On sufficient conditions for rainbow cycles in edge-colored graphs

Shinya Fujita* Bo Ning† Chuandong Xu‡ Shenggui Zhang§

Abstract

Let G be an edge-colored graph. We use $e(G)$ and $c(G)$ to denote the number of edges of G and the number of colors appearing on $E(G)$, respectively. For a vertex $v \in V(G)$, the *color neighborhood* of v is defined as the set of colors assigned to the edges incident to v. A subgraph of G is *rainbow* if all of its edges are assigned with distinct colors. The well-known Mantel's theorem states that a graph G on n vertices contains a triangle if $e(G) \geq \lfloor \frac{n^2}{4} \rfloor + 1$. Rademacher (1941) showed that G contains at least $\lfloor \frac{n}{2} \rfloor$ triangles under the same condition. Li, Ning, Xu and Zhang (2014) proved a rainbow version of Mantel's theorem: An edge-colored graph G has a rainbow triangle if $e(G) + c(G) \geq n(n+1)/2$. In this paper, we first characterize all graphs G satisfying $e(G) + c(G) \geq n(n+1)/2 - 1$ but containing no rainbow triangles. Motivated by Rademacher's theorem, we then characterize all graphs G which satisfy $e(G) + c(G) \geq n(n+1)/2$ but contain only one rainbow triangle. We further obtain two results on color neighborhood conditions for the existence of rainbow short cycles. Our results improve a previous theorem due to Broersma, Li, Woeginger, and Zhang (2005). Moreover, we provide a sufficient condition in terms of color neighborhood for the existence of a specified number of vertex-disjoint rainbow cycles.

Keywords: Edge-colored graph; Rainbow cycle; Color neighborhood; Minimum color degree.

---

[*] School of Data Science, Yokohama City University, 22-2, Seto, Kanazawa-ku, Yokohama, 236-0027, Japan. E-mail: fujita@yokohama-cu.ac.jp (S. Fujita).

[†] Corresponding author. Center for Applied Mathematics, Tianjin University, Tianjin, 300072, P.R. China. E-mail: bo.ning@tju.edu.cn (B. Ning).

[‡] School of Mathematics and Statistics, Xidian University, Xi'an, 710071, P.R. China. E-mail: xuchuandong@xidian.edu.cn (C. Xu).

[§] [a] Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an, Shaanxi, 710072, P.R. China. [b] Xi'an-Budapest Joint Research Center for Combinatorics, Northwestern Polytechnical University, Xi'an, Shaanxi 710129, P.R. China. E-mail: sgzhang@nwpu.edu.cn (S. Zhang).

1 Introduction

Let G be a graph. We use $V(G)$ and $E(G)$ to denote the vertex set and edge set of G, respectively, and call $|G| := |V(G)|$ and $e(G) := |E(G)|$ the *order* and the *size* of G. For a subset S of $V(G)$, we use $G[S]$ to denote the subgraph of G induced by S, and $G-S$ to denote the subgraph $G[V(G)\backslash S]$. When $S=\{v\}$, we use $G-v$ instead of $G-\{v\}$. For disjoint subsets S, S' of $V(G)$, let $G[S,S']$ denote the bipartite subgraph of G induced by S and S', i.e., $G[S,S']$ has classes S, S' and edge set $\{xy \in E(G) : x \in S, y \in S'\}$.

An edge-coloring of G is a mapping $C : E(G) \to \mathbb{N}$, where $\mathbb{N}$ is the set of all natural numbers. When G has such a coloring, we call it an *edge-colored graph*. Let G be an edge-colored graph. We use $C(G)$ to denote the set of colors appearing on the edges of G and let $c(G) := |C(G)|$. For a vertex $v \in V(G)$ and a subgraph H of G, the *color neighborhood* of v in H, denoted by $CN_H(v)$, is defined as the set of colors assigned to the edges from v to $V(H)\backslash\{v\}$. The *color degree* of v in H is denoted by $d^c_H(v) := |CN_H(v)|$; and the *minimum color degree* of G, denoted by $\delta^c(G)$, is equal to $\min\{d^c_G(v) : v \in V(G)\}$. When there is no fear of confusion, we write $CN(v)$ and $d^c(v)$ instead of $CN_G(v)$ and $d^c_G(v)$ for short, respectively. An edge-colored graph is *rainbow* if all of its edges receive distinct colors, and *monochromatic* if all its edges have the same color. We use Bondy and Murty [4], and Chartrand and Zhang [8] for notation and terminology not defined here. For more results on related topics on rainbow subgraphs, we refer the reader to surveys due to Kano and Li [16], and Fujita, Magnant and Ozeki [13, 14].

We first recall some classical result on the existence of short cycles in uncolored graphs. Mantel's theorem (1907) is one important starting point of extremal graph theory, which is stated as every graph G on n vertices contains a triangle if $e(G) \geq \lfloor \frac{n^2}{4} \rfloor$, unless $G \cong K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$. Li et al. [17] obtained a rainbow version of Mantel's theorem.

Theorem 1 (Li, Ning, Xu, and Zhang [17]). *Let G be an edge-colored graph of order $n \geq 3$. If $e(G)+c(G) \geq n(n+1)/2$, then G contains a rainbow C_3.*

The bound for $e(G)+c(G)$ in the above theorem is best possible. To see this, let $\mathcal{G}_0$ be the set of all edge-colored complete graphs which satisfy the following properties (see Figure 1):

1. $K_1 \in \mathcal{G}_0$;

2. For every $G \in \mathcal{G}_0$ of order $n \geq 2$, $c(G) = n-1$ and there is a bipartition $V(G) = V_1 \cup V_2$, such that $G[V_1, V_2]$ is monochromatic and $G[V_i] \in \mathcal{G}_0$ for $i = 1, 2$.

One can check that every graph in $\mathcal{G}_0$ satisfies that $e(G) + c(G) \geq n(n+1)/2 - 1$ but contains no rainbow triangles.

In this paper we firstly characterize all the graphs which satisfy $e(G) + c(G) \geq n(n+1)/2 - 1$ but contain no rainbow triangles. Our result shows that all extremal graphs are included in $\mathcal{G}_0$.

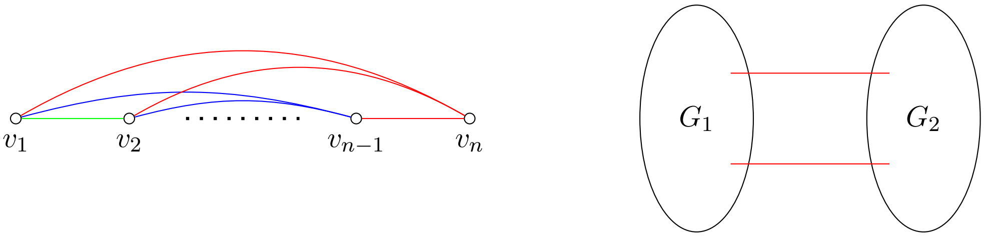

Figure 1: An example in $\mathcal{G}_0$ and the structure of graphs in $\mathcal{G}_0$ for $n \geq 2$.

Theorem 2. *Let G be an edge-colored graph of order n. If $e(G) + c(G) \geq \binom{n+1}{2} - 1$ and G contains no rainbow triangles, then G belongs to $\mathcal{G}_0$.*

In 1941, an extension of Mantel's theorem was obtained by Rademacher in an unpublished manuscript (see [9]). He proved that every graph G on n vertices contains at least $\lfloor n/2 \rfloor$ triangles if $e(G) \geq \lfloor \frac{n^2}{4} \rfloor + 1$. So, one may naturally ask whether there is a rainbow version of Rademacher's theorem. The following example shows that the answer is no.

Let $\mathcal{G}_1$ be the set of all edge-colored complete graphs which satisfy the following properties:

1. The rainbow C_3 is included in $\mathcal{G}_1$;
2. For every $G \in \mathcal{G}_1$ of order $n \geq 4$, $c(G) = n$ and there is a bipartition $V(G) = V_1 \cup V_2$, such that $G[V_1, V_2]$ is monochromatic and $G_1 = G[V_1] \in \mathcal{G}_1$, $G_2 = G[V_2] \in \mathcal{G}_0$.

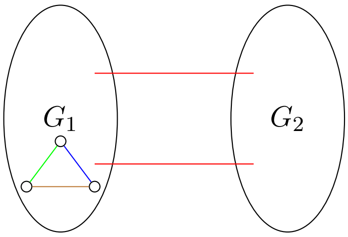

Figure 2: The structure of graphs in $\mathcal{G}_1$ for $n \geq 4$.

Generally, let $\mathcal{G}_k$ ($k \geq 2$) be the set of all edge-colored complete graphs constructed as follows: For every $G \in \mathcal{G}_k$ of order $n \geq 3k$, there is a bipartition $V(G) = V_1 \cup V_2$, such that $G[V_1, V_2]$ is monochromatic and $G[V_1] \in \mathcal{G}_i$, $G[V_2] \in \mathcal{G}_{k-i}$ for some $0 \leq i \leq k$. It is easy to see that for every $G \in \mathcal{G}_k$, $e(G) = c(G) = n(n-1) + (k-1)$ and G contains exactly k rainbow triangles. For $k = 1$, we can show $\mathcal{G}_1$ is exactly the set of graphs which satisfy such properties.

Theorem 3. *Let G be an edge-colored graph of order $n \geq 3$. If $e(G) + c(G) \geq \binom{n+1}{2}$ and G contains exactly one rainbow triangle, then G belongs to $\mathcal{G}_1$.*

Aside from the color number condition in Theorem 1, Li et al. [17] also considered a Dirac-type color degree condition for the existence of rainbow triangles in edge-colored graphs.

Theorem 4 (Li, Ning, Xu, and Zhang [17])**.** *Let G be an edge-colored graph of order $n \geq 5$. If $d^c(v) \geq n/2$ for every vertex $v \in V(G)$ and G contains no rainbow C_3, then the underlying graph of G is $K_{n/2,n/2}$, where n is even.*

Returning to related topics in uncolored graphs, let us recall the Ore-type condition, that is, the condition in terms of the minimum degree sum of non-adjacent vertices in a graph (see e.g. [20]). This kind of condition was introduced as an extension of the minimum degree condition for cycles, thereby yielding affluent results in this area. Motivated by this, when we try to consider some natural extensions from the minimum color degree condition in edge-colored graphs, what kind of color degree condition would be appropriate?

Perhaps the following theorem due to Broersma et al. [5] gives us a reasonable answer to this question.

Theorem 5 (Broersma, Li, Woeginger, and Zhang [5])**.** *Let G be an edge-colored graph of order $n \geq 4$ such that $|CN(u) \cup CN(v)| \geq n-1$ for every pair of vertices u and v in $V(G)$. Then G contains a rainbow C_3 or a rainbow C_4.*

Unlike Ore-type conditions in uncolored graphs, we look at every pair of vertices in the edge-colored graph G under the assumption of Theorem 5. This is because we need to deal with the case that G is an edge-colored complete graph, and even in this special case, problems for finding rainbow cycles are far from trivial in general (unlike the uncolored version). An example is a theorem by Li et al. [18] which states that an edge-colored graph on n vertices contains a rainbow triangle if the color degree sum of every two adjacent vertices is at least $n+1$.

Motivated by Theorem 5, one may naturally ask whether we can find both a rainbow C_3 and a rainbow C_4 under the same condition. The following theorems answer the above question affirmatively in some sense.

Theorem 6. *Let k be a positive integer, and G an edge-colored graph of order $n \geq 105k - 24$ such that $|CN(u) \cup CN(v)| \geq n - 1$ for every pair of vertices u and v in $V(G)$. Then G contains k rainbow C_4's.*

Theorem 7. *Let G be an edge-colored graph of order $n \geq 6$ such that $|CN(u) \cup CN(v)| \geq n - 1$ for every pair of vertices u and v in $V(G)$. Then G contains a rainbow C_3 unless G is a rainbow $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$.*

So far, we have introduced some results on the existence of rainbow short cycles in edge-colored graphs. As observed, we need quite a strong assumption to guarantee the existence of rainbow short cycles. Similarly, when we consider a degree condition for the existence of small cycles in uncolored graphs, it becomes a strong assumption. However, this is not the case if we just want to find a cycle with no restriction on its length in uncolored graphs. In contrast to this uncolored case, the situation might not change drastically even if we just hope for the existence of rainbow cycles with no restriction on their lengths in edge-colored graphs. Yet we could improve the coefficient of n in the assumption of Theorem 5 from 1 to 1/2, if we do not restrict the length of rainbow cycles. Moreover, we could strengthen the conclusion part as "vertex-disjoint" rainbow cycles.

Theorem 8. *Let k be a positive integer. If an edge-colored graph G of order n satisfies $|CN(u) \cup CN(v)| \geq n/2 + 64k + 1$ for every pair of vertices $u, v \in V(G)$, then G contains k vertex-disjoint rainbow cycles.*

By this theorem, we obtain the following corollary, although Theorem 4 already implies it as well.

Corollary 1. *Let k be a positive integer. If an edge-colored graph G of order n satisfies $\delta^c(G) \geq n/2 + 64k + 1$, then G contains k vertex-disjoint rainbow cycles.*

Comparing with the color degree conditions under the assumptions of Theorem 8 and Corollary 1, we can observe that our theorem provides a substantial extension in view of color degree conditions for the existence of vertex-disjoint rainbow cycles.

The organization of this paper is as follows. In Section 2, we prove Theorems 2 and 3. In Section 3, we prove Theorems 6, 7 and 8. We conclude this paper with some remarks and problems.

2 Proofs of Theorems 2 and 3

Before giving the proofs, we first introduce a concept given in [17]. Let v be a vertex in an edge-colored graph G. A color c is *saturated* by v if all the edges with the color c are incident to v. In this case, $c \notin C(G-v)$. As in [17], the *color saturated degree* of v is defined as $d^s(v) := c(G) - c(G-v)$.

Lemma 1 (Li, Ning, Xu, and Zhang [17])**.** *Let G be an edge-colored graph. Then $\sum\limits_{v \in V(G)} d^s(v) \leq 2c(G)$, and the equality holds if and only if G is rainbow.*

Lemma 2. *Let G be an edge-colored graph of order $n \geq 2$. If $e(G) + c(G) = \binom{n+1}{2} - 1$ and G contains no rainbow triangle, then G is complete and contains a vertex u such that $d^s(u) = 1$.*

Proof. We prove this lemma by induction on the order of G. It is trivial that the result holds for $n = 2, 3$. Now assume that it holds for a graph with order smaller than n, where $n \geq 4$.

Claim 1. *For every $v \in V(G)$, $d(v) + d^s(v) \geq n$.*

Proof. Suppose not. Then there exists a vertex $v \in V(G)$ satisfying $d(v) + d^s(v) \leq n - 1$. This implies that $e(G-v) + c(G-v) = e(G) + c(G) - d(v) - d^s(v) \geq \binom{n}{2}$. It follows from Theorem 1 that $G - v$ contains a rainbow triangle, a contradiction. $\square$

Claim 2. *There exists a vertex $u \in V(G)$ such that $d(u) + d^s(u) = n$.*

Proof. Suppose not. Then $d(v)+d^s(v) \geq n+1$ for every $v \in V(G)$. It follows from Lemma 1 that

$$n(n+1) \leq \sum_{v \in V(G)} \big(d(v) + d^s(v)\big) \leq 2e(G) + 2c(G) = n(n+1) - 2,$$

a contradiction. $\square$

It is easy to see that $e(G-u) + c(G-u) = e(G) + c(G) - d(u) - d^s(u) = \binom{n}{2} - 1$. By the induction hypothesis, the graph $G - u$ is complete.

If $d(u) < n - 1$ then $d^s(u) \geq 2$. Let uv, uw be two edges with distinct colors which are saturated by u. By the definition of saturated colors, neither $C(uv)$ nor $C(uw)$ appears in $G - u$. Thus, $uvwu$ is a rainbow triangle, a contradiction. It follows that $d(u) = n - 1$ and $d^s(u) = 1$. Thus, G is complete and $d^s(u) = 1$. This proves Lemma 2. $\square$

A *Gallai coloring* is an edge-coloring of the complete graph K_n such that there are no rainbow triangles in it. (See the references in [15].) The following two classical theorems on Gallai colorings play an important role in the proof of Theorem 2.

Lemma 3 (Gyárfás and Simonyi [15])**.** *Any Gallai coloring can be obtained by substituting complete graphs with Gallai colorings into vertices of 2-edge-colored complete graphs with at least two vertices.*

Lemma 4 (Erdős, Simonovits, and Sós [12])**.** *Any Gallai coloring of K_n can use at most $n-1$ colors.*

Proof of Theorem 2. We prove this result by induction on the order of G. Obviously, the result holds for $n = 1, 2, 3$. Now assume that it holds for any graph with order smaller than $n \geq 4$.

By Theorem 1, we can assume that $e(G) + c(G) = \binom{n+1}{2} - 1$. It follows from Lemma 2 that G is complete. Since $e(G) + c(G) = \binom{n+1}{2} - 1$, $c(G) = n - 1$. Thus the edge-coloring of G is a Gallai coloring with $n-1$ colors. By Lemma 3, the coloring of G can be obtained by substituting complete graphs $H_1, H_2, \ldots, H_k$ with Gallai colorings into vertices of a 2-edge-colored complete graph K_k, where $k \geq 2$, and $|H_i| = n_i$, $i = 1, 2, \ldots, k$. Note that $\sum_{i=1}^{k} n_i = n$. By Lemmas 3 and 4,

$$c(G) \leq \sum_{i=1}^{k} c(H_i) + 2 \leq \sum_{i=1}^{k} (n_i - 1) + 2 = n - k + 2.$$

On the other hand, $c(G) = n - 1$. Thus $k = 2, 3$.

It is easy to see that every 2-edge-colored K_k has a monochromatic cut for $k = 2, 3$. By Lemma 3, there is also a monochromatic cut in G. Let V_1, V_2 be the classes of this monochromatic cut. It follows from Lemma 4 that

$$n - 1 = c(G) \leq c(G[V_1]) + c(G[V_2]) + c(G[V_1, V_2]) \leq (|V_1| - 1) + (|V_2| - 1) + 1 = n - 1.$$

This implies that

$$c(G) = c(G[V_1]) + c(G[V_2]) + c(G[V_1, V_2]),$$

which holds if and only if $C(G[V_1])$, $C(G[V_2])$ and $C(G[V_1, V_2])$ are pairwise disjoint sets. Moreover,

$$c(G[V_1]) = |V_1| - 1, \quad c(G[V_2]) = |V_2| - 1 \quad \text{and} \quad c(G[V_1, V_2]) = 1.$$

By the induction hypothesis, both $G[V_1]$ and $G[V_2]$ belong to $\mathcal{G}_0$. It follows from the definition of $\mathcal{G}_0$ that $G \in \mathcal{G}_0$.

The proof is complete. ☐

The proof of Theorem 3 is based on the following two lemmas.

Lemma 5 (Rademacher [9]). *Let G be a graph with order n and size m. If $m \geq \lfloor \frac{n^2}{4} \rfloor + 1$, then G contains at least $\lfloor \frac{n}{2} \rfloor$ triangles.*

Lemma 6. *Let G be an edge-colored graph of order $n \geq 3$. If $e(G) + c(G) \geq \binom{n+1}{2}$ and G contains exactly one rainbow triangle, then $e(G) + c(G) = \binom{n+1}{2}$ and G is complete.*

Proof. We prove this result by induction on the order of G. It is trivial for $n = 3$. Now we assume that the lemma holds for any graph of order smaller than $n \geq 4$. Denote by $v_1v_2v_3v_1$ the unique rainbow triangle in G. Let $V_1 = \{v_1, v_2, v_3\}$ and $V_2 = V(G)\backslash V_1$.

Claim 1. *G is not rainbow.*

Proof. Suppose that G is rainbow. Then $e(G) = c(G) \geq \frac{n^2}{4} + \frac{n}{4} \geq \lfloor \frac{n^2}{4} \rfloor + 1$. It follows from Lemma 5 that G contains at least $\lfloor n/2 \rfloor \geq 2$ triangles, which are rainbow triangles in G, a contradiction. ☐

Claim 2. $e(G) + c(G) = \binom{n+1}{2}$.

Proof. Suppose that $e(G) + c(G) \geq \binom{n+1}{2} + 1$. Let e be an edge in the unique rainbow triangle of G. Then $G - e$ contains no rainbow triangle, and

$$e(G-e) + c(G-e) \geq (e(G) - 1) + (c(G) - 1) \geq \binom{n+1}{2} - 1.$$

It follows from Theorem 2 that $G - e$ is complete, a contradiction. ☐

Claim 3. *For every $v \in V_1$, $d(v) + d^s(v) \geq n + 1$; for every $v \in V_2$, $d(v) + d^s(v) \geq n$.*

Proof. For every $v \in V_1$, $G - v$ contains no rainbow triangle. It follows from Theorem 1 that $e(G - v) + c(G - v) \leq \binom{n}{2} - 1$. Thus $d(v) + d^s(v) \geq n + 1$.

Suppose that there exists a vertex $u \in V_2$ such that $d(u) + d^s(u) \leq n - 1$. Then $G - u$ contains a unique rainbow triangle and $e(G - u) + c(G - u) \geq \binom{n}{2} + 1$. It follows from the induction hypothesis that $e(G - u) + c(G - u) = \binom{n}{2}$, a contradiction. ☐

Claim 4. *There exists a vertex $u \in V_2$ such that $d(u) + d^s(u) = n$.*

Proof. Suppose not. Then, $d(v) + d^s(v) \geq n + 1$ for every $v \in V_2$. By Claim 3 and Lemma 1,

$$n(n+1) \leq \sum_v \big(d(v) + d^s(v)\big) \leq 2e(G) + 2c(G) = n(n+1).$$

Thus $\sum_v d^s(v) = 2c(G)$. It follows from Lemma 1 that G is rainbow, a contradiction to Claim 1. $\square$

Let u be as in Claim 4. Note that $G-u$ contains exactly one rainbow triangle and

$$e(G-u)+c(G-u)=e(G)+c(G)-d(u)-d^s(u)=\binom{n}{2}.$$

It follows from the induction hypothesis that $G-u$ is complete.

Now we show that $d(u)=n-1$. Suppose that $d(u)<n-1$. Then, we obtain $d^s(u)\geq 2$. Let uv and uw be two edges with distinct colors which are saturated by u. It is easy to see that $uvwu$ is a rainbow triangle distinct from $v_1v_2v_3v_1$, a contradiction. Thus, G is complete, and together with Claim 2, this proves Lemma 6. $\square$

Proof of Theorem 3. We prove this result by induction on the order of G. It is trivial for $n=3$. Now assume that the theorem holds for graphs with order smaller than $n\geq 4$. Denote by $v_1v_2v_3v_1$ the unique rainbow triangle in G.

We show that $C(v_1v_2), C(v_1v_3)$ are saturated by the vertex v_1. It follows from Claim 3 (in the proof of Lemma 6) that $d(v_i)+d^s(v_i)\geq n+1$ for each $i=1,2,3$, and hence $d^s(v_i)\geq 2$ for each $i=1,2,3$. First, suppose that there is exactly one color in $\{C(v_1v_2), C(v_1v_3)\}$, say $C(v_1v_2)$, which is saturated by v_1. Since $d^s(v_1)\geq 2$, we can choose $w\in N(v_1)$ such that $w\neq v_2$, $C(v_1w)\neq C(v_1v_2)$ and $C(v_1w)$ is saturated by v_1. Since $C(v_1v_3)$ is not saturated by v_1, we have $C(v_1w)\neq C(v_1v_3)$, and thus $w\neq v_3$. Now $C(wv_2)\neq C(v_1v_2)$ and $C(wv_2)\neq C(v_1w)$, and $v_1v_2wv_1$ is a rainbow C_3. Hence there are two rainbow C_3's, a contradiction. Suppose that none of $\{C(v_1v_2), C(v_1v_3)\}$ is saturated by v_1. There are $w,x\in N(v_1)$ such that $C(v_1w), C(v_1x)$ are saturated by v_1, so $C(v_1v_2)$, $C(v_1v_3)$, $C(v_1w)$ and $C(v_1x)$ are distinct. Moreover, v_1wxv_1 is a rainbow triangle. Hence there are two rainbow triangles in G, a contradiction. Thus, we have proved that $C(v_1v_2), C(v_1v_3)$ are saturated by the vertex v_1. Similarly, $C(v_2v_1), C(v_2v_3)$ are saturated by v_2, and $C(v_3v_1), C(v_3v_2)$ are saturated by v_3. Notice that $C(v_1v_2)$ is saturated by both v_1 and v_2. Thus, $C(v_1v_2)$ appears only once in G. Similarly, we can see that $C(v_1v_3)$ and $C(v_2v_3)$ appear only once in G.

By Lemma 6, since G is complete, it is easy to see that there is no edge v_iw satisfying $w\in V(G)\backslash\{v_1,v_2,v_3\}$ and $C(v_iw)$ is saturated by v_i, for each $i=1,2,3$.

Let G^* be the edge-colored graph obtained by replacing the color of v_1v_2 by $C(v_1v_3)$. For any vertex $w\in V(G)\backslash\{v_1,v_2,v_3\}$ and $i,j\in\{1,2,3\}$, since wv_iv_jw is not rainbow in G and each color on $v_1v_2v_3v_1$ appears only once, $C(wv_i)=C(wv_j)$. Hence wv_iv_jw is not rainbow in G^*. So, G^* contains no rainbow triangle and $c(G^*) = n - 1$. It follows from Theorem 2 that G^* belongs to $\mathcal{G}_0$. Thus there exists a partition $V = V_1 \cup V_2$ (we can assume $\{v_1, v_2, v_3\} \subseteq V_1$), such that $G^*[V_1, V_2]$ is monochromatic and $G^*[V_i] \in \mathcal{G}_0$ for $i = 1, 2$.

It is easy to see that $G[V_1, V_2]$ is monochromatic and $G[V_2] = G^*[V_2] \in \mathcal{G}_0$. Moreover, $c(G[V_1]) = |G[V_1]|$ and $G[V_1]$ contains only one rainbow triangle. By the induction hypothesis, $G[V_1] \in \mathcal{G}_1$. It follows from the definition of $\mathcal{G}_1$ that $G \in \mathcal{G}_1$.

The proof is complete. $\square$

3 Proofs of Theorems 6, 7 and 8

We need the following lemmas.

Lemma 7. *Let G be an edge-colored graph. Then G contains a spanning bipartite subgraph H such that $2d^c_H(v) + 3d_H(v) \geq d^c_G(v) + d_G(v)$ for every vertex $v \in V(H)$.*

Proof. We choose a spanning bipartite subgraph H of G such that $f(H) := e(H) + \sum_{v\in V(H)} d^c_H(v)$ is as large as possible. We will show that $2d^c_H(v)+3d_H(v) \geq d^c_G(v)+d_G(v)$ for every vertex $v \in V(H)$.

Suppose that the bipartition of H is (X, Y). Then any edge xy of G with $x \in X$ and $y \in Y$ is also an edge of H. Otherwise, $f(H + xy) > f(H)$, contradicting the choice of H. One can see that $d^c_H(x) = |CN_{G[Y]}(x)|$ for $x \in X$, and $d^c_H(y) = |CN_{G[X]}(y)|$ for $y \in Y$.

Suppose that there exists a vertex $u \in V(H)$ such that

$$2d^c_H(u) + 3d_H(u) < d^c_G(u) + d_G(u). \tag{1}$$

Without loss of generality, we may assume $u \in X$. We claim that $|X| \geq 2$. Suppose that $X = \{u\}$. Since $e_G(X, Y) = e_H(X, Y)$, we get $2d^c_H(u) + 3d_H(u) \geq 2d^c_H(u) + 3d_G(u) \geq d^c_G(u) + d_G(u)$, a contradiction. This proves $|X| \geq 2$. Let H' be the spanning bipartite subgraph of G with the bipartition $(X\backslash\{u\}, Y \cup \{u\})$ and edge set $E(H) \cup \{ux \in E(G) : x \in X \setminus \{u\}\} \setminus \{uy \in E(G) : y \in Y\}$. Then

$$e(H') - e(H) = (d_G(u) - d_H(u)) - d_H(u) = d_G(u) - 2d_H(u). \tag{2}$$

On the other hand, we obtain

$$\begin{aligned} d^c_{H'}(u) - d^c_H(u) &= |CN_{G[X]}(u)| - |CN_{G[Y]}(u)| \\ &\geq |CN_G(u)| - 2|CN_{G[Y]}(u)| \\ &= d^c_G(u) - 2d^c_H(u), \end{aligned}$$

and

$$\begin{aligned}\sum_{v\in V(G)\setminus\{u\}}(d^c_{H'}(v)-d^c_H(v)) &= \sum_{v\in X\setminus\{u\}}(d^c_{H'}(v)-d^c_H(v))+\sum_{v\in Y}(d^c_{H'}(v)-d^c_H(v))\\ &\geq \sum_{v\in Y}(d^c_{H'}(v)-d^c_H(v))\\ &= \sum_{v\in Y}(|CN_{G[X\setminus\{u\}]}(v)|-|CN_{G[X]}(v)|)\\ &\geq -\sum_{v\in Y}|CN_{G[\{u\}]}(v)| = -d_H(u).\end{aligned}$$

Thus

$$\begin{aligned}\sum_{v\in V(G)}d^c_{H'}(v)-\sum_{v\in V(G)}d^c_H(v) &= \sum_{v\in V(G)\setminus\{u\}}(d^c_{H'}(v)-d^c_H(v))+(d^c_{H'}(u)-d^c_H(u))\\ &\geq (d^c_G(u)-2d^c_H(u))-d_H(u),\end{aligned}$$

that is,

$$\sum_{v\in V(G)}d^c_{H'}(v)-\sum_{v\in V(G)}d^c_H(v)\geq d^c_G(u)-2d^c_H(u)-d_H(u). \tag{3}$$

By (1), (2) and (3), we get

$$f(H')-f(H)\geq d_G(u)+d^c_G(u)-2d^c_H(u)-3d_H(u)>0,$$

which contradicts the choice of H. The proof is complete. □

Lemma 8 (Čada, Kaneko, Ryjáček, and Yoshimoto [7])**.** *Let G be an edge-colored graph of order n. If G is triangle-free and $\delta^c(G)\geq \frac{n}{3}+1$, then G contains a rainbow C_4.*

Lemma 9. *Let $k\geq 1$ be an integer and G an edge-colored graph of order $n\geq k+3$. If G is triangle-free and $\delta^c(G)\geq \frac{n}{3}+k$, then G contains k rainbow C_4's.*

Proof. We prove this lemma by induction on k. The case $k=1$ is true by Lemma 8. Suppose that the lemma holds for $k-1$. Let v be a vertex of a rainbow C_4 in G, and set $G':=G-v$. Then $\delta^c(G')\geq\delta^c(G)-1\geq\frac{n}{3}+k-1>\frac{|G'|}{3}+(k-1)$. By the induction hypothesis, there are $k-1$ rainbow C_4's in G', and still in G. So, there are k rainbow C_4's in G. □

We point out that Lemma 9 has the following extension. This result can be proved by using Lemma 8 and induction, we omit the proof here.

Proposition 1. *Let $k \geq 1$ be an integer and G an edge-colored graph of order $n \geq 4k$. If G is triangle-free and $\delta^c(G) \geq n/3 + 2(k-1) + 1$, then G contains k vertex-disjoint rainbow C_4's.*

Lemma 10. *Let G be an edge-colored graph of order n such that $\delta^c(G) = n-1$ (so G is complete). For any subset S of $V(G)$ with $|S| = 5$, $G[S]$ contains a rainbow C_4.*

Proof. We prove the lemma by contradiction. Suppose that $G[S]$ contains no rainbow C_4. Let $S = \{x_1, x_2, x_3, x_4, x_5\} \subset V(G)$. Since $\delta^c(G) = n-1$, any two incident edges have distinct colors in G. Thus, we may assume that $G[S]$ contains two monochromatic independent edges, say, $C(x_1x_2) = C(x_3x_4) = 1$. Without loss of generality, set $C(x_1x_5) = 2$ and $C(x_3x_5) = 3$. Since $G[S]$ contains no rainbow C_4 and any two incident edges have distinct colors, we obtain $C(x_2x_3) = 2, C(x_1x_4) = 3$, and moreover, $C(x_2x_4) \notin \{1,2,3\}$, say, $C(x_2x_4) = 4$. Observing the colors on the edges incident to x_2 and x_5, we see that $C(x_2x_5) \notin \{1,2,3,4\}$, so set $C(x_2x_5) = 5$. Consequently, there is a rainbow C_4 with colors $1,3,4,5$ in $G[S \backslash \{x_1\}]$, a contradiction. $\square$

Proof of Theorem 6. When $\delta^c(G) = n-1$, it follows from Lemma 10 that there are k rainbow C_4's in G, since the order $n \geq 105k - 24 \geq 5k$. Thus we may assume that $\delta^c(G) \leq n-2$.

Let u be a vertex with $d^c_G(u) = \delta^c(G)$ and set $t := \delta^c(G)$. Let T be a subset of $N_G(u)$ such that $|T| = t$ and $C(ux) \neq C(uy)$ for every two vertices $x, y \in T$. Without loss of generality, set $T = \{x_1, x_2, \ldots, x_t\}$ and assume that $C(ux_i) = i$ for $i \in \{1, 2, \ldots, t\}$. Set $G_1 = G[T \cup \{u\}]$ and $G_2 = G - G_1$. Since $|G_1| = t+1 \leq n-1$, $V(G_2) \neq \emptyset$.

First, suppose that there are k vertices $z \in V(G_2)$ such that $|CN_{G_1}(z) \setminus CN(u)| \geq 2$. By the choice of T, if $v \in V(G_1)$ is a neighbor of z such that $C(vz) \in CN_{G_1}(z) \setminus CN(u)$, then $v \neq u$. Since $|CN_{G_1}(z) \setminus CN(u)| \geq 2$, choose $x_r, x_s \in T$ with $\{C(x_rz), C(x_sz)\} \subseteq CN_{G_1}(z) \backslash CN(u)$, and ux_rzx_su is a rainbow C_4. Thus, there are k rainbow C_4's.

Now, suppose that $|CN_{G_1}(v) \backslash CN(u)| \leq 1$ holds for at least $n-t-k$ vertices $v \in V(G_2)$. We say that a vertex $v \in V(G_2)$ is *good* if $|CN_{G_1}(v) \setminus CN(u)| \leq 1$.

Claim 1. $|CN_{G_2}(v)| = |G_2| - 1$ *for any good vertex* $v \in V(G_2)$.

Proof. First, $|CN_{G_1}(v) \setminus CN(u)| \leq 1$. It follows from $|CN(u)| = t$ that $|CN(u) \cup CN_{G_1}(v)| \leq t+1$. Note that $|CN(u) \cup CN(v)| \geq n-1$, we have $|CN(v) \backslash CN_{G_1}(v)| \geq n-t-2$. On the other hand, $|CN(v) \backslash CN_{G_1}(v)| \leq |CN_{G_2}(v)| \leq d_{G_2}(v) \leq |G_2| - 1 = n-t-2$. Thus, $|CN_{G_2}(v)| = |G_2| - 1$, where $|G_2| = n-t-1$. $\square$

Denote by H' the subgraph induced by $n-t-k$ good vertices in G_2. By Claim 1, the underlying graph of H' is complete. Furthermore, for any vertex $v \in V(H')$, $d^c_{H'}(v) = |H'|-1$. First suppose that $t \le n-6k$. Note that $|H'| = n-t-k \ge 5k$. Applying Lemma 10 to H', we see that there are k rainbow C_4's in G_2, which are also in G.

Thus we may assume $t \ge n-6k+1$. By Lemma 7, there is a spanning bipartite subgraph H of G such that

$$2d^c_H(v) + 3d_H(v) \ge d^c_G(v) + d_G(v) \tag{4}$$

for every vertex $v \in V(H)$. On the other hand, since H is a subgraph of G, it is not difficult to see that

$$d_H(v) - d^c_H(v) \le d_G(v) - d^c_G(v), \tag{5}$$

and

$$d_G(v) - d^c_G(v) \le d_G(v) - \delta^c(G) \le (n-1) - (n-6k+1) = 6k-2. \tag{6}$$

Together with (5) and (6),

$$d^c_H(v) - d_H(v) \ge 2-6k. \tag{7}$$

Recall that $d^c_G(v) \ge \delta^c(G) = t \ge n-6k+1$, and $d_G(v) \ge d^c_G(v)$. Then, combining (4) with (7), we obtain

$$d^c_H(v) \ge \frac{1}{5}(d^c_G(v) + d_G(v) + 6 - 18k) \ge \frac{2n-30k+8}{5} \ge \frac{n}{3} + k$$

when $n \ge 105k-24$. By Lemma 9, there are k rainbow C_4's in H, which are also k rainbow C_4's in G. The proof of Theorem 6 is complete. $\square$

Proof of Theorem 7. Suppose that G contains no rainbow triangles. First suppose that there exists a vertex, say u, such that $d^c_G(u) \le \frac{n-1}{2}$. For any vertex v which is adjacent to u, $|CN(u) \cup CN(v)| \ge n-1$. This implies that

$$d^c_G(u) + d^c_G(v) = |CN(u) \cup CN(v)| + |CN(u) \cap CN(v)| \ge (n-1)+1 = n.$$

It follows that $d^c_G(v) \ge \frac{n+1}{2}$ for any vertex v adjacent to u. For any vertex v which is not adjacent to u, we also have $|CN(u) \cup CN(v)| \ge n-1$. This implies $d^c_G(u) + d^c_G(v) = |CN(u) \cup CN(v)| + |CN(u) \cap CN(v)| \ge n-1$. It follows that $d^c_G(v) \ge \frac{n-1}{2}$ for any vertex v not adjacent to u.

Set $H := G - u$. Then, we obtain $d^c_H(v) \geq d^c_G(v) - 1 \geq \frac{|H|}{2}$ for any vertex v adjacent to u, and $d^c_H(v) \geq d^c_G(v) \geq \frac{|H|}{2}$ for any vertex v not adjacent to u. By Theorem 4, the underlying graph of H is isomorphic to $K_{\frac{n-1}{2},\frac{n-1}{2}}$, where n is odd. Let (X, Y) be the bipartition of H, where $X = \{x_1, x_2, \ldots, x_t\}$, $Y = \{y_1, y_2, \ldots, y_t\}$, $t = \frac{n-1}{2}$. We claim that $N_G(u) \subseteq X$ or $N_G(u) \subseteq Y$. Suppose that $N_G(u) \cap X \neq \emptyset$ and $N_G(u) \cap Y \neq \emptyset$. Without loss of generality, suppose that $ux_1 \in E(G)$ and $uy_1 \in E(G)$. Since $d^c_G(x_1) \geq \frac{n+1}{2} = d_G(x_1)$ and $d^c_G(y_1) \geq \frac{n+1}{2} = d_G(y_1)$, we have equality in both cases, and thus $C(x_1u) \neq C(x_1y_1)$ and $C(y_1u) \neq C(x_1y_1)$. This implies that $C(x_1u) = C(y_1u)$. We also can derive that all edges incident to u have the same color, that is, $d^c_G(u) = 1$. For two vertices x, u, $|CN(u) \cup CN(x_1)| = |CN(x_1)| = \frac{n+1}{2} < n - 1$ when $n \geq 4$, a contradiction. Thus, we have shown that $N_G(u) \subseteq X$ or $N_G(u) \subseteq Y$. Without loss of generality, suppose that $N_G(u) \subseteq X$. For any vertex $v \in Y$, we have $|CN_G(u) \cup CN_G(v)| = n - 1$ and $|CN_G(v)| = |X| = \frac{n-1}{2}$. Thus, $|CN_G(u)| = \frac{n-1}{2}$ and $CN_G(u) \cap CN_G(v) = \emptyset$. This implies that the underlying graph of G is $K_{\frac{n+1}{2},\frac{n-1}{2}}$. For any two vertices $v_1, v_2 \in Y$, by the condition $|CN(v_1) \cup CN(v_2)| \geq n - 1$, we can derive that any two edges incident to v_1 or v_2 have distinct colors. Since $v_1, v_2 \in Y$ are chosen arbitrarily, G is a rainbow $K_{\frac{n+1}{2},\frac{n-1}{2}}$.

Now assume that $d^c_G(v) \geq \frac{n}{2}$ for any vertex $v \in V(G)$. By Theorem 4, n is even and the underlying graph of G is $K_{\frac{n}{2},\frac{n}{2}}$. Arguing similarly as above, we see that G is a rainbow $K_{\frac{n}{2},\frac{n}{2}}$. The proof is complete. □

Let D be a digraph with the vertex set $V(D)$ and arc set $A(D)$. For $v \in V(D)$, the *out-degree* of v in D, denoted by $d^+_D(v)$, is the number of out arcs from v.

Lemma 11 (Alon [1]). *Every digraph with minimum out-degree at least* $64k$ *contains* k *vertex-disjoint directed cycles.*

Proof of Theorem 8. By contradiction, suppose that G contains no k vertex-disjoint rainbow cycles. Let $G_1, G_2, \cdots, G_r$ be r vertex-disjoint rainbow cycles in G, where $|G_i| \in \{3, 4, 5\}$ (possibly, $r = 0$). We may assume that $G_1, G_2, \ldots, G_r$ are chosen so that r is as large as possible. Obviously, $r \leq k - 1$. Let $H := G_1 \cup G_2 \cup \ldots G_r$, and $G' := G - V(H)$. Note that $0 \leq |H| \leq 5r$.

Now choose $u, v \in V(G')$ with $uv \in E(G)$, and $S_1 = \{x_1, x_2, \ldots, x_{s_1}\} \subset N_{G'}(u) \backslash \{v\}$ and $S_2 = \{y_1, y_2, \ldots, y_{s_2}\} \subset N_{G'}(v) \backslash \{u\}$, so that the following two conditions hold:

(1) for any $1 \leq i < j \leq s_1$, $C(x_iu) \neq C(x_ju), C(x_iu) \neq C(uv)$; for any $1 \leq i < j \leq s_2$, $C(y_iv) \neq C(y_jv), C(y_iv) \neq C(uv)$; and for any $i \in \{1, 2, \ldots, s_1\}$, $j \in \{1, 2 \ldots, s_2\}$, $C(x_iu) \neq C(y_jv)$; and,

(2) subject to (1), $s_1 + s_2$ is maximized.

Since G' contains no rainbow C_3, $S_1 \cap S_2 = \emptyset$. Set $G^* := G[S_1 \cup S_2 \cup \{u, v\}]$. Note that

$$s_1 + s_2 + 1 = |CN_{G'}(u) \cup CN_{G'}(v)| \geq |CN(u) \cup CN(v)| - 2|H| \geq n/2 + 64k + 1 - 2|H|,$$

and

$$|G^*| = s_1 + s_2 + 2 \geq n/2 + 64k + 2 - 2|H|.$$

In what follows, we construct a digraph D from G^* by the following operations:

(a) Set $V(D) = S_1 \cup S_2$;

(b) For any pair of vertices $x_i, x_j \in S_1$ with $x_i x_j \in E(G)$, $x_i x_j \in A(D)$ if $C(x_i x_j) = C(ux_j)$; and $x_j x_i \in A(D)$ if $C(x_i x_j) = C(ux_i)$;

(c) For any pair of vertices $y_i, y_j \in S_2$ with $y_i y_j \in E(G)$, $y_i y_j \in A(D)$ if $C(y_i y_j) = C(vy_j)$; and $y_j y_i \in A(D)$ if $C(y_i y_j) = C(vy_i)$;

(d) For any pair of vertices $x_i \in S_1, y_j \in S_2$ with $x_i y_j \in E(G)$, $C(x_i y_j) \in \{C(ux_i), C(vy_j), C(uv)\}$, or there is a rainbow C_4. If $C(x_i y_j) = C(uv)$, then we do not add an arc to D; if $C(x_i y_j) = C(ux_i)$ then $y_j x_i \in A(D)$; and if $C(x_i y_j) = C(vy_j)$ then $x_i y_j \in A(D)$.

By the construction, note that there is a directed cycle in D if and only if there is a rainbow cycle in G^*. Furthermore, if there are $(k-r)$ vertex-disjoint directed cycles in D, then there are $(k-r)$ vertex-disjoint rainbow cycles in G^*, and together with the r vertex-disjoint rainbow cycles, this contradicts the assumption that G does not contain k vertex-disjoint rainbow cycles. Thus, there are no $(k-r)$ vertex-disjoint directed cycles in D. By Lemma 11, we can see there is a vertex, say $w_1 \in S_1 \cup S_2$, such that $d^+_D(w_1) \leq 64(k-r) - 1$. If $d^+_D(u) \geq 64(k-r) + 1$ for any vertex $u \in V(D) \backslash \{w_1\}$, then $d^+_{D'}(u) \geq 64(k-r)$, in which $D' := D - w_1$. By Lemma 11, there are $k - r$ directed cycles in D, and k rainbow cycles in G, a contradiction. Thus, there are two vertices, say $w_1, w_2 \in S_1 \cup S_2$, such that $d^+_D(w_1) \leq 64(k-r) - 1$ and $d^+_D(w_2) \leq 64(k-r)$.

Claim 1. $|G - (V(G^*) \cup V(H))| \geq n/2 + 64k - 2|H| - 128(k-r) - 1$.

Proof. We divide the proof into two cases.

First, we assume that w_1, w_2 belong to a same set of S_1, S_2, say, $w_1, w_2 \in S_1$. In this case, we know that all edges incident to w_1 or w_2 in G^* can have at most $3+(128(k-r)-1)$ colors, where the term 3 comes from the fact that uw_1, uw_2, together with the possibly existing edge incident to w_1 or w_2 with the color $C(uv)$, correspond to three colors. Since $|CN(w_1) \cup CN(w_2)| \geq \frac{n}{2} + 64k + 1$, there are at least

$$n_1 := n/2 + 64k + 1 - 2|H| - 3 - (128(k-r) - 1) = n/2 + 64k - 2|H| - 128(k-r) - 1$$

colors between $\{w_1, w_2\}$ and $V(G - G^* - H)$ in G. Let C^* be the set of these n_1 colors. Notice that $C^* \subset CN_{G'-G^*}(w_1) \cup CN_{G'-G^*}(w_2)$. For any vertex $w' \in V(G') \backslash V(G^*)$ such that $w_1w', w_2w' \in E(G)$ and $\{C(w_1w'), C(w_2w')\} \cap \{C(uw_1), C(uw_2)\} = \emptyset$, it follows from G' contains no rainbow C_4 that $C(w_1w') = C(w_2w')$. Furthermore, every common neighbor of w_1, w_2 in $G' - G^*$ with the color in C^* must correspond to one new color. Thus, there are at least $n/2 + 64k - 2|H| - 128(k-r) - 1$ vertices in $G - (V(G^*) \cup V(H))$.

Thus, we may assume that w_1, w_2 belong to different sets, say, $w_1 \in S_1$ and $w_2 \in S_2$. In this case, we know that all edges incident to w_1 or w_2 in G^* can have at most $3 + (128(k-r) - 1)$ colors, where the term 3 comes from the fact that uw_1, vw_2, together with the possible existing edge incident to w_1 or w_2 with the color $C(uv)$, correspond to three colors. So, there are at least

$$n/2 + 64k + 1 - 2|H| - 3 - (128(k-r) - 1) = n/2 + 64k - 2|H| - 128(k-r) - 1$$

colors in $C^* = CN_{G'-G^*}(w_1) \cup CN_{G'-G^*}(w_2)$. For any vertex $w' \in V(G') \backslash V(G^*)$ such that $w_1w', w_2w' \in E(G)$ and $\{C(w_1w'), C(w_2w')\} \cap \{C(uw_1), C(vw_2), C(uv)\} = \emptyset$, it follows from G' contains no rainbow C_5 that $C(w_1w') = C(w_2w')$. Thus, every common neighbor of w_1, w_2 in $G' - G^*$ with the color in $C^* \backslash \{C(uw_1), C(vw_2), C(uv)\}$ corresponds to one new color. Thus, there are at least $n/2 + 64k - 2|H| - 128(k-r) - 1$ vertices in $G - (V(G^*) \cup V(H))$. $\square$

By Claim 1,

$$\begin{aligned}
|G| &= |G^*| + |H| + |G - (V(G^*) \cup V(H))| \\
&\geq n/2 + 64k + 2 - 2|H| + |H| + n/2 + 64k - 2|H| - 128(k-r) - 1 \\
&= n + 128k - 3|H| - 128(k-r) + 1 \\
&\geq n + 113r + 1 \\
&\geq n + 1,
\end{aligned}$$

a contradiction. The proof of Theorem 8 is complete. $\square$

Remark 1. *Bermond and Thomassen [2] conjectured that every directed graph with minimum out-degree at least* $2k-1$ *contains* k *vertex-disjoint directed cycles. Alon [1] gave a* *linear bound by proving that* $64k$ *suffices (Lemma 11). Recently, Bucić [6] proved a better bound* $18k$ *towards this conjecture. One may find that if we apply Bucić's new bound instead of Alon's bound to our proof of Theorem 8, then we can improve the constant in the second term of Theorem 8.*

4 Concluding remarks

Extending Mantel's theorem, Erdős [9] proved that a graph of order n and size $\geq \lfloor \frac{n^2}{4} \rfloor + l$ contains at least $l\lfloor n/2 \rfloor$ triangles, provided $l \leq 3 < n/2$. Erdős [10] further conjectured that the same conclusion holds when $l < n/2$. A slightly weaker form of Erdős' conjecture was proved by Lovász and Simonovits [19]. (See also Bollobás [3, pp.302].) One may ask for the rainbow version of Erdős' conjecture. Furthermore, we can pose the following related problem.

Problem 1. *Let* $k \geq 1$ *be an integer. Let* G *be an edge-colored graph of order* n*. Determine an integer valued function* $f(k)$ *as small as possible, such that if* $e(G) + c(G) \geq n(n+1)/2 + f(k)$ *and* n *is sufficiently large, then* G *contains at least* k *rainbow* C_3*'s.*

Recently, Xu et al. [21] proved a rainbow version of Turán's theorem. Maybe it is also interesting to characterize the extremal graphs in their main theorem.

Furthermore, our Lemma 7 is motivated by the following theorem due to Erdős.

Theorem 9 (Erdős [11])**.** *Let* G *be a graph. Then* G *contains a spanning bipartite subgraph* H*, such that* $d_H(v) \geq \frac{1}{2} d_G(v)$ *for all vertices* $v \in V(G)$*.*

We can naturally consider the counterpart of of Erdős' theorem for edge-colored graphs. Indeed, our Lemma 7 can be regarded as our attempt in this viewpoint. Along this line, it might be interesting to consider a degree condition for the existence of rainbow (or properly colored) spanning bipartite subgraphs in edge-colored graphs.

Acknowledgements

The first author is supported by JSPS KAKENHI (No. 15K04979). The second author is supported by NSFC (No. 11601379). The third author is supported by NSFC (No. 11701441) and the Fundamental Research Funds for the Central Universities (No. XJS17027). The fourth author is supported by NSFC (No. 11671320). The authors are very indebted to an anonymous referee for his/her suggestions which largely improve the presentation of this paper.

References

[1] N. Alon, Disjoint directed cycles, *J. Combin. Theory, Ser. B* **68** (1996), no. 2, 167–178.

[2] J.C. Bermond and C. Thomassen, Cycles in digraphs–a survey, *J. Graph Theory* **5** (1981), no.1, 1–43.

[3] B. Bollobás, Extremal Graph Theory, Academic Press, 1978.

[4] J.A. Bondy and U.S.R. Murty, Graph Theory, GTM-244, Springer, Berlin, 2008.

[5] H.J. Broersma, X. Li, G. Woeginger, and S. Zhang, Paths and cycles in colored graphs, *Australas. J. Combin.* **31** (2005), 299–311.

[6] M. Bucić, An improved bound for disjoint directed cycles, *Discrete Math.* **341** (2018), no. 8, 2231–2236.

[7] R. Čada, A. Kaneko, Z. Ryjáček, and K. Yoshimoto, Rainbow cycles in edge-colored graphs, *Discrete Math.* **339** (2016), no.4, 1387–1392.

[8] G. Chartrand and P. Zhang, Chromatic Graph Theory, Chapman & Hall, Landon, 2008

[9] P. Erdős, Some theorems on graphs, *Riveon Lematematika* **9** (1955), 13–17.

[10] P. Erdős, On a theorem of Rademacher-Turán, *Illinois J. Math.* **6** (1962), 122–127.

[11] P. Erdős, On some extremal problems in graph theory, *Israel J. Math.* **3** (1965), 113–116.

[12] P. Erdős, M. Simonovits, and V.T. Sós, Anti-Ramsey theorems, Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. II, pp. 633-643. Colloq. Math. Soc. János Bolyai, Vol. 10, North-Holland, Amsterdam, 1975.

[13] S. Fujita, C. Magnant, and K. Ozeki, Rainbow generalizations of Ramsey theory: a survey, *Graphs Combin.* **26** (2010), no.1, 1–30.

[14] S. Fujita, C. Magnant, and K. Ozeki, Rainbow generalizations of Ramsey theory– a dynamic survey, *Theory Appl. Graphs* **0** (2014), Iss. i, Article 1(electronic, 42 pages).

[15] A. Gyárfás and G. Simonyi, Edge colorings of complete graphs without tricolored triangles, *J. Graph Theory* **46** (2004), no.3, 211–216.

[16] M. Kano and X. Li, Monochromatic and heterochromatic subgraphs in edge-colored graphs-a survey, *Graphs Combin.* **24** (2008), no.4, 237–263.

[17] B. Li, B. Ning, C. Xu, and S. Zhang, Rainbow triangles in edge-colored graphs, *European J. Combin.* **36** (2014), 453–459.

[18] R. Li, B. Ning, and S. Zhang, Color degree sum condition for rainbow triangles in edge-colored graphs, *Graphs Combin.* **32** (2016), 2001–2008.

[19] L. Lovász and M. Simonovits, On the number of complete subgraphs of a graph, in: Proc. Fifth British Combinatorial Conf. (Nash-Williams, C.St.J.A. and Sheehan, J. eds), *Utilitas Math.*, Winnipeg, (1976), 431–441.

[20] D.B. West, Introduction to Graph Theory, Prentice-Hall, 2000.

[21] C. Xu, X. Hu, W. Wang, and S. Zhang, Rainbow cliques in edge-colored graphs, *European J. Combin.* **54** (2016), 193–200.