1) with respect to t at $t = 0$, and it is obvious that (3) implies (4) while (6) follows from (5). So, it remains to prove that (1) implies (3), and (4) implies (1).

(a) (1) implies (3). Let M be Ricci parallel with $\mathcal{R}ic \geq K$. By (1.2) we have

$$|W_s(v)| \leq \mathrm{e}^{-\frac{K}{2}s}|v|, \quad v \in T_xM, \tag{8.3}$$

and

$$\begin{aligned} &\mathrm{d}|W_t(v) - \|_t v|^2 = \mathrm{d}|\Phi_t^{-1}W_t(v) - \Phi_0^{-1}v|^2 \\ &= \langle W_t(v) - \|_t v, \mathcal{R}ic^{\#}(W_t(v))\rangle \mathrm{d}t \leq |W_t(v) - \|_t v| \cdot \|\mathcal{R}ic\|_\infty \mathrm{e}^{-\frac{K}{2}t}|v|. \end{aligned}$$

So,

$$|W_t(v) - v| \leq \frac{\|\mathcal{R}ic\|_\infty}{2}\int_0^t \mathrm{e}^{-\frac{K}{2}s}\mathrm{d}s = \frac{\|\mathcal{R}ic\|_\infty(1-\mathrm{e}^{-\frac{K}{2}t})}{K}, \quad |v| \leq 1.$$

Thus, for $|v_1|, |v_2| = 1$,

$$\begin{aligned} &\big|\mathrm{Hess}_f(W_t(v_1), W_t(v_2)) - \mathrm{Hess}_f(\|_t v_1, \|_t v_2)\big| \\ &\leq \|\mathrm{Hess}_f\|\big(|W_t(v_1)| \cdot |W_t(v_2) - \|_t v_2)| + |W_t(v_1) - \|_t v_1|\big) \\ &\leq \|\mathrm{Hess}_f\|\big(\mathrm{e}^{-\frac{K}{2}t} + 1\big)\big(1 - \mathrm{e}^{-\frac{K}{2}t}\big)\frac{\|\mathcal{R}ic\|_\infty}{K} = \|\mathrm{Hess}_f\|\frac{\|\mathcal{R}ic\|_\infty(1-\mathrm{e}^{-Kt})}{K}. \end{aligned} \tag{8.4}$$

Combining (4.1) with (3.6) for $\zeta = \infty$, (4.2), (8.3) and (8.4), we obtain

$$
\begin{aligned}
&\|\mathrm{Hess}_{P_tf} - P_t\mathrm{Hess}_f\| \\
&\le \sup_{|v_1|,|v_2|\le 1} \big|\mathrm{Hess}_{P_tf}(v_1,v_2) - \mathbb{E}\mathrm{Hess}_f(W_t(v_1),W_t(v_2))\big| + \frac{\|\mathcal{R}ic\|_\infty(1-\mathrm{e}^{-Kt})}{K}P_t\|\mathrm{Hess}_f\| \\
&\le \frac{\|\mathcal{R}ic\|_\infty(1-\mathrm{e}^{-Kt})}{K}P_t\|\mathrm{Hess}_f\| + \|\mathcal{R}\|_\infty \int_0^t \mathrm{e}^{-Ks}\,\mathbb{E}\|\mathrm{Hess}_{P_{t-s}f}\|(X_s)\mathrm{d}s \\
&\le \frac{\|\mathcal{R}ic\|_\infty(1-\mathrm{e}^{-Kt})}{K}P_t\|\mathrm{Hess}_f\| + \|\mathcal{R}\|_\infty \int_0^t \mathrm{e}^{-Ks}P_s\|\mathrm{Hess}_{P_{t-s}f}\|\mathrm{d}s \\
&\le P_t\|\mathrm{Hess}_f\|\bigg(\frac{\|\mathcal{R}ic\|_\infty(1-\mathrm{e}^{-Kt})}{K} + \|\mathcal{R}\|_\infty\int_0^t \mathrm{e}^{-Ks+(\|\mathcal{R}\|_\infty-K)(t-s)}\mathrm{d}s\bigg) \\
&= \Big(\frac{\|\mathcal{R}ic\|_\infty(1-\mathrm{e}^{-Kt})}{K} + \mathrm{e}^{(\|\mathcal{R}\|_\infty-K)t} - \mathrm{e}^{-Kt}\Big)P_t\|\mathrm{Hess}_f\|.
\end{aligned}
$$

So, (8.1) holds.

(b) (4) $\Rightarrow$ (1). For any $x \in M$ and $v_1, v_2 \in T_xM$, let $f \in C_0^\infty(M)$ such that $\nabla f(x) = v_1$ and $\mathrm{Hess}_f(x) = 0$. Since $\mathrm{Hess}_f(x) = 0$ and $\mathrm{Hess}_{P_tf}(x)$ is smooth in $t \ge 0$,

$$\|\mathrm{Hess}_{P_tf}\|(x) \le ct, \quad t \in [0,1]$$

holds for some constant $c > 0$. Combining this with Theorem 5.1, (8.2) and (5.5), we obtain

$$
\begin{aligned}
&\big|(\nabla_{v_2}\mathcal{R}ic)(v_1,v_1)\big| = \lim_{t\downarrow 0}\frac{|P_t\mathrm{Hess}_f - \mathrm{Hess}_{P_tf}|(v_1,v_2)}{t} \\
&\le \lim_{t\downarrow 0}\frac{h(t,x)}{t}\big(P_t\mathrm{Hess}_f\|(x) + \|\mathrm{Hess}_{P_tf}\|(x)\big) \le \lim_{t\downarrow 0}\frac{h(t,x)(c_1\sqrt{t}+ct)}{t} = 0.
\end{aligned}
$$

By the symmetry of $(\nabla_{v_2}\mathcal{R}ic)$, this implies $\nabla\mathcal{R}ic = 0$. Thus, (4) implies (1). □

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