On the Nadaraya-Watson kernel regression estimator for irregularly spaced spatial data

M. EL MACHKOURI, X. FAN and L. REDING

Laboratoire de Mathématiques Raphaël Salem
UMR CNRS 6085, Université de Rouen Normandie, France
mohamed.elmachkouri@univ-rouen.fr
lucas.reding@etu.univ-rouen.fr
Center for Applied Mathematics
Tianjin University
Tianjin, China
fanxiequan@hotmail.com

Abstract

We investigate the asymptotic normality of the Nadaraya-Watson kernel regression estimator for irregularly spaced data collected on a finite region of the lattice $\mathbb{Z}^d$ where d is a positive integer. The results are stated for strongly mixing random fields in the sense of Rosenblatt (1956) and for weakly dependent random fields in the sense of Wu (2005). Only minimal conditions on the bandwidth parameter and simple conditions on the dependence structure of the data are assumed.

AMS Subject Classifications (2000): 62G05, 60J25, 62G07.
Key words and phrases: Nonparametric estimation, Nadaraya-Watson estimator, irregularly spaced spatial data, strong mixing, m-dependence, physical dependence measure, random fields, asymptotic normality.

1 Introduction

In many situations, practicians want to know the relationship between some predictors and a response. If the form of the functional relation is unknown then a nonparametric approach is necessary. This is a natural question and a very important task in statistics. A very popular tool to handle this problem is the Nadaraya-Watson estimator (NWE) introduced by Nadaraya [21] and Watson [29]. In this work, we investigate the asymptotic normality of the NWE in the context of dependent irregularly spaced spatial data. Let d, n and N be positive integers. Let also $(Y_i, X_i)_{i\in\mathbb{Z}^d}$ be a strictly stationary $\mathbb{R}\times\mathbb{R}^N$-valued random field defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that the common law μ of the random variables $(X_i)_{i\in\mathbb{Z}^d}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^N$. We denote by f the unknown probability density function of μ. Let Λ_n be a finite region of $\mathbb{Z}^d$ and let $(\eta_i)_{i\in\mathbb{Z}^d}$ be iid $\mathbb{R}^N$-valued random variables with zero mean and finite variance and independent of $(X_i)_{i\in\mathbb{Z}^d}$. The regression model is characterized by the relation $Y_i = R(X_i, \eta_i)$ for i in Λ_n where R is an unknown functional. In our setting, it is important to note that no regularity condition is imposed on Λ_n which can be very general (irregularly spaced data). The regression function r is defined for any x in $\mathbb{R}^N$ by

$$r(x) = \begin{cases} \mathbb{E}\left[R(x, \eta_0)\right] & \text{if } f(x) \neq 0 \\ \mathbb{E}\left[Y_0\right] & \text{else,} \end{cases}$$

and the NWE r_n of r is defined for any x in $\mathbb{R}^N$ by

$$r_n(x) = \begin{cases} \dfrac{\sum_{i\in\Lambda_n} Y_i \mathrm{K}\left(\frac{x-X_i}{b_n}\right)}{\sum_{i\in\Lambda_n} \mathrm{K}\left(\frac{x-X_i}{b_n}\right)} & \text{if } \sum_{i\in\Lambda_n} \mathrm{K}\left(\frac{x-X_i}{b_n}\right) \neq 0 \\ \dfrac{1}{|\Lambda_n|} \sum_{i\in\Lambda_n} Y_i & \text{else,} \end{cases}$$

where $|\Lambda_n|$ is the number of elements in the region Λ_n, the function $\mathrm{K} : \mathbb{R}^N \longrightarrow \mathbb{R}$ is a probability kernel (that is $\int_{\mathbb{R}^N} \mathrm{K}(t)dt = 1$) and the bandwidth parameter b_n is a positive constant going to zero as n goes to infinity. For time series (i.e. for $d = 1$), the problem which we are concerned has been extensively studied. One can refer, e.g., to Lu and Cheng [18], Masry and Fan [19], Robinson [24], Roussas [27] and many references therein. In the spatial case (i.e. for $d \geqslant 2$), some contributions for strongly mixing random fields were made by Biau and Cadre [1], Carbon et al. [2], Dabo-Niang and Rachdi [3], Dabo-Niang and Yao [4], El Machkouri [7], El Machkouri and Stoica [10], Hallin et al. [12] and Lu and Chen [16, 17]. The main motivation of this work is to provide sufficient simple conditions for the NWE to be asymptotically normal in the context of mixing but also non-mixing random fields. More precisely, we consider strongly mixing random fields in the sense of Rosenblatt [25] and weakly dependent random fields in the sense of Wu [30] (see also [11]). To the best of our knowledge, our work provides the first central limit theorem (Theorem 2) for the NWE under minimal conditions on the bandwidth parameter and irregularly spaced dependent spatial data (i.e. $b_n \longrightarrow 0$ and $|\Lambda_n|b_n^N \longrightarrow \infty$ as $n \longrightarrow \infty$). In particular, our result improves in several directions a previous central limit theorem for the NWE for spatial data established by [1] (see the comments after Corollary 1 below).
The paper is organized as follows. Our main results are stated and discussed in Section 2 whereas proofs of the main results and its preliminary lemmas are deferred to Sections 4 and 5. Finally, Section 3 is devoted to a numerical illustration of the central limit theorem obtained in Section 2.

2 Main results

Given two σ-algebras $\mathcal{U}$ and $\mathcal{V}$, the α-mixing coefficient introduced by Rosenblatt [25] is

$$\alpha(\mathcal{U}, \mathcal{V}) = \sup\{|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|\,,\, A \in \mathcal{U},\, B \in \mathcal{V}\}.$$

Let p be fixed in $[1, +\infty]$. The strong mixing coefficients $(\alpha_{1,p}(n))_{n\geqslant 0}$ associated to $(X_i)_{i\in\mathbb{Z}^d}$ are defined by

$$\alpha_{1,p}(n) = \sup\left\{\alpha(\sigma(X_k), \mathcal{F}_\Gamma),\ k \in \mathbb{Z}^d,\ \Gamma \subset \mathbb{Z}^d,\ |\Gamma| \leqslant p,\ \rho(\Gamma, \{k\}) \geqslant n\right\},$$

where $\mathcal{F}_\Gamma = \sigma(X_i\,;\ i \in \Gamma)$, $|\Gamma|$ is the number of element in Γ and the distance ρ is defined for any subsets Γ_1 and Γ_2 of $\mathbb{Z}^d$ by $\rho(\Gamma_1, \Gamma_2) = \min\{|i-j|,\ i \in \Gamma_1,\ j \in \Gamma_2\}$ with $|i-j| = \max_{1\leqslant s\leqslant d} |i_s - j_s|$ for any $i = (i_1, \ldots, i_d)$ and $j = (j_1, \ldots, j_d)$ in $\mathbb{Z}^d$. We say that the random field $(X_i)_{i\in\mathbb{Z}^d}$ is strongly mixing if $\lim_{n\to\infty} \alpha_{1,p}(n) = 0$. Let m be a positive integer. We are also going to establish our results for Bernoulli fields of the form

$$X_i = G\left(\varepsilon_{i-s}\,;\ s \in \mathbb{Z}^d\right), \quad i \in \mathbb{Z}^d, \tag{1}$$

where $G : (\mathbb{R}^m)^{\mathbb{Z}^d} \longrightarrow \mathbb{R}^N$ is some function and $(\varepsilon_i)_{i\in\mathbb{Z}^d}$ are iid $\mathbb{R}^m$-valued random variables. Let $(\varepsilon_j^{'})_{j\in\mathbb{Z}^d}$ be an iid copy of $(\varepsilon_j)_{j\in\mathbb{Z}^d}$ and let X_i^* be the coupled version of X_i defined by

$$X_i^* = G\left(\varepsilon_{i-s}^*\,;\ s \in \mathbb{Z}^d\right),$$

where $\varepsilon_j^* = \varepsilon_j$ if $j \neq 0$ and $\varepsilon_0^* = \varepsilon_0^{'}$. Note that X_i^* is obtained from X_i by replacing ε_0 by its copy ε_0'. For any positive integer ℓ and any $\mathbb{R}^\ell$-valued random variable $Z \in \mathbb{L}^p(\Omega, \mathcal{F}, \mathbb{P})$ with $p > 0$, we denote $\|Z\|_p := \mathbb{E}\left[\|Z\|^p\right]^{1/p}$ where $\|.\|$ is the Euclidian norm of $\mathbb{R}^\ell$. Following Wu [30] and El Machkouri et al. [11], we define the physical dependence measure

$$\delta_{i,p} := \|X_i - X_i^*\|_p$$

as soon as X_i is p-integrable for $p \geqslant 2$. We say that X is p-stable if $\sum_{i\in\mathbb{Z}^d} \delta_{i,p} < \infty$. Physical dependence measure should be seen as a measure of the dependence of the function G (defined in (1)) in the coordinate zero. In some sense, it quantifies the degree of dependence of outputs on inputs in physical systems and provide a natural framework for a limit theory for stationary random fields (see [11]). In particular, it gives mild and easily verifiable conditions (see condition (A3)(ii) below) because it is directly related to the data-generating mechanism. In mathematical physics, various versions of similar ideas (local perturbation of a configuration) appear. One can refer for example to Liggett [14] or Stroock and Zegarlinski [28]. As an illustration, the reader should keep in mind the following two examples:

- Linear random fields: Let $(\varepsilon_i)_{i\in\mathbb{Z}^d}$ be i.i.d $\mathbb{R}^m$-valued random variables such that ε_i belongs to $\mathbb{L}^p(\Omega, \mathcal{F}, \mathbb{P})$, $p \geq 2$. The linear random field X defined for all i in $\mathbb{Z}^d$ by

  $$X_i = \sum_{s\in\mathbb{Z}^d} A_s \varepsilon_{i-s}$$

  where $A_s = (a_{s,k_1,k_2})_{\substack{1\leqslant k_1\leqslant N\\ 1\leqslant k_2\leqslant m}}$ is a $N \times m$ matrix such that $\sum_{s\in\mathbb{Z}^d} \sum_{k_1=1}^N \sum_{k_2=1}^m a_{s,k_1,k_2}^2 < \infty$ is of the form (1) with a linear functional G. For all i in $\mathbb{Z}^d$,

  $$\delta_{i,p} \leqslant \|\varepsilon_0 - \varepsilon_0^{'}\|_p \times \sqrt{\sum_{k_1=1}^N \sum_{k_2=1}^m a_{i,k_1,k_2}^2}.$$

  So, X is p-stable as soon as $\sum_{i\in\mathbb{Z}^d} \sqrt{\sum_{k_1=1}^N \sum_{k_2=1}^m a_{i,k_1,k_2}^2} < \infty$. Clearly, if H is a Lipschitz continuous function, under the above condition, the subordinated process $Y_i = \mathrm{H}(X_i)$ is also p-stable.

- *Volterra field* : Another class of nonlinear random field is the Volterra process which plays an important role in the nonlinear system theory. Let $i \in \mathbb{Z}^d$ and

$$X_i = \sum_{s_1,s_2 \in \mathbb{Z}^d} a_{s_1,s_2} \varepsilon_{i-s_1} \varepsilon_{i-s_2},$$

where a_{s_1,s_2} are real coefficients with $a_{s_1,s_2} = 0$ if $s_1 = s_2$ and $(\varepsilon_i)_{i\in\mathbb{Z}^d}$ are i.i.d. real random variables with ε_i in $\mathbb{L}^p(\Omega, \mathcal{F}, \mathbb{P})$, $p \geq 2$. By the Burkholder inequality, there exists a constant $C_p > 0$ such that

$$\delta_{i,p} \leq C_p \|\varepsilon_0 - \varepsilon_0^{'}\|_p \|\varepsilon_0\|_p \times \sqrt{\sum_{s\in\mathbb{Z}^d} (a_{s,i} + a_{i,s})^2}$$

So, X is p-stable as soon as $\sum_{i\in\mathbb{Z}^d} \sqrt{\sum_{s\in\mathbb{Z}^d} (a_{s,i} + a_{i,s})^2} < \infty$.

Let $(b_n)_{n\geqslant 1}$ be a sequence of positive real numbers going to zero as n goes to infinity. Denote $\mathrm{K}_n(x, v) = \mathrm{K}\left(\frac{x-v}{b_n}\right)$ for any $(x, v) \in \mathbb{R}^N \times \mathbb{R}^N$ and any integer $n \geqslant 1$. If $x \in \mathbb{R}^N$ and $f_n(x) \neq 0$ then $r_n(x) = \varphi_n(x)/f_n(x)$, where

$$\varphi_n(x) = \frac{\sum_{i\in\Lambda_n} Y_i \mathrm{K}_n(x, X_i)}{|\Lambda_n| b_n^N} \quad \text{and} \quad f_n(x) = \frac{\sum_{i\in\Lambda_n} \mathrm{K}_n(x, X_i)}{|\Lambda_n| b_n^N}.$$

Recall that f_n is the classical Parzen-Rosenblatt estimator of the marginal density f of X_0 (see [8, 9, 22, 26]). Similarly, if φ is the function defined for any $x \in \mathbb{R}^N$ by $\varphi(x) = r(x)f(x)$ then φ_n is an estimator of φ. In the sequel, we consider the following assumptions:

(A1) Assume $b_n \longrightarrow 0$ such that $|\Lambda_n| b_n^N \longrightarrow \infty$ and that K is symmetric, Lipschitz and satisfies $|\mathrm{K}|_\infty := \sup_{t\in\mathbb{R}^N} |\mathrm{K}(t)| < \infty$, $\lim_{\|t\|\to\infty} \|t\| \, |\mathrm{K}(t)| = 0$, $\int_{\mathbb{R}^N} |\mathrm{K}(t)| dt < \infty$ and $\int_{\mathbb{R}^N} \|t\|^2 |\mathrm{K}(t)| dt < \infty$ where $\| \, . \, \|$ is the Euclidian norm on $\mathbb{R}^N$.

(A2) There exists $\kappa > 0$ such that $|f_{0,i}(x, y) - f(x)f(y)| \leq \kappa$ for any (x, y) in $\mathbb{R}^N \times \mathbb{R}^N$ and any i in $\mathbb{Z}^d \setminus \{0\}$, where $f_{0,i}$ is the joint density of (X_0, X_i).

(A3) There exists $\theta > 0$ such that $\mathbb{E}\left[|Y_0|^{2+\theta}\right] < \infty$ and one of the following condition holds:

   (i) $(X_i)_{i\in\mathbb{Z}^d}$ is strongly mixing and $\sum_{n=1}^{\infty} n^{\frac{(2d-1)\theta+6d-2}{2+\theta}} \alpha_{1,\infty}^{\frac{\theta}{2+\theta}}(n) < \infty$;

   (ii) $(X_i)_{i\in\mathbb{Z}^d}$ is of the form (1) and $\sum_{i\in\mathbb{Z}^d} |i|^{\frac{d((3N+2)\theta^2+(10N+8)\theta+8N)}{2\theta(\theta+2)N}} \delta_{i,2}^{\frac{\theta}{2+\theta}} < \infty$.

(A4) There exists $\theta > 0$ such that $\mathbb{E}\left[|Y_0|^{2+\theta}\right] < \infty$ and the function $x \longmapsto \mathbb{E}\left[\Psi_p(|Y_0|)\,|X_0 = x\right]$ is continuous for $p \in \{1, 2, 2+\theta\}$ where $\Psi_p(t) = t^p$ for any real t. Moreover, the functions f and φ are twice differentiable with bounded second partial derivatives.

Assumptions $(A1)$, $(A2)$ and $(A4)$ are classical conditions in nonparametric statistics (see [2], [16]). Moreover, one can notice that if $\theta = \infty$ then $(A3)(i)$ and $(A3)(ii)$ reduce to the conditions obtained in [8] and [9] respectively where the asymptotic normality of the Parzen-Rosenblatt estimator is established.

First, we show that φ_n and f_n are asymptotically unbiaised estimators of φ and f respectively.

Theorem 1 *Assume that f and φ are twice differentiable with bounded second partial derivatives and $\int_{\mathbb{R}^N} \|t\|^2 |\mathrm{K}(t)| dt < \infty$. Then*

$$\sup_{x\in\mathbb{R}^N} |\mathbb{E}[f_n(x)] - f(x)| = O\left(b_n^2\right) \quad \text{and} \quad \sup_{x\in\mathbb{R}^N} |\mathbb{E}[\varphi_n(x)] - \varphi(x)| = O\left(b_n^2\right).$$

Consequently, if $|\Lambda_n| b_n^{N+4} \longrightarrow 0$ as $n \longrightarrow \infty$, then

$$\lim_{n\to\infty} \sqrt{|\Lambda_n| b_n^N} \sup_{x\in\mathbb{R}^N} |\mathbb{E}[f_n(x)] - f(x)| = \lim_{n\to\infty} \sqrt{|\Lambda_n| b_n^N} \sup_{x\in\mathbb{R}^N} |\mathbb{E}[\varphi_n(x)] - \varphi(x)| = 0.$$

Our main result is the following central limit theorem for the NWE.

Theorem 2 *If (A1), (A2), (A3) and (A4) hold, then for any $x \in \mathbb{R}^N$ such that $f(x) > 0$,*

$$\sqrt{|\Lambda_n| b_n^N} \left(r_n(x) - \frac{\mathbb{E}[\varphi_n(x)]}{\mathbb{E}[f_n(x)]} \right) \xrightarrow[n\to\infty]{Law} \mathcal{N}\left(0, \sigma^2(x)\right),$$

where $\sigma^2(x) = \frac{V(x)}{f(x)} \int_{\mathbb{R}^N} \mathrm{K}^2(t) dt$ and $V(x) = \mathbb{E}\left[Y_0^2 | X_0 = x\right] - r^2(x)$.

Using Theorem 1, the condition $|\Lambda_n| b_n^{N+4} \longrightarrow 0$ can be imposed for the control of the bias of the estimator and leads immediately to the following corollary (its proof is left to the reader).

Corollary 1 *If (A1), (A2), (A3) and (A4) hold and $|\Lambda_n| b_n^{N+4} \longrightarrow 0$, then for any $x \in \mathbb{R}^N$ such that $f(x) > 0$,*

$$\sqrt{|\Lambda_n| b_n^N} \left(r_n(x) - r(x) \right) \xrightarrow[n\to\infty]{Law} \mathcal{N}\left(0, \sigma^2(x)\right),$$

where $\sigma^2(x)$ is defined in Theorem 2.

The asymptotic normality of r_n given by Theorem 2 holds under mild conditions on the regions Λ_n and the bandwidth b_n, that is $b_n \longrightarrow 0$ and $|\Lambda_n| b_n^N \longrightarrow \infty$. These conditions on the bandwidth parameter are sometimes called *minimal conditions* since these are required for the asymptotic normality of the Parzen-Rosenblatt estimator f_n when the observations are assumed to be independent (see [22]). To the best of our knowledge, Theorem 2 is the first central limit theorem for the NWE under minimal conditions on the bandwidth and irregularly spaced dependent spatial data. In particular, we improve in several directions Theorem 2.2 in [1] for strongly mixing random fields where the authors considered a set of conditions on the bandwidth parameter and the mixing coefficients interlaced in a complicated way. More precisely, using ours notations, Theorem 2.2 in [1] gives the asymptotic normality of the NWE as soon as $\mathbb{E}[\exp(|Y_0|^\tau)] < \infty$ for some positive real τ, the regions Λ_n are rectangular subsets of $\mathbb{Z}^d$ such that $|\Lambda_n| b_n^{N+2} \longrightarrow 0$ and $|\Lambda_n| b_n^{N(1+2\delta d)} \log(|\Lambda_n|)^{-8d/\tau} \longrightarrow \infty$ for some $0 < \delta < 1/2$, there exists $q_n \longrightarrow \infty$ such that $q_n^{2d} = o\left(b_n^{N(1+2\delta d)} \log(|\Lambda_n|)^{-8d/\tau}\right)$ and $|\Lambda_n| \sum_{n\geqslant 1} n^{d-1} \alpha_{1,\infty}(n q_n) \longrightarrow 0$ and $b_n^{-N\delta} (\log(|\Lambda_n|))^{2/\tau} \sum_{n\geqslant q_n} n^{d-1} \alpha_{1,\infty}^{\delta}(n) \longrightarrow 0$. In particular, it is assumed that $\sum_{n=1}^{\infty} n^{d-1} \alpha_{1,\infty}^{\delta}(n) < \infty$. In order to compare with our results, one can notice that if $\delta = \theta/(2+\theta) \in]0, 1/2[$ and $\mathbb{E}\left[|Y_0|^{2+\theta}\right]$ with $0 < \theta < 2$ then $(A3)(i)$ reduces to $\sum_{n\geqslant 1} n^{d(3-\delta)-1} \alpha_{1,\infty}^{\delta}(n) < \infty$. However, our main result holds even if Y_0 does not have finite exponential moments and also for general regions Λ_n (irregularly spaced spatial data) and under only minimal conditions on the bandwidth ($b_n \longrightarrow 0$ and $|\Lambda_n| b_n^N \longrightarrow \infty$).

3 Numerical illustration

In order to illustrate the asymptotic normality of the NWE provided by Theorem (1), we are going to consider two regression models where the predictors are given by an autoregressive random field $\left(X_{i,j}^{(\mathrm{AR})}\right)_{(i,j)\in\mathbb{Z}^2}$ and a Volterra random field $\left(X_{i,j}^{(\mathrm{Vol})}\right)_{(i,j)\in\mathbb{Z}^2}$ respectively (see Model 1 and Model 2 below). In Model 1, the autoregressive random field $\left(X_{i,j}^{(\mathrm{AR})}\right)_{(i,j)\in\mathbb{Z}^2}$ is defined by

$$X_{i,j}^{(\mathrm{AR})} = 0.7X_{i-1,j}^{(\mathrm{AR})} + 0.15X_{i,j-1}^{(\mathrm{AR})} + \varepsilon_{i,j}, \tag{2}$$

where $(\varepsilon_{i,j})_{(i,j)\in\mathbb{Z}^2}$ are iid real random variables ($N = 1$) with standard normal law. From [13], we know that the stationary solution of (2) is the linear random field given by

$$X_{i,j}^{(\mathrm{AR})} = \sum_{s_1\geqslant 0}\sum_{s_2\geqslant 0}\binom{s_1+s_2}{s_1}(0.7)^{s_1}(0.15)^{s_2}\varepsilon_{i-s_1,j-s_2}. \tag{3}$$

So, we fix a positive integer n_1 and we simulate the $\varepsilon_{i,j}$'s over the grid $[0,2n_1]^2\cap\mathbb{Z}^2$ in order to get the data $X_{i,j}^{(\mathrm{AR})}$ for (i,j) in $[n_1+1,2n_1]^2\cap\mathbb{Z}^2$ following (2) and (3). In Model 2, in order to consider nonlinearity, we define

$$X_{i,j}^{(\mathrm{Vol})} = \sum_{s_1\geqslant 0}\sum_{s_2\geqslant 0}\sum_{t_1>s_1}\sum_{t_2>s_2}\binom{s_1+s_2}{s_1}\binom{t_1-s_1+t_2-s_2}{t_1-s_1}(0.7)^{s_1+t_1}(0.15)^{s_2+t_2}\varepsilon_{i-s_1,j-s_2}\varepsilon_{i-t_1,j-t_2}.$$

Since

$$X_{i,j}^{(\mathrm{Vol})} = \sum_{s_1\geqslant 0}\sum_{s_2\geqslant 0}\binom{s_1+s_2}{s_1}(0.7)^{2s_1}(0.15)^{2s_2}\varepsilon_{i-s_1,j-s_2}\beta_{i-s_1,j-s_2} \tag{4}$$

where

$$\beta_{i,j} = \sum_{t_1>0}\sum_{t_2>0}\binom{t_1+t_2}{t_1}(0.7)^{t_1}(0.15)^{t_2}\varepsilon_{i-t_1,j-t_2}, \tag{5}$$

we fix a positive integer n_2 and we simulate the $\varepsilon_{i,j}$'s over the grid $[0,4n_2]^2\cap\mathbb{Z}^2$ and we get the data $\beta_{i,j}$ for (i,j) in $[2n_2+1,4n_2]^2\cap\mathbb{Z}^2$ using (5) and following the previous implementation of $(X_{i,j}^{(\mathrm{AR})})_{(i,j)\in\mathbb{Z}^2}$. Starting from the data $\varepsilon_{i,j}\beta_{i,j}$ for (i,j) in $[2n_2+1,4n_2]^2$, we simulate in the same way the data $X_{i,j}^{(\mathrm{Vol})}$ for (i,j) in $[3n_2+1,4n_2]^2$ using (4). From the two data sets

$$Y_{i,j}^{(\mathrm{AR})} = \sin\left(X_{i,j}^{(\mathrm{AR})}\right) + \varepsilon_{i,j},\quad (i,j)\in[n_1+1,2n_1]^2 \qquad \text{(Model 1)}$$

and

$$Y_{i,j}^{(\mathrm{Vol})} = \sin\left(X_{i,j}^{(\mathrm{Vol})}\right) + \varepsilon_{i,j},\quad (i,j)\in[3n_2+1,4n_2]^2 \qquad \text{(Model 2)},$$

we consider 500 replications of $\sqrt{2\sqrt{\pi}\hat{f}_{n_1}^{(\mathrm{AR})}(0)n_1^2 b_{n_1}}\, r_{n_1}^{(\mathrm{AR})}(0)$ and $\sqrt{2\sqrt{\pi}\hat{f}_{n_2}^{(\mathrm{Vol})}(0)n_2^2 b_{n_2}}\, r_{n_2}^{(\mathrm{Vol})}(0)$ where

$$\hat{f}_{n_1}^{(\mathrm{AR})}(0) = \frac{1}{n_1^2 b_{n_1}}\sum_{(i,j)\in[n_1+1,2n_1]^2}\mathrm{K}\left(\frac{X_{i,j}^{(\mathrm{AR})}}{b_{n_1}}\right), \qquad \hat{f}_{n_2}^{(\mathrm{Vol})}(0) = \frac{1}{n_2^2 b_{n_2}}\sum_{(i,j)\in[n_2+1,2n_2]^2}\mathrm{K}\left(\frac{X_{i,j}^{(\mathrm{Vol})}}{b_{n_2}}\right),$$

$$r_{n_1}^{(\mathrm{AR})}(0) = \frac{\sum_{(i,j)\in[n_1+1,2n_1]^2} Y_{i,j}^{(\mathrm{AR})}\,\mathrm{K}\left(\frac{X_{i,j}^{(\mathrm{AR})}}{b_{n_1}}\right)}{n_1^2 b_{n_1}\hat{f}_{n_1}^{(\mathrm{AR})}(0)}, \qquad r_{n_2}^{(\mathrm{Vol})}(0) = \frac{\sum_{(i,j)\in[3n_2+1,4n_2]^2} Y_{i,j}^{(\mathrm{Vol})}\,\mathrm{K}\left(\frac{X_{i,j}^{(\mathrm{Vol})}}{b_{n_2}}\right)}{n_2^2 b_{n_2}\hat{f}_{n_2}^{(\mathrm{Vol})}(0)},$$

the kernel K is Gaussian and the bandwidth parameters b_{n_1} and b_{n_2} are selected by cross validation. So, in Figure 1 below, we obtain the histograms for

$$\sqrt{2\sqrt{\pi}\hat{f}_{n_1}^{(\mathrm{AR})}(0)n_1^2 b_{n_1}}\,r_{n_1}^{(\mathrm{AR})}(0) \quad \text{and} \quad \sqrt{2\sqrt{\pi}\hat{f}_{n_2}^{(\mathrm{Vol})}(0)n_2^2 b_{n_2}}\,r_{n_2}^{(\mathrm{Vol})}(0)$$

with $n_1, n_2 \in \{10, 30\}$ along with the standard normal law.

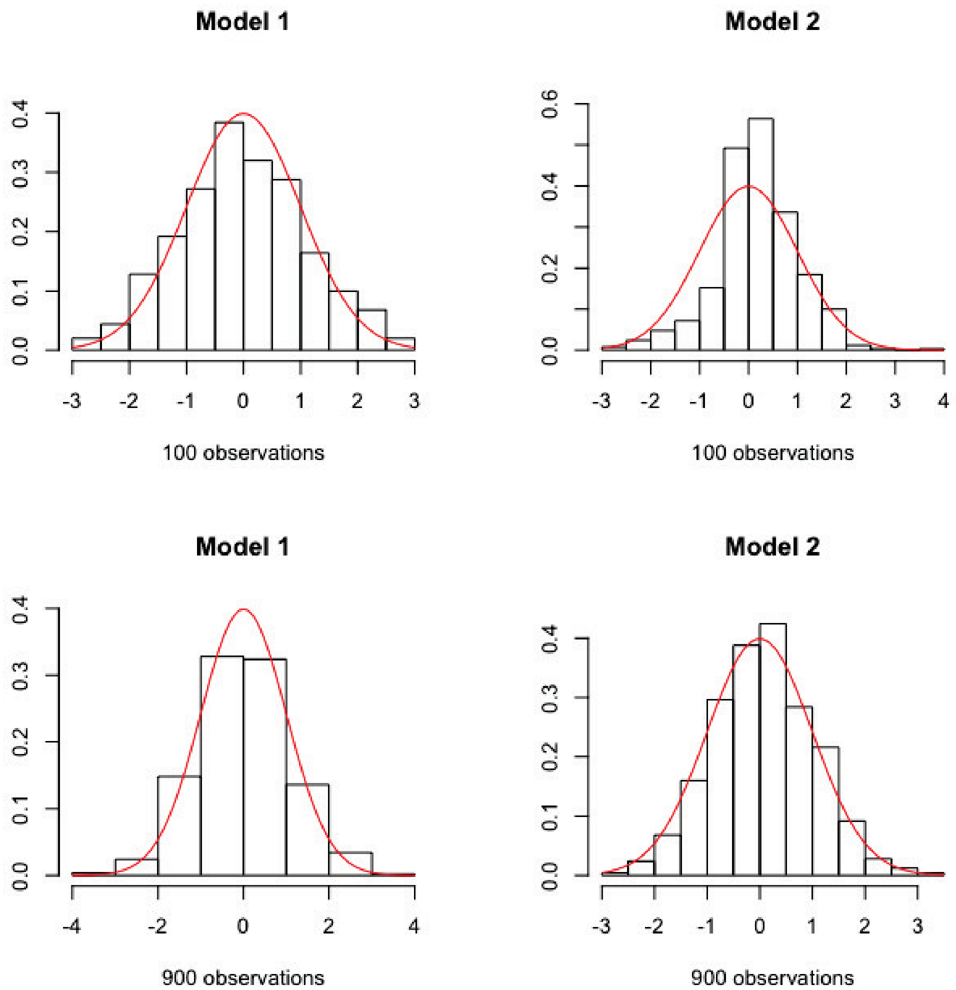

Figure 1: Histograms for $\sqrt{2\sqrt{\pi}\hat{f}_{n_1}^{(\mathrm{AR})}(0)n_1^2 b_{n_1}}\,r_{n_1}^{(\mathrm{AR})}(0)$ (Model 1) and $\sqrt{2\sqrt{\pi}\hat{f}_{n_2}^{(\mathrm{Vol})}(0)n_2^2 b_{n_2}}\,r_{n_2}^{(\mathrm{Vol})}(0)$ (Model 2) along with the standard normal density.

4 Preliminary lemmas

In the sequel, for any sequences $(a_n)_{n\geqslant 1}$ and $(b_n)_{n\geqslant 1}$ of real positive numbers, we denote $a_n \trianglelefteq b_n$ if and only if there exists $\kappa > 0$ (not depending on n) such that $a_n \leqslant \kappa b_n$. For any real x, we define also $\lceil x \rceil = \lfloor x \rfloor + 1$, where $\lfloor x \rfloor$ is the largest integer less than x. We shall need the following technical lemmas.

Lemma 1 *Assume (A1), (A2) and (A4) and let $x \in \mathbb{R}^N$. If $\Phi_1 : \mathbb{R} \longrightarrow \mathbb{R}$ and $\Phi_2 : \mathbb{R} \longrightarrow \mathbb{R}$ are two functions such that $x \longmapsto \mathbb{E}\left[\Phi_1(Y_0)|X_0 = x\right]$ is continuous and the conditions $\sup_{t\in\mathbb{R}^N} |\Phi_2(\mathrm{K}(t))| < \infty$, $\lim_{\|t\|\to\infty} \|t\| \, |\Phi_2(\mathrm{K}(t))| = 0$ and $\int_{\mathbb{R}^N} |\Phi_2(\mathrm{K}(t))| \, dt < \infty$ are satisfied then*

$$\lim_{n\to\infty} \frac{\mathbb{E}\left[\Phi_1(Y_0)\Phi_2\left(\mathrm{K}_n(x, X_0)\right)\right]}{b_n^N} = \mathbb{E}\left[\Phi_1(Y_0)|X_0 = x\right] f(x) \int_{\mathbb{R}^N} \Phi_2\left(\mathrm{K}(v)\right) dv.$$

Moreover, we have also $\sup_{\substack{j\in\mathbb{Z}^d \\ j\neq 0}} \mathbb{E}\left[\left|\mathrm{K}_n(x, X_0)\mathrm{K}_n(x, X_j)\right|\right] \trianglelefteq b_n^{2N}$.

Proof. Let $x \in \mathbb{R}^N$ and let n be a positive integer. It is obvious that

$$\mathbb{E}\left[\Phi_1(Y_0)\Phi_2\left(\mathrm{K}_n(x, X_0)\right)\right] = b_n^N \int_{\mathbb{R}^N} \mathbb{E}\left[\Phi_1\left(Y_0\right) | X_0 = x - vb_n\right] \Phi_2\left(\mathrm{K}(v)\right) f(x - vb_n) dv.$$

By Theorem 1A in [22], we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \mathbb{E}\left[\Phi_1\left(Y_0\right) | X_0 = x - vb_n\right] \Phi_2\left(\mathrm{K}(v)\right) f(x - vb_n) dv$$
$$= \mathbb{E}\left[\Phi_1\left(Y_0\right) | X_0 = x\right] f(x) \int_{\mathbb{R}^N} \Phi_2\left(\mathrm{K}(v)\right) dv.$$

Consequently,

$$\lim_{n\to\infty} \frac{\mathbb{E}\left[\Phi_1\left(Y_0\right)\Phi_2\left(\mathrm{K}_n(x, X_0)\right)\right]}{b_n^N} = \mathbb{E}\left[\Phi_1\left(Y_0\right) | X_0 = x\right] f(x) \int_{\mathbb{R}^N} \Phi_2\left(\mathrm{K}(v)\right) dv. \tag{6}$$

In the other part, keeping in mind assumptions (A1) and (A2) and using (6), we derive

$$\sup_{\substack{j\in\mathbb{Z}^d \\ j\neq 0}} \mathbb{E}\left[\left|\mathrm{K}_n(x, X_0)\mathrm{K}_n(x, X_j)\right|\right] \leqslant \kappa \left(\int_{\mathbb{R}^N} |\mathrm{K}_n(x, u)| \, du\right)^2 + \left(\mathbb{E}\left[|\mathrm{K}_n(x, X_0)|\right]\right)^2 \trianglelefteq b_n^{2N}.$$

The proof of Lemma 1 is complete. □

Lemma 2 *If $(A3)$ holds, then there exists a sequence $(m_n)_{n\geqslant 1}$ of positive integers satisfying*

$$\lim_{n\to\infty} m_n = +\infty, \quad \lim_{n\to\infty} m_n^d b_n^{\frac{\theta N}{4+\theta}} = 0 \quad \text{and} \quad \begin{cases} \lim_{n\to\infty} b_n^{-\frac{\theta N}{2+\theta}} \sum_{|i|>m_n} \alpha_{1,\infty}^{\frac{\theta}{2+\theta}}(|i|) = 0 & \text{if } (A3)(i) \text{ holds} \\ \lim_{n\to\infty} b_n^{-\frac{\theta(N+2)+2N}{2(2+\theta)}} \sum_{|i|>m_n} |i|^d \delta_{i,2}^{\frac{\theta}{2+\theta}} = 0 & \text{if } (A3)(ii) \text{ holds.} \end{cases}$$

Notice that when $|\Lambda_n| b_n^N \longrightarrow \infty$, we have $m_n^d = o\left(|\Lambda_n|\right)$.

Proof. First, we assume $(A3)(i)$. So, we have $\mathbb{E}\left[|Y_0|^{2+\theta}\right] < \infty$ and $\sum_{i\in\mathbb{Z}^d} |i|^{\frac{d(4+\theta)}{2+\theta}} \alpha_{1,\infty}^{\frac{\theta}{2+\theta}}(|i|) < \infty$ for some $\theta > 0$. Let $\gamma > (4+\theta)/(2+\theta)$ be fixed and let $(m_n)_{n\geq 1}$ be defined by

$$m_n = \max\left\{ v_n, \left[b_n^{\frac{-\theta N}{d(4+\theta)}} \left(\sum_{|i|>v_n} |i|^{\frac{d(4+\theta)}{2+\theta}} \alpha_{1,\infty}^{\frac{\theta}{2+\theta}}(|i|) \right)^{\frac{1}{d\gamma}} \right] \right\} \quad \text{and} \quad v_n = \lfloor b_n^{\frac{-\theta N}{2d(4+\theta)}} \rfloor.$$

Since $v_n \longrightarrow \infty$, we have $m_n \longrightarrow \infty$ as n goes to infinity. Moreover,

$$m_n^d b_n^{\frac{\theta N}{4+\theta}} \trianglelefteq \max\left\{ b_n^{\frac{\theta N}{2(4+\theta)}}, \left(\sum_{|i|>v_n} |i|^{\frac{d(4+\theta)}{2+\theta}} \alpha_{1,\infty}^{\frac{\theta}{2+\theta}}(|i|) \right)^{\frac{1}{\gamma}} + b_n^{\frac{\theta N}{4+\theta}} \right\} \xrightarrow[n\to\infty]{} 0.$$

Since $v_n \leqslant m_n$, we have

$$m_n^d b_n^{\frac{\theta N}{4+\theta}} \geqslant \left(\sum_{|i|>m_n} |i|^{\frac{d(4+\theta)}{2+\theta}} \alpha_{1,\infty}^{\frac{\theta}{2+\theta}}(|i|) \right)^{\frac{1}{\gamma}}.$$

Consequently,

$$b_n^{-\frac{\theta N}{2+\theta}} \sum_{|i|>m_n} \alpha_{1,\infty}^{\frac{\theta}{2+\theta}}(|i|) \leqslant \left(m_n^d b_n^{\frac{\theta N}{4+\theta}} \right)^{-\frac{4+\theta}{2+\theta}} \sum_{|i|>m_n} |i|^{\frac{d(4+\theta)}{2+\theta}} \alpha_{1,\infty}^{\frac{\theta}{2+\theta}}(|i|) \leqslant \left(\sum_{|i|>m_n} |i|^{\frac{d(4+\theta)}{2+\theta}} \alpha_{1,\infty}^{\frac{\theta}{2+\theta}}(|i|) \right)^{\frac{\gamma(2+\theta)-4-\theta}{\gamma(2+\theta)}}.$$

Since $\gamma > (4+\theta)/(2+\theta)$, we derive $\lim_{n\to\infty} b_n^{-\frac{\theta N}{2+\theta}} \sum_{|i|>m_n} \alpha_{1,\infty}^{\frac{\theta}{2+\theta}}(|i|) = 0$.

Similarly, assume $(A3)(ii)$ holds and define

$$\tilde{m}_n = \max\left\{ v_n, \left[b_n^{\frac{-\theta N}{d(4+\theta)}} \left(\sum_{|i|>v_n} |i|^{\frac{d((3N+2)\theta^2+(10N+8)\theta+8N)}{2\theta(2+\theta)N}} \delta_{i,2}^{\frac{\theta}{2+\theta}} \right)^{\frac{1}{d\gamma}} \right] \right\} \quad \text{with} \quad \gamma > \frac{((N+2)\theta+2N)(\theta+4)}{2\theta(\theta+2)N}.$$

Then, arguing as before, we derive

$$\lim_{n\to\infty} \tilde{m}_n^d b_n^{\frac{\theta N}{4+\theta}} = 0 \quad \text{and} \quad \lim_{n\to\infty} b_n^{-\frac{\theta(N+2)+2N}{2(2+\theta)}} \sum_{|i|>\tilde{m}_n} |i|^d \delta_{i,2}^{\frac{\theta}{2+\theta}} = 0.$$

The details of the proof are left to the reader. The proof of Lemma 2 is complete. □

For any i in $\mathbb{Z}^d$, any positive integer n and any $x \in \mathbb{R}^N$, we denote

$$\Delta_i = \frac{\mathrm{K}_n(x, X_i) - \mathbb{E}\left[\mathrm{K}_n(x, X_0)\right]}{\sqrt{b_n^N}} \quad \text{and} \quad \Theta_i = \frac{Y_i \mathrm{K}_n(x, X_i) - \mathbb{E}\left[Y_0 \mathrm{K}_n(x, X_0)\right]}{\sqrt{b_n^N}}. \tag{7}$$

Lemma 3 *Assume (A1), (A2) and (A4). If there exists $\theta > 0$ such that $\mathbb{E}\left[|Y_0|^{2+\theta}\right] < \infty$, then* $\max\{\|\Delta_0\|_{2+\theta}^2, \|\Theta_0\|_{2+\theta}^2\} \trianglelefteq b_n^{-\frac{\theta N}{2+\theta}}$.

Proof. Let $\theta > 0$ such that $\mathbb{E}\left[|Y_0|^{2+\theta}\right] < \infty$, we have

$$\|\Delta_0\|_{2+\theta}^2 \leqslant \frac{2\,\|\mathrm{K}_n(x,X_0)\|_{2+\theta}^2}{b_n^N} + \frac{2\left(\mathbb{E}\left[\mathrm{K}_n(x,X_0)\right]\right)^2}{b_n^N}$$

and

$$\|\Theta_0\|_{2+\theta}^2 \leqslant \frac{2\,\|Y_0\mathrm{K}_n(x,X_0)\|_{2+\theta}^2}{b_n^N} + \frac{2\left(\mathbb{E}\left[Y_0\mathrm{K}_n(x,X_0)\right]\right)^2}{b_n^N}.$$

Keeping in mind that $|\mathrm{K}|_\infty := \sup_{t\in\mathbb{R}^N}|\mathrm{K}(t)| < \infty$ and using Lemma 1, we derive

$$\mathbb{E}\left[|\mathrm{K}_n(x,X_0)|^{2+\theta}\right] \trianglelefteq b_n^N, \qquad \left|\mathbb{E}\left[\mathrm{K}_n(x,X_0)\right]\right| \trianglelefteq b_n^N,$$
$$\mathbb{E}\left[|Y_0\mathrm{K}_n(x,X_0)|^{2+\theta}\right] \trianglelefteq b_n^N \quad \text{and} \quad \left|\mathbb{E}\left[Y_0\mathrm{K}_n(x,X_0)\right]\right| \trianglelefteq b_n^N.$$

Consequently, we obtain $\max\{\|\Delta_0\|_{2+\theta}^2, \|\Theta_0\|_{2+\theta}^2\} \trianglelefteq b_n^{-\frac{\theta N}{2+\theta}}$. The proof of Lemma 3 is complete. □

Lemma 4 *Assume (A1), (A2) and (A4). Then,* $\sup_{\substack{j\in\mathbb{Z}^d\\ j\neq 0}} \mathbb{E}\left[|\Delta_0\Delta_j|\right] \trianglelefteq b_n^N$*. Moreover, if* $\mathbb{E}\left[|Y_0|^{2+\theta}\right] < \infty$ *for some* $\theta > 0$ *then*

$$\sup_{\substack{j\in\mathbb{Z}^d\\ j\neq 0}} \mathbb{E}\left[|\Theta_0\Theta_j|\right] \trianglelefteq b_n^{\frac{\theta N}{4+\theta}} \quad \textit{and} \quad \sup_{\substack{j\in\mathbb{Z}^d\\ j\neq 0}} \mathbb{E}\left[|\Theta_0\Delta_j|\right] \trianglelefteq b_n^{\frac{\theta N}{2+\theta}}.$$

Proof. Let $j \neq 0$ in $\mathbb{Z}^d$, then

$$\mathbb{E}\left[|\Delta_0\Delta_j|\right] \leqslant \frac{\mathbb{E}\left[|\mathrm{K}_n(x,X_0)\mathrm{K}_n(x,X_j)|\right] + 3\left(\mathbb{E}\left[|\mathrm{K}_n(x,X_0)|\right]\right)^2}{b_n^N}.$$

Applying Lemma 1, we get

$$\sup_{\substack{j\in\mathbb{Z}^d\\ j\neq 0}} \mathbb{E}\left[|\Delta_0\Delta_j|\right] \trianglelefteq b_n^N.$$

Let $L \geqslant 1$. We have

$$\mathbb{E}\left[|\Theta_0\Theta_j|\right] \leqslant \frac{\mathbb{E}\left[|Y_0Y_j\mathrm{K}_n(x,X_0)\mathrm{K}_n(x,X_j)|\right] + 3\left(\mathbb{E}\left[|Y_0\mathrm{K}_n(x,X_0)|\right]\right)^2}{b_n^N} \tag{8}$$

and

$$\begin{aligned}
\mathbb{E}\left[|Y_0Y_j\mathrm{K}_n(x,X_0)\mathrm{K}_n(x,X_j)|\right] =& \mathbb{E}\left[|Y_0Y_j\,\mathbb{1}_{|Y_0|\leqslant L}\,\mathbb{1}_{|Y_j|\leqslant L}\mathrm{K}_n(x,X_0)\mathrm{K}_n(x,X_j)|\right]\\
&+ \mathbb{E}\left[|Y_0Y_j\,\mathbb{1}_{|Y_0|\leqslant L}\,\mathbb{1}_{|Y_j|>L}\mathrm{K}_n(x,X_0)\mathrm{K}_n(x,X_j)|\right]\\
&+ \mathbb{E}\left[|Y_0Y_j\,\mathbb{1}_{|Y_0|>L}\,\mathbb{1}_{|Y_j|\leqslant L}\mathrm{K}_n(x,X_0)\mathrm{K}_n(x,X_j)|\right]\\
&+ \mathbb{E}\left[|Y_0Y_j\,\mathbb{1}_{|Y_0|>L}\,\mathbb{1}_{|Y_j|>L}\mathrm{K}_n(x,X_0)\mathrm{K}_n(x,X_j)|\right].
\end{aligned}$$

By Cauchy-Schwarz's inequality, we derive

$$\begin{aligned}\mathbb{E}\left[\left|Y_0 Y_j \mathrm{K}_n(x,X_0)\mathrm{K}_n(x,X_j)\right|\right] \leqslant & L^2\mathbb{E}\left[\left|\mathrm{K}_n(x,X_0)\mathrm{K}_n(x,X_j)\right|\right] \\ & + \sqrt{\mathbb{E}\left[Y_0^2\mathrm{K}_n^2(x,X_0)\right]}\sqrt{\mathbb{E}\left[Y_0^2\,\mathbb{1}_{|Y_0|>L}\mathrm{K}_n^2(x,X_0)\right]} \\ & + \sqrt{\mathbb{E}\left[Y_0^2\,\mathbb{1}_{|Y_0|>L}\mathrm{K}_n^2(x,X_0)\right]}\sqrt{\mathbb{E}\left[Y_0^2\mathrm{K}_n^2(x,X_0)\right]} \\ & + \mathbb{E}\left[Y_0^2\,\mathbb{1}_{|Y_0|>L}\mathrm{K}_n^2(x,X_0)\right].\end{aligned}$$

Let $\theta > 0$ such that $\mathbb{E}\left[|Y_0|^{2+\theta}\right] < \infty$. Applying Lemma 1, we get

$$\frac{\mathbb{E}\left[\left|Y_0 Y_j \mathrm{K}_n(x,X_0)\mathrm{K}_n(x,X_j)\right|\right]}{b_n^N} \trianglelefteq L^2 b_n^N + L^{-\theta/2} + L^{-\theta} \trianglelefteq L^2 b_n^N + L^{-\theta/2}. \tag{9}$$

Making the choice $L = b_n^{-\frac{2N}{4+\theta}}$ and combining (8), (9) and Lemma 1, we obtain

$$\sup_{\substack{j\in\mathbb{Z}^d\\ j\neq 0}} \mathbb{E}\left[\left|\Theta_0\Theta_j\right|\right] \trianglelefteq b_n^{\frac{\theta N}{4+\theta}}.$$

Now,

$$\mathbb{E}\left[\left|\Theta_0\Delta_j\right|\right] \leqslant \frac{\mathbb{E}\left[\left|Y_0\mathrm{K}_n(x,X_0)\mathrm{K}_n(x,X_j)\right|\right] + 3\mathbb{E}\left[|Y_0\mathrm{K}_n(x,X_0)|\right]\mathbb{E}\left[|\mathrm{K}_n(x,X_0)|\right]}{b_n^N}.$$

So, if $L' \geqslant 1$ is fixed then

$$\begin{aligned}\mathbb{E}\left[\left|\Theta_0\Delta_j\right|\right] \leqslant & \frac{L'\mathbb{E}\left[\left|\mathrm{K}_n(x,X_0)\mathrm{K}_n(x,X_j)\right|\right]}{b_n^N} + \frac{\mathbb{E}\left[\left|Y_0\right|\mathbb{1}_{|Y_0|>L'}\left|\mathrm{K}_n(x,X_0)\mathrm{K}_n(x,X_j)\right|\right]}{b_n^N} \\ & + \frac{3\mathbb{E}\left[|Y_0\mathrm{K}_n(x,X_0)|\right]\mathbb{E}\left[|\mathrm{K}_n(x,X_0)|\right]}{b_n^N}.\end{aligned}$$

By Cauchy-Schwarz's inequality, we get

$$\mathbb{E}\left[\left|Y_0\right|\mathbb{1}_{|Y_0|>L'}\left|\mathrm{K}_n(x,X_0)\mathrm{K}_n(x,X_j)\right|\right] \leqslant L'^{-\theta/2}\sqrt{\mathbb{E}\left[|Y_0|^{2+\theta}\mathrm{K}_n^2(x,X_0)\right]}\sqrt{\mathbb{E}\left[\mathrm{K}_n^2(x,X_0)\right]}.$$

Appplying Lemma 1 and making the choice $L' = b_n^{-\frac{2N}{2+\theta}}$, we obtain

$$\sup_{\substack{j\in\mathbb{Z}^d\\ j\neq 0}} \mathbb{E}\left[\left|\Theta_0\Delta_j\right|\right] \trianglelefteq L' b_n^N + L'^{-\theta/2} + b_n^N \trianglelefteq b_n^{\frac{\theta N}{2+\theta}}.$$

The proof of Lemma 4 is complete. □

The following proposition is a crucial tool in the proof of the asymptotic normality for the NWE (Theorem 2) when the random field $(X_i)_{i\in\mathbb{Z}^d}$ is of the form (1).

Proposition 1 *Let n and M be two positive integers and let $x \in \mathbb{R}^N$. If Λ is a finite subset of $\mathbb{Z}^d$ and $\Phi : \mathbb{R} \longrightarrow \mathbb{R}$ is a measurable function such that $\|\Phi(Y_0)\|_{2+\theta} < \infty$ for some $\theta \in]0, +\infty]$ then for any family $(c_i)_{i\in\Lambda}$ of real numbers and any $(p, q) \in [2, +\infty[\times]0, +\infty]$ such that $p + q \leqslant 2 + \theta$, we have*

$$\left\|\sum_{i\in\Lambda} c_i W_{i,n}\right\|_p \leqslant 8pM^d|\mathrm{K}|_\infty^{\frac{p}{p+q}} C(p, q)\sqrt{\sum_{i\in\Lambda} c_i^2 b_n^{\frac{-q}{p+q}} \sum_{|i|>M} \delta_{i,p}^{\frac{q}{p+q}}},$$

where

$$W_{i,n} := \Phi(Y_i)\mathrm{K}_n(x, X_i) - \mathbb{E}\left[\Phi(Y_i)\mathrm{K}_n(x, X_i)|\, \mathcal{H}_{i,M}\right], \tag{10}$$

$$C(p, q) = 2^{\frac{2p+q}{p+q}} \|\Phi(Y_0)\|_{p+q} \|\mathrm{K}\|_{\mathrm{Lip}}^{\frac{q}{p+q}} + |\mathrm{K}|_\infty^{\frac{q}{p+q}} \left\| \sup_{\substack{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N \\ x\neq y}} \frac{|\Phi(R(x, \eta_0)) - \Phi(R(y, \eta_0))|}{\|x - y\|} \right\|_p.$$

and

$$\mathcal{H}_{i,M} = \sigma\left(\eta_i, \varepsilon_{i-s}\, ; |s| \leqslant M\right) \quad \textit{and} \quad \|\mathrm{K}\|_{\mathrm{Lip}} = \sup_{\substack{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N \\ x\neq y}} \frac{|\mathrm{K}(x) - \mathrm{K}(y)|}{\|x - y\|}.$$

Proof. Let M and n be two positive integers and let x in $\mathbb{R}^N$ and i in $\mathbb{Z}^d$ be fixed. Recall that $Y_i = R(X_i, \eta_i)$. We follow the same lines as in the proof of Proposition 1 in [11]. Let $2 \leqslant p < 2 + \theta$ and denote by H_n the measurable function such that $W_{i,n} = \mathrm{H}_n(\mathcal{H}_{i,\infty})$ with $\mathcal{H}_{i,\infty} = \sigma\left(\eta_i, \varepsilon_{i-s}\, ; s \in \mathbb{Z}^d\right)$. Then, we define the physical dependence measure coefficient $\delta_{i,p}^{(n)}$ associated to $W_{i,n}$ by $\delta_{i,p}^{(n)} = \|W_{i,n} - W_{i,n}^*\|_p$, where $W_{i,n}^* = \mathrm{H}_n(\mathcal{H}_{i,\infty}^*)$ and $\mathcal{H}_{i,\infty}^* = \sigma\left(\eta_i, \varepsilon_{i-s}^*\, ; s \in \mathbb{Z}^d\right)$ keeping in mind that $\varepsilon_j^* = \varepsilon_j$ if $j \neq 0$ and $\varepsilon_0^* = \varepsilon_0'$. In other words, we obtain $W_{i,n}^*$ from $W_{i,n}$ by just replacing ε_0 by its copy ε_0' (see [30]). Let τ be a bijection from $\mathbb{Z}$ to $\mathbb{Z}^d$ and ℓ in $\mathbb{Z}$ be fixed. We define the projection operateur P_ℓ by $P_\ell f = \mathbb{E}\left[f|\mathcal{F}_\ell\right] - \mathbb{E}\left[f|\mathcal{F}_{\ell-1}\right]$ for any integrable function f, where $\mathcal{F}_\ell = \sigma\left(\varepsilon_{\tau(s)}; s \leq \ell\right)$. Consequently, by stationarity, we have

$$\left\|P_\ell W_{i,n}\right\|_p = \left\|\mathbb{E}\left[W_{0,n}|T^i\mathcal{F}_\ell\right] - \mathbb{E}\left[W_{0,n}|T^i\mathcal{F}_{\ell-1}\right]\right\|_p,$$

where $T^i\mathcal{F}_\ell = \sigma\left(\varepsilon_{\tau(s)-i}; s \leq \ell\right)$. Keeping in mind that $W_{0,n} = \mathrm{H}_n\left(\mathcal{H}_{0,\infty}\right)$, we derive

$$\left\|P_\ell W_{i,n}\right\|_p = \left\|\mathbb{E}\left[\mathrm{H}_n\left(\mathcal{H}_{0,\infty}\right)|\, T^i\mathcal{F}_\ell\right] - \mathbb{E}\left[\mathrm{H}_n\left(\mathcal{H}_{0,\infty}^{(i,\ell)}\right)|\, T^i\mathcal{F}_\ell\right]\right\|_p \leqslant \left\|\mathrm{H}_n\left(\mathcal{H}_{0,\infty}\right) - \mathrm{H}_n\left(\mathcal{H}_{0,\infty}^{(i,\ell)}\right)\right\|_p,$$

where $\mathcal{H}_{0,\infty}^{(i,\ell)} = \sigma\left(\eta, \varepsilon_{\tau(\ell)-i}', \varepsilon_{-s}\, ; s \in \mathbb{Z}^d\backslash\{i - \tau(\ell)\}\right)$. It means that $\mathcal{H}_{0,\infty}^{(i,\ell)}$ is obtained from $\mathcal{H}_{0,\infty}$ by replacing $\varepsilon_{\tau(\ell)-i}$ by its copy $\varepsilon_{\tau(\ell)-i}'$. Consequently, using again the stationarity of the random field and noting that

$$T^{\tau(\ell)-i}\mathcal{H}_{0,\infty} = \sigma\left(\eta_{i-\tau(\ell)}, \varepsilon_{i-\tau(\ell)-s}\, ; s \in \mathbb{Z}^d\right) = \mathcal{H}_{i-\tau(\ell),\infty},$$
$$T^{\tau(\ell)-i}\mathcal{H}_{0,\infty}^{(i,\ell)} = \sigma\left(\eta_{i-\tau(\ell)}, \varepsilon_0', \varepsilon_{i-\tau(\ell)-s}\, ; s \in \mathbb{Z}^d\backslash\{i - \tau(\ell)\}\right) = \mathcal{H}_{i-\tau(\ell),\infty}^*,$$

we obtain

$$\left\|P_\ell W_{i,n}\right\|_p \leqslant \left\|\mathrm{H}_n\left(T^{\tau(\ell)-i}\mathcal{H}_{0,\infty}\right) - \mathrm{H}_n\left(T^{\tau(\ell)-i}\mathcal{H}_{0,\infty}^{(i,\ell)}\right)\right\|_p = \left\|W_{i-\tau(\ell),n} - W_{i-\tau(\ell),n}^*\right\|_p = \delta_{i-\tau(\ell),p}^{(n)}. \tag{11}$$

Moreover, since $W_{i,n} = \sum_{\ell\in\mathbb{Z}} P_\ell W_{i,n}$, we have

$$\left\|\sum_{j\in\Lambda} c_j W_{j,n}\right\|_p = \left\|\sum_{\ell\in\mathbb{Z}}\sum_{j\in\Lambda} c_j P_\ell W_{j,n}\right\|_p .$$

Since $\left(\sum_{j\in\Lambda} c_j P_\ell W_{j,n}\right)_{\ell\in\mathbb{Z}}$ is a martingale difference sequence with respect to the filtration $(\mathcal{F}_\ell)_{\ell\in\mathbb{Z}}$, the Burkholder inequality (see [6], remark 6, page 85) implies

$$\left\|\sum_{j\in\Lambda} c_j W_{j,n}\right\|_p \leq \left(2p\sum_{\ell\in\mathbb{Z}}\left\|\sum_{j\in\Lambda} c_j P_\ell W_{j,n}\right\|_p^2\right)^{\frac{1}{2}} \leq \left(2p\sum_{\ell\in\mathbb{Z}}\left(\sum_{j\in\Lambda}|c_j|\left\|P_\ell W_{j,n}\right\|_p\right)^2\right)^{\frac{1}{2}}. \tag{12}$$

Moreover, by the Cauchy-Schwarz inequality, we have

$$\left(\sum_{j\in\Lambda}|c_j|\left\|P_\ell W_{j,n}\right\|_p\right)^2 \leq \sum_{i\in\Lambda} c_i^2\left\|P_\ell W_{i,n}\right\|_p \times \sum_{j\in\Lambda}\|P_\ell W_{j,n}\|_p. \tag{13}$$

Using (11), we have $\sup_{\ell\in\mathbb{Z}}\sum_{j\in\mathbb{Z}^d}\|P_\ell W_{j,n}\|_p \leq \sum_{j\in\mathbb{Z}^d}\delta_{j,p}^{(n)}$. So, combining (12) and (13), we obtain

$$\left\|\sum_{j\in\Lambda} c_j W_{j,n}\right\|_p \leq \left(2p\sum_{j\in\mathbb{Z}^d}\delta_{j,p}^{(n)}\sum_{i\in\Lambda} c_i^2\sum_{\ell\in\mathbb{Z}}\left\|P_\ell W_{i,n}\right\|_p\right)^{\frac{1}{2}}.$$

Using (11) and keeping in mind that τ is a bijection, we have $\sup_{i\in\mathbb{Z}^d}\sum_{\ell\in\mathbb{Z}}\|P_\ell W_{i,n}\|_p \leq \sum_{j\in\mathbb{Z}^d}\delta_{j,p}^{(n)}$. Hence, we derive

$$\left\|\sum_{j\in\Lambda} c_j W_{j,n}\right\|_p \leq \left(2p\sum_{j\in\Lambda} c_j^2\right)^{\frac{1}{2}}\sum_{j\in\mathbb{Z}^d}\delta_{j,p}^{(n)}. \tag{14}$$

Now, since

$$\mathbb{E}\left[\Phi(Y_i)\mathrm{K}_n(x,X_i)|\mathcal{H}_{i,M}\right]^* = \mathbb{E}\left[\Phi(R(X_i^*,\eta_i))\mathrm{K}_n(x,X_i^*)|\mathcal{H}_{i,M}^*\right],$$

where $\mathcal{H}_{i,M}^* = \sigma\left(\eta_i, \varepsilon_{i-s}^*\,;\,|s|\leq M\right)$, we have

$$W_{i,n}^* = \Phi(R(X_i^*,\eta_i))\mathrm{K}_n(x,X_i^*) - \mathbb{E}\left[\Phi(R(X_i^*,\eta_i))\mathrm{K}_n(x,X_i^*)|\mathcal{H}_{i,M}^*\right].$$

Moreover,

$$\mathbb{E}\left[\Phi(Y_i)\mathrm{K}_n(x,X_i)|\mathcal{H}_{i,M}\right] = \mathbb{E}\left[\Phi(Y_i)\mathrm{K}_n(x,X_i)|\mathcal{H}_{i,M}\vee\mathcal{H}_{i,M}^*\right]$$

and

$$\mathbb{E}\left[\Phi(R(X_i^*,\eta_i))\mathrm{K}_n(x,X_i^*)|\mathcal{H}_{i,M}^*\right] = \mathbb{E}\left[\Phi(R(X_i^*,\eta_i))\mathrm{K}_n(x,X_i^*)|\mathcal{H}_{i,M}^*\vee\mathcal{H}_{i,M}\right].$$

Consequently,

$$\delta_{i,p}^{(n)} = \left\|W_{i,n} - W_{i,n}^*\right\|_p \leqslant 2\left\|\Phi(R(X_i,\eta_i))\mathrm{K}_n(x,X_i) - \Phi(R(X_i^*,\eta_i))\mathrm{K}_n(x,X_i^*)\right\|_p. \tag{15}$$

Let $L > 0$ be fixed. From (15), we derive

$$\delta_{i,p}^{(n)} \leqslant 2\left\|\Phi(R(X_i,\eta_i))\left(\mathrm{K}_n(x,X_i)-\mathrm{K}_n(x,X_i^*)\right)-\left(\Phi(R(X_i^*,\eta_i))-\Phi(R(X_i,\eta_i))\right)\mathrm{K}_n(x,X_i^*)\right\|_p$$
$$\leqslant 2L\left\|\mathrm{K}\right\|_{\mathrm{Lip}}\frac{\delta_{i,p}}{b_n}+4|\mathrm{K}|_\infty L^{-q/p}\left\|\Phi(Y_0)\right\|_{p+q}^{\frac{p+q}{p}}+2|\mathrm{K}|_\infty\left\|\sup_{\substack{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N\\ x\neq y}}\frac{|\Phi(R(x,\eta_0))-\Phi(R(y,\eta_0))|}{\|x-y\|}\right\|_p\delta_{i,p}.$$

Optimizing this last inequality in L, we get

$$\delta_{i,p}^{(n)} \leqslant 2|\mathrm{K}|_\infty^{\frac{p}{p+q}}C(p,q)b_n^{\frac{-q}{p+q}}\delta_{i,p}^{\frac{q}{p+q}}, \tag{16}$$

where

$$C(p,q)=2^{\frac{2p+q}{p+q}}\left\|\Phi(Y_0)\right\|_{p+q}\left\|\mathrm{K}\right\|_{\mathrm{Lip}}^{\frac{q}{p+q}}+|\mathrm{K}|_\infty^{\frac{q}{p+q}}\left\|\sup_{\substack{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N\\ x\neq y}}\frac{|\Phi(R(x,\eta_0))-\Phi(R(y,\eta_0))|}{\|x-y\|}\right\|_p.$$

Now, by stationarity, we have $\delta_{i,p}^{(n)}=\left\|W_{i,n}-W_{i,n}^*\right\|_p \leqslant 2\left\|W_{0,n}\right\|_p$. Let $\ell \geqslant 0$ be a fixed integer. We denote by Γ_ℓ the set of all j in $\mathbb{Z}^d$ such that $|j|=\ell$ and we define

$$a_\ell := \sum_{j=0}^{\ell}|\Gamma_j| = 1+2d\sum_{j=1}^{\ell}(2j+1)^{d-1}.$$

If $u=(u_1,\dots,u_d)$ and $v=(v_1,\dots,v_d)$ are distinct elements of $\mathbb{Z}^d$, the notation $u <_{\mathrm{lex}} v$ means that either $u_1 < v_1$ or for some k in $\{2,\dots,d\}$, $u_k < v_k$ and $u_s = v_s$ for $1 \leqslant s < k$ (lexicographic order). Let $\tau_0 :]0,+\infty[\cap\mathbb{Z} \longrightarrow \mathbb{Z}^d$ be the bijection defined by

- $\tau_0(1)=0$,
- $\tau_0(s)\in\Gamma_\ell$ if $a_{\ell-1} < s \leqslant a_\ell$ and $\ell > 0$,
- $\tau_0(s) <_{\mathrm{lex}} \tau_0(t)$ if $a_{\ell-1} < s < t \leqslant a_\ell$ and $\ell > 0$.

Let $\mathcal{G}_M=\sigma\left(\eta_0,\varepsilon_{\tau_0(s)}\,;1\leqslant s\leqslant M\right)$ and recall that $\mathcal{H}_{0,M}=\sigma\left(\eta_0,\varepsilon_{-s}\,;|s|\leqslant M\right)$. Since $1 \le s \le a_M$ if and only if $|\tau_0(s)| \le M$, we have $\mathcal{G}_{a_M}=\mathcal{H}_{0,M}$. Consequently,

$$W_{0,n}=\sum_{\ell>a_M}D_\ell \quad \text{where} \quad D_\ell=\mathbb{E}\left[\Phi(R(X_0,\eta_0))\mathrm{K}_n(x,X_0)|\mathcal{G}_\ell\right]-\mathbb{E}\left[\Phi(R(X_0,\eta_0))\mathrm{K}_n(x,X_0)|\mathcal{G}_{\ell-1}\right].$$

Since $(D_\ell)_{\ell\geqslant1}$ is a martingale difference sequence with respect to the filtration $(\mathcal{G}_\ell)_{\ell\geqslant1}$, we apply Burkholder's inequality ([6], remark 6, page 85) and we obtain

$$\left\|W_{0,n}\right\|_p \leqslant \left(2p\sum_{\ell>a_M}\|D_\ell\|_p^2\right)^{1/2}. \tag{17}$$

Let $L>0$ be fixed. Denoting $X_{0,\ell}^{'}=G\left(\eta_0,\varepsilon_{\tau_0(\ell)}^{'},\varepsilon_{-s}\,;s\in\mathbb{Z}^d\backslash\{-\tau_0(\ell)\}\right)$, we have

$$\mathbb{E}\left[\Phi(R(X_0,\eta_0))\mathrm{K}_n(x,X_0)|\mathcal{G}_{\ell-1}\right]=\mathbb{E}\left[\Phi(R(X_{0,\ell}^{'},\eta_0))\mathrm{K}_n(x,X_{0,\ell}^{'})|\mathcal{G}_\ell\right].$$

Then

$$\begin{aligned}
\|D_\ell\|_p &\leqslant \left\|\Phi(R(X_0,\eta_0))\mathrm{K}_n(x,X_0)-\Phi(R(X_{0,\ell}',\eta_0))\mathrm{K}_n(x,X_{0,\ell}')\right\|_p\\
&= \left\|\Phi(R(X_0,\eta_0))\left(\mathrm{K}_n(x,X_0)-\mathrm{K}_n(x,X_{0,\ell}')\right)-\left(\Phi(R(X_{0,\ell}',\eta_0))-\Phi(R(X_0,\eta_0))\right)\mathrm{K}_n(x,X_{0,\ell}')\right\|_p\\
&\leqslant \frac{L\,\|\mathrm{K}\|_{\mathrm{Lip}}}{b_n}\left\|X_0-X_{0,\ell}'\right\|_p+2|\mathrm{K}|_\infty L^{-q/p}\,\|\Phi(Y_0)\|_{p+q}^{\frac{p+q}{p}}+|\mathrm{K}|_\infty\left\|\sup_{\substack{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N\\ x\neq y}}\frac{|\Phi(R(x,\eta_0))-\Phi(R(y,\eta_0))|}{\|x-y\|}\right\|_p\left\|X_0-X_{0,\ell}'\right\|_p.
\end{aligned}$$

Optimizing in L, we obtain

$$\|D_\ell\|_p \leqslant |\mathrm{K}|_\infty^{\frac{p}{p+q}}C(p,q)b_n^{\frac{-q}{p+q}}\left\|X_0-X_{0,\ell}'\right\|_p^{\frac{q}{p+q}}.$$

Moreover, by stationarity, we have

$$\begin{aligned}
\left\|X_0-X_{0,\ell}'\right\|_p &= \left\|G\left(\varepsilon_{-s}\,;s\in\mathbb{Z}^d\right)-G\left(\varepsilon_{\tau_0(\ell)}',\varepsilon_{-s}\,;s\in\mathbb{Z}^d\backslash\{-\tau_0(\ell)\}\right)\right\|_p\\
&= \left\|G\left(\varepsilon_{-\tau_0(\ell)-s}\,;s\in\mathbb{Z}^d\right)-G\left(\varepsilon_0',\varepsilon_{-\tau_0(\ell)-s}\,;s\in\mathbb{Z}^d\backslash\{-\tau_0(\ell)\}\right)\right\|_p\\
&= \left\|X_{-\tau_0(\ell)}-X_{-\tau_0(\ell)}^*\right\|_p\\
&= \delta_{-\tau_0(\ell),p}.
\end{aligned}$$

So, we derive

$$\|D_\ell\|_p \leqslant |\mathrm{K}|_\infty^{\frac{p}{p+q}}C(p,q)b_n^{\frac{-q}{p+q}}\delta_{-\tau_0(\ell),p}^{\frac{q}{p+q}}. \tag{18}$$

Combining (17) and (18), we obtain

$$\left\|W_{0,n}\right\|_p \leqslant |\mathrm{K}|_\infty^{\frac{p}{p+q}}\sqrt{2p}C(p,q)b_n^{\frac{-q}{p+q}}\sum_{\ell>a_M}\delta_{-\tau_0(\ell),p}^{\frac{q}{p+q}} \leqslant |\mathrm{K}|_\infty^{\frac{p}{p+q}}\sqrt{2p}C(p,q)b_n^{\frac{-q}{p+q}}\sum_{|j|>M}\delta_{j,p}^{\frac{q}{p+q}} \tag{19}$$

and

$$\sup_{i\in\mathbb{Z}^d}\delta_{i,p}^{(n)} \leqslant 2\left\|W_{0,n}\right\|_p \leqslant 2\sqrt{2p}|\mathrm{K}|_\infty^{\frac{p}{p+q}}C(p,q)b_n^{\frac{-q}{p+q}}\sum_{|j|>M}\delta_{j,p}^{\frac{q}{p+q}}. \tag{20}$$

So, from (16) and (20), we get

$$\sum_{i\in\mathbb{Z}^d}\delta_{i,p}^{(n)} \leqslant 2\sqrt{2p}|\mathrm{K}|_\infty^{\frac{p}{p+q}}C(p,q)b_n^{\frac{-q}{p+q}}(M^d+1)\sum_{|j|>M}\delta_{j,p}^{\frac{q}{p+q}}. \tag{21}$$

Finally, combining (14) and (21), we derive

$$\left\|\sum_{i\in\Lambda}c_iW_{i,n}\right\|_p \leq 8pM^d|\mathrm{K}|_\infty^{\frac{p}{p+q}}C(p,q)\left(\sum_{i\in\Lambda}c_i^2\right)^{\frac{1}{2}}b_n^{\frac{-q}{p+q}}\sum_{|j|>M}\delta_{j,p}^{\frac{q}{p+q}}.$$

The proof of Proposition 1 is complete. □

Now, we denote by $\mathbb{V}(Z)$ the variance of any square-integrable $\mathbb{R}$-valued random variable Z.

Lemma 5 *Assume (A1) – (A4). For any $x \in \mathbb{R}^N$ such that $f(x) > 0$, we have*

$$\lim_{n\to\infty} |\Lambda_n| b_n^N \mathbb{V}\left[f_n(x)\right] = f(x) \int_{\mathbb{R}^N} \mathrm{K}^2(t)dt,$$

$$\lim_{n\to\infty} |\Lambda_n| b_n^N \mathbb{V}\left[\varphi_n(x)\right] = \mathbb{E}\left[Y_0^2|X_0 = x\right] f(x) \int_{\mathbb{R}^N} \mathrm{K}^2(t)dt,$$

$$\lim_{n\to\infty} |\Lambda_n| b_n^N Cov\left[\varphi_n(x), f_n(x)\right] = r(x)f(x) \int_{\mathbb{R}^N} \mathrm{K}^2(t)dt.$$

Proof. Let $n \geqslant 1$ and $x \in \mathbb{R}^N$ such that $f(x) > 0$ be fixed. Then,

$$|\Lambda_n| b_n^N \mathbb{V}\left[f_n(x)\right] = \mathbb{E}\left[\left(\frac{\sum_{i\in\Lambda_n} \Delta_i}{\sqrt{|\Lambda_n|}}\right)^2\right] = \mathbb{E}\left[\Delta_0^2\right] + \frac{1}{|\Lambda_n|} \sum_{\substack{j\in\mathbb{Z}^d\\ j\neq 0}} |\Lambda_n \cap (\Lambda_n - j)| \, \mathbb{E}\left[\Delta_0 \Delta_j\right].$$

Consequently,

$$\left| |\Lambda_n| b_n^N \mathbb{V}\left[f_n(x)\right] - \mathbb{E}\left[\Delta_0^2\right] \right| \leqslant \sum_{\substack{j\in\mathbb{Z}^d\\ j\neq 0}} \left| \mathbb{E}\left[\Delta_0 \Delta_j\right] \right|, \tag{22}$$

where

$$\mathbb{E}\left[\Delta_0^2\right] = \frac{\mathbb{E}\left[\mathrm{K}_n^2(x, X_0)\right] - \left(\mathbb{E}\left[\mathrm{K}_n(x, X_0)\right]\right)^2}{b_n^N}.$$

Applying Lemmas 1 and 4, we get

$$\lim_{n\to\infty} \mathbb{E}\left[\Delta_0^2\right] = f(x) \int_{\mathbb{R}^N} \mathrm{K}^2(v)dv \tag{23}$$

and

$$\sup_{\substack{j\in\mathbb{Z}^d\\ j\neq 0}} \left| \mathbb{E}\left[\Delta_0 \Delta_j\right] \right| \trianglelefteq b_n^N \leqslant b_n^{\frac{\theta N}{4+\theta}}, \tag{24}$$

where $\theta > 0$ is given by (A3). Similarly, we have

$$|\Lambda_n| b_n^N \mathbb{V}\left[\varphi_n(x)\right] = \mathbb{E}\left[\left(\frac{\sum_{i\in\Lambda_n} \Theta_i}{\sqrt{|\Lambda_n|}}\right)^2\right] = \mathbb{E}\left[\Theta_0^2\right] + \frac{1}{|\Lambda_n|} \sum_{\substack{j\in\mathbb{Z}^d\\ j\neq 0}} |\Lambda_n \cap (\Lambda_n - j)| \, \mathbb{E}\left[\Theta_0 \Theta_j\right].$$

So, we derive

$$\left| |\Lambda_n| b_n^N \mathbb{V}\left[\varphi_n(x)\right] - \mathbb{E}\left[\Theta_0^2\right] \right| \leqslant \sum_{\substack{j\in\mathbb{Z}^d\\ j\neq 0}} \left| \mathbb{E}\left[\Theta_0 \Theta_j\right] \right|, \tag{25}$$

where

$$\mathbb{E}\left[\Theta_0^2\right] = \frac{\mathbb{E}\left[Y_0^2 \mathrm{K}_n^2(x, X_0)\right] - \left(\mathbb{E}\left[Y_0 \mathrm{K}_n(x, X_0)\right]\right)^2}{b_n^N}.$$

Applying again Lemma 1, we obtain

$$\lim_{n\to\infty} \mathbb{E}\left[\Theta_0^2\right] = \mathbb{E}\left[Y_0^2 \,|\, X_0 = x\right] f(x) \int_{\mathbb{R}^N} \mathrm{K}^2(v)dv. \tag{26}$$

By Lemma 4, we have also

$$\sup_{\substack{j\in\mathbb{Z}^d\\ j\neq 0}}\left|\mathbb{E}\left[\Theta_0\Theta_j\right]\right| \trianglelefteq b_n^{\frac{\theta N}{4+\theta}}. \tag{27}$$

Arguing as before, we write

$$|\Lambda_n| b_n^N \operatorname{Cov}(\varphi_n(x), f_n(x)) = \mathbb{E}[\Theta_0\Delta_0] + \frac{1}{|\Lambda_n|}\sum_{\substack{j\in\mathbb{Z}^d\\ j\neq 0}} |\Lambda_n \cap (\Lambda_n - j)|\,\mathbb{E}\left[\Theta_0\Delta_j\right].$$

Consequently,

$$\left||\Lambda_n| b_n^N \operatorname{Cov}(\varphi_n(x), f_n(x)) - \mathbb{E}[\Theta_0\Delta_0]\right| \leqslant \sum_{\substack{j\in\mathbb{Z}^d\\ j\neq 0}}\left|\mathbb{E}\left[\Theta_0\Delta_j\right]\right|, \tag{28}$$

where, using Lemma 1, we have

$$\mathbb{E}[\Theta_0\Delta_0] = \frac{\mathbb{E}\left[Y_0 \mathrm{K}_n^2(x, X_0)\right] - \mathbb{E}[Y_0\mathrm{K}_n(x,X_0)]\,\mathbb{E}[\mathrm{K}_n(x,X_0)]}{b_n^N} \xrightarrow[n\to\infty]{} r(x)f(x)\int_{\mathbb{R}^N}\mathrm{K}^2(v)dv. \tag{29}$$

By Lemma 4, we have also

$$\sup_{\substack{j\in\mathbb{Z}^d\\ j\neq 0}}\left|\mathbb{E}\left[\Theta_0\Delta_j\right]\right| \trianglelefteq b_n^{\frac{\theta N}{2+\theta}} \leqslant b_n^{\frac{\theta N}{4+\theta}}. \tag{30}$$

Now, we assume that $(A3)(i)$ holds and we introduce the letter Ξ which can be replaced in the sequel by either Δ or Θ. By Rio's inequality (see [23]), for $j \neq 0$, we have

$$\left|\mathbb{E}\left[\Xi_0\Xi_j\right]\right| \leqslant 2\int_0^{2\alpha_{1,\infty}(|j|)} Q_{\Xi_0}^2(u)du \quad \text{where } Q_{\Xi_0}(u) = \inf\left\{ t \geqslant 0 \,\middle|\, \mathbb{P}(|\Xi_0| > t) \leqslant u\right\}.$$

Using Lemma 3 and noting that $Q_{\Xi_0}(u) \leqslant u^{-\frac{1}{2+\theta}}\|\Xi_0\|_{2+\theta}$, we derive

$$\left|\mathbb{E}\left[\Xi_0\Xi_j\right]\right| \trianglelefteq \alpha_{1,\infty}^{\frac{\theta}{2+\theta}}(|j|)\,\|\Xi_0\|_{2+\theta}^2 \trianglelefteq b_n^{-\frac{\theta N}{2+\theta}}\alpha_{1,\infty}^{\frac{\theta}{2+\theta}}(|j|). \tag{31}$$

Combining (24), (27) and (31) and using Lemma 2, we obtain

$$\sum_{\substack{j\in\mathbb{Z}^d\\ j\neq 0}}\left|\mathbb{E}\left[\Xi_0\Xi_j\right]\right| \trianglelefteq m_n^d b_n^{\frac{\theta N}{4+\theta}} + b_n^{-\frac{\theta N}{2+\theta}}\sum_{|j|>m_n}\alpha_{1,\infty}^{\frac{\theta}{2+\theta}}(|j|) \xrightarrow[n\to\infty]{} 0, \tag{32}$$

where m_n is given by Lemma 2. Combining (22), (23), (25), (26) and (32), we get

$$\lim_{n\to\infty}|\Lambda_n| b_n^N \mathbb{V}[f_n(x)] = f(x)\int_{\mathbb{R}^N}\mathrm{K}^2(t)dt \quad \text{and} \quad \lim_{n\to\infty}|\Lambda_n| b_n^N \mathbb{V}[\varphi_n(x)] = \mathbb{E}\left[Y_0^2 \mid X_0 = x\right] f(x)\int_{\mathbb{R}^N}\mathrm{K}^2(t)dt.$$

Applying again Rio's inequality, we have

$$\left|\mathbb{E}\left[\Theta_0\Delta_j\right]\right| \leqslant 2\int_0^{2\alpha_{1,\infty}(|j|)} Q_{\Theta_0}(u)Q_{\Delta_j}(u)du$$

and by Lemma 3, we derive

$$Q_{\Theta_0}(u) \leqslant u^{-\frac{1}{2+\theta}} \|\Theta_0\|_{2+\theta} \leqslant u^{-\frac{1}{2+\theta}} b_n^{-\frac{\theta N}{2(2+\theta)}} \quad \text{and} \quad Q_{\Delta_j}(u) \leqslant u^{-\frac{1}{2+\theta}} \|\Delta_0\|_{2+\theta} \leqslant u^{-\frac{1}{2+\theta}} b_n^{-\frac{\theta N}{2(2+\theta)}}$$

for any $u \in]0,1[$. Consequently, we obtain

$$\left|\mathbb{E}\left[\Theta_0\Delta_j\right]\right| \trianglelefteq b_n^{-\frac{\theta N}{2+\theta}} \alpha_{1,\infty}^{\frac{\theta}{2+\theta}}(|j|). \tag{33}$$

Combining (30) and (33), we derive

$$\sum_{\substack{j\in\mathbb{Z}^d\\ j\neq 0}} \left|\mathbb{E}\left[\Theta_0\Delta_j\right]\right| \trianglelefteq m_n^d b_n^{\frac{\theta N}{4+\theta}} + b_n^{-\frac{\theta N}{2+\theta}} \sum_{|j|>m_n} \alpha_{1,\infty}^{\frac{\theta}{2+\theta}}(|j|) \xrightarrow[n\to\infty]{} 0, \tag{34}$$

where m_n is given by Lemma 2. Finally, combining (28), (29) and (34), we obtain

$$\lim_{n\to\infty} |\Lambda_n| b_n^N \operatorname{Cov}\left[\varphi_n(x), f_n(x)\right] = r(x)f(x)\int_{\mathbb{R}^N} \mathrm{K}^2(t)dt.$$

From now on, we assume $(A3)(ii)$ holds. Keeping in mind that Ξ stands for either Δ or Θ, we define $\overline{\Xi}_i = \mathbb{E}\left[\Xi_i|\mathcal{H}_{i,m_n}\right]$ for any i in $\mathbb{Z}^d$. Note that $(\overline{\Xi}_i)_{i\in\mathbb{Z}^d}$ is a $2m_n$-dependent random field (it means that if $|i-j|>2m_n$ then $\overline{\Xi}_i$ and $\overline{\Xi}_j$ are independent) and

$$\left|\mathbb{E}\left[\left(\sum_{i\in\Lambda_n}\Xi_i\right)^2\right] - \mathbb{E}\left[\left(\sum_{i\in\Lambda_n}\overline{\Xi}_i\right)^2\right]\right| \leqslant \left\|\sum_{i\in\Lambda_n}\left(\Xi_i - \overline{\Xi}_i\right)\right\|_2^2 + 2\left\|\sum_{i\in\Lambda_n}\overline{\Xi}_i\right\|_2 \left\|\sum_{i\in\Lambda_n}\left(\Xi_i - \overline{\Xi}_i\right)\right\|_2. \tag{35}$$

Using Proposition 1 and Lemma 2, we obtain

$$|\Lambda_n|^{-1/2}\left\|\sum_{i\in\Lambda_n}\left(\Xi_i - \overline{\Xi}_i\right)\right\|_2 \trianglelefteq b_n^{-\frac{\theta(N+2)+2N}{2(2+\theta)}} \sum_{|j|>m_n} |j|^d \delta_{j,2}^{\frac{\theta}{2+\theta}} \xrightarrow[n\to\infty]{} 0. \tag{36}$$

In the other part, since $(\overline{\Xi}_i)_{i\in\mathbb{Z}^d}$ is $2m_n$-dependent, we have

$$\frac{1}{|\Lambda_n|}\mathbb{E}\left[\left(\sum_{i\in\Lambda_n}\overline{\Xi}_i\right)^2\right] = \mathbb{E}\left[\overline{\Xi}_0^2\right] + \frac{1}{|\Lambda_n|}\sum_{\substack{j\in\mathbb{Z}^d\setminus\{0\}\\ |j|\leqslant 2m_n}} |\Lambda_n \cap (\Lambda_n - j)|\,\mathbb{E}\left[\overline{\Xi}_0\overline{\Xi}_j\right]$$

and consequently

$$\left|\frac{1}{|\Lambda_n|}\mathbb{E}\left[\left(\sum_{i\in\Lambda_n}\overline{\Xi}_i\right)^2\right] - \mathbb{E}\left[\overline{\Xi}_0^2\right]\right| \leqslant (2m_n+1)^d \sup_{\substack{j\in\mathbb{Z}^d\\ j\neq 0}}\left|\mathbb{E}\left[\overline{\Xi}_0\overline{\Xi}_j\right]\right|. \tag{37}$$

Moreover, using (19) and Lemma 2 and noting that $\|\Xi_0\|_2 \trianglelefteq 1$, we have also

$$\left|\mathbb{E}\left[\overline{\Xi}_0^2\right] - \mathbb{E}\left[\Xi_0^2\right]\right| \leqslant 2\|\Xi_0\|_2 \left\|\overline{\Xi}_0 - \Xi_0\right\|_2 \trianglelefteq b_n^{-\frac{\theta(N+2)+2N}{2(2+\theta)}} \sum_{|j|>m_n} \delta_{j,2}^{\frac{\theta}{2+\theta}} \xrightarrow[n\to\infty]{} 0.$$

So, using (23) and (26), we derive

$$\lim_{n\to\infty} \mathbb{E}\left[\overline{\Xi}_0^2\right] = \xi(x)f(x)\int_{\mathbb{R}^N} \mathrm{K}^2(t)dt, \tag{38}$$

where $\xi(x) = 1$ if $\Xi = \Delta$ and $\xi(x) = \mathbb{E}\left[Y_0^2|X_0 = x\right]$ if $\Xi = \Theta$. Similarly, using (19), we obtain

$$m_n^d \sup_{\substack{j\in\mathbb{Z}^d\\ j\neq 0}} \left| |\mathbb{E}\left[\overline{\Xi}_0\overline{\Xi}_j\right]| - |\mathbb{E}\left[\Xi_0\Xi_j\right]| \right| \leqslant 2m_n^d \|\Xi_0\|_2 \left\|\overline{\Xi}_0 - \Xi_0\right\|_2 \trianglelefteq b_n^{-\frac{\theta(N+2)+2N}{2(2+\theta)}} \sum_{|j|>m_n} |j|^d \delta_{j,2}^{\frac{\theta}{2+\theta}} \xrightarrow[n\to\infty]{} 0.$$

Using Lemma 4, we have $\sup_{\substack{j\in\mathbb{Z}^d\\ j\neq 0}} |\mathbb{E}\left[\Xi_0\Xi_j\right]| \trianglelefteq b_n^{\frac{\theta N}{4+\theta}}$ and consequently, by Lemma 2, we get

$$m_n^d \sup_{\substack{j\in\mathbb{Z}^d\\ j\neq 0}} \left|\mathbb{E}\left[\overline{\Xi}_0\overline{\Xi}_j\right]\right| \trianglelefteq m_n^d b_n^{\frac{\theta N}{4+\theta}} + b_n^{-\frac{\theta(N+2)+2N}{2(2+\theta)}} \sum_{|j|>m_n} |j|^d \delta_{j,2}^{\frac{\theta}{2+\theta}} \xrightarrow[n\to\infty]{} 0. \tag{39}$$

Combining (37), (38) and (39), we obtain

$$\lim_{n\to\infty} \frac{1}{|\Lambda_n|}\mathbb{E}\left[\left(\sum_{i\in\Lambda_n} \overline{\Xi}_i\right)^2\right] = \xi(x)f(x)\int_{\mathbb{R}^N} \mathrm{K}^2(t)dt. \tag{40}$$

Combining (35), (36) and (40), we get

$$\lim_{n\to\infty} \frac{1}{|\Lambda_n|}\mathbb{E}\left[\left(\sum_{i\in\Lambda_n} \Xi_i\right)^2\right] = \xi(x)f(x)\int_{\mathbb{R}^N} \mathrm{K}^2(t)dt.$$

So, we have shown

$$\lim_{n\to\infty} |\Lambda_n|b_n^N \mathbb{V}\left[f_n(x)\right] = f(x)\int_{\mathbb{R}^N} \mathrm{K}^2(t)dt \quad \text{and} \quad \lim_{n\to\infty} |\Lambda_n|b_n^N \mathbb{V}\left[\varphi_n(x)\right] = \mathbb{E}\left[Y_0^2|X_0 = x\right] f(x)\int_{\mathbb{R}^N} \mathrm{K}^2(t)dt.$$

Now, it sufficies to prove

$$\lim_{n\to\infty} |\Lambda_n|b_n^N \mathrm{Cov}\left[\varphi_n(x), f_n(x)\right] = r(x)f(x)\int_{\mathbb{R}^N} \mathrm{K}^2(t)dt$$

when $(A3)(ii)$ holds. If we define

$$\overline{f}_n(x) = \frac{1}{|\Lambda_n|b_n^N} \sum_{i\in\Lambda_n} \mathbb{E}\left[\mathrm{K}_n(x, X_i)|\mathcal{H}_{i,m_n}\right] \quad \text{and} \quad \overline{\varphi}_n(x) = \frac{1}{|\Lambda_n|b_n^N} \sum_{i\in\Lambda_n} \mathbb{E}\left[Y_i\mathrm{K}_n(x, X_i)|\mathcal{H}_{i,m_n}\right]$$

then

$$|\Lambda_n|b_n^N \mathrm{Cov}\left[\varphi_n(x), f_n(x)\right] = C_1 + C_2 + C_3 + C_4,$$

where

$$\begin{aligned}
C_1 &= |\Lambda_n| b_n^N \mathbb{E}\left[\left(\varphi_n(x) - \overline{\varphi}_n(x)\right)\left(f_n(x) - \overline{f}_n(x)\right)\right] \\
C_2 &= |\Lambda_n| b_n^N \mathbb{E}\left[\left(\varphi_n(x) - \overline{\varphi}_n(x)\right)\left(\overline{f}_n(x) - \mathbb{E}\left[\overline{f}_n(x)\right]\right)\right] \\
C_3 &= |\Lambda_n| b_n^N \mathbb{E}\left[\left(\overline{\varphi}_n(x) - \mathbb{E}\left[\overline{\varphi}_n(x)\right]\right)\left(f_n(x) - \overline{f}_n(x)\right)\right] \\
C_4 &= |\Lambda_n| b_n^N \mathbb{E}\left[\left(\overline{\varphi}_n(x) - \mathbb{E}\left[\overline{\varphi}_n(x)\right]\right)\left(\overline{f}_n(x) - \mathbb{E}\left[\overline{f}_n(x)\right]\right)\right].
\end{aligned}$$

Using Proposition 1, we have

$$|C_1| \leqslant \frac{1}{\sqrt{|\Lambda_n|}} \left\| \sum_{i\in\Lambda_n} (\Theta_i - \overline{\Theta}_i) \right\|_2 \times \frac{1}{\sqrt{|\Lambda_n|}} \left\| \sum_{i\in\Lambda_n} (\Delta_i - \overline{\Delta}_i) \right\|_2 \trianglelefteq \left(b_n^{-\frac{\theta(N+2)+2N}{2(2+\theta)}} \sum_{|i|>m_n} |i|^d \delta_{i,2}^{\frac{\theta}{2+\theta}} \right)^2 = o(1).$$

From (40), we have $|\Lambda_n|^{-1/2} \left\| \sum_{j\in\Lambda_n} \overline{\Delta}_j \right\|_2 \trianglelefteq 1$ and $|\Lambda_n|^{-1/2} \left\| \sum_{j\in\Lambda_n} \overline{\Theta}_j \right\|_2 \trianglelefteq 1$. So,

$$|C_2| \leqslant \frac{1}{\sqrt{|\Lambda_n|}} \left\| \sum_{i\in\Lambda_n} (\Theta_i - \overline{\Theta}_i) \right\|_2 \times \frac{1}{\sqrt{|\Lambda_n|}} \left\| \sum_{j\in\Lambda_n} \overline{\Delta}_j \right\|_2 \trianglelefteq b_n^{-\frac{\theta(N+2)+2N}{2(2+\theta)}} \sum_{|i|>m_n} |i|^d \delta_{i,2}^{\frac{\theta}{2+\theta}} = o(1)$$

and

$$|C_3| \leqslant \frac{1}{\sqrt{|\Lambda_n|}} \left\| \sum_{i\in\Lambda_n} (\Delta_i - \overline{\Delta}_i) \right\|_2 \times \frac{1}{\sqrt{|\Lambda_n|}} \left\| \sum_{j\in\Lambda_n} \overline{\Theta}_j \right\|_2 \trianglelefteq b_n^{-\frac{\theta(N+2)+2N}{2(2+\theta)}} \sum_{|i|>m_n} |i|^d \delta_{i,2}^{\frac{\theta}{2+\theta}} = o(1).$$

Finally, since

$$C_4 = \mathbb{E}\left[\overline{\Theta}_0 \overline{\Delta}_0\right] + \frac{1}{|\Lambda_n|} \sum_{\substack{j\in\mathbb{Z}^d \setminus \{0\} \\ |j|\leqslant 2m_n}} |\Lambda_n \cap (\Lambda_n - j)| \mathbb{E}\left[\overline{\Theta}_0 \overline{\Delta}_j\right],$$

we obtain

$$\left| C_4 - \mathbb{E}\left[\overline{\Theta}_0 \overline{\Delta}_0\right] \right| \leqslant (2m_n + 1)^d \sup_{\substack{j\in\mathbb{Z}^d \\ j\neq 0}} \left| \mathbb{E}\left[\overline{\Theta}_0 \overline{\Delta}_j\right] \right|. \tag{41}$$

Using (19) and keeping in mind that $\|\Delta_0\|_2 \trianglelefteq 1$ and $\|\Theta_0\|_2 \trianglelefteq 1$, we have also

$$\left| \mathbb{E}\left[\overline{\Theta}_0 \overline{\Delta}_0\right] - \mathbb{E}\left[\Theta_0 \Delta_0\right] \right| \leqslant \|\Theta_0\|_2 \left\|\overline{\Delta}_0 - \Delta_0\right\|_2 + \|\Delta_0\|_2 \left\|\overline{\Theta}_0 - \Theta_0\right\|_2 \trianglelefteq b_n^{-\frac{\theta(N+2)+2N}{2(2+\theta)}} \sum_{|i|>m_n} \delta_{i,2}^{\frac{\theta}{2+\theta}} = o(1).$$

Moreover, using Lemma 1, we have

$$\mathbb{E}\left[\Theta_0 \Delta_0\right] = \frac{1}{b_n^N} \left(\mathbb{E}\left[Y_0 \mathrm{K}_n^2(x, X_0)\right] - \mathbb{E}\left[\mathrm{K}_n(x, X_0)\right] \mathbb{E}\left[Y_0 \mathrm{K}_n(x, X_0)\right] \right) \xrightarrow[n\to\infty]{} r(x) f(x) \int_{\mathbb{R}^N} \mathrm{K}^2(t) dt$$

and consequently, we obtain

$$\lim_{n\to\infty} \mathbb{E}\left[\overline{\Theta}_0 \overline{\Delta}_0\right] = r(x) f(x) \int_{\mathbb{R}^N} \mathrm{K}^2(t) dt. \tag{42}$$

Now, using (19) and Lemma 2, we have

$$m_n^d \sup_{\substack{j\in\mathbb{Z}^d \\ j\neq 0}} \left| |\mathbb{E}\left[\overline{\Theta}_0 \overline{\Delta}_j\right]| - |\mathbb{E}\left[\Theta_0 \Delta_j\right]| \right| \leqslant m_n^d \left(\|\Theta_0\|_2 \left\|\overline{\Delta}_0 - \Delta_0\right\|_2 + \|\Delta_0\|_2 \left\|\overline{\Theta}_0 - \Theta_0\right\|_2 \right) \trianglelefteq b_n^{-\frac{\theta(N+2)+2N}{2(2+\theta)}} \sum_{|j|>m_n} |j|^d \delta_{j,2}^{\frac{\theta}{2+\theta}} = o(1).$$

Using Lemma 4, we have $\sup_{\substack{j\in\mathbb{Z}^d\\ j\neq 0}} \left|\mathbb{E}\left[\Theta_0\Delta_j\right]\right| \leq b_n^{\frac{\theta N}{2+\theta}} \leqslant b_n^{\frac{\theta N}{4+\theta}}$ and consequently, by Lemma 2, we get

$$m_n^d \sup_{\substack{j\in\mathbb{Z}^d\\ j\neq 0}} \left|\mathbb{E}\left[\overline{\Theta}_0\overline{\Delta}_j\right]\right| \leq m_n^d b_n^{\frac{\theta N}{4+\theta}} + b_n^{-\frac{\theta(N+2)+2N}{2(2+\theta)}} \sum_{|j|>m_n} |j|^d \delta_{j,2}^{\frac{\theta}{2+\theta}} \xrightarrow[n\to\infty]{} 0. \tag{43}$$

Combining (41), (42) and (43), we obtain

$$C_4 \xrightarrow[n\to\infty]{} r(x)f(x)\int_{\mathbb{R}^N} \mathrm{K}^2(t)dt.$$

The proof of Lemma 5 is complete. □

5 Proofs of Theorems

In this section, we present the proofs of Theorems 1 and 2.

Proof of Theorem 1. Let $n \geqslant 1$ be fixed. Since K is symmetric such that $\int_{\mathbb{R}^N} \|v\|^2 |\mathrm{K}(v)|dv < \infty$ and f and φ are twice differentiable with bounded second partial derivatives, by Taylor's formula, we get

$$\sup_{x\in\mathbb{R}^N}\left|\mathbb{E}\left[f_n(x)\right] - f(x)\right| = \sup_{x\in\mathbb{R}^N}\left|\int_{\mathbb{R}^N} (f(x-vb_n)-f(x))\,\mathrm{K}(v)dv\right| \leq b_n^2\int_{\mathbb{R}^N}\|v\|^2|\mathrm{K}(v)|dv$$

and

$$\sup_{x\in\mathbb{R}^N}\left|\mathbb{E}\left[\varphi_n(x)\right] - \varphi(x)\right| = \sup_{x\in\mathbb{R}^N}\left|\int_{\mathbb{R}^N} (\varphi(x-vb_n)-\varphi(x))\,\mathrm{K}(v)dv\right| \leq b_n^2\int_{\mathbb{R}^N}\|v\|^2|\mathrm{K}(v)|dv.$$

The proof of Theorem 1 is complete. □

Proof of Theorem 2. Let $n \geqslant 1$ and $x \in \mathbb{R}^N$ such that $f(x) > 0$ be fixed. Then

$$r_n(x) - \frac{\mathbb{E}\left[\varphi_n(x)\right]}{\mathbb{E}\left[f_n(x)\right]} = \frac{(\varphi_n(x)-\mathbb{E}\left[\varphi_n(x)\right])\mathbb{E}\left[f_n(x)\right] - (f_n(x)-\mathbb{E}\left[f_n(x)\right])\mathbb{E}\left[\varphi_n(x)\right]}{f_n(x)\mathbb{E}\left[f_n(x)\right]}.$$

Combining Lemma 5 and Theorem 1, we obtain that $f_n(x)$ converges in probability to $f(x)$ as $n \longrightarrow \infty$. Moreover, we have also $\lim_{n\to\infty}\frac{\mathbb{E}[\varphi_n(x)]}{\mathbb{E}[f_n(x)]} = r(x)$. So, using Slutsky's lemma and assumptions (A1) – (A4), it is sufficient to prove that

$$\lambda_1\sqrt{|\Lambda_n|b_n^N}\left(\varphi_n(x)-\mathbb{E}\left[\varphi_n(x)\right]\right) + \lambda_2\sqrt{|\Lambda_n|b_n^N}\left(f_n(x)-\mathbb{E}\left[f_n(x)\right]\right) \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}\left(0,\rho^2(x)\right),$$

where $\rho^2(x) = \left(\lambda_1^2\mathbb{E}\left[Y_0^2|X_0=x\right] + 2\lambda_1\lambda_2 r(x) + \lambda_2^2\right) \times f(x)\int_{\mathbb{R}^N}\mathrm{K}^2(t)dt$ for any $(\lambda_1,\lambda_2)\in\mathbb{R}^2$.

Let $(\lambda_1,\lambda_2)\in\mathbb{R}^2$ be fixed. Then

$$\lambda_1\sqrt{|\Lambda_n|b_n^N}\left(\varphi_n(x)-\mathbb{E}\left[\varphi_n(x)\right]\right) + \lambda_2\sqrt{|\Lambda_n|b_n^N}\left(f_n(x)-\mathbb{E}\left[f_n(x)\right]\right) = |\Lambda_n|^{-1/2}\sum_{i\in\Lambda_n} U_i,$$

where

$$U_i = \lambda_1\Theta_i + \lambda_2\Delta_i = \frac{(\lambda_1 Y_i + \lambda_2)\,\mathrm{K}_n(x,X_i) - \mathbb{E}\left[(\lambda_1 Y_0 + \lambda_2)\,\mathrm{K}_n(x,X_0)\right]}{\sqrt{b_n^N}}$$

with Θ_i and Δ_i defined by (7). For the asymptotic normality of r_n when $(X_i)_{i\in\mathbb{Z}^d}$ is of the form (1), we are going to use an approximation by $2m_n$-dependent random fields. So, recall that $\mathcal{H}_{i,m_n} = \sigma\left(\eta_i, \varepsilon_{i-s}; |s| \leqslant m_n\right)$ and define

$$\overline{\Delta}_i = \mathbb{E}\left[\Delta_i|\mathcal{H}_{i,m_n}\right], \quad \overline{\Theta}_i = \mathbb{E}\left[\Theta_i|\mathcal{H}_{i,m_n}\right] \quad \text{and} \quad \overline{U}_i = \lambda_1\overline{\Theta}_i + \lambda_2\overline{\Delta}_i = \mathbb{E}\left[U_i|\mathcal{H}_{i,m_n}\right].$$

By construction, $(\overline{U}_i)_{i\in\mathbb{Z}^d}$ is $2m_n$-dependent (it means that if $|i-j| > 2m_n$ then $\overline{U}_i$ and $\overline{U}_j$ are independent). So, if $(A3)(ii)$ holds, applying Proposition 1 with $\Phi(t) = \lambda_1 t + \lambda_2$ for any $t \in \mathbb{R}$, then

$$|\Lambda_n|^{-1/2}\left\|\sum_{i\in\Lambda_n}\left(U_i - \overline{U}_i\right)\right\|_2 \preceq b_n^{-\frac{\theta(N+2)+2N}{2(2+\theta)}} \sum_{|i|>m_n} |i|^d \delta_{i,2}^{\frac{\theta}{2+\theta}} \xrightarrow[n\to\infty]{} 0 \quad \text{(by Lemma 2).}$$

Consequently, when $(X_i)_{i\in\mathbb{Z}^d}$ is of the form (1) it sufficies to establish the asymptotic normality of $|\Lambda_n|^{-1/2}\sum_{i\in\Lambda_n}\overline{U}_i$. From now on, we denote

$$Z_i = \begin{cases} U_i & \text{if } (X_i)_{i\in\mathbb{Z}^d} \text{ is strongly mixing} \\ \overline{U}_i & \text{if } (X_i)_{i\in\mathbb{Z}^d} \text{ is of the form (1)} \end{cases}$$

and

$$M_n = \begin{cases} m_n & \text{if } (X_i)_{i\in\mathbb{Z}^d} \text{ is strongly mixing} \\ 2m_n & \text{if } (X_i)_{i\in\mathbb{Z}^d} \text{ is of the form (1).} \end{cases}$$

Lemma 6 $\lim_{n\to\infty}\mathbb{E}\left[Z_0^2\right] = \rho^2(x)$.

Proof. We have

$$\mathbb{E}\left[U_0^2\right] = \frac{\mathbb{E}\left[(\lambda_1 Y_0 + \lambda_2)^2\,\mathrm{K}_n^2(x,X_0)\right] - \left(\mathbb{E}\left[(\lambda_1 Y_0 + \lambda_2)\,\mathrm{K}_n(x,X_0)\right]\right)^2}{b_n^N}.$$

Applying Lemma 1, we get

$$\lim_{n\to\infty}\mathbb{E}\left[U_0^2\right] = \mathbb{E}\left[(\lambda_1 Y_0 + \lambda_2)^2 \,|\, X_0 = x\right] f(x)\int_{\mathbb{R}^N}\mathrm{K}^2(t)dt = \rho^2(x).$$

Moreover, by Proposition 1 and Lemma 2 and using $\|U_0\|_2 \preceq 1$, we derive

$$\left|\mathbb{E}\left[\overline{U}_0^2\right] - \mathbb{E}\left[U_0^2\right]\right| \leqslant 2\,\|U_0\|_2\left\|U_0 - \overline{U}_0\right\|_2 \preceq b_n^{-\frac{\theta(N+2)+2N}{2(2+\theta)}} \sum_{|i|>M_n} |i|^d \delta_{i,2}^{\frac{\theta}{2+\theta}} \xrightarrow[n\to\infty]{} 0.$$

So, $\lim_{n\to\infty}\mathbb{E}\left[\overline{U}_0^2\right] = \rho^2(x)$. The proof of Lemma 6 is complete. □

Let $(\xi_i)_{i\in\mathbb{Z}^d}$ be i.i.d. normal random variables independent of $(X_i)_{i\in\mathbb{Z}^d}$ and $(\eta_i)_{i\in\mathbb{Z}^d}$. Assume that $\mathbb{E}[\xi_0]=0$ and $\mathbb{E}\left[\xi_0^2\right]=\mathbb{E}\left[Z_0^2\right]$. For any $i\in\mathbb{Z}^d$, we define

$$T_i=\frac{Z_i}{\sqrt{|\Lambda_n|}}\quad\text{and}\quad\gamma_i=\frac{\xi_i}{\sqrt{|\Lambda_n|}}.$$

Let g be the unique function from $[1,|\Lambda_n|]\cap\mathbb{Z}$ to Λ_n such that $g(k)<_{\text{lex}}g(\ell)$ for $1\leqslant k<\ell\leqslant|\Lambda_n|$, where $<_{\text{lex}}$ is the lexicographic order on $\mathbb{Z}^d$. For all integer $1\leqslant k\leqslant|\Lambda_n|$, we put

$$S_{g(k)}(T)=\sum_{s=1}^{k}T_{g(s)}\quad\text{and}\quad S^c_{g(k)}(\gamma)=\sum_{s=k}^{|\Lambda_n|}\gamma_{g(s)}$$

with the convention $S_{g(0)}(T)=S^c_{g(|\Lambda_n|+1)}(\gamma)=0$. Let ψ be any measurable function from $\mathbb{R}$ to $\mathbb{R}$. For any $1\leqslant k\leqslant\ell\leqslant|\Lambda_n|$, we introduce the notation $\psi_{k,\ell}=\psi\left(S_{g(k)}(T)+S^c_{g(\ell)}(\gamma)\right)$. Let $h:\mathbb{R}\to\mathbb{R}$ be a four times continuously differentiable function such that $\max_{0\leqslant i\leqslant 4}\|h^{(i)}\|_\infty\leqslant 1$. It suffices to prove $\lim_{n\to\infty}|L_n|=0$, where

$$L_n:=\mathbb{E}\left[h\left(\sum_{i\in\Lambda_n}\frac{Z_i}{\sqrt{|\Lambda_n|}}\right)\right]-\mathbb{E}\left[h\left(\sum_{i\in\Lambda_n}\frac{\xi_i}{\sqrt{|\Lambda_n|}}\right)\right].$$

Using Lindeberg's idea [15] (see also [5]), we have

$$\begin{aligned}L_n&=\mathbb{E}\left[h_{|\Lambda_n|,|\Lambda_n|+1}-h_{0,1}\right]=\sum_{k=1}^{|\Lambda_n|}\mathbb{E}\left[h_{k,k+1}-h_{k-1,k}\right]\\&=\sum_{k=1}^{|\Lambda_n|}\Big(\mathbb{E}\left[h_{k,k+1}-h_{k-1,k+1}\right]-\mathbb{E}\left[h_{k-1,k}-h_{k-1,k+1}\right]\Big).\end{aligned}$$

Applying Taylor's formula, we get

$$L_n=\sum_{k=1}^{|\Lambda_n|}\left(\mathbb{E}\left[T_{g(k)}h'_{k-1,k+1}+\frac{1}{2}T^2_{g(k)}h''_{k-1,k+1}+v_k\right]-\mathbb{E}\left[\gamma_{g(k)}h'_{k-1,k+1}+\frac{1}{2}\gamma^2_{g(k)}h''_{k-1,k+1}+w_k\right]\right),$$

where $|v_k|\leqslant T^2_{g(k)}\left(1\wedge|T_{g(k)}|\right)$ and $|w_k|\leqslant\gamma^2_{g(k)}\left(1\wedge|\gamma_{g(k)}|\right)$. Since $\gamma^2_{g(k)}$ and $h''_{k-1,k+1}$ are independent, $\mathbb{E}\left[\gamma_{g(k)}h'_{k-1,k+1}\right]=0$ and $\mathbb{E}\left[\gamma^2_{g(k)}\right]=\frac{\mathbb{E}[Z_0^2]}{|\Lambda_n|}$, we obtain

$$L_n=\sum_{k=1}^{|\Lambda_n|}\left(\mathbb{E}\left[T_{g(k)}h'_{k-1,k+1}\right]+\frac{1}{2}\mathbb{E}\left[\left(T^2_{g(k)}-\frac{\mathbb{E}\left[Z_0^2\right]}{|\Lambda_n|}\right)h''_{k-1,k+1}\right]+\mathbb{E}\left[v_k-w_k\right]\right).$$

Since ξ_0 is a gaussian random variable with zero mean and variance $\mathbb{E}\left[Z_0^2\right]$ and $\mathbb{E}\left[Z_0^2\right]\leqslant\mathbb{E}\left[U_0^2\right]$, we have

$$\sum_{k=1}^{|\Lambda_n|}\mathbb{E}\left[|w_k|\right]\leqslant\frac{\mathbb{E}\left[|\xi_0|^3\right]}{\sqrt{|\Lambda_n|}}\trianglelefteq\frac{\left(\mathbb{E}\left[U_0^2\right]\right)^{3/2}}{\sqrt{|\Lambda_n|}}.$$

By Lemma 1, we have $\mathbb{E}\left[U_0^2\right] \trianglelefteq 1$ and consequently, we obtain $\lim_{n\to\infty}\sum_{k=1}^{|\Lambda_n|}\mathbb{E}\left[|w_k|\right] = 0$. Let $d_n := \left(|\Lambda_n|b_n^N\right)^{\frac{-\theta}{2(\theta+1)}}$. Then,

$$\sum_{k=1}^{|\Lambda_n|}\mathbb{E}\left[|v_k|\right] \leqslant d_n\mathbb{E}\left[Z_0^2\right] + \mathbb{E}\left[Z_0^2\,\mathbb{1}_{|Z_0|>d_n\sqrt{|\Lambda_n|}}\right] \leqslant d_n\mathbb{E}\left[U_0^2\right] + \frac{\mathbb{E}\left[|U_0|^{2+\theta}\right]}{d_n^\theta|\Lambda_n|^{\theta/2}}.$$

Using Lemma 3, we get

$$\sum_{k=1}^{|\Lambda_n|}\mathbb{E}\left[|v_k|\right] \trianglelefteq d_n + \frac{1}{d_n^\theta\left(|\Lambda_n|b_n^N\right)^{\theta/2}} = 2d_n \xrightarrow[n\to\infty]{} 0.$$

Now, we have to prove that

$$\lim_{n\to\infty}\sum_{k=1}^{|\Lambda_n|}\left(\mathbb{E}\left[T_{g(k)}h'_{k-1,k+1}\right] + \mathbb{E}\left[\left(\frac{Z_{g(k)}^2 - \mathbb{E}\left[Z_0^2\right]}{2|\Lambda_n|}\right)h''_{k-1,k+1}\right]\right) = 0. \tag{44}$$

For any integers $n \geqslant 1$ and $1 \leqslant k \leqslant |\Lambda_n|$, we define

$$\mathrm{E}_k^{(n)} = \left\{ j\in\Lambda_n \,\middle|\, j <_{\text{lex}} g(k) \text{ and } |j-g(k)| > M_n \right\} \quad \text{and} \quad S_{g(k)}^{(M_n)}(T) = \sum_{i\in \mathrm{E}_k^{(n)}} T_i.$$

For any $1 \leqslant k < \ell \leqslant |\Lambda_n|$ and any function ψ from $\mathbb{R}$ to $\mathbb{R}$, we define also $\psi_{k-1,\ell}^{(M_n)} = \psi\left(S_{g(k)}^{(M_n)}(T) + S_{g(\ell)}^c(\gamma)\right)$. Using Taylor's formula, we have

$$T_{g(k)}h'_{k-1,k+1} = T_{g(k)}h'^{(M_n)}_{k-1,k+1} + T_{g(k)}\left(S_{g(k-1)}(T) - S_{g(k)}^{(M_n)}(T)\right)h''^{(M_n)}_{k-1,k+1} + v'_k$$

with

$$|v'_k| \leqslant 2\left|T_{g(k)}\left(S_{g(k-1)}(T) - S_{g(k)}^{(M_n)}(T)\right)\left(1\wedge|S_{g(k-1)}(T) - S_{g(k)}^{(M_n)}(T)|\right)\right|.$$

In order to obtain (44), we have to prove

$$\lim_{n\to\infty}\sum_{k=1}^{|\Lambda_n|}\mathbb{E}\left[T_{g(k)}h'^{(M_n)}_{k-1,k+1}\right] = 0, \tag{45}$$

$$\lim_{n\to\infty}\sum_{k=1}^{|\Lambda_n|}\mathbb{E}\left[T_{g(k)}\left(S_{g(k-1)}(T) - S_{g(k)}^{(M_n)}(T)\right)h''^{(M_n)}_{k-1,k+1}\right] = 0 \tag{46}$$

$$\lim_{n\to\infty}\sum_{k=1}^{|\Lambda_n|}\mathbb{E}\left[|v'_k|\right] = 0, \tag{47}$$

and

$$\lim_{n\to\infty}\frac{1}{|\Lambda_n|}\sum_{k=1}^{|\Lambda_n|}\mathbb{E}\left[\left(Z_{g(k)}^2 - \mathbb{E}\left[Z_0^2\right]\right)h''_{k-1,k+1}\right] = 0. \tag{48}$$

First, we are going to prove (45). Since γ is independent of T, then $\mathbb{E}\left[T_{g(k)}h'\left(S_{g(k+1)}^c(\gamma)\right)\right] = 0$. Consequently, if π is a one to one map from $[1, |\mathrm{E}_k^{(n)}|]\cap\mathbb{Z}$ to $\mathrm{E}_k^{(n)}$ such that $|\pi(i) - g(k)| \leq |\pi(i-1) - g(k)|$ then

$$\mathbb{E}\left[T_{g(k)}h'^{(M_n)}_{k-1,k+1}\right] = \mathbb{E}\left[T_{g(k)}\left(h'^{(M_n)}_{k-1,k+1} - h'\left(S_{g(k+1)}^c(\gamma)\right)\right)\right] = \sum_{i=1}^{|\mathrm{E}_k^{(n)}|}\mathrm{Cov}\left(T_{g(k)}, \beta_i - \beta_{i-1}\right),$$

where $\beta_i = h^{'}\left(S_{\pi(i)}(T) + S^c_{g(k+1)}(\gamma)\right)$ and $S_{\pi(0)}(T) = 0$. If $(X_i)_{i\in\mathbb{Z}^d}$ is strongly mixing then, using Rio's inequality ([23], Theorem 1.1) and keeping in mind that $|\pi(i) - g(k)| \leq |\pi(i-1) - g(k)|$, we get

$$\left|\mathbb{E}\left[T_{g(k)} h^{'(M_n)}_{k-1,k+1}\right]\right| \leqslant 2\sum_{i=1}^{|\mathrm{E}^{(n)}_k|}\int_0^{2\alpha_{1,\infty}(|\pi(i)-g(k)|)} Q_{T_{g(k)}}(u) Q_{\beta_i-\beta_{i-1}}(u)du.$$

For any $u \in]0,1[$, noting that $h^{'}$ is Lipschitz, we have

$$Q_{T_{g(k)}}(u) \leqslant \frac{u^{-\frac{1}{2+\theta}}\|Z_0\|_{2+\theta}}{\sqrt{|\Lambda_n|}} \quad \text{and} \quad Q_{\beta_i-\beta_{i-1}}(u) \leqslant \frac{u^{-\frac{1}{2+\theta}}\|Z_0\|_{2+\theta}}{\sqrt{|\Lambda_n|}}.$$

Moreover, by Lemma 3, we have $\|Z_0\|^2_{2+\theta} \leqslant \|U_0\|^2_{2+\theta} \trianglelefteq b_n^{-\frac{\theta N}{2+\theta}}$ and consequently, we obtain

$$\sum_{k=1}^{|\Lambda_n|}\left|\mathbb{E}\left[T_{g(k)} h^{'(M_n)}_{k-1,k+1}\right]\right| \trianglelefteq \frac{b_n^{-\frac{\theta N}{2+\theta}}}{|\Lambda_n|}\sum_{k=1}^{|\Lambda_n|}\sum_{i=1}^{|\mathrm{E}^{(n)}_k|}\alpha_{1,\infty}^{\frac{\theta}{2+\theta}}\left(|\pi(i) - g(k)|\right) \leqslant b_n^{-\frac{\theta N}{2+\theta}}\sum_{|i|>M_n}\alpha_{1,\infty}^{\frac{\theta}{2+\theta}}\left(|i|\right).$$

Using Lemma 2, we get (45).

The following lemma is a simple consequence of Lemma 4 (its proof is left to the reader).

Lemma 7 $\sup_{\substack{j\in\mathbb{Z}^d\\ j\neq 0}}\mathbb{E}\left[|U_0U_j|\right] \trianglelefteq b_n^{\frac{\theta N}{4+\theta}}$.

Since $(X_i)_{i\in\mathbb{Z}^d}$ is assumed to be strongly mixing, we have $Z_i = U_i$ for any $i \in \mathbb{Z}^d$. Using Lemma 2 and Lemma 4, we have

$$\sum_{k=1}^{|\Lambda_n|}\mathbb{E}\left[|v_k'|\right] \leqslant 2\mathbb{E}\left[|Z_0|\left(\sum_{\substack{|i|\leqslant M_n\\ i\neq 0}}|Z_i|\right)\left(1\wedge\sum_{\substack{|i|\leqslant M_n\\ i\neq 0}}\frac{|Z_i|}{\sqrt{|\Lambda_n|}}\right)\right] \leqslant 2\sum_{\substack{|i|\leqslant M_n\\ i\neq 0}}\mathbb{E}\left[|U_0U_i|\right] \trianglelefteq M_n^d b_n^{\frac{\theta N}{4+\theta}} \xrightarrow[n\to\infty]{} 0$$

and

$$\sum_{k=1}^{|\Lambda_n|}\left|\mathbb{E}\left[T_{g(k)}\left(S_{g(k-1)}(T) - S^{(M_n)}_{g(k)}(T)\right)h^{''(M_n)}_{k-1,k+1}\right]\right| \leqslant \sum_{\substack{|i|\leqslant M_n\\ i\neq 0}}\mathbb{E}\left[|U_0U_i|\right] \trianglelefteq M_n^d b_n^{\frac{\theta N}{4+\theta}} \xrightarrow[n\to\infty]{} 0.$$

So, we obtain (46) and (47).

Now, it suffices to prove (48). Let $\beta \geqslant 1$ be a positive integer. In the sequel, for any $j \in \mathbb{Z}^d$, the notation $\mathbb{E}_\beta[Z_j]$ stands for the conditional expectation of Z_j with respect to the σ-algebra $\sigma\left(Z_i\,;\ i <_{\text{lex}} j \text{ and } |i-j| \geqslant \beta\right)$. Then,

$$\frac{1}{|\Lambda_n|}\sum_{k=1}^{|\Lambda_n|}\left|\mathbb{E}\left[\left(Z^2_{g(k)} - \mathbb{E}\left[Z_0^2\right]\right)h''_{k-1,k+1}\right]\right| \leqslant I_1 + I_2,$$

where

$$I_1 = \frac{1}{|\Lambda_n|}\sum_{k=1}^{|\Lambda_n|}\left|\mathbb{E}\left[\left(Z^2_{g(k)} - \mathbb{E}_\beta[Z^2_{g(k)}]\right)h''_{k-1,k+1}\right]\right| \text{ and } I_2 = \frac{1}{|\Lambda_n|}\sum_{k=1}^{|\Lambda_n|}\left|\mathbb{E}\left[\left(\mathbb{E}_\beta[Z^2_{g(k)}] - \mathbb{E}\left[Z_0^2\right]\right)h''_{k-1,k+1}\right]\right|.$$

The next result can be found in [20].

Lemma 8 *Let $\mathcal{U}$ and $\mathcal{V}$ be two σ-algebras and let X be a random variable which is measurable with respect to $\mathcal{U}$. If $1 \leqslant p \leqslant r \leqslant \infty$, then*

$$\|\mathbb{E}[X|\mathcal{V}] - \mathbb{E}[X]\|_p \leqslant 2(2^{1/p}+1)\,(\alpha(\mathcal{U},\mathcal{V}))^{\frac{1}{p}-\frac{1}{r}}\,\|X\|_r.$$

Assume that $(X_i)_{i\in\mathbb{Z}^d}$ is strongly mixing. Using Lemma 8 with $p = 1$ and $r = (2+\theta)/2$ and keeping in mind that $\|Z_0\|_{2+\theta}^2 = \|U_0\|_{2+\theta}^2 \trianglelefteq b_n^{-\frac{\theta N}{2+\theta}}$, we have

$$I_2 \leqslant \left\|\mathbb{E}_\beta[Z_0^2] - \mathbb{E}\left[Z_0^2\right]\right\|_1 \leqslant 6\alpha_{1,\infty}^{\frac{\theta}{2+\theta}}(\beta)\,\|Z_0\|_{2+\theta}^2 \trianglelefteq 6b_n^{-\frac{\theta N}{2+\theta}}\alpha_{1,\infty}^{\frac{\theta}{2+\theta}}(\beta).$$

Now, we make the choice

$$\beta = \left\lceil b_n^{-\frac{\theta N}{(2d-1)\theta+6d-2}}\right\rceil. \tag{49}$$

Consequently, using (A3)(i), we obtain

$$I_2 \trianglelefteq \beta^{\frac{(2d-1)\theta+6d-2}{2+\theta}}\alpha_{1,\infty}^{\frac{\theta}{2+\theta}}(\beta) \xrightarrow[n\to\infty]{} 0.$$

In the other part, noting that $\mathbb{E}\left[\left(Z_{g(k)}^2 - \mathbb{E}_\beta[Z_{g(k)}^2]\right)h_{k-1,k+1}^{\prime\prime(\beta)}\right] = 0$, we have

$$\mathbb{E}\left[\left(Z_{g(k)}^2 - \mathbb{E}_\beta[Z_{g(k)}^2]\right)h_{k-1,k+1}^{\prime\prime}\right] = \mathbb{E}\left[\left(Z_{g(k)}^2 - \mathbb{E}_\beta[Z_{g(k)}^2]\right)\left(h_{k-1,k+1}^{\prime\prime} - h_{k-1,k+1}^{\prime\prime(\beta)}\right)\right].$$

So, we obtain

$$I_1 \leqslant \mathbb{E}\left[\left(2 \wedge \left|\sum_{\substack{|i|\leqslant\beta\\ i<_{\text{lex}}0}}\frac{Z_i}{\sqrt{|\Lambda_n|}}\right|\right)\left(Z_0^2 + \mathbb{E}_\beta[Z_0^2]\right)\right].$$

If $L > 0$, then

$$I_1 \leqslant \frac{L}{\sqrt{|\Lambda_n|}}\sum_{\substack{|i|\leqslant\beta\\ i\neq 0}}\mathbb{E}\left[|Z_0Z_i|\right] + 2\mathbb{E}\left[Z_0^2\,\mathbb{1}_{|Z_0|>L}\right] + 2\left\|\mathbb{E}_\beta\left[Z_0^2\right] - \mathbb{E}\left[Z_0^2\right]\right\|_1 + \left\|\sum_{\substack{|i|\leqslant\beta\\ i<_{\text{lex}}0}}\frac{Z_i}{\sqrt{|\Lambda_n|}}\right\|_2\mathbb{E}\left[Z_0^2\right].$$

Recall that $Z_i = U_i$ for any i in $\mathbb{Z}^d$. Since $\mathbb{E}\left[U_0^2\right] \trianglelefteq 1$ and $\|U_0\|_{2+\theta}^2 \trianglelefteq b_n^{-\frac{\theta N}{2+\theta}}$, we derive from Lemma 7 that

$$I_1 \trianglelefteq \frac{\beta^d L b_n^{\frac{\theta N}{4+\theta}}}{\sqrt{|\Lambda_n|}} + L^{-\theta}b_n^{-\frac{\theta N}{2}} + \beta^{\frac{(2d-1)\theta+6d-2}{2+\theta}}\alpha_{1,\infty}^{\frac{\theta}{2+\theta}}(\beta) + \left\|\sum_{\substack{|i|\leqslant\beta\\ i<_{\text{lex}}0}}\frac{Z_i}{\sqrt{|\Lambda_n|}}\right\|_2.$$

Now, we make the choice

$$L = \frac{|\Lambda_n|^{\frac{1}{2(1+\theta)}}}{\beta^{\frac{d}{1+\theta}}\,b_n^{\frac{\theta(\theta+6)N}{2(1+\theta)(4+\theta)}}} \tag{50}$$

and we obtain

$$I_1 \trianglelefteq \left(|\Lambda_n|b_n^N\right)^{\frac{-\theta}{2(1+\theta)}} \times b_n^{\frac{\theta^2(2+\theta)(d-1)N}{(1+\theta)(4+\theta)((2d-1)\theta+6d-2)}} + \beta^{\frac{(2d-1)\theta+6d-2}{2+\theta}}\alpha_{1,\infty}^{\frac{\theta}{2+\theta}}(\beta) + \left\|\sum_{\substack{|i|\leqslant\beta\\ i<_{\text{lex}}0}}\frac{U_i}{\sqrt{|\Lambda_n|}}\right\|_2.$$

Moreover,

$$\left\| \sum_{\substack{|i|\leqslant\beta \\ i<_{\text{lex}}0}} \frac{U_i}{\sqrt{|\Lambda_n|}} \right\|_2^2 \leqslant \frac{(2\beta+1)^d\, \mathbb{E}\left[U_0^2\right]}{|\Lambda_n|} + \frac{1}{|\Lambda_n|} \sum_{\substack{j\in\mathbb{Z}^d \\ j\neq 0}} \left| [-\beta,\beta]^d \cap \left([-\beta,\beta]^d - j\right)\right| \left|\mathbb{E}\left[U_0U_j\right]\right|$$

$$\trianglelefteq \frac{\beta^d}{|\Lambda_n|}\left(\mathbb{E}\left[U_0^2\right] + \sum_{\substack{j\in\mathbb{Z}^d \\ j\neq 0}} \left|\mathbb{E}\left[U_0U_j\right]\right|\right).$$

Using (32) and (34), we have $\sum_{\substack{j\in\mathbb{Z}^d \\ j\neq 0}} \left|\mathbb{E}\left[U_0U_j\right]\right| = o(1)$. Consequently, we get

$$\left\| \sum_{\substack{|i|\leqslant\beta \\ i<_{\text{lex}}0}} \frac{U_i}{\sqrt{|\Lambda_n|}} \right\|_2^2 \trianglelefteq \frac{\beta^d}{|\Lambda_n|} \trianglelefteq \frac{1}{|\Lambda_n| b_n^{\frac{d\theta N}{(2d-1)\theta+6d-2}}} \leqslant \frac{1}{|\Lambda_n| b_n^N}.$$

So, we obtain

$$I_1 \trianglelefteq \left(|\Lambda_n| b_n^N\right)^{\frac{-\theta}{2(1+\theta)}} \times b_n^{\frac{\theta^2(2+\theta)(d-1)N}{(1+\theta)(4+\theta)((2d-1)\theta+6d-2)}} + \beta^{\frac{(2d-1)\theta+6d-2}{2+\theta}} \alpha_{1,\infty}^{\frac{\theta}{2+\theta}}(\beta) + \frac{1}{\sqrt{|\Lambda_n| b_n^N}} \xrightarrow[n\to\infty]{} 0.$$

Finally, if $(X_i)_{i\in\mathbb{Z}^d}$ is strongly mixing, then (48) holds. In order to complete the proof of Theorem 2, we only need to prove (45), (46), (47) and (48) when $(X_i)_{i\in\mathbb{Z}^d}$ is of the form (1). So, assume that $(X_i)_{i\in\mathbb{Z}^d}$ is of the form (1) and $(A3)(ii)$ holds. Then $(Z_i)_{i\in\mathbb{Z}^d} = (\overline{U}_i)_{i\in\mathbb{Z}^d}$ is M_n-dependent. Consequently, $\mathbb{E}\left[T_{g(k)} h_{k-1,k+1}^{\prime(M_n)}\right] = 0$ and (45) follows.

Lemma 9 $\sup_{\substack{j\in\mathbb{Z}^d \\ j\neq 0}} \mathbb{E}\left[|\overline{U}_0\overline{U}_j|\right] = o\left(M_n^{-d}\right)$.

Proof. We have

$$\sup_{\substack{j\in\mathbb{Z}^d \\ j\neq 0}} \left|\mathbb{E}\left[|\overline{U}_0\overline{U}_j|\right] - \mathbb{E}\left[|U_0U_j|\right]\right| \leqslant 2\left\|U_0\right\|_2 \left\|U_0 - \overline{U}_0\right\|_2.$$

Combining (19), Lemma 2 and Lemma 7 and keeping in mind $\|U_0\|_2 \trianglelefteq 1$, we obtain

$$M_n^d \sup_{\substack{j\in\mathbb{Z}^d \\ j\neq 0}} \mathbb{E}\left[|\overline{U}_0\overline{U}_j|\right] \trianglelefteq M_n^d b_n^{\frac{\theta N}{4+\theta}} + b_n^{-\frac{\theta(N+2)+2N}{2(2+\theta)}} \sum_{|j|>M_n} |j|^d \delta_{j,2}^{\frac{\theta}{2+\theta}} = \xrightarrow[n\to\infty]{} 0.$$

The proof of Lemma 9 is complete. □

Applying Lemma 9, we have

$$\sum_{k=1}^{|\Lambda_n|} \mathbb{E}\left[|v_k'|\right] \leqslant 2\mathbb{E}\left[|Z_0|\left(\sum_{\substack{|i|\leqslant M_n \\ i\neq 0}} |Z_i|\right)\left(1 \wedge \sum_{\substack{|i|\leqslant M_n \\ i\neq 0}} \frac{|Z_i|}{\sqrt{|\Lambda_n|}}\right)\right] \leqslant 2 \sum_{\substack{|i|\leqslant M_n \\ i\neq 0}} \mathbb{E}\left[|\overline{U}_0\overline{U}_i|\right] \xrightarrow[n\to\infty]{} 0$$

and

$$\sum_{k=1}^{|\Lambda_n|} \left| \mathbb{E}\left[T_{g(k)} \left(S_{g(k-1)}(T) - S_{g(k)}^{(M_n)}(T) \right) h_{k-1,k+1}^{''(M_n)} \right] \right| \leqslant \sum_{\substack{|i| \leqslant M_n \\ i \neq 0}} \mathbb{E}\left[|\overline{U}_0 \overline{U}_i| \right] \xrightarrow[n\to\infty]{} 0.$$

So, we obtain (46) and (47). Moreover, we have

$$\mathbb{E}\left[\left(Z_{g(k)}^2 - \mathbb{E}\left[Z_0^2\right] \right) h_{k-1,k+1}^{''(M_n)} \right] = \mathbb{E}\left[\left(\overline{U}_{g(k)}^2 - \mathbb{E}\left[\overline{U}_0^2\right] \right) h_{k-1,k+1}^{''(M_n)} \right] = 0.$$

Consequently,

$$\begin{aligned} \frac{1}{|\Lambda_n|} \sum_{k=1}^{|\Lambda_n|} \left| \mathbb{E}\left[\left(Z_{g(k)}^2 - \mathbb{E}[Z_0^2] \right) h_{k-1,k+1}'' \right] \right| &= \frac{1}{|\Lambda_n|} \sum_{k=1}^{|\Lambda_n|} \left| \mathbb{E}\left[\left(\overline{U}_{g(k)}^2 - \mathbb{E}\left[\overline{U}_0^2\right] \right) \left(h_{k-1,k+1}'' - h_{k-1,k+1}^{''(M_n)} \right) \right] \right| \\ &\leqslant \mathbb{E}\left[\left(2 \wedge \left| \sum_{\substack{|i| \leqslant M_n \\ i <_{\text{lex}} 0}} \frac{\overline{U}_i}{\sqrt{|\Lambda_n|}} \right| \right) \left(\overline{U}_0^2 + \mathbb{E}\left[\overline{U}_0^2\right] \right) \right]. \end{aligned}$$

As before, if $L' > 0$, then using $\|\overline{U}_0\|_{2+\theta}^2 \leqslant \|U_0\|_{2+\theta}^2 \trianglelefteq b_n^{-\frac{\theta N}{2+\theta}}$, we get

$$\frac{1}{|\Lambda_n|} \sum_{k=1}^{|\Lambda_n|} \left| \mathbb{E}\left[\left(Z_{g(k)}^2 - \mathbb{E}[Z_0^2] \right) h_{k-1,k+1}'' \right] \right| \trianglelefteq \frac{M_n^d L' \sup_{\substack{j \in \mathbb{Z}^d \\ j \neq 0}} \mathbb{E}\left[|\overline{U}_0 \overline{U}_j| \right]}{\sqrt{|\Lambda_n|}} + L'^{-\theta} b_n^{-\frac{\theta N}{2}} + \mathbb{E}\left[U_0^2\right] \left\| \sum_{\substack{|i| \leqslant M_n \\ i <_{\text{lex}} 0}} \frac{\overline{U}_i}{\sqrt{|\Lambda_n|}} \right\|_2 .$$

Applying Lemma 9 and keeping in mind that $\mathbb{E}\left[U_0^2\right] \trianglelefteq 1$ then

$$\begin{aligned} \left\| \sum_{\substack{|i| \leqslant M_n \\ i <_{\text{lex}} 0}} \frac{\overline{U}_i}{\sqrt{|\Lambda_n|}} \right\|_2^2 &\leqslant \frac{(2M_n + 1)^d \, \mathbb{E}\left[\overline{U}_0^2\right]}{|\Lambda_n|} + \frac{1}{|\Lambda_n|} \sum_{\substack{|j| \leqslant M_n \\ j \neq 0}} \left| [-M_n, M_n]^d \cap \left([-M_n, M_n]^d - j \right) \right| \left| \mathbb{E}\left[\overline{U}_0 \overline{U}_j \right] \right| \\ &\trianglelefteq \frac{M_n^d}{|\Lambda_n|} \left(\mathbb{E}\left[U_0^2\right] + M_n^d \sup_{\substack{j \in \mathbb{Z}^d \\ j \neq 0}} \mathbb{E}\left[|\overline{U}_0 \overline{U}_j| \right] \right) \trianglelefteq \frac{M_n^d}{|\Lambda_n|}. \end{aligned}$$

Then,

$$\frac{1}{|\Lambda_n|} \sum_{k=1}^{|\Lambda_n|} \left| \mathbb{E}\left[\left(Z_{g(k)}^2 - \mathbb{E}[Z_0^2] \right) h_{k-1,k+1}'' \right] \right| \trianglelefteq \frac{M_n^d L' \sup_{\substack{j \in \mathbb{Z}^d \\ j \neq 0}} \mathbb{E}\left[|\overline{U}_0 \overline{U}_j| \right]}{\sqrt{|\Lambda_n|}} + L'^{-\theta} b_n^{-\frac{\theta N}{2}} + \frac{M_n^{d/2}}{\sqrt{|\Lambda_n|}}.$$

For

$$L' = \frac{|\Lambda_n|^{\frac{1}{2(1+\theta)}}}{\left(M_n^d \sup_{\substack{j \in \mathbb{Z}^d \\ j \neq 0}} \mathbb{E}\left[|\overline{U}_0 \overline{U}_j| \right] \right)^{\frac{1}{1+\theta}} b_n^{\frac{\theta N}{2(1+\theta)}}},$$

we get

$$\frac{1}{|\Lambda_n|}\sum_{k=1}^{|\Lambda_n|}\left|\mathbb{E}\left[\left(Z_{g(k)}^2-\mathbb{E}[Z_0^2]\right)h''_{k-1,k+1}\right]\right| \trianglelefteq \frac{\left(M_n^d \sup_{\substack{j\in\mathbb{Z}^d\\ j\neq 0}}\mathbb{E}\left[|\overline{U}_0\overline{U}_j|\right]\right)^{\frac{\theta}{1+\theta}}}{(|\Lambda_n|b_n^N)^{\frac{\theta}{2(1+\theta)}}}+\frac{M_n^{d/2}}{\sqrt{|\Lambda_n|}}.$$

Finally, using again Lemma 9 and keeping in mind that $M_n^d = o(|\Lambda_n|)$ (see Lemma 2) we derive (48). The proof of Theorem 2 is complete. □

Acknowledgements

We are grateful for the usefull suggestions of two anonymous reviewers which allow us to improve the presentation of this work. M. El Machkouri would like to thank the hospitality of the University of Tianjin where a part of this work was done. X. Fan has been partially supported by the National Natural Science Foundation of China (Grant nos. 11601375).

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