A Cascadic Multigrid Method for Nonsymmetric Eigenvalue Problem*

Meiling Yue†, Hehu Xie‡ and Manting Xie§

Abstract

In this paper, a cascadic multigrid method is proposed to solve nonsymmetric eigenvalue problems. Based on the multilevel correction method, the proposed method transforms a nonsymmetric eigenvalue problem solving on the finest finite element space to linear smoothing steps on a sequence of multilevel finite element spaces and some nonsymmetric eigenvalue problems solving on a very low dimensional space. Choosing the sequence of finite element spaces and the number of smoothing steps appropriately, we obtain the optimal convergence rate with the optimal computing complexity. Some numerical examples are provided to validate the theoretical results and the efficiency of this proposed scheme.

Keywords. Nonsymmetric eigenvalue problem, cascadic multigrid, multilevel correction method, finite element method.

AMS subject classifications. 35Q99, 65N30, 65M12, 65M70.

1 Introduction

In modern science and industry, eigenvalue problems appear in many fields such as nanosciences (electronic structure calculations [24, 40]) and engineering (aero-elasticity, chemical engineering [7, 13]) et al. Nonsymmetric eigenvalue problems play important roles in convection-diffusion problems in fluid mechanics, environmental problems and so on (cf. [5, 12]). The analysis of the stability for nonlinear partial differential equations always leads to nonsymmetric eigenvalue problems. However, extensions of the methods for self-adjoint eigenvalue problems to the nonsymmetric ones are not trivial. A two-level method and multilevel correction methods for nonsymmetric eigenvalue problems have been proposed in [17] and [23, 35], respectively. In [22, 29], the authors use a polynomial-preserving gradient recovery (PPR) technique [39] to improve the convergence rate for both symmetric and nonsymmetric eigenvalue problems. In this paper, we aim to construct a cascadic multigrid method for nonsymmetric eigenvalue problems.

The cascadic multigrid method, proposed by [3, 8] and analyzed by [26], is a useful method for solving boundary value problems. It is based on a hierarchy of nested meshes. From the coarsest level to the finest one, the approximate solution on the previous level acts as the initial value of a simple iterative solver (a smoother). However, the algebraic error of initial value from the previous level would accumulate. In cascadic multigrid method, the algebraic error on coarser levels can be decreased by increasing the number of the iteration steps for the smoothing process. Fortunately, the smaller dimensions of problems on coarser levels lead to the optimality of this method. For more information about the cascadic multigrid method, please refer to [6, 27, 28] and the references cited therein.

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†School of Science, Beijing Technology and Business University, Beijing 100048, P. R. China (yuemeiling@lsec.cc.ac.cn)

‡LSEC, NCMIS, Institute of Computational Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China, and School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing, 100049, P. R. China (hhxie@lsec.cc.ac.cn)

§Corresponding author. Center for Applied Mathematics, Tianjin University, Tianjin 300072, P. R. China (mtxie@tju.edu.cn)

Recently, the multilevel correction method for eigenvalue problems has been proposed in [19, 20, 31] and applied in many useful eigenvalue problems, such as nonlinear eigenvalue problems [15, 16], biharmonic eigenvalue problem [37], Fredholm integral eigenvalue problems [36], Bose-Einstein Condensates [33], Kohn-Sham equation [14] and so on. Especially, multilevel method has been applied to nonsymmetric eigenvalue problem, such as [19], Helmholtz transmission eigenvalue problems [10, 30], interior transmission eigenvalue problems [32] and Steklov eigenvalue problems in inverse scattering [38]. Combining multilevel correction scheme and cascadic multigrid method, Han, Xie and Xu [11] present a cascadic multigrid method for self-adjoint eigenvalue problems. This type of cascadic multigrid method can obtain the optimal convergence rate with the optimal scale of computational work.

The purpose of this paper is to construct a cascadic multigrid method to solve the nonsymmetric eigenvalue problem and its adjoint eigenvalue problem. With this method, solving nonsymmetric eigenvalue problems will not be much more expensive than solving corresponding source problems. Based on multilevel correction method, the nonsymmetric eigenvalue problem solving can be reduced to a series of smoothing steps on the sequence of meshes and nonsymmetric eigenvalue problems solving on a very low dimensional space. Similarly to the cascadic multigrid for the boundary value problem, we regard the previous eigenpair approximation as the initial value in each smoothing step and choose suitable numbers of smoothing steps on different levels. Finally, the optimal convergence rate and the optimal computing complexity of the cascadic multigrid can be achieved.

An outline of the paper goes as follows. In Section 2, we introduce the finite element method for nonsymmetric eigenvalue problems. A type of cascadic multigrid method based on the multilevel correction scheme is presented and analyzed in Section 3. Section 4 is devoted to giving the estimation of computational work for the proposed method. In Section 5, three numerical examples are presented to validate the efficiency of the proposed method. Finally, some concluding remarks are given in the last section.

2 Discretization by finite element method

In this section, we introduce the concerned nonsymmetric eigenvalue problem and its corresponding finite element method. The standard notation for the Sobolev spaces $W^{s,p}(\Omega)$ and their associated norms $\|\cdot\|_{s,p,\Omega}$ and seminorms $|\cdot|_{s,p,\Omega}$ will be used (see, e.g. [1, 4]). For $p=2$, we denote $H^s(\Omega)=W^{s,2}(\Omega)$, $H_0^1(\Omega)=\{v\in H^1(\Omega):\ v|_{\partial\Omega}=0\}$, where $v|_{\partial\Omega}=0$ is in the sense of trace, and $\|\cdot\|_{s,\Omega}=\|\cdot\|_{s,2,\Omega}$ for simplicity. In this paper, $\|\cdot\|_{1,\Omega}$ and $\|\cdot\|_{0,\Omega}$ are abbreviated to $\|\cdot\|_1$ and $\|\cdot\|_0$, respectively. In this paper, the letter C (with or without subscripts) denotes a generic positive constant which may be different at its different occurrences through the paper. For convenience, the symbols $\lesssim$, $\gtrsim$ and $\approx$ will be used in this paper. These $x_1\lesssim y_1, x_2\gtrsim y_2$ and $x_3\approx y_3$, mean that $x_1\leq \widetilde{C}_1y_1$, $x_2\geq \widetilde{c}_2y_2$ and $\widetilde{c}_3x_3\leq y_3\leq \widetilde{C}_3x_3$ for some constants $\widetilde{C}_1,\widetilde{c}_2,\widetilde{c}_3$ and $\widetilde{C}_3$ that are independent of mesh size.

2.1 Nonsymmetric eigenvalue problems

In this paper, we consider the following nonsymmetric eigenvalue problem: Find $\lambda\in\mathcal{C}$ and u such that

$$\left\{\begin{array}{rcll} -\nabla\cdot(\mathcal{A}\nabla u)+\mathbf{b}\cdot\nabla u+\rho u &=& \lambda u, & \text{in } \Omega,\\ u &=& 0, & \text{on } \partial\Omega,\\ \int_\Omega |u|^2 d\Omega &=& 1, & \end{array}\right. \tag{2.1}$$

where $\Omega\subset\mathcal{R}^d$ $(d=2,3)$ is a bounded polygonal domain with boundary $\partial\Omega$, $\mathcal{A}$ is a uniformly bounded symmetric positive definite matrix function defined on Ω, $\mathbf{b}\in(W^{1,\infty}(\Omega))^d$ is a bounded real or complex vector function on Ω and $\rho\in L^\infty(\Omega)$ is a bounded function on Ω.

For the aim of finite element discretization, we define the corresponding variational form for (2.1) as follows: Find $(\lambda,u)\in\mathcal{C}\times V$ such that $\|u\|_0=1$ and

$$a(u,v)=\lambda(u,v),\quad \forall v\in V, \tag{2.2}$$

where $V := H_0^1(\Omega)$ and

$$a(\phi,\psi) := \int_\Omega \big(\mathcal{A}\nabla\phi\cdot\overline{\nabla\psi} + (\mathbf{b}\cdot\nabla\phi)\overline{\psi} + \rho\phi\overline{\psi}\big)\mathrm{d}\Omega,$$
$$(\phi,\psi) := \int_\Omega \phi\overline{\psi}\mathrm{d}\Omega,$$

with $\phi,\psi\in V$ and bar denoting the complex conjugate of a function.

For convenience, we define a $H_0^1(\Omega)$ inner product as follows

$$a_s(\phi,\psi) := \int_\Omega \mathcal{A}\nabla\phi\cdot\overline{\nabla\psi}\mathrm{d}\Omega,$$

and the following ellipticity holds

$$\frac{1}{C_a^2}\|\phi\|_1^2 \le a_s(\phi,\phi),\quad \forall\phi\in V. \tag{2.3}$$

For the nonsymmetric eigenvalue problem (2.2), there exists the corresponding adjoint eigenvalue problem (cf. [2]): Find $\lambda^*\in\mathcal{C}$ and u^* such that

$$\left\{\begin{array}{rcll} -\nabla\cdot(\mathcal{A}\nabla u^*) - \nabla\cdot(\overline{\mathbf{b}}u^*) + \overline{\rho}u^* &=& \lambda^* u^*, & \text{in } \Omega,\\ u^* &=& 0, & \text{on } \partial\Omega,\\ \int_\Omega |u^*|^2\mathrm{d}\Omega &=& 1. & \end{array}\right. \tag{2.4}$$

Here, (2.1) and (2.4) connect with each other according to $\lambda^*=\overline{\lambda}$. Using the unified notation, we define the variational form for (2.4) as follows: Find $(\lambda,u^*)\in\mathcal{C}\times V$ such that $\|u^*\|_0=1$ and

$$a(v,u^*) = (v,\lambda^* u^*) = \lambda(v,u^*),\quad \forall v\in V. \tag{2.5}$$

In this paper, the conjugate bilinear form $a(\cdot,\cdot)$ is assumed to satisfy (cf. [35])

$$\|w\|_1 \lesssim \sup_{v\in V}\frac{|a(w,v)|}{\|v\|_1},\quad \text{and}\quad \|w\|_1 \lesssim \sup_{v\in V}\frac{|a(v,w)|}{\|v\|_1},\quad \forall w\in V. \tag{2.6}$$

Furthermore, we suppose $a(\cdot,\cdot)$ is V-elliptic, i.e.

$$\|v\|_1^2 \lesssim \mathrm{Re}\ a(v,v),\quad \forall v\in V,$$

where Re denotes the real part of a complex number. We will use Im to denote the imaginary part of a complex number in the following parts.

For simplicity, we only consider the nondefective eigenvalues (the ascent equals to 1) of the nonsymmetric eigenvalue problem. Thus, the algebraic multiplicity equals to the geometric multiplicity and the generalized eigenspace is the same as the eigenspace. More details about the nonsymmetric eigenvalue problems, please refer to [2, 34, 35].

2.2 Finite element method

Now, we introduce the finite element method (cf. [2, 4]) for the nonsymmetric eigenvalue problem (2.2) and its corresponding adjoint problem (2.5).

First, we decompose the computing domain $\Omega\subset\mathcal{R}^d$ $(d=2,3)$ into shape-regular triangles or rectangles for $d=2$ (tetrahedrons or hexahedrons for $d=3$) and the diameter of a cell $K\in\mathcal{T}_h$ is denoted by h_K. The mesh diameter h describes the maximum diameter of all cells $K\in\mathcal{T}_h$. Based on the mesh $\mathcal{T}_h$, we construct the conforming finite element space denoted by $V_h\subset V$. For simplicity, we only consider the linear Lagrange conforming finite element space which is defined as follows

$$V_h = \big\{v_h\in C(\overline{\Omega})\ \big|\ v_h|_K\in\mathcal{P}_1(K),\ \ \forall K\in\mathcal{T}_h\big\}\cap H_0^1(\Omega), \tag{2.7}$$

where $\mathcal{P}_1(K)$ denotes the space of polynomials of degree ≤ 1. From (2.6), the finite element space V_h satisfies the following conditions

$$\|w_h\|_1 \lesssim \sup_{v_h\in V_h} \frac{|a(w_h,v_h)|}{\|v_h\|_1} \quad \text{and} \quad \|w_h\|_1 \lesssim \sup_{v_h\in V_h} \frac{|a(v_h,w_h)|}{\|v_h\|_1}, \quad \forall w_h \in V_h. \tag{2.8}$$

The standard finite element method for (2.2) is to solve the following eigenvalue problem: Find $(\lambda_h, u_h) \in \mathcal{C} \times V_h$ such that $\|u_h\|_0 = 1$ and

$$a(u_h, v_h) = \lambda_h(u_h, v_h), \quad \forall v_h \in V_h. \tag{2.9}$$

We give the discretization of the adjoint problem (2.5) in the same finite element space: Find $(\lambda_h, u_h^*) \in \mathcal{C} \times V_h$ such that $\|u_h^*\|_0 = 1$ and

$$a(v_h, u_h^*) = \lambda_h(v_h, u_h^*), \quad \forall v_h \in V_h. \tag{2.10}$$

Hereafter, we use the triple (λ_h, u_h, u_h^*) to denote the approximate eigenpair of the nonsymmetric eigenvalue problems (2.2) and (2.5).

Let $M(\lambda)$ and $M^*(\lambda)$ denote two eigenspaces corresponding to the eigenvalue λ of (2.2) and (2.5), respectively,

$$\begin{aligned} M(\lambda) &= \{u \in V : u \text{ is an eigenfunction of (2.2) corresponding to } \lambda\}, \\ M^*(\lambda) &= \{u^* \in V : u^* \text{ is an eigenfunction of (2.5) corresponding to } \lambda\}. \end{aligned}$$

Then, we introduce the following notation for error estimation

$$\begin{aligned} \delta_h(\lambda) &:= \sup_{u\in M(\lambda),\|u\|_0=1} \inf_{v_h\in V_h} \|u - v_h\|_1, \\ \delta_h^*(\lambda) &:= \sup_{u^*\in M^*(\lambda),\|u^*\|_0=1} \inf_{v_h\in V_h} \|u^* - v_h\|_1, \\ \eta(V_h) &:= \sup_{f\in V,\|f\|_0=1} \inf_{v_h\in V_h} \|Tf - v_h\|_1, \\ \eta^*(V_h) &:= \sup_{f\in V,\|f\|_0=1} \inf_{v_h\in V_h} \|T_*f - v_h\|_1, \end{aligned}$$

where the operators $T, T_* \in \mathcal{L}(V)$ are defined by

$$a(Tw, v) = (w, v) = a(w, T_*v), \quad \forall w, v \in V.$$

Since the ascent of the nonsymmetric eigenvalue problem equals to 1, we have the following error estimates.

Theorem 2.1. *([2, Section 8]) When the mesh size h is small enough, for finite element solution (λ_h, u_h) and (λ_h^*, u_h^*), there exist the exact solution (λ, u) and (λ^*, u^*), such that the following error estimates hold*

$$\begin{aligned} \|u - u_h\|_1 &\leq C_{e1}\delta_h(\lambda), &(2.11)\\ \|u^* - u_h^*\|_1 &\leq C_{e1}^*\delta_h^*(\lambda), &(2.12)\\ \|u - u_h\|_0 &\leq C_{e0}\eta^*(V_h)\|u - u_h\|_1 \leq C_{e0}C_{e1}\eta^*(V_h)\delta_h(\lambda), &(2.13)\\ \|u^* - u_h^*\|_0 &\leq C_{e0}^*\eta(V_h)\|u^* - u_h^*\|_1 \leq C_{e0}^*C_{e1}^*\eta(V_h)\delta_h^*(\lambda), &(2.14)\\ |\lambda - \lambda_h| &\leq C_{e\lambda}\|u - u_h\|_1\|u^* - u_h^*\|_1 \leq C_{e\lambda}C_{e1}C_{e1}^*\delta_h(\lambda)\delta_h^*(\lambda), &(2.15) \end{aligned}$$

where $C_{e1}, C_{e0}, C_{e1}^*, C_{e0}^*, C_{e\lambda}$ are constants depending on the eigenvalue distribution but independent of the mesh size h.

Lemma 2.1 ([17]). *Assume $(\lambda, u) \in \mathcal{C}\times V$ and $(\lambda, u^*) \in \mathcal{C}\times V$ satisfy (2.2) and (2.5), respectively, and suppose ψ, $\psi^* \in V$ such that $(\psi, \psi^*) \neq 0$. Let us define*

$$\widehat{\lambda} = \frac{a(\psi, \psi^*)}{(\psi, \psi^*)}.$$

Then we have following expansion

$$\widehat{\lambda} - \lambda = \frac{a(\psi - u, \psi^* - u^*) - \lambda(\psi - u, \psi^* - u^*)}{(\psi, \psi^*)}.$$

3 Cascadic multigrid method

In this section, a type of cascadic multigrid method for nonsymmetric eigenvalue problems will be proposed. At first, we generate a coarse mesh $\mathcal{T}_H$ with the mesh size H and the low dimentional linear finite element space V_H is defined on $\mathcal{T}_H$. Then, suppose $\mathcal{T}_{h_1}$ (produced from $\mathcal{T}_H$ by regular refinements) is given and let $\mathcal{T}_{h_k}$ be obtained from $\mathcal{T}_{h_{k-1}}$ via some regular refinements such that

$$\frac{1}{\beta}h_{k-1} \leq h_k \leq \frac{1}{\beta_1}h_{k-1}, \tag{3.1}$$

where the positive numbers β and β_1 denote the refinement indices and are larger than 1 (always equal 2). Based on this sequence of meshes, we construct the corresponding nested linear finite element spaces by (2.7)

$$V_H \subseteq V_{h_1} \subset V_{h_2} \subset \cdots \subset V_{h_n}.$$

For the following symmetric linear boundary value problem

$$a_s(w_h, v_h) = (f, v_h), \quad \forall v_h \in V_h,$$

we introduce a smoothing operator $S_h : V_h \to V_h$. For generality, assume the concerned smoother S_h satisfies the following estimates

$$\left\{\begin{array}{rcl} \|S_h^m w_h\|_1 & \leq & \frac{C_S}{m^\alpha}\frac{1}{h}\|w_h\|_0, \\ \|S_h^m w_h\|_1 & \leq & \|w_h\|_1, \\ \|S_h^m(w_h+v_h)\|_1 & \leq & \|S_h^m w_h\|_1 + \|S_h^m v_h\|_1, \end{array}\right. \tag{3.2}$$

where C_S is a constant independent of h, α is some positive number depending on S_h, m denotes the number of smoothing steps. It is proved in [9, 25] that the symmetric Gauss-Seidel, the SSOR, the damped Jacobi and the Richardson iteration are smoothers satisfying (3.2) with $\alpha = 1/2$ and the conjugate-gradient iteration is the one with $\alpha = 1$ (cf. [26, 27]).

For simplicity, we denote the smoothing process as

$$\breve{w}_h = \texttt{Smooth}(V_h, f, \xi_h, m, S_h),$$

where ξ_h denotes the initial value, m is the number of smoothing steps, S_h is the smoother and $\breve{w}_h$ is the output of this smoothing process.

3.1 Cascadic multigrid method

We now proceed to give the cascadic multigrid method for nonsymmetric eigenvalue problems. For simplicity, the desired eigenvalue is assumed to be nondefective (the ascent equals to 1) and the computing domain is convex. From the error estimate theory of finite element method [4, 21], there exist constants $C_\delta, C_\delta^*, C_\eta, C_\eta^* > 0$ such that for $k = 1, \cdots, n$

$$\delta_{h_k}(\lambda) \leq C_\delta h_k, \quad \delta_{h_k}^*(\lambda) \leq C_\delta^* h_k \quad \text{and} \quad \eta(V_{h_k}) \leq C_\eta h_k, \quad \eta^*(V_{h_k}) \leq C_\eta^* h_k. \tag{3.3}$$

Then from (3.1) and (3.3), there exist constants $C_{\delta\delta}$, $C_{\delta\delta}^*$, $C_{\eta\eta}$, $C_{\eta\eta}^* > 0$ such that for $k = 1, \cdots, n-1$

$$\begin{aligned} &\delta_{h_k}(\lambda) \leq C_{\delta\delta}\beta\delta_{h_{k+1}}(\lambda), \quad \delta_{h_k}^*(\lambda) \leq C_{\delta\delta}^*\beta\delta_{h_{k+1}}^*(\lambda), \\ &\eta(V_{h_k}) \leq C_{\eta\eta}\beta\eta(V_{h_{k+1}}), \quad \eta^*(V_{h_k}) \leq C_{\eta\eta}^*\beta\eta(V_{h_{k+1}}). \end{aligned} \tag{3.4}$$

Furthermore, we have constants $C_{\delta\eta}$, $C_{\delta\eta}^* > 0$ such that for $k = 1, \cdots, n$

$$\delta_{h_k}(\lambda) \leq C_{\delta\eta}\eta(V_{h_k}) \quad \text{and} \quad \delta_{h_k}^*(\lambda) \leq C_{\delta\eta}^*\eta^*(V_{h_k}). \tag{3.5}$$

Remark 3.1. *The relation (3.4) is reasonable since the lower bound result $\delta_{h_k}(\lambda)$ is given in [21].*

Assume we have obtained an eigenpair approximation $(\widetilde{\lambda}_{h_k}, \widetilde{u}_{h_k}, \widetilde{u}^*_{h_k}) \in \mathcal{C} \times V_{h_k} \times V_{h_k}$. Now we introduce a cascadic type of one correction step to improve the accuracy of current approximation.

Algorithm 3.1. *Cascadic Type of One Correction Step*

1. *Define the following symmetric positive definite linear problems:*

   Find $\widehat{u}_{h_{k+1}} \in V_{h_{k+1}}$ *such that for any* $v_{h_{k+1}} \in V_{h_{k+1}}$

$$a_s(\widehat{u}_{h_{k+1}}, v_{h_{k+1}}) = \widetilde{\lambda}_{h_k}(\widetilde{u}_{h_k}, v_{h_{k+1}}) - (\mathbf{b} \cdot \nabla \widetilde{u}_{h_k}, v_{h_{k+1}}) - (\rho \widetilde{u}_{h_k}, v_{h_{k+1}}). \tag{3.6}$$

   Find $\widehat{u}^*_{h_{k+1}} \in V_{h_{k+1}}$ *such that for any* $v_{h_{k+1}} \in V_{h_{k+1}}$

$$a_s(v_{h_{k+1}}, \widehat{u}^*_{h_{k+1}}) = \widetilde{\lambda}_{h_k}(v_{h_{k+1}}, \widetilde{u}^*_{h_k}) + (v_{h_{k+1}}, \nabla \cdot (\overline{\mathbf{b}} \widetilde{u}^*_{h_k})) - (v_{h_{k+1}}, \overline{\rho} \widetilde{u}^*_{h_k}). \tag{3.7}$$

   Solve (3.6) and (3.7) by smoothing process to obtain new eigenfunction approximations

$$\check{u}_{h_{k+1}} = \texttt{Smooth}(V_{h_{k+1}}, \widetilde{\lambda}_{h_k} \widetilde{u}_{h_k} - \mathbf{b} \cdot \nabla \widetilde{u}_{h_k} - \rho \widetilde{u}_{h_k}, \widetilde{u}_{h_k}, m_{k+1}, S_{h_{k+1}}),$$
$$\check{u}^*_{h_{k+1}} = \texttt{Smooth}(V_{h_{k+1}}, \overline{\widetilde{\lambda}}_{h_k} \widetilde{u}^*_{h_k} + \nabla \cdot (\overline{\mathbf{b}} \widetilde{u}^*_{h_k}) - \overline{\rho} \widetilde{u}^*_{h_k}, \widetilde{u}^*_{h_k}, m_{k+1}, S_{h_{k+1}}).$$

2. *Define two new finite element spaces* $V_{H,h_{k+1}} = V_H + \mathrm{span}\{\check{u}_{h_{k+1}}\}$ *and* $V^*_{H,h_{k+1}} = V_H + \mathrm{span}\{\check{u}^*_{h_{k+1}}\}$*, and solve following eigenvalue problems:*

   Find $(\widetilde{\lambda}_{h_{k+1}}, \widetilde{u}_{h_{k+1}}) \in \mathcal{C} \times V_{H,h_{k+1}}$ *such that* $\|\widetilde{u}_{h_{k+1}}\|_0 = 1$ *and*

$$a(\widetilde{u}_{h_{k+1}}, v_{H,h_{k+1}}) = \widetilde{\lambda}_{h_{k+1}}(\widetilde{u}_{h_{k+1}}, v_{H,h_{k+1}}), \quad \forall v_{H,h_{k+1}} \in V_{H,h_{k+1}}. \tag{3.8}$$

   Find $(\widetilde{\lambda}_{h_{k+1}}, \widetilde{u}^*_{h_{k+1}}) \in \mathcal{C} \times V^*_{H,h_{k+1}}$ *such that* $\|\widetilde{u}^*_{h_{k+1}}\|_0 = 1$ *and*

$$a(v^*_{H,h_{k+1}}, \widetilde{u}^*_{h_{k+1}}) = \widetilde{\lambda}_{h_{k+1}}(v^*_{H,h_{k+1}}, \widetilde{u}^*_{h_{k+1}}), \quad \forall v^*_{H,h_{k+1}} \in V^*_{H,h_{k+1}}. \tag{3.9}$$

Summarize the above two steps by

$$(\widetilde{\lambda}_{h_{k+1}}, \widetilde{u}_{h_{k+1}}, \widetilde{u}^*_{h_{k+1}}) = \texttt{CascadicCorrection}(V_H, V_{h_{k+1}}, \widetilde{\lambda}_{h_k}, \widetilde{u}_{h_k}, \widetilde{u}^*_{h_k}, m_{k+1}, S_{h_{k+1}}).$$

Based on this type of one correction step, we can build a cascadic multigrid scheme for nonsymmetric eigenvalue problem (2.2) and its adjoint eigenvalue problem (2.4) in the next algorithm.

Algorithm 3.2. *Cascadic Multigrid Method*

1. *Solve thee following eigenvalue problems in* V_{h_1} *to obtain the initial eigenpair approximation:*

   Find $(\widetilde{\lambda}_{h_1}, \widetilde{u}_{h_1}) \in \mathcal{C} \times V_{h_1}$ *such that* $\|\widetilde{u}_{h_1}\|_0 = 1$ *and*

$$a(\widetilde{u}_{h_1}, v_{h_1}) = \widetilde{\lambda}_{h_1}(\widetilde{u}_{h_1}, v_{h_1}), \quad \forall v_{h_1} \in V_{h_1}.$$

   Find $(\widetilde{\lambda}_{h_1}, \widetilde{u}^*_{h_1}) \in \mathcal{C} \times V_{h_1}$ *such that* $\|\widetilde{u}^*_{h_1}\|_0 = 1$ *and*

$$a(v_{h_1}, \widetilde{u}^*_{h_1}) = \widetilde{\lambda}_{h_1}(v_{h_1}, \widetilde{u}^*_{h_1}), \quad \forall v_{h_1} \in V_{h_1}.$$

2. *For* $k = 1, \cdots, n-1$*, do the following iteration*

$$(\widetilde{\lambda}_{h_{k+1}}, \widetilde{u}_{h_{k+1}}, \widetilde{u}^*_{h_{k+1}}) = \texttt{CascadicCorrection}(V_H, V_{h_{k+1}}, \widetilde{\lambda}_{h_k}, \widetilde{u}_{h_k}, \widetilde{u}^*_{h_k}, m_{k+1}, S_{h_{k+1}}).$$

Finally, we obtain the approximation $(\widetilde{\lambda}_{h_n}, \widetilde{u}_{h_n}, \widetilde{u}^*_{h_n}) \in \mathcal{C} \times V_{h_n} \times V_{h_n}$ *in the finest level of space* V_{h_n}.

The error estimate of the above cascadic multigrid method needs some auxiliary results. Hence, we first propose an auxiliary multilevel correction method in the next subsection and then give the error estimate of Algorithm 3.2 in subsection 3.3.

3.2 Auxiliary multilevel correction method

In order to analyze the convergence of Algorithm 3.2, we introduce an auxiliary algorithm and then show its super approximate property in this subsection. Assume we have obtained the approximation $(\underset{\sim}{\lambda}_{h_k}, \underset{\sim}{u}_{h_k}, \underset{\sim}{u}^*_{h_k}) \in \mathcal{C} \times V_{h_k} \times V_{h_k}$. Then define an auxiliary one correction step as
abc
follows.

Algorithm 3.3. *Auxiliary One Correction Step*

1. *Define the following auxiliary source problems:*

   Find $\underline{u}_{h_{k+1}} \in V_{h_{k+1}}$ *such that for any* $v_{h_{k+1}} \in V_{h_{k+1}}$

$$a_s(\underline{u}_{h_{k+1}}, v_{h_{k+1}}) = \underset{\sim}{\lambda}_{h_k}(\underset{\sim}{u}_{h_k}, v_{h_{k+1}}) - (\mathbf{b}\cdot\nabla \underset{\sim}{u}_{h_k}, v_{h_{k+1}}) - (\rho \underset{\sim}{u}_{h_k}, v_{h_{k+1}}). \tag{3.10}$$

   Find $\underline{u}^*_{h_{k+1}} \in V_{h_{k+1}}$ *such that for any* $v_{h_{k+1}} \in V_{h_{k+1}}$

$$a_s(v_{h_{k+1}}, \underline{u}^*_{h_{k+1}}) = \underset{\sim}{\lambda}_{h_k}(v_{h_{k+1}}, \underset{\sim}{u}^*_{h_k}) + (v_{h_{k+1}}, \nabla\cdot(\overline{\mathbf{b}}\underset{\sim}{u}^*_{h_k})) - (v_{h_{k+1}}, \overline{\rho}\underset{\sim}{u}^*_{h_k}). \tag{3.11}$$

2. *Define two new finite element spaces* $\underset{\sim}{V}_{H,h_{k+1}} = V_H + \mathrm{span}\{\underline{u}_{h_{k+1}}\} + \mathrm{span}\{\check{u}_{h_{k+1}}\}$ *and* $\underset{\sim}{V}^*_{H,h_{k+1}} = V_H + \mathrm{span}\{\underline{u}^*_{h_{k+1}}\} + \mathrm{span}\{\check{u}^*_{h_{k+1}}\}$, *and solve following eigenvalue problems:*

   Find $(\underset{\sim}{\lambda}_{h_{k+1}}, \underset{\sim}{u}_{h_{k+1}}) \in \mathcal{C} \times \underset{\sim}{V}_{H,h_{k+1}}$ *such that* $\|\underset{\sim}{u}_{h_{k+1}}\|_0 = 1$ *and*

$$a(\underset{\sim}{u}_{h_{k+1}}, \underset{\sim}{v}_{H,h_{k+1}}) = \underset{\sim}{\lambda}_{h_{k+1}}(\underset{\sim}{u}_{h_{k+1}}, \underset{\sim}{v}_{H,h_{k+1}}), \quad \forall \underset{\sim}{v}_{H,h_{k+1}} \in \underset{\sim}{V}_{H,h_{k+1}}. \tag{3.12}$$

   Find $(\underset{\sim}{\lambda}_{h_{k+1}}, \underset{\sim}{u}^*_{h_{k+1}}) \in \mathcal{C} \times \underset{\sim}{V}^*_{H,h_{k+1}}$ *such that* $\|\underset{\sim}{u}^*_{h_{k+1}}\|_0 = 1$ *and*

$$a(\underset{\sim}{v}^*_{H,h_{k+1}}, \underset{\sim}{u}^*_{h_{k+1}}) = \underset{\sim}{\lambda}_{h_{k+1}}(\underset{\sim}{v}^*_{H,h_{k+1}}, \underset{\sim}{u}^*_{h_{k+1}}), \quad \forall \underset{\sim}{v}^*_{H,h_{k+1}} \in \underset{\sim}{V}^*_{H,h_{k+1}}. \tag{3.13}$$

Summarize the above two steps by defining

$$(\underset{\sim}{\lambda}_{h_{k+1}}, \underset{\sim}{u}_{h_{k+1}}, \underset{\sim}{u}^*_{h_{k+1}}) = \texttt{AuxiliaryCorrection}(V_H, V_{h_{k+1}}, \underset{\sim}{\lambda}_{h_k}, \underset{\sim}{u}_{h_k}, \underset{\sim}{u}^*_{h_k}, \check{u}_{h_{k+1}}, \check{u}^*_{h_{k+1}}).$$

Algorithm 3.4. *Auxiliary Multilevel Correction Method*

1. *Solve the following eigenvalue problems in* V_{h_1}*:*

   Find $(\underset{\sim}{\lambda}_{h_1}, \underset{\sim}{u}_{h_1}) \in \mathcal{C} \times V_{h_1}$ *such that* $\|\underset{\sim}{u}_{h_1}\|_0 = 1$ *and*

$$a(\underset{\sim}{u}_{h_1}, v_{h_1}) = \underset{\sim}{\lambda}_{h_1}(\underset{\sim}{u}_{h_1}, v_{h_1}), \quad \forall v_{h_1} \in V_{h_1}.$$

   Find $(\underset{\sim}{\lambda}_{h_1}, \underset{\sim}{u}^*_{h_1}) \in \mathcal{C} \times V_{h_1}$ *such that* $\|\underset{\sim}{u}^*_{h_1}\|_0 = 1$ *and*

$$a(v_{h_1}, \underset{\sim}{u}^*_{h_1}) = \underset{\sim}{\lambda}_{h_1}(v_{h_1}, \underset{\sim}{u}^*_{h_1}), \quad \forall v_{h_1} \in V_{h_1}.$$

2. *For* $k = 1, \cdots, n-1$, *do the following iteration*

$$(\underset{\sim}{\lambda}_{h_{k+1}}, \underset{\sim}{u}_{h_{k+1}}, \underset{\sim}{u}^*_{h_{k+1}}) = \texttt{AuxiliaryCorrection}(V_H, V_{h_{k+1}}, \underset{\sim}{\lambda}_{h_k}, \underset{\sim}{u}_{h_k}, \underset{\sim}{u}^*_{h_k}, \check{u}_{h_{k+1}}, \check{u}^*_{h_{k+1}}).$$

Finally, Algorithm 3.4 output the approximation $(\underset{\sim}{\lambda}_{h_n}, \underset{\sim}{u}_{h_n}, \underset{\sim}{u}^*_{h_n}) \in \mathcal{C} \times V_{h_n} \times V_{h_n}$.

Remark 3.2. *The only aim of introducing Algorithms 3.3 and 3.4 is to analyze the convergence of Algorithm 3.2. They will not be used for computing.*

Before analyzing the convergence of Algorithm 3.2, we show a super approximate property between the eigenpair approximation obtained by Algorithm 3.4 and the standard finite element solution.

Theorem 3.1. *Assume $(\underset{\sim}{\lambda}_{h_k}, \underset{\sim}{u}_{h_k}, \underset{\sim}{u}^*_{h_k})$ is the output of Algorithm 3.4, $(\lambda_{h_k}, u_{h_k}, u^*_{h_k})$ is the standard finite element solution in V_{h_k}, $k = 1, \cdots, n$. If the coarse finite element space V_H satisfies*

$$\hat{C}_1\eta^*(V_H) < 1 \quad \text{and} \quad \hat{C}_1^*\eta(V_H) < 1, \tag{3.14}$$

for $k = 1, \cdots, n$, the following estimates hold

$$\begin{aligned} \|u_{h_k} - \underset{\sim}{u}_{h_k}\|_1 &\leq C_1\eta^*(V_{h_k})\delta_{h_k}(\lambda), &(3.15)\\ \|u_{h_k} - \underset{\sim}{u}_{h_k}\|_0 &\leq C_{e0}C_1\eta^*(V_H)\eta^*(V_{h_k})\delta_{h_k}(\lambda), &(3.16)\\ \|u^*_{h_k} - \underset{\sim}{u}^*_{h_k}\|_1 &\leq C_1^*\eta(V_{h_k})\delta^*_{h_k}(\lambda), &(3.17)\\ \|u^*_{h_k} - \underset{\sim}{u}^*_{h_k}\|_0 &\leq C^*_{e0}C_1^*\eta(V_H)\eta(V_{h_k})\delta^*_{h_k}(\lambda). &(3.18)\end{aligned}$$

Here, the constants C_1 and C_1^* are defined as follows

$$C_1 = \frac{\hat{C}_1}{1 - \hat{C}_1\eta^*(V_H)} \quad \text{and} \quad C_1^* = \frac{\hat{C}_1^*}{1 - \hat{C}_1^*\eta(V_H)},$$

where

$$\begin{aligned} \hat{C}_1 &= C_rC^*_{\eta\eta}C_{\delta\delta}\beta^2, \quad C_r = \widetilde{C}C_aC_m\widetilde{C}_m,\\ \hat{C}_1^* &= C_r^*C_{\eta\eta}C^*_{\delta\delta}\beta^2, \quad C_r^* = \widetilde{C}C_aC_m\widetilde{C}^*_m,\\ C_m &= \max\{1, |\underset{\sim}{\lambda}_{h_k}| + \|\rho\|_{0,\infty} + \|\mathbf{b}\|_{1,\infty}\},\\ \widetilde{C}_m &= \max\{2C_{e0}C_{e1} + 2C_{e\lambda}C_{e1}C^*_{e1}C^*_{\delta\eta}, 2C_{e0} + 2C_{e\lambda}C^*_{e1}C^*_{\delta\eta}\},\\ \widetilde{C}^*_m &= \max\{2C^*_{e0}C^*_{e1} + 2C_{e\lambda}C^*_{e1}C_{e1}C_{\delta\eta}, 2C^*_{e0} + 2C_{e\lambda}C_{e1}C_{\delta\eta}\},\end{aligned}$$

and the constant $\widetilde{C} \geq 1$ is similar to C_{e1} in (2.11).

The estimate between eigenvalue approximations λ_{h_k} and $\underset{\sim}{\lambda}_{h_k}$ satisfies

$$\big|\lambda_{h_k} - \underset{\sim}{\lambda}_{h_k}\big| \leq C_{e\lambda}\|u_{h_k} - \underset{\sim}{u}_{h_k}\|_1\|u^*_{h_k} - \underset{\sim}{u}^*_{h_k}\|_1. \tag{3.19}$$

Proof. Actually, eigenvalue problem (3.12) defined in $\underset{\sim}{V}_{H,h_{k+1}}$ is a finite dimensional approximation of (2.9) in $V_{h_{k+1}}$. Similarly to Theorem 2.1 (see [2]), there exists a constant $\widetilde{C} \geq 1$ (similar to C_{e1} in (2.11)) such that

$$\begin{aligned} \|u_{h_{k+1}} - \underset{\sim}{u}_{h_{k+1}}\|_1 &\leq \widetilde{C}\inf_{\underset{\sim}{v}_{H,h_{k+1}} \in \underset{\sim}{V}_{H,h_{k+1}}}\|u_{h_{k+1}} - \underset{\sim}{v}_{H,h_{k+1}}\|_1\\ &\leq \widetilde{C}\|u_{h_{k+1}} - \underline{u}_{h_{k+1}}\|_1. &(3.20)\end{aligned}$$

Hence, we first estimate $\|u_{h_{k+1}} - \underline{u}_{h_{k+1}}\|_1$.

Setting $w_{h_{k+1}} = u_{h_{k+1}} - \underline{u}_{h_{k+1}} \in V_{h_{k+1}}$, using (2.9) to minus (3.10), we get

$$\begin{aligned} &a_s(u_{h_{k+1}} - \underline{u}_{h_{k+1}}, w_{h_{k+1}})\\ =\; &\lambda_{h_{k+1}}(u_{h_{k+1}}, w_{h_{k+1}}) - (\mathbf{b}\cdot\nabla u_{h_{k+1}}, w_{h_{k+1}}) - (\rho u_{h_{k+1}}, w_{h_{k+1}})\\ &-\underset{\sim}{\lambda}_{h_k}(\underset{\sim}{u}_{h_k}, w_{h_{k+1}}) + (\mathbf{b}\cdot\nabla\underset{\sim}{u}_{h_k}, w_{h_{k+1}}) + (\rho\underset{\sim}{u}_{h_k}, w_{h_{k+1}})\\ =\; &((\lambda_{h_{k+1}}u_{h_{k+1}} - \underset{\sim}{\lambda}_{h_k}\underset{\sim}{u}_{h_k}) - \rho(u_{h_{k+1}} - \underset{\sim}{u}_{h_k}), w_{h_{k+1}}) + (u_{h_{k+1}} - \underset{\sim}{u}_{h_k}, \nabla\cdot(\overline{\mathbf{b}}w_{h_{k+1}})).\end{aligned}$$

Then using the triangle inequality and the Hölder inequality, we have

$$\begin{aligned} &a_s(u_{h_{k+1}} - \underline{u}_{h_{k+1}}, w_{h_{k+1}})\\ \leq\; &C_m(|\lambda_{h_{k+1}} - \underset{\sim}{\lambda}_{h_k}| + \|u_{h_{k+1}} - \underset{\sim}{u}_{h_k}\|_0)\|w_{h_{k+1}}\|_1\\ \leq\; &C_m(|\lambda_{h_{k+1}} - \lambda_{h_k}| + |\lambda_{h_k} - \underset{\sim}{\lambda}_{h_k}| + \|u_{h_{k+1}} - u_{h_k}\|_0 + \|u_{h_k} - \underset{\sim}{u}_{h_k}\|_0)\|w_{h_{k+1}}\|_1\\ =\; &C_m(|\lambda_{h_{k+1}} - \lambda_{h_k}| + \|u_{h_{k+1}} - u_{h_k}\|_0 + \varepsilon_{h_k})\|w_{h_{k+1}}\|_1, &(3.21)\end{aligned}$$

where $C_m = \max\{1, |\underset{\sim}{\lambda}_{h_k}| + \|\rho\|_{0,\infty} + \|\mathbf{b}\|_{1,\infty}\}$ and $\varepsilon_{h_k} = |\lambda_{h_k} - \underset{\sim}{\lambda}_{h_k}| + \|u_{h_k} - \underset{\sim}{u}_{h_k}\|_0$.

Combining the ellipticity (2.3) and (3.21), we have

$$\|u_{h_{k+1}} - \underset{\sim}{u}_{h_{k+1}}\|_1 \leq C_a C_m (|\lambda_{h_{k+1}} - \lambda_{h_k}| + \|u_{h_{k+1}} - u_{h_k}\|_0 + \varepsilon_{h_k}). \tag{3.22}$$

Noting $\underset{\sim}{V}_{H,h_k} \subset V_{h_k} \subset V_{h_{k+1}} \subset V$, from Theorem 2.1 and (3.5), the following inequalities hold

$$\begin{aligned}
\|u_{h_{k+1}} - u_{h_k}\|_1 &\leq \|u_{h_{k+1}} - u\|_1 + \|u - u_{h_k}\|_1 \leq 2C_{e1}\delta_{h_k}(\lambda), &(3.23)\\
\|u_{h_{k+1}} - u_{h_k}\|_0 &\leq C_{e0}\eta^*(V_{h_k})\|u_{h_{k+1}} - u_{h_k}\|_1 \leq 2C_{e0}C_{e1}\eta^*(V_{h_k})\delta_{h_k}(\lambda), &(3.24)\\
|\lambda_{h_{k+1}} - \lambda_{h_k}| &\leq 2C_{e\lambda}C_{e1}C_{e1}^*\delta_{h_k}(\lambda)\delta_{h_k}^*(\lambda)\\
&\leq 2C_{e\lambda}C_{e1}C_{e1}^*C_{\delta\eta}^*\eta^*(V_{h_k})\delta_{h_k}(\lambda), &(3.25)\\
\|u_{h_k} - \underset{\sim}{u}_{h_k}\|_0 &\leq C_{e0}\eta^*(\underset{\sim}{V}_{H,h_k})\|u_{h_k} - \underset{\sim}{u}_{h_k}\|_1\\
&\leq C_{e0}\eta^*(V_H)\|u_{h_k} - \underset{\sim}{u}_{h_k}\|_1, &(3.26)\\
\|u_{h_k}^* - \underset{\sim}{u}_{h_k}^*\|_1 &\leq \|u_{h_k}^* - u^*\|_1 + \|u^* - \underset{\sim}{u}_{h_k}^*\|_1\\
&\leq 2C_{e1}^*\delta_H^*(\lambda) \leq 2C_{e1}^*C_{\delta\eta}^*\eta^*(V_H), &(3.27)\\
|\lambda_{h_k} - \underset{\sim}{\lambda}_{h_k}| &\leq C_{e\lambda}\|u_{h_k} - \underset{\sim}{u}_{h_k}\|_1\|u_{h_k}^* - \underset{\sim}{u}_{h_k}^*\|_1\\
&\leq 2C_{e\lambda}C_{e1}^*C_{\delta\eta}^*\eta^*(V_H)\|u_{h_k} - \underset{\sim}{u}_{h_k}\|_1, &(3.28)
\end{aligned}$$

where we use the following inequality

$$\eta^*(\underset{\sim}{V}_{H,h_k}) := \sup_{f\in L^2(\Omega),\|f\|_0=1}\ \inf_{v\in \underset{\sim}{V}_{H,h_k}} \|T_*f - v\|_1 \leq \eta^*(V_H).$$

Combining the definition of ε_{h_k}, (3.26) and (3.28), we have

$$\varepsilon_{h_k} = \|u_{h_k} - \underset{\sim}{u}_{h_k}\|_0 + |\lambda_{h_k} - \underset{\sim}{\lambda}_{h_k}| \leq (C_{e0} + 2C_{e\lambda}C_{e1}^*C_{\delta\eta}^*)\eta^*(V_H)\|u_{h_k} - \underset{\sim}{u}_{h_k}\|_1.$$

From (3.22), (3.24), (3.25) and the above estimate, the following estimate holds for $k = 1, \cdots, n-1$,

$$\|u_{h_{k+1}} - \underset{\sim}{u}_{h_{k+1}}\|_1 \leq C_a C_m \widetilde{C}_m(\eta^*(V_{h_k})\delta_{h_k}(\lambda) + \eta^*(V_H)\|u_{h_k} - \underset{\sim}{u}_{h_k}\|_1), \tag{3.29}$$

where $\widetilde{C}_m = \max\{2C_{e0}C_{e1} + 2C_{e\lambda}C_{e1}C_{e1}^*C_{\delta\eta}^*, C_{e0} + 2C_{e\lambda}C_{e1}^*C_{\delta\eta}^*\}$.

Then combining (3.20) and (3.29), we have

$$\|u_{h_{k+1}} - \underset{\sim}{u}_{h_{k+1}}\|_1 \leq C_r\big(\eta^*(V_{h_k})\delta_{h_k}(\lambda) + \eta^*(V_H)\|u_{h_k} - \underset{\sim}{u}_{h_k}\|_1\big), \tag{3.30}$$

where $C_r = \widetilde{C}C_aC_m\widetilde{C}_m$.

According to (2.9) and the first step of Algorithm 3.4, we have $\underset{\sim}{u}_{h_1} = u_{h_1}$, $\underset{\sim}{\lambda}_{h_1} = \lambda_{h_1}$. Together with (3.30), the following inequality holds

$$\|u_{h_2} - \underset{\sim}{u}_{h_2}\|_1 \leq C_r\eta^*(V_{h_1})\delta_{h_1}(\lambda). \tag{3.31}$$

From (3.4), (3.30), (3.31) and recursive argument, we have following estimates

$$\begin{aligned}
\|u_{h_k} - \underset{\sim}{u}_{h_k}\|_1 &\leq C_r\sum_{j=2}^{k}(C_r\eta^*(V_H))^{k-j}\eta^*(V_{h_{j-1}})\delta_{h_{j-1}}(\lambda)\\
&\leq C_r\sum_{j=2}^{k}(C_r\eta^*(V_H))^{k-j}(C_{\eta\eta}^*\beta)^{k-j+1}\eta^*(V_{h_k})(C_{\delta\delta}\beta)^{k-j+1}\delta_{h_k}(\lambda)\\
&\leq C_rC_{\eta\eta}^*C_{\delta\delta}\beta^2\Big(\sum_{j=2}^{k}\big(C_r\eta^*(V_H)C_{\eta\eta}^*C_{\delta\delta}\beta^2\big)^{k-j}\Big)\eta^*(V_{h_k})\delta_{h_k}(\lambda)\\
&\leq \frac{C_rC_{\eta\eta}^*C_{\delta\delta}\beta^2}{1 - C_rC_{\eta\eta}^*C_{\delta\delta}\eta^*(V_H)\beta^2}\eta^*(V_{h_k})\delta_{h_k}(\lambda). &(3.32)
\end{aligned}$$

Therefore, take $\hat{C}_1 = C_rC_{\eta\eta}^*C_{\delta\delta}\beta^2$ and $C_1 = \hat{C}_1/(1 - \hat{C}_1\eta^*(V_H))$, (3.32) is the desired result (3.15) under the condition $\hat{C}_1\eta^*(V_H) < 1$. The estimate (3.16) is the direct results of Theorem 2.1 and (3.15). The estimates (3.17) and (3.18) for the adjoint problem can be proved in the similar way. Then the desired result (3.19) is the direct of Theorem 2.1, Lemma 2.1 and (3.15)-(3.18). □

Noting $V_{H,h_k} \subset \underset{\sim}{V}_{H,h_k}$ and $V^*_{H,h_k} \subset \underset{\sim}{V}^*_{H,h_k}$, we have following estimates which are useful in our analysis.

Lemma 3.1. *([2, Lemma 3.5]) Under the conditions of Theorem 3.1, the following error estimates hold for $k = 1, \cdots, n$*

$$\begin{aligned}
\|\widetilde{u}_{h_k} - \underset{\sim}{u}_{h_k}\|_1 &\leq C_2\|\check{u}_{h_k} - \underset{\sim}{u}_{h_k}\|_1, &(3.33)\\
\|\widetilde{u}_{h_k} - \underset{\sim}{u}_{h_k}\|_0 &\leq C_{e0}\eta^*(V_H)\|\widetilde{u}_{h_k} - \underset{\sim}{u}_{h_k}\|_1, &(3.34)\\
\|\widetilde{u}^*_{h_k} - \underset{\sim}{u}^*_{h_k}\|_1 &\leq C^*_2\|\check{u}^*_{h_k} - \underset{\sim}{u}^*_{h_k}\|_1, &(3.35)\\
\|\widetilde{u}^*_{h_k} - \underset{\sim}{u}^*_{h_k}\|_0 &\leq C^*_{e0}\eta(V_H)\|\widetilde{u}^*_{h_k} - \underset{\sim}{u}^*_{h_k}\|_1, &(3.36)\\
|\widetilde{\lambda}_{h_k} - \underset{\sim}{\lambda}_{h_k}| &\leq C_{e\lambda}\|\widetilde{u}_{h_k} - \underset{\sim}{u}_{h_k}\|_1\|\widetilde{u}^*_{h_k} - \underset{\sim}{u}^*_{h_k}\|_1, &(3.37)
\end{aligned}$$

where $C_2 = \widetilde{C}\check{C}$, $C^*_2 = \widetilde{C}^*\check{C}^*$, $\check{C} > 0$ and $\check{C}^* > 0$ are some constants.

Proof. Combining (3.8), (3.12) and $V_{H,h_k} \subset \underset{\sim}{V}_{H,h_k}$, $\widetilde{u}_{h_k}$ is an approximation of $\underset{\sim}{u}_{h_k}$. From Theorem 2.1 and the constructions of V_{H,h_k} and $\underset{\sim}{V}_{H,h_k} = V_{H,h_k} + \text{span}\{\underset{\sim}{u}_{h_k}\}$, similar to (3.20), there exists a constant $\check{C} > 0$ such that

$$\begin{aligned}
&\|\widetilde{u}_{h_k} - \underset{\sim}{u}_{h_k}\|_1 \leq \widetilde{C} \inf_{v_{H,h_k} \in V_{H,h_k}} \|v_{H,h_k} - \underset{\sim}{u}_{h_k}\|_1\\
&\leq \widetilde{C}\check{C} \inf_{v_{H,h_k} \in V_{H,h_k}} \|v_{H,h_k} - \underset{\sim}{u}_{h_k}\|_1 \leq \widetilde{C}\check{C}\|\check{u}_{h_k} - \underset{\sim}{u}_{h_k}\|_1, \qquad (3.38)
\end{aligned}$$

which is the desired result (3.33). Furthermore, the estimate (3.34) can be deduced from Theorem 2.1 and the following inequality

$$\eta^*(V_{H,h_k}) := \sup_{f \in L^2(\Omega), \|f\|_0 = 1} \inf_{v \in V_{H,h_k}} \|T_* f - v\|_1 \leq \eta^*(V_H).$$

Similarly, we can prove the inequalities (3.35) and (3.36). Then the desired estimate (3.37) can be obtained from (2.15), Lemma 2.1 and (3.33)-(3.36). □

Now, we are in the position to give error estimates between the outputs of Algorithms 3.2 and 3.4.

Theorem 3.2. *Assume the smoothers satisfy the smoothing property (3.2). Under the conditions of Theorem 3.1, we have following estimates*

$$\begin{aligned}
\|\widetilde{u}_{h_n} - \underset{\sim}{u}_{h_n}\|_1 &\leq C_4 \sum_{k=2}^{n} \frac{(C_3)^{n-k}}{m_k^\alpha}\delta_{h_{k-1}}(\lambda), &(3.39)\\
\|\widetilde{u}^*_{h_n} - \underset{\sim}{u}^*_{h_n}\|_1 &\leq C^*_4 \sum_{k=2}^{n} \frac{(C^*_3)^{n-k}}{m_k^\alpha}\delta^*_{h_{k-1}}(\lambda), &(3.40)\\
\left|\widetilde{\lambda}_{h_n} - \underset{\sim}{\lambda}_{h_n}\right| &\leq C_{e\lambda}\|\widetilde{u}_{h_n} - \underset{\sim}{u}_{h_n}\|_1\|\widetilde{u}^*_{h_n} - \underset{\sim}{u}^*_{h_n}\|_1, &(3.41)
\end{aligned}$$

where the constants C_3, C^*_3, C_4 and C^*_4 are defined as follows

$$\begin{aligned}
C_3 &= 1 + 2C_aC_m\big(C_{e\lambda}C^*_{e1}\delta^*_H(\lambda) + C_{e0}\eta^*(V_H)\big),\\
C^*_3 &= 1 + 2C_aC_m\big(C_{e\lambda}C_{e1}\delta_H(\lambda) + C^*_{e0}\eta(V_H)\big),\\
C_4 &= C_2C_S\widetilde{\widetilde{C}}_mC^*_\eta\beta, \quad C^*_4 = C^*_2C_S\widetilde{\widetilde{C}}^*_mC_\eta\beta
\end{aligned}$$

with

$$\begin{aligned}
\widetilde{\widetilde{C}}_m &= C_aC_m\widetilde{C}_m\big(1 + C_1\eta^*(V_H)\big) + 2C_{e0}C_{e1} + C_{e0}C_1\eta^*(V_H),\\
\widetilde{\widetilde{C}}^*_m &= C_aC_m\widetilde{C}^*_m\big(1 + C^*_1\eta(V_H)\big) + 2C^*_{e0}C^*_{e1} + C^*_{e0}C^*_1\eta(V_H).
\end{aligned}$$

Proof. Define $e_{h_k} := \widetilde{u}_{h_k} - \underline{u}_{h_k}$ $(k = 1, \cdots, n)$. Using (3.33) in Lemma 3.1 and the triangle inequality, we have

$$\begin{aligned}\|e_{h_{k+1}}\|_1 &= \|\widetilde{u}_{h_{k+1}} - \underline{u}_{h_{k+1}}\|_1 \le C_2\|\check{u}_{h_{k+1}} - \underline{u}_{h_{k+1}}\|_1\\ &\le C_2\big(\|\check{u}_{h_{k+1}} - \widehat{u}_{h_{k+1}}\|_1 + \|\widehat{u}_{h_{k+1}} - \underline{u}_{h_{k+1}}\|_1\big).\end{aligned} \tag{3.42}$$

We first estimate the second term of (3.42). Set $w_{h_{k+1}} = \widehat{u}_{h_{k+1}} - \underline{u}_{h_{k+1}} \in V_{h_{k+1}}$, using (3.6) to minus (3.10), we get

$$\begin{aligned}& a_s(\widehat{u}_{h_{k+1}} - \underline{u}_{h_{k+1}}, w_{h_{k+1}})\\ =\ & \widetilde{\lambda}_{h_k}(\widetilde{u}_{h_k}, w_{h_{k+1}}) - (\mathbf{b}\cdot\nabla\widetilde{u}_{h_k}, w_{h_{k+1}}) - (\rho\widetilde{u}_{h_k}, w_{h_{k+1}})\\ & -\underline{\lambda}_{h_k}(\underline{u}_{h_k}, w_{h_{k+1}}) + (\mathbf{b}\cdot\nabla\underline{u}_{h_k}, w_{h_{k+1}}) + (\rho\underline{u}_{h_k}, w_{h_{k+1}})\\ =\ & \big((\widetilde{\lambda}_{h_k}\widetilde{u}_{h_k} - \underline{\lambda}_{h_k}\underline{u}_{h_k}) - \rho(\widetilde{u}_{h_k} - \underline{u}_{h_k}), w_{h_{k+1}}\big) + (\widetilde{u}_{h_k} - \underline{u}_{h_k}, \nabla\cdot(\overline{\mathbf{b}}w_{h_{k+1}})).\end{aligned}$$

Then using the triangle inequality and Hölder inequality, we have the following estimate

$$a_s(\widehat{u}_{h_k} - \underline{u}_{h_{k+1}}, w_{h_{k+1}}) \le C_m(|\widetilde{\lambda}_{h_k} - \underline{\lambda}_{h_k}| + \|\widetilde{u}_{h_k} - \underline{u}_{h_k}\|_0)\|w_{h_{k+1}}\|_1,$$

where $C_m = \max\{1, |\underline{\lambda}_{h_k}| + \|\rho\|_{0,\infty} + \|\mathbf{b}\|_{1,\infty}\}$. Since the ellipticity (2.3) of $a_s(\cdot,\cdot)$, we have

$$\|\widehat{u}_{h_k} - \underline{u}_{h_{k+1}}\|_1 \le C_a C_m(|\widetilde{\lambda}_{h_k} - \underline{\lambda}_{h_k}| + \|\widetilde{u}_{h_k} - \underline{u}_{h_k}\|_0). \tag{3.43}$$

Similar to the proof of Theorem 3.1, from Theorem 2.1 and $V_{H,h_k} \subset \underline{V}_{H,h_k}$, the following inequalities hold

$$\begin{aligned}&|\widetilde{\lambda}_{h_k} - \underline{\lambda}_{h_k}| \le C_{e\lambda}C_{e1}^*\delta_H^*(\lambda)\|\widetilde{u}_{h_k} - \underline{u}_{h_k}\|_1,\\ &\|\widetilde{u}_{h_k} - \underline{u}_{h_k}\|_0 \le C_{e0}\eta^*(V_H)\|\widetilde{u}_{h_k} - \underline{u}_{h_k}\|_1.\end{aligned}$$

Then the combination of (3.43) and above inequalties leads to

$$\begin{aligned}\|\widehat{u}_{h_{k+1}} - \underline{u}_{h_{k+1}}\|_1 &\le C_aC_m\big(C_{e\lambda}C_{e1}^*\delta_H^*(\lambda) + C_{e0}\eta^*(V_H)\big)\|\widetilde{u}_{h_k} - \underline{u}_{h_k}\|_1\\ &= C_aC_m\big(C_{e\lambda}C_{e1}^*\delta_H^*(\lambda) + C_{e0}\eta^*(V_H)\big)\|e_{h_k}\|_1.\end{aligned} \tag{3.44}$$

Secondly, we turn to estimate the first term of (3.42). According to (3.6) in Algorithm 3.1, we obtain

$$\|\widehat{u}_{h_{k+1}} - \check{u}_{h_{k+1}}\|_1 = \|S_{h_{k+1}}^{m_{k+1}}(\widehat{u}_{h_{k+1}} - \widetilde{u}_{h_k})\|_1. \tag{3.45}$$

From (3.2), (3.44), (3.45) and the triangle inequality, the following estimates hold

$$\begin{aligned}& \|\widehat{u}_{h_{k+1}} - \check{u}_{h_{k+1}}\|_1 = \|S_{h_{k+1}}^{m_{k+1}}(\widehat{u}_{h_{k+1}} - \widetilde{u}_{h_k})\|_1\\ \le\ & \|S_{h_{k+1}}^{m_{k+1}}(\widehat{u}_{h_{k+1}} - \underline{u}_{h_k})\|_1 + \|S_{h_{k+1}}^{m_{k+1}}(\underline{u}_{h_k} - \widetilde{u}_{h_k})\|_1\\ \le\ & \|S_{h_{k+1}}^{m_{k+1}}(\widehat{u}_{h_{k+1}} - \underline{u}_{h_{k+1}})\|_1 + \|S_{h_{k+1}}^{m_{k+1}}(\underline{u}_{h_{k+1}} - \underline{u}_{h_k})\|_1 + \|\underline{u}_{h_k} - \widetilde{u}_{h_k}\|_1\\ \le\ & \|\widehat{u}_{h_{k+1}} - \underline{u}_{h_{k+1}}\|_1 + \frac{C_S}{m_{k+1}^\alpha}\frac{1}{h_{k+1}}\|\underline{u}_{h_{k+1}} - \underline{u}_{h_k}\|_0 + \|\underline{u}_{h_k} - \widetilde{u}_{h_k}\|_1\\ =\ & \|\widehat{u}_{h_{k+1}} - \underline{u}_{h_{k+1}}\|_1 + \|e_{h_k}\|_1 + \frac{C_S}{m_{k+1}^\alpha}\frac{1}{h_{k+1}}\|\underline{u}_{h_{k+1}} - \underline{u}_{h_k}\|_0.\end{aligned} \tag{3.46}$$

From (3.15) and (3.29), we have

$$\begin{aligned}& \|\underline{u}_{h_{k+1}} - u_{h_{k+1}}\|_0 \le \|\underline{u}_{h_{k+1}} - u_{h_{k+1}}\|_1\\ \le\ & C_aC_m\widetilde{C}_m\Big(\eta^*(V_{h_k})\delta_{h_k}(\lambda) + \eta^*(V_H)\|u_{h_k} - \underline{u}_{h_k}\|_1\Big)\\ \le\ & C_aC_m\widetilde{C}_m\big(1 + C_1\eta^*(V_H)\big)\eta^*(V_{h_k})\delta_{h_k}(\lambda).\end{aligned} \tag{3.47}$$

Then combining (3.16), (3.24), (3.47) and triangle inequality leads to following inequalities

$$\|\underline{u}_{h_{k+1}} - \underline{u}_{h_k}\|_0 \le \|\underline{u}_{h_{k+1}} - u_{h_{k+1}}\|_0 + \|u_{h_{k+1}} - u_{h_k}\|_0 + \|u_{h_k} - \underline{u}_{h_k}\|_0$$

$$\leq \quad \widetilde{\widetilde{C}}_m \eta^*(V_{h_k})\delta_{h_k}(\lambda), \tag{3.48}$$

where $\widetilde{\widetilde{C}}_m = C_a C_m \widetilde{C}_m\big(1 + C_1\eta^*(V_H)\big) + 2C_{e0}C_{e1} + C_{e0}C_1\eta^*(V_H)$.

From (3.48) and (3.46), we have

$$\|\widehat{u}_{h_{k+1}} - \check{u}_{h_{k+1}}\|_1 \leq \|\widehat{u}_{h_{k+1}} - \underline{u}_{h_{k+1}}\|_1 + \|e_{h_k}\|_1 + \frac{C_S\widetilde{\widetilde{C}}_m}{m_{k+1}^\alpha}\frac{\eta^*(V_{h_k})}{h_{k+1}}\delta_{h_k}(\lambda).$$

Together with (3.1) and (3.3), the following estimate holds

$$\|\widehat{u}_{h_{k+1}} - \check{u}_{h_{k+1}}\|_1 \leq \|\widehat{u}_{h_{k+1}} - \underline{u}_{h_{k+1}}\|_1 + \|e_{h_k}\|_1 + \frac{C_S\widetilde{\widetilde{C}}_m C_\eta^*\beta}{m_{k+1}^\alpha}\delta_{h_k}(\lambda). \tag{3.49}$$

Finally, combining (3.42), (3.44) and (3.49), we obtain

$$\|e_{h_{k+1}}\|_1 \leq C_3\|e_{h_k}\|_1 + \frac{C_4}{m_{k+1}^\alpha}\delta_{h_k}(\lambda), \quad k = 1, \cdots, n-1, \tag{3.50}$$

where $C_3 = C_2\Big(1 + 2C_aC_m\big(C_{e\lambda}C_{e1}^*\delta_H^*(\lambda) + C_{e0}\eta^*(V_H)\big)\Big)$ and $C_4 = C_2C_S\widetilde{\widetilde{C}}_mC_\eta^*\beta$.

According to Algorithms 3.2 and 3.4, $e_{h_1} = \widetilde{u}_{h_1} - \underline{u}_{h_1} = 0$. Using the inequality (3.50) and recursive argument, the following estimates hold

$$\begin{aligned}
\|e_{h_n}\|_1 &\leq C_3\|e_{h_{n-1}}\|_1 + \frac{C_4}{m_n^\alpha}\delta_{h_{n-1}}(\lambda)\\
&\leq (C_3)^2\|e_{h_{n-2}}\|_1 + C_3\frac{C_4}{m_{n-1}^\alpha}\delta_{h_{n-2}}(\lambda) + \frac{C_4}{m_n^\alpha}\delta_{h_{n-1}}(\lambda)\\
&\leq C_4\sum_{k=2}^{n}(C_3)^{n-k}\frac{1}{m_k^\alpha}\delta_{h_{k-1}}(\lambda).
\end{aligned}$$

This is the desired result (3.39), and (3.40) can be proved in the similar way. Furthermore, (3.41) can be obtained similar to (2.15) by Lemma 2.1. □

3.3 Error estimate of cascadic multigrid method

In order to control the error in the final level, from Theorem 3.2 we know that the number of iterations in coarse spaces should be larger than finer spaces. We assume the number of iterations m_k in each level satisfies the following inequality

$$\Big(\frac{h_k}{h_n}\Big)^\zeta \leq \frac{m_k^\alpha}{m_n^\alpha} \leq \sigma\Big(\frac{h_k}{h_n}\Big)^\zeta, \qquad k = 2, \cdots, n-1, \tag{3.51}$$

where $\sigma > 1$ and $\zeta > 1$ are some appropriate constants.

Theorem 3.3. *Assume the coarse mesh size H is small enough such that the conditions (3.14), $\beta^{1-\zeta}(1 + C_5H) < 1$ and $\beta^{1-\zeta}(1 + C_5^*H) < 1$ hold with C_5 and C_5^* being defined as follows*

$$C_5 = 2C_aC_m(C_{e\lambda}C_{e1}^*C_\delta^* + C_{e0}C_\eta^*), \quad C_5^* = 2C_aC_m(C_{e\lambda}C_{e1}C_\delta + C_{e0}^*C_\eta).$$

For any given $\gamma \in (0, 1]$, we have following error estimates

$$\begin{aligned}
\|u_{h_n} - \widetilde{u}_{h_n}\|_1 &\leq 2\gamma h_n, &(3.52)\\
\|u_{h_n}^* - \widetilde{u}_{h_n}^*\|_1 &\leq 2\gamma h_n, &(3.53)\\
|\lambda_{h_n} - \widetilde{\lambda}_{h_n}| &\leq C_{e\lambda}\|u_{h_n} - \widetilde{u}_{h_n}\|_1\|u_{h_n}^* - \widetilde{u}_{h_n}^*\|_1 \leq 4C_{e\lambda}\gamma^2h_n^2, &(3.54)
\end{aligned}$$

provided the following conditions

$$m_n \geq \max\left\{\Big(\frac{C_\zeta}{\gamma}\Big)^{\frac{1}{\alpha}}, \Big(\frac{C_\zeta^*}{\gamma}\Big)^{\frac{1}{\alpha}}\right\}, \quad \max\left\{C_1C_\delta\eta^*(V_{h_n}), C_1^*C_\delta^*\eta(V_{h_n})\right\} \leq \gamma, \tag{3.55}$$

where

$$C_\zeta = C_4C_\delta\beta/(1-\beta^{1-\zeta}(1+C_5H)), \qquad C_\zeta^* = C_4^*C_\delta^*\beta/(1-\beta^{1-\zeta}(1+C_5^*H)).$$

Proof. According to (3.1), (3.3), (3.39) and (3.51), we have

$$\begin{aligned}
\|\widetilde{u}_{h_n} - \underline{u}_{h_n}\|_1 &\le C_4\sum_{k=2}^{n}(C_3)^{n-k}\frac{1}{m_k^\alpha}\delta_{h_{k-1}}(\lambda)\\
&\le C_4\sum_{k=2}^{n}(C_3)^{n-k}\frac{1}{m_n^\alpha}\Big(\frac{h_k}{h_n}\Big)^{-\zeta}(C_\delta\beta h_k)\\
&\le C_4C_\delta\beta\sum_{k=2}^{n}(C_3)^{n-k}\frac{h_n}{m_n^\alpha}\beta^{(n-k)(1-\zeta)}\\
&= C_4C_\delta\beta\frac{h_n}{m_n^\alpha}\sum_{k=0}^{n-2}\big(\beta^{1-\zeta}C_3\big)^k. \qquad (3.56)
\end{aligned}$$

From the definition in Theorem 3.2 and the error estimate (3.3), the following estimate for C_3 holds

$$\begin{aligned}
C_3 &= 1+2C_aC_m\big(C_{e\lambda}C_{e1}^*\delta_H^*(\lambda)+C_{e0}\eta^*(V_H)\big)\\
&\le 1+2C_aC_m\big(C_{e\lambda}C_{e1}^*C_\delta^*H+C_{e0}C_\eta^*H\big)\\
&= 1+C_5H, \qquad (3.57)
\end{aligned}$$

where $C_5 = 2C_aC_m(C_{e\lambda}C_{e1}^*C_\delta^*+C_{e0}C_\eta^*)$. Hence combining (3.56) and (3.57) leads to the following inequality

$$\|\widetilde{u}_{h_n} - \underline{u}_{h_n}\|_1 \le C_4C_\delta\beta\frac{h_n}{m_n^\alpha}\sum_{k=0}^{n-2}\big(\beta^{1-\zeta}(1+C_5H)\big)^k. \qquad (3.58)$$

When H is small enough such that $\beta^{1-\zeta}(1+C_5H)<1$, (3.58) leads to the following inequality

$$\|\widetilde{u}_{h_n} - \underline{u}_{h_n}\|_1 \le C_4C_\delta\beta\frac{h_n}{m_n^\alpha}\frac{1}{1-\beta^{1-\zeta}(1+C_5H)} = \frac{C_\zeta}{m_n^\alpha}h_n, \qquad (3.59)$$

where $C_\zeta = C_4C_\delta\beta/(1-\beta^{1-\zeta}(1+C_5H))$.

Then from the condition (3.55) and (3.59), we have the estimate $\|\widetilde{u}_{h_n} - \underline{u}_{h_n}\|_1 \le \gamma h_n$. Since H satisfies the condition (3.14), the estimate (3.15) holds. Then combing (3.15) and the condition (3.55), we have the following inequalities

$$\begin{aligned}
\|u_{h_n} - \widetilde{u}_{h_n}\|_1 &\le \|u_{h_n} - \underline{u}_{h_n}\|_1 + \|\underline{u}_{h_n} - \widetilde{u}_{h_n}\|_1\\
&\le \gamma h_n + \gamma h_n = 2\gamma h_n,
\end{aligned}$$

which is the desired result (3.52), and (3.53) can be derived in the similar way. Finally, the desired estimate (3.54) is the direct result of (2.15), (3.52) and (3.53). ☐

Combining Theorems 2.1 and 3.3, the final error estimate of Algorithm 3.2 is presented in the following theorem.

Theorem 3.4. *Under the conditions of Theorem 3.3, we have following error estimates for Algorithm 3.2*

$$\begin{aligned}
\|u - \widetilde{u}_{h_n}\|_1 &\le (C_{e1}C_\delta + 2\gamma)h_n,\\
\|u - \widetilde{u}_{h_n}^*\|_1 &\le (C_{e1}^*C_\delta^* + 2\gamma)h_n,\\
|\lambda - \widetilde{\lambda}_{h_n}| &\le C_{e\lambda}(C_{e1}C_{e1}^*C_\delta C_\delta^* + 4\gamma^2)h_n^2.
\end{aligned}$$

4 Estimate of computational work

In order to estimate the computational work for Algorithm 3.2, we denote the dimension of V_{h_k} as N_k, $k = 1, \cdots, n$. According to (3.1), the following property holds

$$N_k \approx \Big(\frac{h_k}{h_n}\Big)^{-d} N_n \le \Big(\frac{1}{\beta_1}\Big)^{d(n-k)} N_n, \quad k = 1, \cdots, n. \tag{4.1}$$

Now, we give the complexity analysis for Algorithm 3.2.

Theorem 4.1. *Assume the computational work of solving eigenvalue problems in the coarse spaces V_H and V_{h_1} are M_H and M_{h_1}, respectively.*

(a) If $\zeta/\alpha < d$, the total computational work of Algorithm 3.2 can be bounded by $\mathcal{O}(N_n + M_{h_1} + M_H \ln(N_n))$, and furthermore $\mathcal{O}(N_n)$ provided $M_H \ll N_n$ and $M_{h_1} \le N_n$;

(b) If $\zeta/\alpha = d$, the total computational work can be bounded by $\mathcal{O}(N_n \ln(N_n) + M_{h_1} + M_H \ln(N_n))$. Similarly, if $M_H \ll N_n$ and $M_{h_1} \le N_n$, the total computational work will be $\mathcal{O}(N_n \ln(N_n))$.

Proof. Let W denote the whole computational work of Algorithm 3.2, W_k be the work on the k-th level for $k = 1, \cdots, n$. Based on Algorithms 3.1 and 3.2,

$$W_1 = 2M_{h_1}, \quad W_k = 2(m_k N_k + M_H), \ k = 2, \cdots, n.$$

Therefore, together with (3.1), (3.51) and (4.1), we have

$$\begin{aligned} W &= \sum_{k=1}^{n} W_k \lesssim M_{h_1} + \sum_{k=2}^{n} m_k N_k + n M_H \\ &\lesssim M_{h_1} + \sum_{k=2}^{n} m_k N_k + \log_{\beta_1}(N_n) M_H \\ &\lesssim M_{h_1} + M_H \ln(N_n) + m_n \sigma^{1/\alpha} N_n \sum_{k=2}^{n} \Big(\frac{1}{\beta_1}\Big)^{(n-k)(d-\zeta/\alpha)}. \end{aligned}$$

Then when $d - \zeta/\alpha > 0$,

$$W \lesssim M_{h_1} + M_H \ln(N_n) + N_n,$$

and when $d - \zeta/\alpha = 0$,

$$W \lesssim M_{h_1} + M_H \ln(N_n) + N_n \ln(N_n).$$

Hence, W can be bounded by $\mathcal{O}(N_n)$ and $\mathcal{O}(N_n \ln(N_n))$, respectively, provided $M_H \ll N_n$ and $M_{h_1} \le N_n$. □

Remark 4.1. *If we choose the conjugate gradient method as the smoother, then $\alpha = 1$ and the complexity of Algorithm 3.2 can be $\mathcal{O}(N_n + M_{h_1} + M_H \ln(N_n))$ or $\mathcal{O}(N_n)$ provided $M_H \ll N_n$ and $M_{h_1} \le N_n$ for both $d = 2, 3$ with $1 < \zeta < d$.*

When the symmetric Gauss-Seidel, SSOR, damped Jacobi or Richardson iteration is chosen, the $\alpha = 1/2$. Then the complexity of Algorithm 3.2 can be $\mathcal{O}(N_n + M_{h_1} + M_H \ln(N_n))$ ($\mathcal{O}(N_n)$ provided $M_H \ll N_n$ and $M_{h_1} \le N_n$) only for $d = 3$ with $1 < \zeta < 3/2$. In the case of $d = 2$, we can only choose $\zeta = 1$ and obtain the error estimates $\|u_{h_n} - \widetilde{u}_{h_n}\|_1 \lesssim h_n |\ln(h_n)|$ and $\|u^*_{h_n} - \widetilde{u}^*_{h_n}\|_1 \lesssim h_n |\ln(h_n)|$. The computational work only be $\mathcal{O}(N_n \ln(N_n) + M_{h_1} + M_H \ln(N_n))$ ($\mathcal{O}(N_n \ln(N_n))$ provided $M_H \ll N_n$ and $M_{h_1} \le N_n$).

5 Numerical results

In this section, three numerical examples are presented to illustrate the efficiency of the cascadic multigrid method for nonsymmetric eigenvalue problems. Here, we consider the following nonsymmetric eigenvalue problem

$$\begin{cases} -\Delta u + \mathbf{b}\cdot\nabla u &= \lambda u, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

with $\mathbf{b} = [b_1, b_2]^T \in \mathcal{C}^2$ being a constant vector in Ω. We solve the nonsymmetric eigenvalue problem on a unit square $\Omega = (0,1)\times(0,1)$ in the first two examples, and choose a real constant vector $\mathbf{b} = [1, 1/2]^T$ in the first example and a complex constant vector $\mathbf{b} = [1+2\mathrm{i}, 1/2-1\mathrm{i}]^T$ in the second example. The case with L-shape domain $\Omega = (-1,1)\times(-1,1)/[0,1)\times(-1,0]$ and $\mathbf{b} = [1, 1/2]^T$ will be considered in the third example. These examples come from [12, 35] and they are in the nondefective case.

Typically, when the computing domain Ω is a unit square $(0,1)\times(0,1)$ and $\mathbf{b}$ is a constant vector, the exact solutions of both the nonsymmetric eigenvalue problem and its adjoint problem can be described as follows (cf. [12, 35])

$$\begin{aligned} \lambda_{k,\ell} &= \frac{b_1^2 + b_2^2}{4} + (k^2 + \ell^2)\pi^2, \\ u_{k,\ell} &= \exp\Big(\frac{b_1 x_1 + b_2 x_2}{2}\Big)\sin(k\pi x_1)\sin(\ell\pi x_2), \\ u_{k,\ell}^* &= \exp\Big(-\frac{\overline{b_1} x_1 + \overline{b_2} x_2}{2}\Big)\sin(k\pi x_1)\sin(\ell\pi x_2), \end{aligned}$$

where $k, \ell \in \mathcal{N}^+$. All through the example, we choose the conjugate-gradient iteration as the smoothing operator ($\alpha = 1$) and define the number of iteration steps by

$$m_k = \lceil 2\sigma\beta^{\zeta(n-k)} \rceil \quad \text{for } k = 2, \cdots, n$$

with $\sigma = 2$, $\zeta = 1.01$ and $\lceil r \rceil$ denoting the smallest integer larger than r.

For the last example, the conjugate-gradient iteration is also adopted as the smoother ($\alpha = 1$) and the number of iteration steps is defined by

$$m_k = \lceil 2\sigma \times 1.4^{\zeta(n-k)/d} \rceil, \quad k = 2, \cdots, n \tag{5.2}$$

with $\sigma = 2$ and $\zeta = 1.01$.

5.1 Nonsymmetric eigenvalue problem with real constant vector b

In the first example, we choose $\mathbf{b} = [1, 1/2]^T$ and $\Omega = (0,1)\times(0,1)$. The sequence of linear finite element spaces are constructed on the series of meshes which are produced by regular refinement (connecting the midpoints of each edge) with $\beta = \beta_1 = 2$ from an initial mesh. Figure 1 shows the initial mesh $\mathcal{T}_H$ which is generated by Delaunay method.

For comparison, we also use the direct method to solve this nonsymmetric eigenvalue problem. Figure 2 gives the corresponding numerical results for the first eigenvalue $\lambda = 5/16 + 2\pi^2$, the corresponding right eigenfunction u and left one u^*. From Figure 2, we find that the proposed cascadic multigrid scheme can obtain the optimal error estimates as the direct finite element method, which is in consistent with Theorem 3.3. While, the computational work of Algorithm 3.2 is optimal.

Furthermore, we also check the efficiency of our cascadic multigrid scheme for several eigenvalues. The first six eigenvalues: $5/16 + [2\pi^2, 5\pi^2, 5\pi^2, 8\pi^2, 10\pi^2, 10\pi^2]$ are investigated. The corresponding numerical results are shown in Figure 3 which also exhibits the optimal convergence rate of the cascadic multigrid scheme.

Remark 5.1. *Although in the theoretical analysis, the desired eigenvalue is assumed to be nondefective, we find that the proposed cascadic multigrid method can also compute multiple eigenvalues. For more details, please refer to [32].*

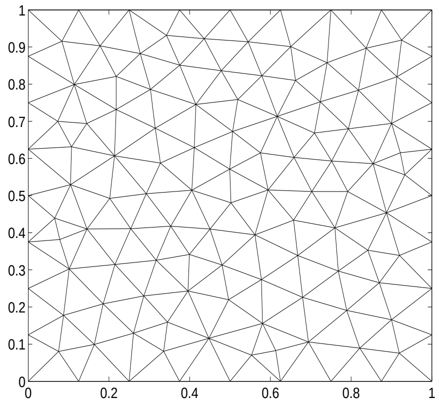

Figure 1: The initial mesh for the unit square

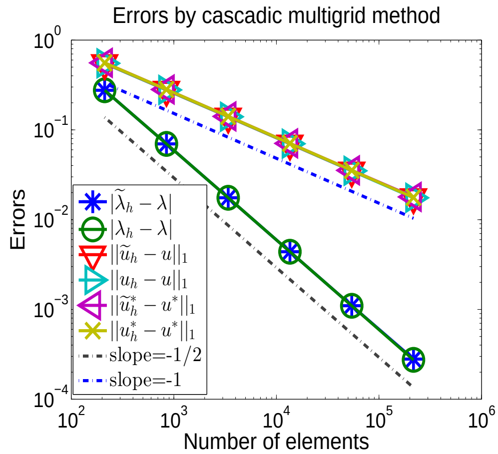

Figure 2: The errors for the first eigenvalue λ and its corresponding right eigenfunction u and left one u^* with $\Omega = (0,1) \times (0,1)$ and $\mathbf{b} = [1, 1/2]^T$, where $(\widetilde{\lambda}_h, \widetilde{u}_h, \widetilde{u}_h^*)$ is the solution of Algorithm 3.2 and (λ_h, u_h, u_h^*) is the solution of the direct eigenvalue solving method

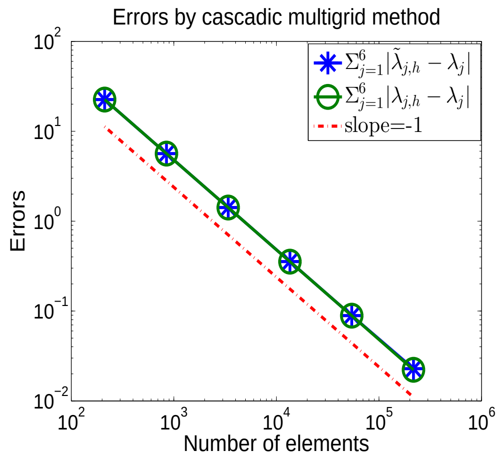

Figure 3: The errors for the first six eigenvalues with $\Omega = (0,1) \times (0,1)$ and $\mathbf{b} = [1, 1/2]^T$, where $\widetilde{\lambda}_{j,h}, j = 1, \cdots, 6$ are solutions of Algorithm 3.2 and $\lambda_{j,h}, j = 1, \cdots, 6$ are solutions of the direct eigenvalue solving method

5.2 Nonsymmetric eigenvalue problem with complex constant vector b

In this subsection, the nonsymmetric term is assumed to be a complex constant vector $\mathbf{b} = [1 + 2\mathrm{i}, 1/2 - 1\mathrm{i}]^T$ and we solve the nonsymmetric eigenvalue problems (5.1) on the unit square $\Omega = (0, 1) \times (0, 1)$. The initial mesh is shown in Figure 1

In this example, we first present numerical results for the first eigenpair in Figure 4 which also confirms the results in Theorem 3.3.

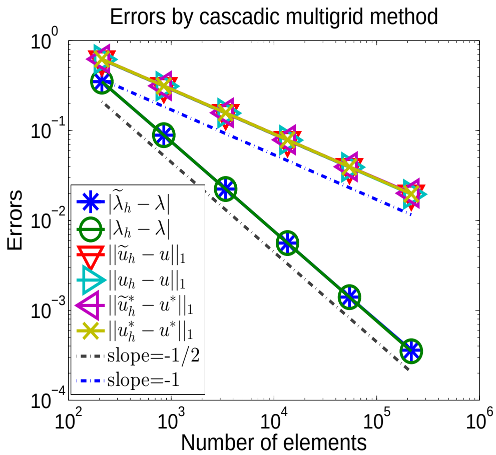

Figure 4: The errors for the first eigenvalue λ and its corresponding right eigenfunction u and left one u^* with $\Omega = (0, 1) \times (0, 1)$ and $\mathbf{b} = [1 + 2\mathrm{i}, 1/2 - 1\mathrm{i}]^T$, where $(\widetilde{\lambda}_h, \widetilde{u}_h, \widetilde{u}_h^*)$ is the solution of Algorithm 3.2 and (λ_h, u_h, u_h^*) is the solution of the direct eigenvalue solving method

Similarly, we also check the efficiency of the cascadic multigrid method for the first six eigenvalues. Figure 5 gives the corresponding numerical results which also shows the optimal convergence rate of the cascadic scheme. Furthermore, this example shows that the cascadic scheme also works well for the nonsymmetric eigenvalue problem with the complex vector.

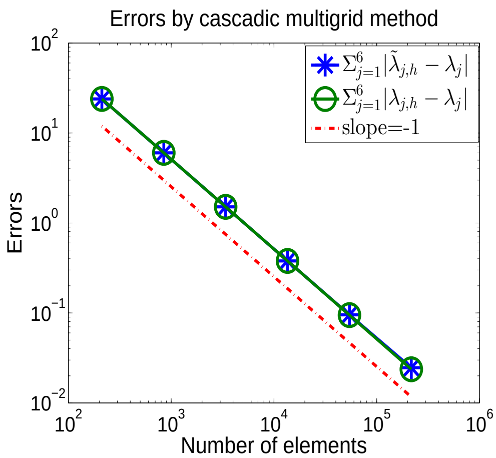

Figure 5: The errors for the first six eigenvalues with $\Omega = (0, 1) \times (0, 1)$ and $\mathbf{b} = [1 + 2\mathrm{i}, 1/2 - 1\mathrm{i}]^T$, where $\widetilde{\lambda}_{j,h}, j = 1, \cdots, 6$ are the solutions of Algorithm 3.2 and $\lambda_{j,h}, j = 1, \cdots, 6$ are the solutions of the direct eigenvalue solving method

5.3 Nonsymmetric eigenvalue problem on L-shape domain

In the last example, we consider the nonsymmetric eigenvalue problem (5.1) defined on the L-shape domain $\Omega = (-1, 1) \times (-1, 1)/[0, 1) \times (-1, 0]$ with $\mathbf{b} = [1, 1/2]^T$. The re-entrant corner on Ω causes the singularity of the first eigenfunction. Consequently, the convergence order for the first eigenvalue approximation is less than 2 by the linear finite element method which is the one for regular eigenfunctions. Since the exact eigenvalue is unknown, we choose an adequately accurate approximation $\lambda = 9.95240442893276$ obtained by extrapolation method [18] as the exact first eigenvalue for our numerical tests.

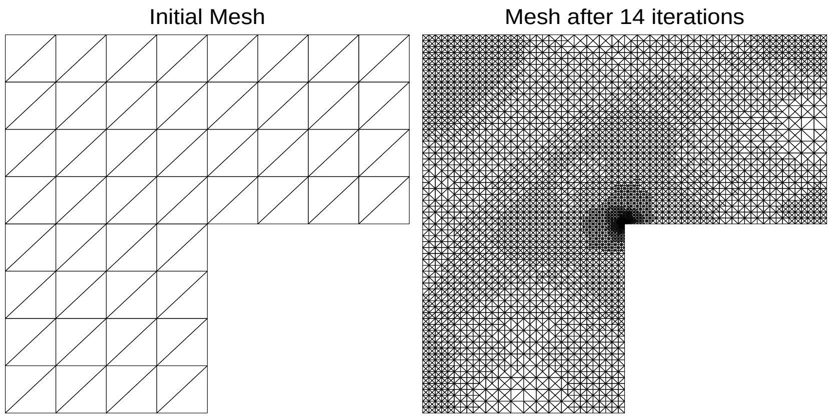

Figure 6: The initial mesh (left) and the one after 14 adaptive iterations (right) for the L-shape domain

For this singular example, we choose the a posteriori error estimator given in [34] and the Dörfler's marking strategy with index $\theta = 0.4$. The number of iteration m_k is defined in (5.2). The initial mesh and the one after 14 adaptive iterations are shown in Figure 6.

Figure 7 presents the corresponding numerical results which also shows the optimal convergence rate of the cascadic multigrid scheme. Furthermore, this example reveals that our cascadic multigrid method also works well on the adaptively refined meshes if the number of iteration of the smoothing is well selected.

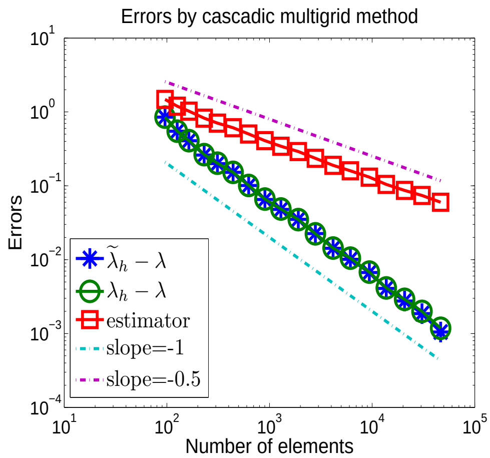

Figure 7: The errors for the first eigenvalue with $\Omega = (-1,1) \times (-1,1)/[0,1) \times (-1,0]$ and $\mathbf{b} = [1, 1/2]^T$, where $\widetilde{\lambda}_h$ is the solution of Algorithm 3.2 and λ_h is the solution of the direct eigenvalue solving method

Remark 5.2. *In this example, since the computational domain is concave, Theorems 3.1-3.4 don't work here. However, the aim of this example is to show the algorithms in this paper can also work on the sequence of adaptively refined meshes.*

6 Concluding remarks

In this paper, a type of cascadic multigrid method is designed to solve nonsymmetric eigenvalue problems based on the cascadic multigrid for boundary value problems and the multilevel correction scheme for eigenvalue problems. Furthermore, when the number of smoothing steps is chosen appropriately, our method can reach the optimal convergence rate with the optimal computing complexity. Three numerical experiments validate the optimality and show that the proposed algorithms can also compute multiple eigenvalues and solve the eigenvalue problems with complex vector. In the future, we will extend our approach to the first-principle calculations, such as the Kohn-Sham equation, Hartree-Fock equation and so on.

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