A conformal Korn inequality on Hölder domains

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Abstract. In this paper, we derive a sharp conformal Korn inequality on Hölder domains, we also provide an example for the sharpness of the result. This shows that the conformal Korn inequality has a different behavior than the classical one on Hölder domains.

1 Introduction and main result

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Let Ω be a bounded, connected domain in $\mathbb{R}^n$, $n \geq 2$. For every vector filed $\mathbf{v} = (v_1, v_2, \dots, v_n) \in W^{1,p}(\Omega, \mathbb{R}^n)$, $1 < p < \infty$, let $D\mathbf{v}$ denote its *differential matrix*, $\epsilon(\mathbf{v})$ denote the *symmetric part* of $D\mathbf{v}$, namely, $\epsilon(\mathbf{v}) = (\epsilon_{i,j}(\mathbf{v}))_{1\leq i,j\leq n}$ with

$$\epsilon_{i,j}(\mathbf{v}) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right),$$

and $\kappa(\mathbf{v}) = (\kappa_{i,j}(\mathbf{v}))_{1\leq i,j\leq n}$ denote the *anti-symmetric part* of $D\mathbf{v}$ as $(D\mathbf{v} - D\mathbf{v}^T)/2$.

The Korn inequality states that, if $\kappa(\mathbf{v})$ has vanishing integral on Ω, then it holds

$$\|D\mathbf{v}\|_{L^p(\Omega)} \leq C_K \|\epsilon(\mathbf{v})\|_{L^p(\Omega)}.$$

This is equivalent to that, for any $\mathbf{v} \in W^{1,p}(\Omega, \mathbb{R}^n)$,

$$(K_p) \qquad \inf_{S=-S^T}\left(\int_\Omega |D\mathbf{v} - S|^p \mathrm{d}x\right)^{1/p} \leq C\left(\int_\Omega |\epsilon(\mathbf{v})|^p \mathrm{d}x\right)^{1/p}.$$

The Korn inequality (K_p) is a fundamental tool in the theory of elasticity and fluid mechanics; we refer the reader to [1, 3, 6, 7, 8, 10, 13, 19] and the references therein.

The validity of Korn's inequality is closely related to the geometry of the domains under consideration. Let us review some earlier results. Friedrichs [8] studied the Korn inequality (K_2) on domains with a finite number of corners or edges on $\partial\Omega$, Nitsche [16] proved the Korn inequality (K_2) on Lipschitz domains, while Kondratiev and Oleinik [13] considered the Korn inequality (K_p) on star-shaped domains. Acosta, Durán and Muschietti [3] established Korn's inequality on John domains, while later the John condition was proved in [11] to be essentially necessary.

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There is a stronger version of the Korn inequality, which states

$$\inf_{\ell(\mathbf{w})=0}\left(\int_\Omega |D\mathbf{v}-D\mathbf{w}|^p \mathrm{d}x\right)^{1/p} \le C\left(\int_\Omega |\ell(\mathbf{v})|^p \mathrm{d}x\right)^{1/p},$$

where

$$\ell(\mathbf{v}) := \epsilon(\mathbf{v}) - \frac{\mathrm{div}(\mathbf{v})}{n} I_{n\times n}.$$

The term $\ell(\mathbf{v})$ is called as *trace-free part*, also known as anti-conformal part (cf. [17]). In what follows we shall call this version as conformal Korn inequality.

Reshetnyak [18] first established this version on star-sharped domains in $\mathbb{R}^n$, $n \ge 3$, and Dain [5] established this version (slightly different) on Lipschitz domains in $\mathbb{R}^n$, $n \ge 3$. Dain [5] showed the conformal Korn inequality does not hold in $\mathbb{R}^2$, and pointed out the conformal Korn inequality has applications in general relativity. A historical survey about the trace-free Korn inequality can be found in [4].

In a recent interesting work, López-García [15] derived the conformal Korn inequality together with its weighted versions on John domains. To be precise, the following result was proved in [15].

Theorem A. ([15, Theorem 1.1]) *Let $1 < p < \infty$ and $0 \le \beta < \infty$. Suppose that Ω is a bounded John domain in $\mathbb{R}^n$ with $n \ge 3$. Then there exists a positive constant C such that, for any vector field $\mathbf{v} \in W^{1,p}(\Omega, \mathbb{R}^n, \rho^{\beta p})$,*

$$\inf_{\ell(\mathbf{w})=0}\left(\int_\Omega |D\mathbf{v}-D\mathbf{w}|^p \rho^{\beta p}\mathrm{d}x\right)^{1/p} \le C\left(\int_\Omega |\ell(\mathbf{v})|^p \rho^{\beta p}\mathrm{d}x\right)^{1/p},$$

where $\rho(x)$ is the distance from x to the boundary of Ω, namely, $\rho(x) := \mathrm{dist}(x, \partial\Omega)$.

Motivated by these studies, and also previous study of Korn's inequality on irregular domains (cf. [1, 2, 6, 12, 14]), in this paper, we study the conformal Korn inequality on Hölder domains. We shall prove that

thm2 **Theorem 1.1.** *Let $1 < p < \infty$, $0 < \alpha \le 1$, $0 \le \beta < \infty$ and $a \le \beta + 2(\alpha - 1)$. Suppose $\Omega \subset \mathbb{R}^n$ with $n \ge 3$ is a Hölder α domain.*

(i) *Then there exists a positive constant C such that, for any vector field $\mathbf{v} \in W^{1,p}(\Omega, \mathbb{R}^n, \rho^{\beta p})$,*

$$\inf_{\ell(\mathbf{w})=0}\left(\int_\Omega |D\mathbf{v}-D\mathbf{w}|^p \rho^{\beta p}\mathrm{d}x\right)^{1/p} \le C\left(\int_\Omega |\ell(\mathbf{v})|^p \rho^{a p}\mathrm{d}x\right)^{1/p}.$$

(ii) *If $0 < \alpha < 1$, there exists a Hölder α domain in $\mathbb{R}^n$ with $n \ge 3$ such that the above inequality does not hold provided $a > \beta + 2(\alpha - 1)$.*

An interesting point here is that, while on the RHS of the above inequality the sharp power of the weight ρ is $a = \beta + 2(\alpha - 1)$, in the classical Korn inequality the sharp power is $\beta + (\alpha - 1)$ (cf. [2]). In particular, for $0 < \alpha < 1$, the sharp power differs a factor of $\alpha - 1$. For the proof, we shall make use of recent developments from [15] and examples from [2] together with some new observations regarding the trace-free part.

This paper is organized as follows. In section 2, we recall some basic notions and notation used in this paper, including weighted Lebesgue spaces, weighted Sobolev spaces and Hölder domains. Section 3 is devoted to the proof of Theorem 1.1 (i). In last section, we remark that the range of a in Theorem 1.1 (i) is optimal via a counterexample.

Finally, we make some conventions on notation. For any set $E \subset \mathbb{R}^n$, we use $|E|$ to denote the *n-dimensional Lebesgue measure* of E and χ_E its the *characteristic function*. The symbol $\delta_{\alpha,\beta}$ denotes the *Kronecker delta*, namely, $\delta_{\alpha,\beta} = 1$ if $\alpha = \beta$ and $\delta_{\alpha,\beta} = 0$ otherwise. The letter C will denote a *positive constant* that may vary from line to line but will remain independent of the main variables. We also use $C_{(\alpha,\beta,\dots,\gamma)}$ to denote a positive constant depending on the indicated parameters $\alpha, \beta, \dots, \gamma$. For any index $1 \le p \le \infty$, q denotes the *conjugate index* of p, namely, $1/p + 1/q = 1$.

2 Notions

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In this section, we recall some basic notions and notation with respect to Lebesgue spaces, Sobolev spaces and Hölder domains.

For any $1 \le p < \infty$ and $0 \le \beta < \infty$, the *weighted Lebesgue space* $L^p(\Omega, \mathbb{R}, \rho^{\beta p})$ consists of those measurable functions $f : \Omega \to \mathbb{R}$ with finite norm given by

$$\|f\|_{L^p(\Omega,\mathbb{R},\rho^{\beta p})} := \left(\int_\Omega |f|^p \rho^{\beta p} \mathrm{d}x \right)^{1/p}.$$

Analogously, the *weighted Sobolev space* $W^{1,p}(\Omega, \mathbb{R}, \rho^{\beta p})$ is defined to be the set of all weakly differentiable functions $f : \Omega \to \mathbb{R}$ with finite norm given by

$$\|f\|_{W^{1,p}(\Omega,\mathbb{R},\rho^{\beta p})} := \left(\int_\Omega |f|^p \rho^{\beta p} \mathrm{d}x + \sum_{i=1}^n \int_\Omega \left| \frac{\partial f}{\partial x_i} \right|^p \rho^{\beta p} \mathrm{d}x \right)^{1/p}.$$

On the other hand, in order to define the spaces $L^p(\Omega, \mathbb{R}^{n\times n}, \rho^{\beta p})$ and $W^{1,p}(\Omega, \mathbb{R}^n, \rho^{\beta p})$, we need the following notion concerning $\|\cdot\|_r$ with $1 \le r \le \infty$.

For any vector $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ and $1 \le r \le \infty$, the *function* $\|\tilde{f}\|_r : \mathbb{R}^n \to \mathbb{R}$ is defined by setting, for any $x \in \mathbb{R}^n$,

$$\left\|\tilde{f}\right\|_r(x) := \begin{cases} \left(\displaystyle\sum_{1\le i,j\le n} \left|\tilde{f}_{i,j}(x)\right|^r \right)^{1/r}, & 1 \le r < \infty; \\ \displaystyle\max_{1\le i,j\le n} \left|\tilde{f}_{i,j}(x)\right|, & r = \infty. \end{cases}$$

It is easy to see that, for any $r_1, r_2 \in [1, \infty]$, there exists a positive constant $C := C_{(r_1,r_2)}$ such that, for any vector $\tilde{f}$,

norm (2.1)
$$\frac{1}{C}\left\|\tilde{f}\right\|_{r_1} \le \left\|\tilde{f}\right\|_{r_2} \le C\left\|\tilde{f}\right\|_{r_1}. \tag{2.1}$$

For any $1 \le p < \infty$ and $0 \le \beta < \infty$, the *space* $L^p(\Omega, \mathbb{R}^{n\times n}, \rho^{\beta p})$ consists of those measurable vectors $\tilde{f} : \Omega \to \mathbb{R}^{n\times n}$ with finite norm given by

$$\left\|\tilde{f}\right\|_{L^p(\Omega,\mathbb{R}^{n\times n},\rho^{\beta p})} := \left(\int_\Omega \left\|\tilde{f}\right\|_p^p \rho^{\beta p}\mathrm{d}x\right)^{1/p}.$$

Analogously, the *space* $W^{1,p}(\Omega, \mathbb{R}^n, \rho^{\beta p})$ is defined to be the set of all weakly differentiable vector fields $\mathbf{v} : \Omega \to \mathbb{R}^n$ with finite norm given by

$$\|\mathbf{v}\|_{W^{1,p}(\Omega,\mathbb{R}^n,\rho^{\beta p})} := \left(\sum_{i=1}^n \int_\Omega |v_i|^p \rho^{\beta p}\mathrm{d}x + \int_\Omega \|D\mathbf{v}\|_p^p \rho^{\beta p}\mathrm{d}x\right)^{1/p}.$$

We end this section with the definition concerning Hölder α domain. A domain $\Omega \subset \mathbb{R}^n$ is called a *Hölder α domain*, $0 < \alpha \le 1$, if the boundary of Ω is locally the graph of a Hölder α function in an appropriate co-ordinate system. Here a function φ is called a Hölder α function if there exists a positive constant K such that $|\varphi(x) - \varphi(y)| \le K|x - y|^\alpha$ for any x, y.

3 Proof of Theorem 1.1

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To show Theorem 1.1, we shall follow the strategy from [15], where the conformal Korn inequality was proved on John domains. As Hölder domains may have exterior cusps, we need some necessary new estimates. Let us begin with some notions and auxiliary lemmas.

The kernel of the operator ℓ, denoted by $\Sigma := \{\mathbf{v} : \mathbb{R}^n \to \mathbb{R}^n \mid \ell(\mathbf{v}) = 0\}$, has a finite dimension equal to $(n + 1)(n + 2)/2$ and a vector field $\mathbf{v} \in \Sigma$ if and only if

$$\mathbf{v}(x) = a + A(x - y) + \lambda(x - y) + \left\{\langle b, x - y\rangle(x - y) - \frac{|x - y|^2 b}{2}\right\},$$

where A is a skew-symmetric matrix, $a, b \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. The vector $y \in \mathbb{R}^n$ is arbitrary but must be fixed to have uniqueness for this representation.

Next define the *space* $\mathcal{V}$ whose elements are the differential matrix of the vector fields in Σ, that is to say,

$$\tilde{\varphi} \in \mathcal{V} \Leftrightarrow \tilde{\varphi}(x) = A + \lambda I + \sum_{i=1}^n b_i H_i(x - y),$$

where, for any $1 \le i \le n$, the vector $H_i = \{(H_i)_{j,k}\}_{n\times n} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is defined by setting, for any $(z_1, z_2, \ldots, z_n) \in \mathbb{R}^n$,

H (3.1)
$$(H_i)_{j,k}(z_1, z_2, \ldots, z_n) := z_i\delta_{j,k} + z_j\delta_{k,i} - z_k\delta_{i,j}. \tag{3.1}$$

For example, when $n = 3$, we have

$$H_1(z) = \begin{pmatrix} z_1 & -z_2 & -z_3 \\ z_2 & z_1 & 0 \\ z_3 & 0 & z_1 \end{pmatrix}, \quad H_2(z) = \begin{pmatrix} z_2 & z_1 & 0 \\ -z_1 & z_2 & -z_3 \\ 0 & z_3 & z_2 \end{pmatrix}, \quad H_3(z) = \begin{pmatrix} z_3 & 0 & z_1 \\ 0 & z_3 & z_2 \\ -z_1 & -z_2 & z_3 \end{pmatrix}.$$

Observe that $\mathcal{V}$ is a function space with finite dimension. Thus we denote by $h := n(n-1)/2+1+n$ the dimension of $\mathcal{V}$.

In addition, for any $1 \le q < \infty$ and $0 \le \beta < \infty$, the *subspace* $\mathcal{W}$ of $L^q(\Omega, \mathbb{R}^{n\times n}, \rho^{-\beta q})$ is defined by

$$\mathcal{W} := \{\tilde{g} \in L^q(\Omega, \mathbb{R}^{n\times n}, \rho^{-\beta q}) \mid \int_\Omega (\tilde{g} : \tilde{\varphi}) \mathrm{d}x = 0 \text{ for any } \tilde{\varphi} \in \mathcal{V}\},$$

where $\tilde{g} : \tilde{\varphi}$ is the *product coordinate by coordinate* of $\tilde{g}$ and $\tilde{\varphi}$ as follows

$$\tilde{g}(x) : \tilde{\varphi}(x) := \sum_{1\le i,j\le n} \tilde{g}_{i,j}(x)\tilde{\varphi}_{i,j}(x).$$

For any $1 \le p < \infty$ and $0 \le \beta < \infty$, it is obvious that $\rho^\beta \in L^p(\Omega, \mathbb{R})$ implies $L^q(\Omega, \mathbb{R}^{n\times n}, \rho^{-\beta q}) \subset L^1(\Omega, \mathbb{R}^{n\times n})$. Moreover, since Ω is bounded, it follows that $\mathcal{V} \subset L^1(\Omega, \mathbb{R}^{n\times n})$. Hence, $\mathcal{W}$ is well-defined.

Using the following Whitney decomposition, we will define a subspace of $\mathcal{W}$.

W **Lemma 3.1.** ([9, Appendix J]) *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Then there exists a collection $\{Q_t\}_{t\in\Gamma}$ of closed dyadic cubes whose interiors are pairwise disjoint such that*

(i) $\Omega = \bigcup_{t\in\Gamma} Q_t$;

(ii) $\mathrm{diam}(Q_t) \le \mathrm{dist}(Q_t, \partial\Omega) \le 4\mathrm{diam}(Q_t)$;

(iii) $\mathrm{diam}(Q_s)/4 \le \mathrm{diam}(Q_t) \le 4\mathrm{diam}(Q_s)$, *if* $Q_s \bigcap Q_t \ne \emptyset$.

Given a Whitney decomposition $\{Q_t\}_{t\in\Gamma}$ of Ω, we refer by an extended Whitney decomposition of Ω to the collection of n-dimensional rectangles $\{\Omega_t\}_{t\in\Gamma}$ defined by

rec (3.2) $$\Omega_t := Q_t' \times (y_{t,1} - l_t/2, y_{t,2}),$$

where $Q_t = Q_t' \times (y_{t,1}, y_{t,2})$ and l_t is the common length of the sides of Q_t. Obviously, $y_{t,2} = y_{t,1} + l_t$. In what follows we shall call this decomposition as the Whitney R-decomposition. Observe that this collection of n-dimensional rectangles satisfies the following finite overlapped property

$$\chi_\Omega \le \sum_{t\in\Gamma} \chi_{\Omega_t} \le 12^n \chi_\Omega.$$

From Lemma 3.1 (ii), we deduce that there exists a positive constant C such that, for any $x \in \Omega_t$,

equal (3.3) $$\frac{1}{C}\mathrm{diam}(\Omega_t) \le \rho(x) \le C\mathrm{diam}(\Omega_t).$$

Given a Whitney R-decomposition $\{\Omega_t\}_{t\in\Gamma}$ of Ω, the *subspace* $\mathcal{W}_0$ of $\mathcal{W}$ is defined by

$$\mathcal{W}_0 := \{\tilde{g} \in \mathcal{W} \mid \operatorname{supp}(\tilde{g}) \bigcap \Omega_t \neq \emptyset \text{ only for a finite number of } t \in \Gamma\}.$$

lma2 **Lemma 3.2.** *The subspace* $\mathcal{W}_0 \oplus \rho^{\beta p}\mathcal{V}$ *is dense in* $L^q(\Omega, \mathbb{R}^{n\times n}, \rho^{-\beta q})$.

Proof. The proof of this lemma is same as that of [15, Lemma 3.5]. Note that although [15, Lemma 3.5] was stated and proved for the cubes, its proof also works with our choice of n-dimensional rectangles. □

Below, we will use some notions concerning graph theory to define a Hardy-type operator; see [15] for more details. A *rooted tree* is a connected graph in which any two vertices are connected by exactly one simple path, and a *root* is simply a distinguished vertex $a \in \Gamma$. For this kind of graph, it is possible to define a *partial order* "$\preceq$" in all vertices as follows: $s \preceq t$ if and only if the unique path connecting t with the root a passes through s. Moreover, the *height* or *level* of any $t \in \Gamma$ is the number of vertices in $\{s \in V \mid s \preceq t \text{ and } s \neq t\}$. The *parent* of a vertex $t \in \Gamma$ is the vertex s satisfying that $s \preceq t$ and its height is one unit smaller than the height of t. We denote by t_p the parent of t. It can be seen that each $t \in \Gamma$ different from the root has a unique parent, but several elements in Γ could have the same parent.

def1 **Definition 3.3.** Given a Whitney R-decomposition $\{\Omega_t\}_{t\in\Gamma}$ of a domain $\Omega \subset \mathbb{R}^n$, where Γ is a tree with root a, we take a collection of open pairwise disjoint n-dimensional rectangles $\{B_t\}_{t\neq a}$ with sides parallel to the axis such that $B_t \subseteq \Omega_t \bigcap \Omega_{t_p}$ and $|\Omega_t| \leq C_{(n)}|B_t|$ for any $t \in \Gamma$. Next the *Hardy-type operator* $\mathcal{H}$ for all functions g in $L^1(\Omega)$ is defined by setting, for any $x \in \Omega$,

$$\mathcal{H}(g)(x) := \sum_{a\neq t\in\Gamma} \frac{\chi_{B_t}(x)}{|W_t|} \int_{W_t} |g| \mathrm{d}y,$$

where $W_t := \bigcup_{s\succeq t} \Omega_s$.

Remark 3.4. We claim that t and s are adjacent vertices implies $\Omega_t \cap \Omega_s \neq \varnothing$ in Definition 3.3. In fact, if t and s are adjacent vertices, then s is the parent of t (without loss of generality). It follows that there exists a B_t such that $B_t \subseteq \Omega_t \cap \Omega_s$ and hence $\Omega_t \cap \Omega_s \neq \varnothing$.

lma6 **Lemma 3.5.** *Let* $\{\Omega_t\}_{t\in\Gamma}$ *be a Whitney R-decomposition of* Ω. *Then, for any* $\tilde{g} \in \mathcal{W}_0$, *there exists a collection of vectors* $\{\tilde{g}_t\}_{t\in\Gamma}$ *with the following properties*:

(i) $\tilde{g} = \sum_{t\in\Gamma} \tilde{g}_t$;

(ii) $\operatorname{supp}(\tilde{g}_t) \subset \Omega_t$ *for any* $t \in \Gamma$;

(iii) $\tilde{g}_t \in \mathcal{W}_0$ *for any* $t \in \Gamma$.

Moreover, for every $1 < q < \infty$, *there exists a positive constant C independent of* $\tilde{g}$ *such that, for any* $x \in \Omega_t$,

norm' (3.4)
$$\|\tilde{g}_t\|_q(x) \leq C\left(\|\tilde{g}\|_q(x) + \frac{\operatorname{diam}(W_t)}{\operatorname{diam}(\Omega_t)} \frac{|W_t|}{|\Omega_t|} \mathcal{H}(\|\tilde{g}\|_1)(x)\right).$$

Proof. The proof of this lemma is same as that of [15, Lemma 4.5]. Note that although [15, Lemma 4.5] was stated and proved for the cubes, its proof also works with our choice of n-dimensional rectangles. □

Before presenting the following lemma, we need to introduce a special Hölder α domain Ω_φ from [14]. For a positive Hölder α function $\varphi : (-3l/2, 3l/2)^{n-1} \to \mathbb{R}$ with $0 < \alpha \leq 1$ and $l > 0$, the domain $\Omega_\varphi \subset \mathbb{R}^n$ is defined by the graph of function φ as follows

0 (3.5)
$$\Omega_\varphi := \left\{(x, y) \in (-l/2, l/2)^{n-1} \times \mathbb{R} \mid 0 < y < \varphi(x)\right\}. \tag{3.5}$$

Moreover, we may assume that $\varphi \geq 2l$ but $\varphi \ngeq 3l$, and $0 < l \leq 1$. Thus Hölder α domain Ω is locally as Ω_φ. Nevertheless, the distance to the boundary of Ω is not necessarily equivalent to the distance to the graph of φ defined over $(-l/2, l/2)^{n-1}$. Thus, in order to solve this problem, we assume that Ω is locally an expanded version of Ω_φ as follows

$$\Omega_{\varphi,E} := \left\{(x, y) \in (-3l/2, 3l/2)^{n-1} \times \mathbb{R} \mid y < \varphi(x)\right\}.$$

With this new approach of the problem, the distance to $\partial\Omega$ is equivalent to G over Ω_φ where

$$G := \left\{(x, y) \in (-3l/2, 3l/2)^{n-1} \times \mathbb{R} \mid y = \varphi(x)\right\}$$

We use ρ_G to denote the distance to G.

lma7 **Lemma 3.6.** *Let $1 < q < \infty$, $0 \leq \beta < \infty$ and $0 < \alpha \leq 1$. There exists a Whitney R-decomposition $\{\Omega_t\}_{t\in\Gamma}$ of Ω_φ with following properties*:

(i) *the Hardy-type operator $\mathcal{H}$ is bounded on $L^q(\Omega_\varphi, \mathbb{R}, \rho_G^{-\beta q})$*;

(ii) *there exist some positive constants C_1, C_2 and C_3 such that*

$$\frac{|W_t|}{|\Omega_t|} \leq C_1[\operatorname{diam}(\Omega_t)]^{\alpha-1} \text{ and } \frac{\operatorname{diam}(W_t)}{\operatorname{diam}(\Omega_t)} \leq C_2 + C_3[\operatorname{diam}(\Omega_t)]^{\alpha-1}.$$

Proof. The proof of this lemma can be obtained from the proof of [14, Lemma 6.1]. □

The following lemma shows a decomposition of $\tilde{g} \in \mathcal{W}_0$ based on the Whitney R-decomposition $\{\Omega_t\}_{t\in\Gamma}$ of Ω_φ.

lma5 **Lemma 3.7.** *Let $\{\Omega_t\}_{t\in\Gamma}$ be a Whitney R-decomposition of Ω_φ. Then, for any $\tilde{g} \in \mathcal{W}_0$, there exists a collection of vectors $\{\tilde{g}_t\}_{t\in\Gamma}$ with the following properties*:

(i) $\tilde{g} = \sum_{t\in\Gamma} \tilde{g}_t$;

(ii) $\operatorname{supp}(\tilde{g}_t) \subset \Omega_t$ *for any* $t \in \Gamma$;

(iii) $\tilde{g}_t \in \mathcal{W}_0$ *for any* $t \in \Gamma$.

Moreover, for every $1 < q < \infty$*, there exists a positive constant* C *independent of* $\tilde{g}$ *such that*

N2
$$\|\tilde{g}_t\|_{L^q(\Omega_t,\mathbb{R}^{n\times n},\rho^{-\beta q})} \le C\left(1 + [\mathrm{diam}(\Omega_t)]^{\alpha-1} + [\mathrm{diam}(\Omega_t)]^{2(\alpha-1)}\right)\|\tilde{g}\|_{L^q(\Omega_t,\mathbb{R}^{n\times n},\rho^{-\beta q})}. \tag{3.6}$$

Proof. To show this lemma, by (i) (ii) and (iii) of Lemma 3.5, it suffices to prove (3.6). One may use (3.4), Minkowski's inequality, Lemma 3.6 and (2.1) to deduce that

$$\begin{aligned}
\|\tilde{g}_t\|_{L^q(\Omega_t,\mathbb{R}^{n\times n},\rho^{-\beta q})} &= \left(\int_{\Omega_t}\|\tilde{g}_t\|_q^q\rho^{-\beta q}\mathrm{d}x\right)^{1/q}\\
&\le C\left[\int_{\Omega_t}\left(\|\tilde{g}\|_q + \frac{\mathrm{diam}(W_t)}{\mathrm{diam}(\Omega_t)}\frac{|W_t|}{|\Omega_t|}\mathcal{H}(\|\tilde{g}\|_1)\right)^q\rho^{-\beta q}\mathrm{d}x\right]^{1/q}\\
&\le C\left(\|\tilde{g}\|_{L^q(\Omega_t,\mathbb{R}^{n\times n},\rho^{-\beta q})} + \frac{\mathrm{diam}(W_t)}{\mathrm{diam}(\Omega_t)}\frac{|W_t|}{|\Omega_t|}\|\mathcal{H}(\|\tilde{g}\|_1)\|_{L^q(\Omega_t,\mathbb{R},\rho^{-\beta q})}\right)\\
&\le C\left(1 + \frac{\mathrm{diam}(W_t)}{\mathrm{diam}(\Omega_t)}\frac{|W_t|}{|\Omega_t|}\right)\|\tilde{g}\|_{L^q(\Omega_t,\mathbb{R}^{n\times n},\rho^{-\beta q})}\\
&\le C\left(1 + [\mathrm{diam}(\Omega_t)]^{\alpha-1} + [\mathrm{diam}(\Omega_t)]^{2(\alpha-1)}\right)\|\tilde{g}\|_{L^q(\Omega_t,\mathbb{R}^{n\times n},\rho^{-\beta q})},
\end{aligned}$$

which is desired. The proof is completed. □

lma4 **Lemma 3.8.** *Let* $1 < p < \infty$*. Suppose* $\Omega \subset \mathbb{R}^n$ *is a n-dimensional rectangle as in* (3.2) *with sides parallel to the axis. Then there exists a positive constant* C *independent of* Ω *such that, for any vector field* $\mathbf{u} \in W^{1,p}(\Omega,\mathbb{R}^n)$*,*

$$\inf_{\ell(\mathbf{w})=0}\left(\int_\Omega |D\mathbf{u} - D\mathbf{w}|^p\mathrm{d}x\right)^{1/p} \le C\left(\int_\Omega |\ell(\mathbf{u})|^p\mathrm{d}x\right)^{1/p}.$$

Proof. Let Ω_0 be the n-dimensional rectangle $(0,1)^{n-1}\times(-1/2,1)$. Hence, any other n-dimensional rectangle as in (3.2) with sides parallel to the axis can be obtained by a translation and dilation of Ω_0. Thus, using the same argument as in the proof of [15, Corollary 3.1], we can easily carry out the proof of this lemma. The proof will be omitted. □

lma1 **Lemma 3.9.** ([15, Lemma 3.2]) *For any* $1 \le q < \infty$ *and* $0 \le \beta < \infty$*, the space* $L^q(\Omega,\mathbb{R}^{n\times n},\rho^{-\beta q})$ *can be written as* $\mathcal{W} \oplus \rho^{\beta p}\mathcal{V}$*. Moreover, for any* $\tilde{g} + \rho^{\beta p}\varphi \in \mathcal{W} \oplus \rho^{\beta p}\mathcal{V}$*, it follows that*

$$\|\tilde{g}\|_{L^q(\Omega,\mathbb{R}^{n\times n},\rho^{-\beta q})} \le C_1\left\|\tilde{g} + \rho^{\beta p}\varphi\right\|_{L^q(\Omega,\mathbb{R}^{n\times n},\rho^{-\beta q})},$$

where

$$C_1 = 1 + \sum_{i=1}^h \left\|\tilde{\psi}_i\right\|_{L^p(\Omega,\mathbb{R}^{n\times n},\rho^{\beta p})}\left\|\tilde{\psi}_i\right\|_{L^q(\Omega,\mathbb{R}^{n\times n},\rho^{\beta q})}.$$

The collection $\{\tilde{\psi}_i\}_{1\le i\le h}$ *in the previous identity is an arbitrary orthonormal basis of* $\mathcal{V}$ *with respect to the inner product*

product
$$\langle\tilde{\psi},\tilde{\varphi}\rangle_\Omega = \int_\Omega(\tilde{\psi} : \tilde{\varphi})\rho^{\beta p}\mathrm{d}x. \tag{3.7}$$

We end this section with the proof of Theorem 1.1 (i).

Proof of Theorem 1.1 (i). **STEP 1.** We first prove Theorem 1.1 under the case Ω_φ. The argument presented in this step partly follows López-García [15, Theorem 1.1]. We claim that there exists a positive constant C such that, for any vector field $\mathbf{u} \in W^{1,p}(\Omega_\varphi, \mathbb{R}^n, \rho_G^{\beta p})$ satisfying $\langle D\mathbf{u}, \mathcal{V}\rangle_{\Omega_\varphi} = 0$,

$$\left(\int_{\Omega_\varphi} |D\mathbf{u}|^p \rho_G^{\beta p} \mathrm{d}x\right)^{1/p} \le C\left(\int_{\Omega_\varphi} |\ell(\mathbf{u})|^p \rho_G^{ap} \mathrm{d}x\right)^{1/p}.$$

Assuming that this claim holds for the moment, we give the proof of this theorem (i).

First, let $\mathbf{v}$ be an arbitrary vector field in $W^{1,p}(\Omega_\varphi, \mathbb{R}^n, \rho_G^{\beta p})$ and $\{\tilde{\psi}_i\}_{1\le i\le h}$ an orthonormal basis of $\mathcal{V}$ with respect to the inner product (3.7). Next, take $\mathbf{w} \in \Sigma$ such that $D\mathbf{w} = \sum_{i=1}^h \alpha_i \tilde{\psi}_i$, where $\alpha_i = \int_{\Omega_\varphi} (D\mathbf{v} : \tilde{\psi}_i)\rho_G^{\beta p} \mathrm{d}x$. Then it follows that, for any $\tilde{\varphi} \in \mathcal{V}$,

$$\int_{\Omega_\varphi} (D\mathbf{v} : \tilde{\varphi})\rho_G^{\beta p} \mathrm{d}x = \int_{\Omega_\varphi} (D\mathbf{w} : \tilde{\varphi})\rho_G^{\beta p} \mathrm{d}x.$$

Noticing that $0 \le \beta < \infty$ and the fact that Ω_φ is bounded, $\mathbf{w}$ also belongs to $W^{1,p}(\Omega_\varphi, \mathbb{R}^n, \rho_G^{\beta p})$. Finally, by taking $\mathbf{u} := \mathbf{v} - \mathbf{w}$, we obtain that

$$\inf_{\ell(\mathbf{w})=0} \left(\int_{\Omega_\varphi} |D\mathbf{v} - D\mathbf{w}|^p \rho_G^{\beta p} \mathrm{d}x\right)^{1/p} \le \left(\int_{\Omega_\varphi} |D\mathbf{u}|^p \rho_G^{\beta p} \mathrm{d}x\right)^{1/p} \le C\left(\int_{\Omega_\varphi} |\ell(\mathbf{v})|^p \rho_G^{ap} \mathrm{d}x\right)^{1/p}.$$

It remains to prove the claim. Applying the fact that $L^q(\Omega_\varphi, \mathbb{R}^{n\times n}, \rho_G^{-\beta q})$ is the dual space of $L^p(\Omega_\varphi, \mathbb{R}^{n\times n}, \rho_G^{\beta p})$ and Lemma 3.2, the claim will be proved by showing that there exists a positive constant C independent of $\mathbf{u}$ such that, for any $\tilde{g} + \rho_G^{\beta p}\tilde{\psi} \in \mathcal{W}_0 \oplus \rho_G^{\beta p}\mathcal{V}$ satisfying $\|\tilde{g} + \rho_G^{\beta p}\tilde{\psi}\|_{L^q(\Omega_\varphi, \mathbb{R}^{n\times n}, \rho_G^{-\beta q})} \le 1$,

q2 (3.8)
$$\left|\int_{\Omega_\varphi} \left[D\mathbf{u} : \left(\tilde{g} + \rho_G^{\beta p}\tilde{\psi}\right)\right] \mathrm{d}x\right| \le C\left(\int_{\Omega_\varphi} |\ell(\mathbf{u})|^p \rho_G^{ap} \mathrm{d}x\right)^{1/p}.$$

For the $\tilde{g} \in \mathcal{W}_0$ mentioned above, we can use Lemma 3.7 to obtain

$$\left|\int_{\Omega_\varphi} \left[D\mathbf{u} : \left(\tilde{g} + \rho_G^{\beta p}\tilde{\psi}\right)\right] \mathrm{d}x\right| = \left|\int_{\Omega_\varphi} (D\mathbf{u} : \tilde{g})\mathrm{d}x\right| = \left|\int_{\Omega_\varphi} \left(D\mathbf{u} : \sum_{t\in\Gamma} \tilde{g}_t\right) \mathrm{d}x\right| \le \sum_{t\in\Gamma} \left|\int_{\Omega_t} (D\mathbf{u} : \tilde{g}_t)\, \mathrm{d}x\right|,$$

where the first "=" is due to $\langle D\mathbf{u}, \mathcal{V}\rangle_{\Omega_\varphi} = 0$ and the last "≤" is due to the finiteness of the sum "$\sum_{t\in\Gamma}$" stated in Lemma 3.7. Noticing that $\int_{\Omega_t} (D\mathbf{w} : \tilde{g}_t)\mathrm{d}x = 0$ for any $\mathbf{w} \in \Sigma$, from the above inequality, Hölder inequality, (3.3), Lemma 3.8 and (3.6), it follows that

$$\left|\int_{\Omega_\varphi} \left[D\mathbf{u} : \left(\tilde{g} + \rho_G^{\beta p}\tilde{\psi}\right)\right] \mathrm{d}x\right| \le \sum_{t\in\Gamma} \|\tilde{g}_t\|_{L^q(\Omega_t, \mathbb{R}^{n\times n}, \rho_G^{-\beta q})} \inf_{\ell(\mathbf{w})=0} \left(\int_{\Omega_t} |D\mathbf{u} - D\mathbf{w}|^p \rho_G^{\beta p} \mathrm{d}x\right)^{1/p}$$

$$
\begin{aligned}
&\leq C\sum_{t\in\Gamma}[\mathrm{diam}(\Omega_t)]^{\beta}\|\tilde{g}_t\|_{L^q(\Omega_t,\mathbb{R}^{n\times n},\rho_G^{-\beta q})}\inf_{\ell(\mathbf{w})=0}\left(\int_{\Omega_t}|D\mathbf{u}-D\mathbf{w}|^p\mathrm{d}x\right)^{1/p}\\
&\leq C\sum_{t\in\Gamma}[\mathrm{diam}(\Omega_t)]^{\beta}\|\tilde{g}_t\|_{L^q(\Omega_t,\mathbb{R}^{n\times n},\rho_G^{-\beta q})}\left(\int_{\Omega_t}|\ell(\mathbf{u})|^p\mathrm{d}x\right)^{1/p}\\
&\leq C\sum_{t\in\Gamma}[\mathrm{diam}(\Omega_t)]^{\beta+2(\alpha-1)}\|\tilde{g}\|_{L^q(\Omega_t,\mathbb{R}^{n\times n},\rho_G^{-\beta q})}\left(\int_{\Omega_t}|\ell(\mathbf{u})|^p\mathrm{d}x\right)^{1/p}\\
&\quad+C\sum_{t\in\Gamma}[\mathrm{diam}(\Omega_t)]^{\beta+\alpha-1}\cdots+C\sum_{t\in\Gamma}[\mathrm{diam}(\Omega_t)]^{\beta}\cdots\\
&=:C(\mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3).
\end{aligned}
$$

For I_1, one may use (3.3), Hölder's inequality, the fact that Ω_t intersects no more 12^n n-dimensional rectangles in $\{\Omega_t\}_{t\in\Gamma}$, Lemma 3.9, and $\rho_G^{\beta+2(\alpha-1)}\leq C_{(\Omega_\varphi)}\rho_G^{a}$ (which is guaranteed by $a\leq\beta+2(\alpha-1)$) to deduce that

$$
\begin{aligned}
\mathrm{I}_1&\leq C\sum_{t\in\Gamma}\|\tilde{g}\|_{L^q(\Omega_t,\mathbb{R}^{n\times n},\rho_G^{-\beta q})}\left(\int_{\Omega_t}|\ell(\mathbf{u})|^p\rho_G^{[\beta+2(\alpha-1)]p}\mathrm{d}x\right)^{1/p}\\
&\leq C\left(\sum_{t\in\Gamma}\|\tilde{g}\|^q_{L^q(\Omega_t,\mathbb{R}^{n\times n},\rho_G^{-\beta q})}\right)^{1/q}\left(\sum_{t\in\Gamma}\int_{\Omega_t}|\ell(\mathbf{u})|^p\rho_G^{[\beta+2(\alpha-1)]p}\mathrm{d}x\right)^{1/p}\\
&\leq C\left\|\tilde{g}+\rho_G^{\beta p}\tilde{\psi}\right\|_{L^q(\Omega_\varphi,\mathbb{R}^{n\times n},\rho_G^{-\beta q})}\left(\sum_{t\in\Gamma}\int_{\Omega_t}|\ell(\mathbf{u})|^p\rho_G^{[\beta+2(\alpha-1)]p}\mathrm{d}x\right)^{1/p}\\
&\leq C\left(\int_{\Omega_\varphi}|\ell(\mathbf{u})|^p\rho_G^{[\beta+2(\alpha-1)]p}\mathrm{d}x\right)^{1/p}\\
&\leq C\left(\int_{\Omega_\varphi}|\ell(\mathbf{u})|^p\rho_G^{ap}\mathrm{d}x\right)^{1/p}.
\end{aligned}
$$

Analogously, for I_2 and I_3, we obtain that

$$
\begin{aligned}
\mathrm{I}_2&\leq C\sum_{t\in\Gamma}\|\tilde{g}\|_{L^q(\Omega_t,\mathbb{R}^{n\times n},\rho_G^{-\beta q})}\left(\int_{\Omega_t}|\ell(\mathbf{u})|^p\rho_G^{(\beta+\alpha-1)p}\mathrm{d}x\right)^{1/p}\\
&\leq C\left(\int_{\Omega_\varphi}|\ell(\mathbf{u})|^p\rho_G^{(\beta+\alpha-1)p}\mathrm{d}x\right)^{1/p}\leq C\left(\int_{\Omega_\varphi}|\ell(\mathbf{u})|^p\rho_G^{ap}\mathrm{d}x\right)^{1/p}
\end{aligned}
$$

and

$$
\mathrm{I}_3\leq C\left(\int_{\Omega_\varphi}|\ell(\mathbf{u})|^p\rho_G^{\beta p}\mathrm{d}x\right)^{1/p}\leq C\left(\int_{\Omega_\varphi}|\ell(\mathbf{u})|^p\rho_G^{ap}\mathrm{d}x\right)^{1/p}.
$$

Combining the estimates of I_1, I_2 and I_3, we have

$$
\left|\int_{\Omega_\varphi}\left[D\mathbf{u}:\left(\tilde{g}+\rho_G^{\beta p}\tilde{\psi}\right)\right]\mathrm{d}x\right|\leq C\left(\int_{\Omega_\varphi}|\ell(\mathbf{u})|^p\rho_G^{ap}\mathrm{d}x\right)^{1/p},
$$

which completes the proof of Step 1.

STEP 2. We are now turning to the proof of Theorem 1.1 (i) under the case Hölder α domain by borrowing a decomposition technique from the proof of [14, Theorem 6.1]. Indeed we may assume that $\partial\Omega$ can be covered by a finite collection of open sets $\{E_i\}_{i=1}^k$ such that $\Omega_{\varphi_i} := E_i \bigcap \Omega$ is in the form (3.5). Now, take a Lipschitz domain $\Omega_{\varphi_0} \subset\subset \Omega$ satisfying $\bigcup_{i=0}^k \Omega_{\varphi_i} = \Omega$.

Let us define the finite collection $\{\Omega_{\varphi_i}\}_{i=0}^k$. The tree structure of the index set $\{0, 1, \ldots, k\}$ is defined in such a way that two nodes i and j are connected by an edge if and only if one of those is the root $a = 0$. Thus, the partial order is given by $i \preccurlyeq j$ if and only if $i = 0$ or $i = j$.

With the help of the tree structure, by checking the proof of [15, Lemma 4.5], we know that, for any $\tilde{g} \in \mathcal{W}$, there exists a collection of vectors $\{\tilde{g}_i\}_{i=0}^k$ with the following properties:

(i) $\tilde{g} = \sum_{i=0}^k \tilde{g}_i$;

(ii) $\operatorname{supp}(\tilde{g}_i) \subset \Omega_{\varphi_i}$;

(iii) $\tilde{g}_i \in \mathcal{W}$;

(iv) $\|\tilde{g}_i\|_{L^q(\Omega,\mathbb{R}^{n\times n},\rho^{-\beta q})} \leq C\|\tilde{g}\|_{L^q(\Omega,\mathbb{R}^{n\times n},\rho^{-\beta q})}$.

Next, similarly to Step 1, it suffices to prove that there exists a positive constant C independent of $\mathbf{v}$ such that, for any $\tilde{g} + \rho^{\beta p}\tilde{\psi} \in \mathcal{W} \oplus \rho^{\beta p}\mathcal{V}$ satisfying $\|\tilde{g} + \rho^{\beta p}\tilde{\psi}\|_{L^q(\Omega,\mathbb{R}^{n\times n},\rho^{-\beta q})} \leq 1$,

$$\left|\int_\Omega \left[D\mathbf{v} : \left(\tilde{g} + \rho^{\beta p}\tilde{\psi}\right)\right] \mathrm{d}x\right| \leq C\left(\int_\Omega |\ell(\mathbf{v})|^p \rho^{ap} \mathrm{d}x\right)^{1/p}.$$

Invoking the decomposition mentioned above to $\tilde{g}$, we arrive at

$$\left|\int_\Omega \left[D\mathbf{v} : \left(\tilde{g} + \rho^{\beta p}\tilde{\psi}\right)\right] \mathrm{d}x\right| = \left|\int_\Omega (D\mathbf{v} : \tilde{g}) \mathrm{d}x\right| \leq \sum_{i=0}^k \left|\int_{\Omega_{\varphi_i}} (D\mathbf{v} : \tilde{g}_i)\, \mathrm{d}x\right|,$$

where the first "=" is due to $\langle D\mathbf{v}, \mathcal{V}\rangle_\Omega = 0$.

For $i = 0$, it follows from the proof of Theorem A, and $a \leq \beta + 2(\alpha - 1)$ that

$$\begin{aligned}\left|\int_{\Omega_{\varphi_0}} (D\mathbf{v} : \tilde{g}_0)\, \mathrm{d}x\right| &\leq C\|\tilde{g}_0\|_{L^q(\Omega,\mathbb{R}^{n\times n},\rho^{-\beta q})}\left(\int_{\Omega_{\varphi_0}} |\ell(\mathbf{v})|^p \rho^{\beta p} \mathrm{d}x\right)^{1/p} \\ &\leq C\|\tilde{g}_0\|_{L^q(\Omega,\mathbb{R}^{n\times n},\rho^{-\beta q})}\left(\int_\Omega |\ell(\mathbf{v})|^p \rho^{ap} \mathrm{d}x\right)^{1/p}.\end{aligned}$$

For $1 \leq i \leq k$, we know by the proof of Step 1 that

$$\left|\int_{\Omega_{\varphi_i}} (D\mathbf{v} : \tilde{g}_i)\, \mathrm{d}x\right| \leq C\|\tilde{g}_i\|_{L^q(\Omega,\mathbb{R}^{n\times n},\rho^{-\beta q})}\left(\int_\Omega |\ell(\mathbf{v})|^p \rho^{ap} \mathrm{d}x\right)^{1/p}.$$

Combining the three inequalities above and using (iv) lead to

$$\left|\int_\Omega \left[D\mathbf{v} : \left(\tilde{g} + \rho^{\beta p}\tilde{\psi}\right)\right] \mathrm{d}x\right| \leq C\|\tilde{g}\|_{L^q(\Omega,\mathbb{R}^{n\times n},\rho^{-\beta q})}\left(\int_\Omega |\ell(\mathbf{v})|^p \rho^{ap} \mathrm{d}x\right)^{1/p} \leq C\left(\int_\Omega |\ell(\mathbf{v})|^p \rho^{ap} \mathrm{d}x\right)^{1/p},$$

where the last "≤" is due to Lemma 3.9. This completes the proof of Step 2 and hence Theorem 1.1 (i). □

4 A counterexample

s5

We conclude this paper by proving Theorem 1.1 (ii), i.e., the range of a in Theorem 1.1 is optimal.

For some $1 < \gamma < \infty$ (may be close to 1^+), consider the following Hölder $1/\gamma$ domain (see Figure 1 below)

$$\Omega := \left\{(x, y, z) \in \mathbb{R}^3 \mid 0 < x < 1 \text{ and } (y^2 + z^2)^{1/2} < x^\gamma\right\}.$$

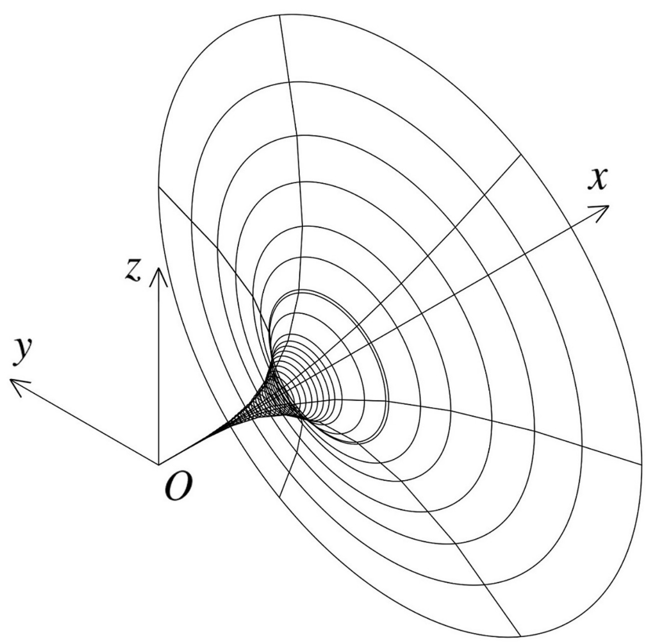

Figure 1

By an argument similar to that used in the proofs of [15, Corollaries 3.6 & 3.7], we know that Theorem 1.1 implies

qqq (4.1)
$$\int_\Omega |D\mathbf{u}|^p \rho^{\beta p} \mathrm{d}x \le C\left(\int_\Omega |\ell(\mathbf{u})|^p \rho^{ap} dx + \int_Q |\mathbf{u}|^p \mathrm{d}x\right),$$

where $Q \subset\subset \Omega$ is a cube. Take a vector field

$$\mathbf{u} := \left(x^s - \frac{s(s-1)}{2} x^{s-2}(y^2 + z^2), sx^{s-1}y, sx^{s-1}z\right),$$

where the constant s is arbitrary for the moment and will be fixed later. Thus, its differential matrix $D\mathbf{u}$ and trace-free part $\ell(\mathbf{v})$ are as follows,

$$D\mathbf{u} = \begin{pmatrix} sx^{s-1} - \frac{s(s-1)(s-2)}{2} x^{s-3}(y^2 + z^2) & -s(s-1)x^{s-2}y & -s(s-1)x^{s-2}z \\ s(s-1)x^{s-2}y & sx^{s-1} & 0 \\ s(s-1)x^{s-2}z & 0 & sx^{s-1} \end{pmatrix};$$

$$\ell(\mathbf{u}) = \begin{pmatrix} -\frac{s(s-1)(s-2)}{3}x^{s-3}(y^2+z^2) & 0 & 0 \\ 0 & \frac{s(s-1)(s-2)}{6}x^{s-3}(y^2+z^2) & 0 \\ 0 & 0 & \frac{s(s-1)(s-2)}{6}x^{s-3}(y^2+z^2) \end{pmatrix}.$$

On the one hand, for any fixed cube $Q \subset\subset \Omega$, one has $\int_Q |\mathbf{u}|^p \mathrm{d}x < \infty$.

On the other hand, a straightforward calculation gives

$$\begin{aligned}
\int_\Omega |\ell(\mathbf{u})|^p \rho^{ap} \mathrm{d}\Omega &= C_{(s,p)} \int_0^1 \mathrm{d}x \int\int_{(y^2+z^2)^{1/2}<x^\gamma} x^{(s-3)p}(y^2+z^2)^p \rho(x,y,z)^{ap} \mathrm{d}y\mathrm{d}z \\
&= C_{(s,p)} \int_0^1 x^{(s-3)p}\mathrm{d}x \int_0^{2\pi} \mathrm{d}\theta \int_0^{x^\gamma} t^{2p} \rho(x, t\cos\theta, t\sin\theta)^{ap} t\mathrm{d}t \\
&= C_{(s,p)} \int_0^1 x^{(s-3)p}\mathrm{d}x \int_0^{2\pi} \mathrm{d}\theta \int_0^{x^\gamma} t^{2p} \rho(x,0,t)^{ap} t\mathrm{d}t \\
&\le C_{(s,p)} \int_0^1 x^{(s-3)p}\mathrm{d}x \int_0^{x^\gamma} t^{2p}(x^\gamma - t)^{ap} t\mathrm{d}t \\
&= C_{(s,p,a)} \int_0^1 x^{(s-3)p+(2p+ap+2)\gamma}\mathrm{d}x
\end{aligned}$$

and

$$\begin{aligned}
\int_\Omega |D\mathbf{u}|^p \rho^{\beta p} \mathrm{d}\Omega &\ge C_{(s,p)} \int_0^1 \mathrm{d}x \int\int_{(y^2+z^2)^{1/2}<x^\gamma} x^{(s-1)p} \rho(x,y,z)^{\beta p} \mathrm{d}y\mathrm{d}z \\
&= C_{(s,p)} \int_0^1 x^{(s-1)p}\mathrm{d}x \int_0^{x^\gamma} \rho(x,0,t)^{\beta p} t\mathrm{d}t \\
&= C_{(s,p)} \int_0^1 x^{(s-1)p}\mathrm{d}x \int_0^{x^\gamma} (x^\gamma - t)^{\beta p} \left(\frac{\rho(x,0,t)}{x^\gamma - t}\right)^{\beta p} t\mathrm{d}t \\
&\ge C_{(s,p)} \int_0^1 x^{(s-1)p}\mathrm{d}x \int_0^{x^\gamma} (x^\gamma - t)^{\beta p} \left(\frac{\rho(1,0,0)}{1^\gamma - 0}\right)^{\beta p} t\mathrm{d}t \\
&= C_{(s,p,\beta)} \int_0^1 x^{(s-1)p}\mathrm{d}x \int_0^{x^\gamma} (x^\gamma - t)^{\beta p} t\mathrm{d}t \\
&= C_{(s,p,\beta)} \int_0^1 x^{(s-1)p+(\beta p+2)\gamma}\mathrm{d}x.
\end{aligned}$$

If $a > \beta + 2(1/\gamma - 1)$, we can take s such that

$$(s-3)p + (2p + ap + 2)\gamma + 1 > 0 \ \text{ and } \ (s-1)p + (\beta p + 2)\gamma + 1 \le 0,$$

which, together with the two above inequalities, implies the left hand of (4.1) is infinite while the right one is finite. This proves that, if $a > \beta + 2(1/\gamma - 1)$, the conformal Korn inequality can not be valid. This completes the proof of Theorem 1.1 (ii).

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References

ADL2006 [1] G. Acosta, R. G. Durán, A. L. Lombardi, Weighted Poincaré and Korn inequalities for Hölder α domains, Math. Methods Appl. Sci. 29 (2006) 387-400. [1, 2]

ADL2013 [2] G. Acosta, R. G. Durán, F. López-García, Korn inequality and divergence opeator: counterexamples and optimality of weighted estimates, Proc. Amer. Math. Soc. 141 (2013) 217-232. [2, 3]

ADM2006 [3] G. Acosta, R. G. Durán, M. A. Muschietti, Solutions of the divergence operator on John domains, Adv. Math. 206 (2006) 373-401. [1]

BCD2017 [4] D. Breit, A. Cianchi, L. Diening, Trace-free Korn inequalities in Orlicz spaces, SIAM J. Math. Anal. 49 (2017) 2496-2526. [2]

Da06 [5] S. Dain, Generalized Korn's inequality and conformal Killing vectors, Calc. Var. Partial Differential Equations 25 (2006) 535-540. [2]

DL2010 [6] R. G. Durán, F. López-García, Solutions of the divergence and Korn inequalities on domains with an external cusp, Ann. Acad. Sci. Fenn. Math. 35 (2010) 421-438. [1, 2]

DL1976 [7] G. Duvaut, J.-L. Lions, Inequalities in Mechanics and Physics, Springer, 1976. [1]

F1947 [8] K. O. Friedrichs, On the boundary-value problems of the theory of elasticity and Korn's inequality, Ann. of Math. (2) 48 (1947) 441-471. [1]

GTM249 [9] L. Grafakos, Classical Fourier Analysis, Second edition, Graduate Texts in Mathematics Vol. 249 (Springer, New York, 2009). [5]

H1995 [10] C. O. Horgan, Korn's inequalities and their applications in continuum mechanics, SIAM Rev. 37 (1995) 491-511. [1]

JK2017 [11] R. Jiang, A. Kauranen, Korn's inequality and John domains, Calc. Var. Partial Differential Equations 56 (2017) no. 4, Art. 109, 18 pp. [1]

JK2015 [12] R. Jiang, A. Kauranen, Korn inequality on irregular domains, J. Math. Anal. Appl. 423 (2015) 41-59. [2]

KO1989 [13] V. A. Kondratiev, O. A. Oleinik, On Korn's inequalities, C. R. Acad. Sci. Paris Sér. I Math. 308 (1989) 483-487. [1]

L2014 [14] F. López-García, A decomposition technique for integrable functions with applications to the divergence problem, J. Math. Anal. Appl. 418 (2014) 79-99. [2, 7, 11]

L [15] F. López-García, Weighted generalized Korn inequalities on John domains, Math. Methods Appl. Sci. 41 (2018) 8003-8018. [2, 3, 4, 6, 7, 8, 9, 11, 12]

N1981 [16] J. A. Nitsche, On Korn's second inequality, RAIRO J. Numer. Anal. 15 (1981) 237-248. [1]

Rei76 [17] H. M. Reimann, Ordinary differential equations and quasiconformal mappings, Invent. Math. 33 (1976) 247-270. [2]

Re70 [18] Yu. G. Reshetnyak, Estimates for certain differential operators with finite-dimensional kernel, Sibirsk. Mat. Ž. 11 (1970) 414-428. [2]

T2001 [19] A. Tiero, On inequalities of Korn, Friedrichs, Magenes-Stampacchia-Nečas and Babuška-Aziz, Z. Anal. Anwend. 20 (2001) 215-222. [1]

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