PARAMETRIC MARCINKIEWICZ INTEGRALS WITH ROUGH KERNELS ACTING ON WEAK MUSIELAK-ORLICZ HARDY SPACES

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ABSTRACT. Let $\varphi : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$ satisfy that $\varphi(x, \cdot)$, for any given $x \in \mathbb{R}^n$, is an Orlicz function and $\varphi(\cdot, t)$ is a Muckenhoupt A_∞ weight uniformly in $t \in (0, \infty)$. The weak Musielak-Orlicz Hardy space $WH^\varphi(\mathbb{R}^n)$ is defined to be the set of all tempered distributions such that their grand maximal functions belong to the weak Musielak-Orlicz space $WL^\varphi(\mathbb{R}^n)$. For $\rho \in (0, \infty)$ and measurable function f on $\mathbb{R}^n$, the parametric Marcinkiewicz integral μ_Ω^ρ related to the Littlewood-Paley g-function is defined by

$$\mu_\Omega^\rho(f)(x) := \left(\int_0^\infty \left| \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} f(y)\, dy \right|^2 \frac{dt}{t^{2\rho+1}} \right)^{1/2},$$

where Ω is homogeneous of degree zero satisfying the cancellation condition.

In this paper, we discuss the boundedness of parametric Marcinkiewicz integral μ_Ω^ρ with rough kernel from weak Musielak-Orlicz Hardy space $WH^\varphi(\mathbb{R}^n)$ to weak Musielak-Orlicz space $WL^\varphi(\mathbb{R}^n)$. These results are new even for classical weighted weak Hardy space of Quek and Yang, and probably new for classical weak Hardy space of Fefferman and Soria.

1. INTRODUCTION

Suppose that S^{n-1} is the unit sphere in $\mathbb{R}^n$ ($n \ge 2$) equipped with normalized Lebesgue measure $d\sigma$. A function $\Omega(x)$ defined on $\mathbb{R}^n$ is said to be in $L^q(S^{n-1})$ with $q \ge 1$, if $\Omega(x)$ satisfies the following conditions:

$$\Omega(\lambda x) = \Omega(x) \text{ for any } x \in \mathbb{R}^n \text{ and } \lambda \in (0, \infty),$$

$$\int_{S^{n-1}} \Omega(x)\, d\sigma(x') = 0$$

and

$$\int_{S^{n-1}} |\Omega(x)|^q\, d\sigma(x') < \infty,$$

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where $x' := x/|x|$ for any $x \neq \mathbf{0}$. For $\rho \in (0, \infty)$ and measurable function f on $\mathbb{R}^n$, the *parametric Marcinkiewicz integral* μ_Ω^ρ is defined by

$$\mu_\Omega^\rho(f)(x) := \left(\int_0^\infty \left| \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} f(y)\, dy \right|^2 \frac{dt}{t^{2\rho+1}} \right)^{1/2}.$$

The Marcinkiewicz integral μ_Ω^1 was introduced by Stein [22] in 1958. He showed that, if $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$ with $\alpha \in (0, 1]$, then μ_Ω^1 is bounded on $L^p(\mathbb{R}^n)$ with $p \in (1, 2]$ and bounded from $L^1(\mathbb{R}^n)$ to weak $L^1(\mathbb{R}^n)$. In 1960, Hörmander [6] proved that, if $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$ with $\alpha \in (0, 1]$, then μ_Ω^ρ is bounded on $L^p(\mathbb{R}^n)$ provided that $p \in (1, \infty)$ and $\rho \in (0, \infty)$. Notice that all these results mentioned above hold true depending on some smoothness condition of Ω. However, in 2009, Shi and Jiang [23] obtained the following celebrated result that μ_Ω^ρ is bounded on $L^p_\omega(\mathbb{R}^n)$ without any smoothness condition of Ω, where $\omega \in A_p$ and A_p denotes the Muckenhoupt weight class.

Theorem A. ([23, Theorem 1.1]) *Let $\rho \in (0, \infty)$, $p, q \in (1, \infty)$, $q' := q/(q-1)$ and $\Omega \in L^q(S^{n-1})$. If $\omega^{q'} \in A_p$, then μ_Ω^ρ is bounded on $L^p_\omega(\mathbb{R}^n)$.*

On the other hand, in the past four decades, there has been an increasing interest in developing the theory of Hardy space. Originally Hardy space appeared in complex analysis in the study of analytic function on the unit disk. And its theory was one-dimensional. The higher dimensional Euclidean theory of Hardy space was developed by Fefferman and Stein [4] who proved a variety of characterizations for them. As everyone knows, many important operators are better behaved on Hardy space $H^p(\mathbb{R}^n)$ than on Lebesgue space $L^p(\mathbb{R}^n)$ in the range $p \in (0, 1]$. For example, when $p \in (0, 1]$, the Riesz transforms are bounded on Hardy space $H^p(\mathbb{R}^n)$, but not on the corresponding Lebesgue space $L^p(\mathbb{R}^n)$. Therefore, one can consider $H^p(\mathbb{R}^n)$ to be a very natural replacement for $L^p(\mathbb{R}^n)$ when $p \in (0, 1]$. Moreover, when studying the endpoint estimate for variant important operators, the weak Hardy space $WH^p(\mathbb{R}^n)$ naturally appear and prove to be a good substitute of Hardy space $H^p(\mathbb{R}^n)$ with $p \in (0, 1]$. For instance, if $\delta \in (0, 1]$, T is a δ-Calderón-Zygmund operator and $T^*(1) = 0$, where T^* denotes the adjoint operator of T, it is known that T is bounded on $H^p(\mathbb{R}^n)$ for any $p \in (n/(n+\delta), 1]$ (see [1]), but T may be not bounded on $H^{\frac{n}{n+\delta}}(\mathbb{R}^n)$; however, Liu [13] proved that T is bounded from $H^{\frac{n}{n+\delta}}(\mathbb{R}^n)$ to $WH^{\frac{n}{n+\delta}}(\mathbb{R}^n)$.

Recently, Ky [12] introduced a new Musielak-Orlicz Hardy space $H^\varphi(\mathbb{R}^n)$, which unifies the classical Hardy space [4], the weighted Hardy space [24], the Orlicz Hardy space [8, 9, 10, 11], and the weighted Orlicz Hardy space. Its s-patial and time variables may not be separable. Later, Liang et al. [18] further introduced a weak Musielak-Orlicz Hardy space $WH^\varphi(\mathbb{R}^n)$, which covers both the weak Hardy space [5], the weighted weak Hardy space [21], the weak Orlicz Hardy space and the weighted weak Orlicz Hardy space, as special cases. Recently, some new characterizations of of $WH^\varphi(\mathbb{R}^n)$ by means of maximal functions, atoms, molecules and Littlewood-Paley functions, and the boundedness of Calderón-Zygmund operators in the critical case were obtained in [18]. Apart from interesting theoretical considerations, the motivation to study Musielak-Orlicz-type space comes from applications to elasticity, fluid dynamics, image processing, PDEs and the calculus of variation (see, for example, [2]). More Musielak-Orlicz-type spaces are referred to [3, 7, 17, 19, 20, 25]. It should be pointed out that the monograph [25] provides a detailed and complete survey of recent developments on the real-variable theory of Musielak-Orlicz Hardy-type function spaces and lays the foundation for further applications.

Motivated by all of the facts mentioned above, a natural and interesting question arises, namely, whether the parametric Marcinkiewicz integral μ_Ω^ρ is bounded from weak Musielak-Orlicz Hardy space $WH^\varphi(\mathbb{R}^n)$ to weak Musielak-Orlicz space $WL^\varphi(\mathbb{R}^n)$ under weaker smoothness condition assumed on Ω. In this paper we shall answer this problem affirmatively. Here, what is worth mentioning is that our results are new even for classical weighted weak Hardy space and probably new for classical weak Hardy space.

Precisely, this paper is organized as follows. In the next section, we recall some notions concerning Muckenhoupt weight, growth function and weak Musielak-Orlicz Hardy space. Then we present the boundedness of μ_Ω^ρ from $WH^\varphi(\mathbb{R}^n)$ to $WL^\varphi(\mathbb{R}^n)$ (see Theorem 2.7, Theorem 2.8, Corollary 2.9 and Theorem 2.10 below). In Section 3, with the help of some auxiliary lemmas and atomic decomposition theory of $WH^\varphi(\mathbb{R}^n)$, the proofs of main results are presented.

Finally, we make some conventions on notation. Let $\mathbb{Z}_+ := \{1, 2, \ldots\}$ and $\mathbb{N} := \{0\} \cup \mathbb{Z}_+$. For any $\beta := (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$, let $|\beta| := \beta_1 + \cdots + \beta_n$ and $\partial^\beta := (\frac{\partial}{\partial x_1})^{\beta_1} \cdots (\frac{\partial}{\partial x_n})^{\beta_n}$. Throughout this paper the letter C will denote a *positive constant* that may vary from line to line but will remain independent of the main variables. The *symbol* $P \lesssim Q$ stands for the inequality $P \le CQ$. If $P \lesssim Q \lesssim P$, we then write $P \sim Q$. For any sets $E, F \subset \mathbb{R}^n$, we use $E^\complement$ to denote the set $\mathbb{R}^n \setminus E$, $|E|$ its *n-dimensional Lebesgue measure*, χ_E its *characteristic function* and $E + F$ the *algebraic sum* $\{x + y : \ x \in E, y \in F\}$. For any $s \in \mathbb{R}$, $\lfloor s \rfloor$ denotes the unique integer such that $s - 1 < \lfloor s \rfloor \le s$. If there are no special instructions, any space $\mathcal{X}(\mathbb{R}^n)$ is denoted simply by $\mathcal{X}$. For instance, $L^2(\mathbb{R}^n)$ is simply denoted by L^2. For any set $E \subset \mathbb{R}^n$, $t \in [0, \infty)$ and measurable function $\varphi(\cdot, t)$, let $\varphi(E, t) := \int_E \varphi(x, t)\, dx$ and $\{|f| > t\} := \{x \in \mathbb{R}^n : \ |f(x)| > t\}$. For any $x \in \mathbb{R}^n$, $r \in (0, \infty)$ and $\alpha \in (0, \infty)$, we use $B(x, r)$ to denote the ball $\{y \in \mathbb{R}^n : \ |y - x| < r\}$ and $\alpha B(x, r)$ to denote $B(x, \alpha r)$ as usual.

2. Notions and main results

In this section, we first recall the definition concerning the weak Musielak-Orlicz Hardy space WH^φ, and then present the boundedness of parametric Marcinkiewicz integral μ_Ω^ρ from weak Musielak-Orlicz Hardy space WH^φ to weak Musielak-Orlicz space WL^φ.

Recall that a nonnegative function φ on $\mathbb{R}^n \times [0, \infty)$ is called a *Musielak-Orlicz function* if, for any $x \in \mathbb{R}^n$, $\varphi(x, \cdot)$ is an Orlicz function on $[0, \infty)$ and, for any $t \in [0, \infty)$, $\varphi(\cdot, t)$ is measurable on $\mathbb{R}^n$. Here a function $\phi : [0, \infty) \to [0, \infty)$ is called an *Orlicz function*, if it is nondecreasing, $\phi(0) = 0$, $\phi(t) > 0$ for any $t \in (0, \infty)$, and $\lim_{t\to\infty} \phi(t) = \infty$.

Given a Musielak-Orlicz function φ on $\mathbb{R}^n \times [0, \infty)$, φ is said to be of *uniformly lower* (resp. *upper*) *type* p with $p \in \mathbb{R}$, if there exists a positive constant $C := C_\varphi$ such that, for any $x \in \mathbb{R}^n$, $t \in [0, \infty)$ and $s \in (0, 1]$ (resp. $s \in [1, \infty)$),

$$\varphi(x, st) \le C s^p \varphi(x, t).$$

The *critical uniformly lower type index* of φ is defined by

$$i(\varphi) := \sup\{p \in \mathbb{R} : \ \varphi \text{ is of uniformly lower type } p\}. \tag{2.1}$$

Observe that $i(\varphi)$ may not be attainable, namely, φ may not be of uniformly lower type $i(\varphi)$ (see [16, p. 415] for more details).

Definition 2.1. (i) Let $q \in [1, \infty)$. A locally integrable function $\varphi(\cdot, t) : \mathbb{R}^n \to [0, \infty)$ is said to satisfy the *uniformly Muckenhoupt condition* $\mathbb{A}_q$, denoted by $\varphi \in \mathbb{A}_q$, if there exists a positive constant C such that, for any ball $B \subset \mathbb{R}^n$ and $t \in (0, \infty)$, when $q = 1$,

$$\frac{1}{|B|}\int_B \varphi(x, t)\,dx \left\{\operatorname*{ess\,sup}_{x\in B}\,[\varphi(x, t)]^{-1}\right\} \le C$$

and, when $q \in (1, \infty)$,

$$\frac{1}{|B|}\int_B \varphi(x, t)\,dx \left\{\frac{1}{|B|}\int_B [\varphi(x, t)]^{-\frac{1}{q-1}}\,dx\right\}^{q-1} \le C.$$

(ii) Let $q \in (1, \infty]$. A locally integrable function $\varphi(\cdot, t) : \mathbb{R}^n \to [0, \infty)$ is said to satisfy the *uniformly reverse Hölder condition* $\mathbb{RH}_q$, denoted by $\varphi \in \mathbb{RH}_q$, if there exists a positive constant C such that, for any ball $B \subset \mathbb{R}^n$ and $t \in (0, \infty)$, when $q \in (1, \infty)$,

$$\left\{\frac{1}{|B|}\int_B \varphi(x, t)\,dx\right\}^{-1}\left\{\frac{1}{|B|}\int_B [\varphi(x, t)]^q\,dx\right\}^{1/q} \le C$$

and, when $q = \infty$,

$$\left\{\frac{1}{|B|}\int_B \varphi(x, t)\,dx\right\}^{-1} \operatorname*{ess\,sup}_{x\in B}\,\varphi(x, t) \le C.$$

Define $\mathbb{A}_\infty := \bigcup_{q\in[1,\infty)} \mathbb{A}_q$. It is well known that if $\varphi \in \mathbb{A}_q$ with $q \in (1, \infty]$, then $\varphi^\varepsilon \in \mathbb{A}_{\varepsilon q+1-\varepsilon} \subset \mathbb{A}_q$ for any $\varepsilon \in (0, 1]$ and $\varphi^\eta \in \mathbb{A}_q$ for some $\eta \in (1, \infty)$. Also, if $\varphi \in \mathbb{A}_q$ with $q \in (1, \infty)$, then $\varphi \in \mathbb{A}_r$ for any $r \in (q, \infty)$ and $\varphi \in \mathbb{A}_d$ for some $d \in (1, q)$. Thus, the *critical weight index* of $\varphi \in \mathbb{A}_\infty$ is defined as follows:

$$q(\varphi) := \inf\{q \in [1, \infty) : \ \varphi \in \mathbb{A}_q\}. \tag{2.2}$$

For uniformly Muckenhoupt (resp. reverse Hölder) condition, we have the following property as the classical case.

Lemma 2.2. ([12, Lemma 4.5]) *Let $\varphi \in \mathbb{A}_q$ with $q \in [1, \infty)$. Then there exists a positive constant C such that, for any ball $B \subset \mathbb{R}^n$, $\lambda \in (1, \infty)$ and $t \in (0, \infty)$,*

$$\varphi(\lambda B, t) \le C\lambda^{nq}\varphi(B, t).$$

Lemma 2.3. ([15, Lemma 3.5]) *Let $r \in (1, \infty)$. Then $\varphi^r \in \mathbb{A}_\infty$ if and only if $\varphi \in \mathbb{RH}_r$.*

Definition 2.4. ([12, Definition 2.1]) A function $\varphi : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$ is called a *growth function* if the following conditions are satisfied:

(i) φ is a Musielak-Orlicz function;

(ii) $\varphi \in \mathbb{A}_\infty$;

(iii) φ is of uniformly lower type p for some $p \in (0, 1]$ and of uniformly upper type 1.

Throughout the paper, we always assume that φ is a growth function.

Recall that the *weak Musielak-Orlicz space* WL^φ is defined to be the space of all measurable functions f such that, for some $\lambda \in (0, \infty)$,

$$\sup_{t\in(0,\infty)} \varphi\left(\{|f| > t\}, \frac{t}{\lambda}\right) < \infty$$

equipped with the quasi-norm

$$\|f\|_{WL^\varphi} := \inf\left\{\lambda \in (0, \infty) : \sup_{t\in(0,\infty)} \varphi\left(\{|f| > t\}, \frac{t}{\lambda}\right) \le 1\right\}.$$

In what follows, we denote by $\mathcal{S}$ the *space of all Schwartz functions* and by $\mathcal{S}'$ its *dual space* (namely, the *space of all tempered distributions*). For any $m \in \mathbb{N}$, let $\mathcal{S}_m$ be the *space* of all $\psi \in \mathcal{S}$ satisfying $\|\psi\|_{\mathcal{S}_m} \le 1$, where

$$\|\psi\|_{\mathcal{S}_m} := \sup_{\substack{\alpha\in\mathbb{N}^n\\ |\alpha|\le m+1}} \sup_{x\in\mathbb{R}^n} (1 + |x|)^{(m+2)(n+1)} |\partial^\alpha \psi(x)|.$$

Then, for any $m \in \mathbb{N}$ and $f \in \mathcal{S}'$, the *non-tangential grand maximal function* f^*_m of f is defined by setting, for all $x \in \mathbb{R}^n$,

$$f^*_m(x) := \sup_{\psi\in\mathcal{S}_m} \sup_{\substack{|y-x|<t\\ t\in(0,\infty)}} |f * \psi_t(y)|,$$

where, for any $t \in (0, \infty)$, $\psi_t(\cdot) := t^{-n}\psi(\frac{\cdot}{t})$. When

$$m = m(\varphi) := \left\lfloor n\left(\frac{q(\varphi)}{i(\varphi)} - 1\right)\right\rfloor, \tag{2.3}$$

we denote f^*_m simply by f^*, where $q(\varphi)$ and $i(\varphi)$ are as in (2.2) and (2.1), respectively.

Definition 2.5. ([18, Definition 2.3]) Let φ be a growth function as in Definition 2.4. The *weak Musielak-Orlicz Hardy space* WH^φ is defined as the space of all $f \in \mathcal{S}'$ such that $f^* \in WL^\varphi$ endowed with the quasi-norm

$$\|f\|_{WH^\varphi} := \|f^*\|_{WL^\varphi}.$$

Remark 2.6. Let ω be a classic Muckenhoupt weight and ϕ an Orlicz function.

(i) If $\varphi(x, t) := \omega(x)\phi(t)$ for all $(x, t) \in \mathbb{R}^n \times [0, \infty)$, then WH^φ goes back to the weighted weak Orlicz Hardy space WH^ϕ_ω, and particularly, when $\omega \equiv 1$, the corresponding unweighted space is also obtained.

(ii) If $\varphi(x, t) := \omega(x)t^p$ for all $(x, t) \in \mathbb{R}^n \times [0, \infty)$ with $p \in (0, 1]$, then WH^φ goes back to the weighted weak Hardy space WH^p_ω, and particularly, when $\omega \equiv 1$, the corresponding unweighted space is also obtained.

Before stating our main results, we recall some notions about Ω. For any $q \in [1, \infty)$ and $\alpha \in (0, 1]$, a function $\Omega \in L^q(S^{n-1})$ is said to satisfy the $L^{q,\alpha}$-*Dini condition* if

$$\int_0^1 \frac{\omega_q(\delta)}{\delta^{1+\alpha}}\, d\delta < \infty,$$

where

$$\omega_q(\delta) := \sup_{\|\gamma\|<\delta} \left(\int_{S^{n-1}} |\Omega(\gamma x') - \Omega(x')|^q \, d\sigma(x') \right)^{1/q}$$

and γ denotes a rotation on S^{n-1} with $\|\gamma\| := \sup_{y' \in S^{n-1}} |\gamma y' - y'|$. For any $\alpha, \beta \in (0, 1]$ with $\beta < \alpha$, it is trivial to see that if Ω satisfies the $L^{q,\alpha}$-Dini condition, then it also satisfies the $L^{q,\beta}$-Dini condition. We thus denote by $\mathrm{Din}^q_\alpha(S^{n-1})$ the class of all functions which satisfy the $L^{q,\beta}$-Dini conditions for all $\beta < \alpha$. For any $\alpha \in (0, 1]$, we define

$$\mathrm{Din}^\infty_\alpha(S^{n-1}) := \bigcap_{q \geq 1} \mathrm{Din}^q_\alpha(S^{n-1}).$$

A routine computation gives rise to

$$\mathrm{Din}^r_\alpha(S^{n-1}) \subset \mathrm{Din}^q_\alpha(S^{n-1}) \ \text{ if } 1 \leq q < r \leq \infty,$$

and

$$\mathrm{Din}^q_\alpha(S^{n-1}) \subset \mathrm{Din}^q_\beta(S^{n-1}) \ \text{ if } 0 < \beta < \alpha \leq 1.$$

The main results of this paper are as follows, the proofs of which are given in Section 3.

Theorem 2.7. *Let $\rho \in (0, \infty)$, $\alpha \in (0, 1]$, $\beta := \min\{1/2, \alpha\}$ and φ be a growth function as in Definition 2.4 with $p \in (n/(n+\beta), 1)$. Suppose that $\Omega \in L^r(S^{n-1}) \cap \mathrm{Din}^1_\alpha(S^{n-1})$ with $r \in (1, \infty]$. If q and φ satisfy one of the following conditions:*

(i) *$r \in (1, 1/p]$ and $\varphi^{r'} \in \mathbb{A}_{p\beta/[n(1-p)]}$;*
(ii) *$r \in (1/p, \infty]$ and $\varphi^{1/(1-p)} \in \mathbb{A}_{p\beta/[n(1-p)]}$,*

then μ^ρ_Ω is bounded from WH^φ to WL^φ.

Theorem 2.8. *Let $\rho \in (0, \infty)$, $\alpha \in (0, 1]$, $\beta := \min\{1/2, \alpha\}$ and φ be a growth function as in Definition 2.4 with $p \in (n/(n+\beta), 1]$. Suppose that $\Omega \in \mathrm{Din}^q_\alpha(S^{n-1})$ with $q \in (1, \infty)$. If $\varphi^{q'} \in \mathbb{A}_{(p+p\beta/n-1/q)q'}$, then μ^ρ_Ω is bounded from WH^φ to WL^φ.*

Corollary 2.9. *Let $\rho \in (0, \infty)$, $\alpha \in (0, 1]$, $\beta := \min\{1/2, \alpha\}$ and φ be a growth function as in Definition 2.4 with $p \in (n/(n+\beta), 1]$. Suppose that $\Omega \in \mathrm{Din}^\infty_\alpha(S^{n-1})$. If $\varphi \in \mathbb{A}_{p(1+\beta/n)}$, then μ^ρ_Ω is bounded from WH^φ to WL^φ.*

Theorem 2.10. *Let $\rho \in (0, \infty)$, $\Omega \in L^q(S^{n-1})$ with $q \in (1, \infty]$, and φ be a growth function as in Definition 2.4 with $p := 1$ and $\varphi^{q'} \in \mathbb{A}_1$. If there exists a positive constant C such that, for any $y, h \in \mathbb{R}^n$ and $M, t \in (0, \infty)$,*

$$\int_{|x| \geq M|y|} \left| \frac{\Omega(x-y)}{|x-y|^n} - \frac{\Omega(x)}{|x|^n} \right| \varphi(x+h, t)\, dx \leq \frac{C}{M} \varphi(y+h, t), \tag{2.4}$$

then μ^ρ_Ω is bounded from WH^φ to WL^φ.

Remark 2.11. (i) It is worth noting that Corollary 2.9 can be regarded as the limit case of Theorem 2.8 by letting $q \to \infty$.

(ii) Theorem 2.7, Theorem 2.8 and Corollary 2.9 jointly answer the question: when $\Omega \in \mathrm{Din}_\alpha^q(S^{n-1})$ with $q = 1$, $q \in (1, \infty)$ or $q = \infty$, respectively, what kind of additional conditions on φ and Ω can deduce the boundedness of μ_Ω^ρ from WH^φ to WL^φ?

(iii) Let ω be a classic Muckenhoupt weight and ϕ an Orlicz function.

(a) When $\varphi(x, t) := \omega(x)\phi(t)$ for all $(x, t) \in \mathbb{R}^n \times [0, \infty)$, we have $WH^\varphi = WH_\omega^\phi$. In this case, Theorem 2.7, Theorem 2.8, Corollary 2.9 and Theorem 2.10 hold true for the weighted weak Orlicz Hardy space. Even when $\omega \equiv 1$, the corresponding unweighted results are also new.

(b) When $\varphi(x, t) := \omega(x)\, t^p$ for all $(x, t) \in \mathbb{R}^n \times [0, \infty)$, we have $WH^\varphi = WH_\omega^p$. In this case, Theorem 2.7, Theorem 2.8, Corollary 2.9 and Theorem 2.10 are new for the weighted weak Hardy space. Even when $\omega \equiv 1$, the corresponding unweighted results are probably new.

3. Proofs of main results

To show main results, we need some notions and auxiliary lemmas.

Definition 3.1. ([12, Definition 2.4]) Let φ be a growth function as in Definition 2.4.

(i) A triplet (φ, q, s) is said to be *admissible*, if $q \in (q(\varphi), \infty]$ and $s \in [m(\varphi), \infty) \cap \mathbb{N}$, where $q(\varphi)$ and $m(\varphi)$ are as in (2.2) and (2.3), respectively.

(ii) For an admissible triplet (φ, q, s), a measurable function a is called a (φ, q, s)-*atom* if there exists some ball $B \subset \mathbb{R}^n$ such that the following conditions are satisfied:

(a) a is supported on B;

(b) $\|a\|_{L_\varphi^q(B)} \le \|\chi_B\|_{L^\varphi}^{-1}$, where

$$\|a\|_{L_\varphi^q(B)} := \begin{cases} \sup\limits_{t \in (0, \infty)} \left[\dfrac{1}{\varphi(B, t)} \displaystyle\int_B |a(x)|^q \varphi(x, t)\, dx\right]^{1/q}, & q \in [1, \infty), \\ \|a\|_{L^\infty}, & q = \infty, \end{cases}$$

and

$$\|\chi_B\|_{L^\varphi} := \inf\left\{\lambda \in (0, \infty) :\ \varphi\left(B, \lambda^{-1}\right) \le 1\right\};$$

(c) $\int_{\mathbb{R}^n} a(x) x^\gamma dx = 0$ for any $\gamma \in \mathbb{N}^n$ with $|\gamma| \le s$.

Definition 3.2. ([18, Definition 3.2]) For an admissible triplet (φ, q, s), the *weak atomic Musielak-Orlicz Hardy space* $WH_{\mathrm{at}}^{\varphi, q, s}$ is defined as the space of all $f \in \mathcal{S}'$ satisfying that there exist a sequence of (φ, q, s)-atoms, $\{a_{i,j}\}_{i \in \mathbb{Z}, j \in \mathbb{Z}_+}$, associated with balls $\{B_{i,j}\}_{i \in \mathbb{Z}, j \in \mathbb{Z}_+}$, and a positive constant C such that $\sum_{j \in \mathbb{Z}_+} \chi_{B_{i,j}}(x) \le C$ for any $x \in \mathbb{R}^n$ and $i \in \mathbb{Z}$, and $f = \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}_+} \lambda_{i,j} a_{i,j}$ in $\mathcal{S}'$, where $\lambda_{i,j} := \widetilde{C} 2^i \|\chi_{B_{i,j}}\|_{L^\varphi}$ for any $i \in \mathbb{Z}$ and $j \in \mathbb{Z}_+$, and $\widetilde{C}$ is a positive constant independent of f, i and j.

Moreover, define

$$\|f\|_{WH^{\varphi,\,q,\,s}_{\mathrm{at}}} := \inf\left\{\inf\left\{\lambda\in(0,\,\infty):\ \sup_{i\in\mathbb{Z}}\left\{\sum_{j\in\mathbb{Z}_+}\varphi\left(B_{i,j},\,\frac{2^i}{\lambda}\right)\right\}\le 1\right\}\right\},$$

where the first infimum is taken over all decompositions of f as above.

Lemma 3.3. ([18, Theorem 3.5]) *Let $(\varphi,\,q,\,s)$ be an admissible triplet. Then*

$$WH^{\varphi} = WH^{\varphi,\,q,\,s}_{\mathrm{at}}$$

with equivalent quasi-norms.

Lemma 3.4. *For any $\alpha\in(0,\,1]$ and $q\in[1,\,\infty)$, suppose that Ω satisfies $L^{q,\,\alpha}$-Dini condition. Let $\rho\in(0,\,\infty)$, $\beta:=\min\{1/2,\,\alpha\}$ and b be a multiple of a (φ,∞,s)-atom associated with some ball $B(x_0,\,r)\subset\mathbb{R}^n$, where $x_0\in\mathbb{R}^n$ and $r\in(0,\,\infty)$.*

(i) *If $q=1$, then there exists a positive constant C independent of b such that, for any $R\in[2r,\,\infty)$,*

$$\int_{R\le|x-x_0|<2R}\left|\mu^{\rho}_{\Omega}(b)(x)\right|\,dx\le C\|b\|_{L^\infty}R^n\left(\frac{r}{R}\right)^{n+\beta}.$$

(ii) *If $q\in(1,\,\infty)$, then there exists a positive constant C independent of b such that, for any $R\in[2r,\,\infty)$ and $t\in(0,\,\infty)$,*

$$\int_{R\le|x-x_0|<2R}\left|\mu^{\rho}_{\Omega}(b)(x)\right|\varphi(x,\,t)\,dx$$
$$\le C\|b\|_{L^\infty}\left[\varphi^{q'}(B(x_0,\,2R),\,t)\right]^{1/q'}R^{n/q}\left(\frac{r}{R}\right)^{n+\beta}.$$

Proof. The proof of this lemma, the details of which we omit, can be completed by the method analogous to that used in the proof of [14, Lemma 4.4]. □

Proof of Theorem 2.7. We need only consider the case $r\in(1,\,\infty)$, since the case $r=\infty$ can be derived from the case $r=2$. Indeed, when $r=\infty$, a routine computation gives rise to $2>1/p$. If Theorem 2.7 holds true for $r=2$, by $\Omega\in L^\infty(S^{n-1})\subset L^2(S^{n-1})$, $2>1/p$ and $\varphi^{1/(1-p)}\in\mathbb{A}_{p\beta/[n(1-p)]}$, we know that Theorem 2.7 holds true for $q=\infty$. We are now turning to the proof of Theorem 2.7 under case $r\in(1,\,\infty)$. We claim that, in either case (i) or (ii) of Theorem 2.7, there exists some $d\in(1,\,p\beta/[n(1-p)])$ such that

$$\varphi^{r'}\in\mathbb{A}_d \text{ and } \varphi^{1/(1-p)}\in\mathbb{A}_d. \tag{3.1}$$

We only prove (3.1) under case (ii) since the proof under case (i) is similar. By $\varphi^{1/(1-p)}\in\mathbb{A}_{p\beta/n(1-p)}$, we see that there exists some $d\in(1,\,p\beta/[n(1-p)])$ such that $\varphi^{1/(1-p)}\in\mathbb{A}_d$. On the other hand, notice that $r'<1/(1-p)$, then $\varphi^{r'}\in\mathbb{A}_d$ as claimed.

Let $(\varphi,\,\infty,\,s)$ be an admissible triplet. By Lemma 3.3, we know that, for any $f\in WH^{\varphi}=WH^{\varphi,\,\infty,\,s}_{\mathrm{at}}$, there exists a sequence of multiples of $(\varphi,\,\infty,\,s)$-atoms,

$\{b_{i,j}\}_{i\in\mathbb{Z},\, j\in\mathbb{Z}_+}$, associated with balls $\{B_{i,j}\}_{i\in\mathbb{Z},\, j\in\mathbb{Z}_+}$, such that

$$f=\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}_+} b_{i,j} \text{ in } \mathcal{S}',$$

$\sum_{j\in\mathbb{Z}_+}\chi_{B_{i,j}}(x)\lesssim 1$ for any $x\in\mathbb{R}^n$ and $i\in\mathbb{Z}$, $\|b_{i,j}\|_{L^\infty}\lesssim 2^i$ for any $i\in\mathbb{Z}$ and $j\in\mathbb{Z}_+$, and

$$\|f\|_{WH^\varphi}\sim\inf\left\{\lambda\in(0,\,\infty):\ \sup_{i\in\mathbb{Z}}\left\{\sum_{j\in\mathbb{Z}_+}\varphi\left(B_{i,j},\,\frac{2^i}{\lambda}\right)\right\}\le 1\right\}.$$

Thus, our problem reduces to prove that, for any $\gamma,\,\lambda\in(0,\,\infty)$ and $f\in WH^\varphi$,

$$\varphi\left(\{|\mu_\Omega^\rho(f)|>\gamma\}\,,\,\frac{\gamma}{\lambda}\right)\lesssim\sup_{i\in\mathbb{Z}}\left\{\sum_{j\in\mathbb{Z}_+}\varphi\left(B_{i,j},\,\frac{2^i}{\lambda}\right)\right\}.$$

To show this inequality, without loss of generality, we may assume that there exists $i_0\in\mathbb{Z}$ such that $\gamma=2^{i_0}$. Let us rewrite

$$f=\sum_{i=-\infty}^{i_0-1}\sum_{j\in\mathbb{Z}_+} b_{i,j}+\sum_{i=i_0}^{\infty}\sum_{j\in\mathbb{Z}_+} b_{i,j}=:F_1+F_2.$$

We estimate F_1 first. From Theorem A with $\Omega\in L^r(S^{n-1})$ and $\varphi^{r'}\in\mathbb{A}_d$ (see (3.1)), Minkowski's inequality, the fact that $\sum_{j\in\mathbb{Z}_+}\chi_{B_{i,j}}(x)\lesssim 1$ for any $x\in\mathbb{R}^n$ and $i\in\mathbb{Z}$, and the uniformly upper type 1 property of φ, we deduce that, for any $\lambda\in(0,\,\infty)$,

$$\begin{aligned}
&\varphi\left(\left\{|\mu_\Omega^\rho(F_1)|>2^{i_0}\right\},\,\frac{2^{i_0}}{\lambda}\right) \qquad (3.2)\\
&=\int_{\{|\mu_\Omega^\rho(F_1)|>2^{i_0}\}}\varphi\left(x,\,\frac{2^{i_0}}{\lambda}\right)dx\\
&\lesssim 2^{-di_0}\int_{\mathbb{R}^n}|F_1(x)|^d\varphi\left(x,\,\frac{2^{i_0}}{\lambda}\right)dx\\
&\lesssim 2^{-di_0}\left\{\sum_{i=-\infty}^{i_0-1}\left[\int_{\mathbb{R}^n}\left|\sum_{j\in\mathbb{Z}_+}b_{i,j}(x)\right|^d\varphi\left(x,\,\frac{2^{i_0}}{\lambda}\right)dx\right]^{1/d}\right\}^d\\
&\lesssim 2^{-di_0}\left\{\sum_{i=-\infty}^{i_0-1}2^i\left[\sum_{j\in\mathbb{Z}_+}\varphi\left(B_{i,j},\,\frac{2^{i_0}}{\lambda}\right)\right]^{1/d}\right\}^d\\
&\lesssim 2^{-di_0}\left\{\sum_{i=-\infty}^{i_0-1}2^i\left[2^{i_0-i}\sum_{j\in\mathbb{Z}_+}\varphi\left(B_{i,j},\,\frac{2^i}{\lambda}\right)\right]^{1/d}\right\}^d
\end{aligned}$$

$$\lesssim 2^{(1-d)i_0}\left(\sum_{i=-\infty}^{i_0-1} 2^{(1-1/d)i}\right)^d \sup_{i\in\mathbb{Z}}\left\{\sum_{j\in\mathbb{Z}_+}\varphi\left(B_{i,j},\, \frac{2^i}{\lambda}\right)\right\}$$
$$\sim \sup_{i\in\mathbb{Z}}\left\{\sum_{j\in\mathbb{Z}_+}\varphi\left(B_{i,j},\, \frac{2^i}{\lambda}\right)\right\},$$

which is desired.

Next let us deal with F_2. Denote the center of $B_{i,j}$ by $x_{i,j}$ and the radius by $r_{i,j}$. Set

$$A_{i_0} := \bigcup_{i=i_0}^{\infty}\bigcup_{j\in\mathbb{Z}_+}\widetilde{B_{i,j}}\,, \tag{3.3}$$

where $\widetilde{B_{i,j}} := B(x_{i,j},\, 2(3/2)^{(i-i_0)/(n+\beta)}r_{i,j})$. To show that

$$\varphi\left(\left\{|\mu_\Omega^\rho(F_2)| > 2^{i_0}\right\},\, \frac{2^{i_0}}{\lambda}\right) \lesssim \sup_{i\in\mathbb{Z}}\left\{\sum_{j\in\mathbb{Z}_+}\varphi\left(B_{i,j},\, \frac{2^i}{\lambda}\right)\right\},$$

we cut $\{|\mu_\Omega^\rho(F_2)| > 2^{i_0}\}$ into A_{i_0} defined in (3.3) and $\{x\in (A_{i_0})^\complement :\ |\mu_\Omega^\rho(F_2)(x)| > 2^{i_0}\}$.

For A_{i_0}, from Lemma 2.2 with $\varphi\in\mathbb{A}_{p(1+\beta/n)}$ (since $\varphi^{1/(1-p)}\in\mathbb{A}_{p\beta/[n(1-p)]}$), and the uniformly lower type p property of φ, it follows that, for any $\lambda\in(0,\infty)$,

$$\begin{aligned}\varphi\left(A_{i_0},\, \frac{2^{i_0}}{\lambda}\right) &\le \sum_{i=i_0}^{\infty}\sum_{j\in\mathbb{Z}_+}\varphi\left(\widetilde{B_{i,j}},\, \frac{2^{i_0}}{\lambda}\right) \\ &\lesssim \sum_{i=i_0}^{\infty}\sum_{j\in\mathbb{Z}_+}\left(\frac{3}{2}\right)^{(i-i_0)p}\varphi\left(B_{i,j},\, \frac{2^{i_0}}{\lambda}\right) \\ &\lesssim \sum_{i=i_0}^{\infty}\sum_{j\in\mathbb{Z}_+}\left(\frac{3}{4}\right)^{(i-i_0)p}\varphi\left(B_{i,j},\, \frac{2^{i}}{\lambda}\right) \\ &\lesssim \sup_{i\in\mathbb{Z}}\left\{\sum_{j\in\mathbb{Z}_+}\varphi\left(B_{i,j},\, \frac{2^i}{\lambda}\right)\right\},\end{aligned} \tag{3.4}$$

which is also desired.

It remains to estimate $(A_{i_0})^\complement$. Applying the inequality $\|\cdot\|_{\ell^1}\le\|\cdot\|_{\ell^p}$ with $p\in(0,1)$, we conclude that, for any $\lambda\in(0,\infty)$,

$$\begin{aligned}&\varphi\left(\left\{x\in(A_{i_0})^\complement :\ |\mu_\Omega^\rho(F_2)(x)| > 2^{i_0}\right\},\, \frac{2^{i_0}}{\lambda}\right) \\ &\le 2^{-i_0p}\int_{(A_{i_0})^\complement}|\mu_\Omega^\rho(F_2)(x)|^p\,\varphi\left(x,\, \frac{2^{i_0}}{\lambda}\right)dx \\ &\le 2^{-i_0p}\sum_{i=i_0}^{\infty}\sum_{j\in\mathbb{Z}_+}\int_{(\widetilde{B_{i,j}})^\complement}\left|\mu_\Omega^\rho\left(b_{i,j}\right)(x)\right|^p\varphi\left(x,\, \frac{2^{i_0}}{\lambda}\right)dx.\end{aligned} \tag{3.5}$$

Below, we will give the estimate of integral

$$\mathrm{I} := \int_{(\widetilde{B_{i,j}})^{\complement}} \left|\mu_{\Omega}^{\rho}\left(b_{i,j}\right)(x)\right|^{p} \varphi\left(x, \frac{2^{i_0}}{\lambda}\right) dx.$$

For any $k \in \mathbb{N}$, let

$$E_k := \left(2^{k+1}\widetilde{B_{i,j}}\right) \setminus \left(2^{k}\widetilde{B_{i,j}}\right).$$

It follows from Hölder's inequality that, for any $\lambda \in (0, \infty)$,

$$\mathrm{I} \le \sum_{k=0}^{\infty} \left[\int_{E_k} \left|\mu_{\Omega}^{\rho}\left(b_{i,j}\right)(x)\right| dx\right]^{p} \left\{\int_{E_k} \left[\varphi\left(x, \frac{2^{i_0}}{\lambda}\right)\right]^{\frac{1}{1-p}} dx\right\}^{1-p}.$$

On the one hand, by Lemma 2.3 with $\varphi^{1/(1-p)} \in \mathbb{A}_d \subset \mathbb{A}_\infty$ (see (3.1)), we have $\varphi \in \mathbb{RH}_{1/(1-p)}$. Thus, thanks to Lemma 2.2 with $\varphi^{1/(1-p)} \in \mathbb{A}_d$, and $\varphi \in \mathbb{RH}_{1/(1-p)}$, it follows that, for any $\lambda \in (0, \infty)$,

$$\begin{aligned}\left\{\int_{E_k} \left[\varphi\left(x, \frac{2^{i_0}}{\lambda}\right)\right]^{\frac{1}{1-p}} dx\right\}^{1-p} &\le \left[\varphi^{\frac{1}{1-p}}\left(2^{k+1}\widetilde{B_{i,j}}, \frac{2^{i_0}}{\lambda}\right)\right]^{1-p} \\ &\lesssim \left[\varphi^{\frac{1}{1-p}}\left(B_{i,j}, \frac{2^{i_0}}{\lambda}\right)\right]^{1-p} \left[2^k\left(\frac{3}{2}\right)^{\frac{i-i_0}{n+\beta}}\right]^{nd(1-p)} \\ &\lesssim \left(r_{i,j}\right)^{-np} \varphi\left(B_{i,j}, \frac{2^{i_0}}{\lambda}\right) \left[2^k\left(\frac{3}{2}\right)^{\frac{i-i_0}{n+\beta}}\right]^{nd(1-p)}.\end{aligned}$$

On the other hand, since $d < p\beta/[n(1-p)]$, we may choose an $\widetilde{\alpha} \in (0, \alpha)$ such that $d < p\widetilde{\beta}/[n(1-p)]$, where $\widetilde{\beta} := \min\{1/2, \widetilde{\alpha}\}$. By the assumption $\Omega \in \mathrm{Din}_{\alpha}^{1}(S^{n-1})$, we know that Ω satisfies the $L^{1,\widetilde{\alpha}}$-Dini condition. Then Lemma 3.4(i) yields that

$$\int_{E_k} \left|\mu_{\Omega}^{\rho}\left(b_{i,j}\right)(x)\right| dx \lesssim 2^{i}\left(r_{i,j}\right)^{n} \left[2^k\left(\frac{3}{2}\right)^{\frac{i-i_0}{n+\beta}}\right]^{-\widetilde{\beta}}.$$

The above three inequalities give us that, for any $\lambda \in (0, \infty)$,

$$\mathrm{I} \lesssim 2^{ip}\varphi\left(B_{i,j}, \frac{2^{i_0}}{\lambda}\right) \sum_{k=0}^{\infty} \left[2^k\left(\frac{3}{2}\right)^{\frac{i-i_0}{n+\beta}}\right]^{nd-ndp-p\widetilde{\beta}}.$$

Substituting this inequality into (3.5) and using the uniformly lower type p property of φ, we obtain that, for any $\lambda \in (0, \infty)$,

$$\begin{aligned}&\varphi\left(\left\{x \in (A_{i_0})^{\complement} : \left|\mu_{\Omega}^{\rho}(F_2)(x)\right| > 2^{i_0}\right\}, \frac{2^{i_0}}{\lambda}\right) \qquad (3.6)\\ &\quad\lesssim 2^{-i_0 p} \sum_{i=i_0}^{\infty} \sum_{j\in\mathbb{Z}_+} 2^{ip}\varphi\left(B_{i,j}, \frac{2^{i_0}}{\lambda}\right) \sum_{k=0}^{\infty} \left[2^k\left(\frac{3}{2}\right)^{\frac{i-i_0}{n+\beta}}\right]^{nd-ndp-p\widetilde{\beta}}\end{aligned}$$

$$\lesssim \sup_{i\in\mathbb{Z}}\left\{\sum_{j\in\mathbb{Z}_+}\varphi\left(B_{i,j},\,\frac{2^i}{\lambda}\right)\right\}\sum_{i=i_0}^{\infty}\sum_{k=0}^{\infty}\left[2^k\left(\frac{3}{2}\right)^{\frac{i-i_0}{n+\beta}}\right]^{nd-ndp-p\widetilde{\beta}}$$
$$\sim \sup_{i\in\mathbb{Z}}\left\{\sum_{j\in\mathbb{Z}_+}\varphi\left(B_{i,j},\,\frac{2^i}{\lambda}\right)\right\},$$

where the last "$\sim$" is due to $d < p\widetilde{\beta}/[n(1-p)]$.

Finally, combining (3.2), (3.4) and (3.6), we obtain the desired inequality. This finishes the proof of Theorem 2.7. □

Proof of Theorem 2.8. We need only consider the case $p < 1$. The proof of the case $p = 1$ is similar and easier. Once we prove Lemma 3.4(ii), the proof of this theorem is quite similar to that of Theorem 2.7, the major change being the substitution of

$$\mathrm{I} \le \sum_{k=0}^{\infty}\left[\int_{E_k}|\mu_\Omega^\rho(b_{i,j})(x)|\,\varphi\left(x,\,\frac{2^{i_0}}{\lambda}\right)dx\right]^p\left[\int_{E_k}\varphi\left(x,\,\frac{2^{i_0}}{\lambda}\right)dx\right]^{1-p}$$

for

$$\mathrm{I} \le \sum_{k=0}^{\infty}\left[\int_{E_k}|\mu_\Omega^\rho(b_{i,j})(x)|\,dx\right]^p\left\{\int_{E_k}\left[\varphi\left(x,\,\frac{2^{i_0}}{\lambda}\right)\right]^{\frac{1}{1-p}}dx\right\}^{1-p}.$$

But to limit the length of this paper, we leave the details to the interested reader. □

Proof of Corollary 2.9. By $\varphi \in \mathbb{A}_{p(1+\beta/n)}$, we see that there exists some $d \in (1,\,\infty)$ such that $\varphi^d \in \mathbb{A}_{p(1+\beta/n)}$. For any $q \in (1,\,\infty)$, by the fact that $p > n/(n+\beta)$, some tedious manipulation yields $(p+p\beta/n-1/q)q' > p(1+\beta/n)$ and hence $\varphi^d \in \mathbb{A}_{(p+p\beta/n-1/q)q'}$. Thus, we may choose $q := d/(d-1)$ such that

$$\varphi^{q'} = \varphi^d \in \mathbb{A}_{(p+\beta/n-1/q)q'}$$

and hence Corollary 2.9 follows from Theorem 2.8. □

Proof of Theorem 2.10. Since the proof of Theorem 2.10 is similar to that of Theorem 2.7, we use the same notation as those used in the proof of Theorem 2.7. Rather that give a completed proof, we just give out the necessary modifications with respect to the estimate of $(A_{i_0})^\complement$. Reset

$$A_{i_0} := \bigcup_{i=i_0}^{\infty}\bigcup_{j\in\mathbb{Z}_+}\widetilde{B_{i,j}},$$

where $\widetilde{B_{i,j}} := B(x_{i,j},\,2(3/2)^{(i-i_0)/n}r_{i,j})$. For any $\lambda \in (0,\,\infty)$, we have

$$\varphi\left(\left\{x \in (A_{i_0})^\complement:\ |\mu_\Omega^\rho(F_2)(x)| > 2^{i_0}\right\},\,\frac{2^{i_0}}{\lambda}\right)$$
$$\le 2^{-i_0}\int_{(A_{i_0})^\complement}|\mu_\Omega^\rho(F_2)(x)|\,\varphi\left(x,\,\frac{2^{i_0}}{\lambda}\right)dx$$

$$\leq 2^{-i_0}\sum_{i=i_0}^{\infty}\sum_{j\in\mathbb{Z}_+}\int_{(\widetilde{B_{i,j}})^\complement}\left|\mu_\Omega^\rho\left(b_{i,j}\right)(x)\right|\varphi\left(x,\,\frac{2^{i_0}}{\lambda}\right)dx$$
$$=: 2^{-i_0}\sum_{i=i_0}^{\infty}\sum_{j\in\mathbb{Z}_+}\mathrm{I}.$$

For any $\lambda\in(0,\,\infty)$, let us write

$$\mathrm{I}\leq\int_{(\widetilde{B_{i,j}})^\complement}\left(\int_0^{|x-x_{i,j}|+r_{i,j}}\left|\int_{|x-y|\leq t}\frac{\Omega(x-y)}{|x-y|^{n-\rho}}b_{i,j}(y)\,dy\right|^2\frac{dt}{t^{2\rho+1}}\right)^{1/2}\varphi\left(x,\,\frac{2^{i_0}}{\lambda}\right)dx$$
$$+\int_{(\widetilde{B_{i,j}})^\complement}\left(\int_{|x-x_{i,j}|+r_{i,j}}^{\infty}\cdots\right)^{1/2}\varphi\left(x,\,\frac{2^{i_0}}{\lambda}\right)dx=:\mathrm{I}_1+\mathrm{I}_2.$$

Below, we will estimate I_1 and I_2, respectively.

For I_1, noticing that $x\in(\widetilde{B_{i,j}})^\complement$ and $y\in B_{i,j}$, we know that

$$|x-y|\sim|x-x_{i,j}|\sim|x-x_{i,j}|+r_{i,j},$$

which, associated with the mean value theorem, implies that

$$\left|\frac{1}{|x-y|^{2\rho}}-\frac{1}{(|x-x_{i,j}|+r_{i,j})^{2\rho}}\right|\lesssim\frac{r_{i,j}}{|x-y|^{2\rho+1}}.$$

From Minkowski's inequality for integrals, the above inequality and Hölder's inequality, it follows that, for any $\lambda\in(0,\,\infty)$,

$$\mathrm{I}_1\leq\int_{(\widetilde{B_{i,j}})^\complement}\left[\int_{B_{i,j}}\left|\frac{\Omega(x-y)}{|x-y|^{n-\rho}}b_{i,j}(y)\right|\left(\int_{|x-y|}^{|x-x_{i,j}|+r_{i,j}}\frac{dt}{t^{2\rho+1}}\right)^{1/2}dy\right]\varphi\left(x,\,\frac{2^{i_0}}{\lambda}\right)dx$$
$$\lesssim 2^i\left(r_{i,j}\right)^{1/2}\int_{(\widetilde{B_{i,j}})^\complement}\left(\int_{B_{i,j}}\frac{|\Omega(x-y)|}{|x-y|^{n+1/2}}\,dy\right)\varphi\left(x,\,\frac{2^{i_0}}{\lambda}\right)dx$$
$$\sim 2^i\left(r_{i,j}\right)^{1/2}\sum_{k=0}^{\infty}\int_{B_{i,j}}\left[\int_{E_k}\frac{|\Omega(x-y)|}{|x-y|^{n+1/2}}\varphi\left(x,\,\frac{2^{i_0}}{\lambda}\right)dx\right]dy$$
$$\lesssim 2^i\left(r_{i,j}\right)^{1/2}\sum_{k=0}^{\infty}\int_{B_{i,j}}\left(\int_{E_k}\frac{|\Omega(x-y)|^q}{|x-y|^{n+1/2}}\,dx\right)^{1/q}$$
$$\times\left(\int_{E_k}\frac{1}{|x-y|^{n+1/2}}\left[\varphi\left(x,\,\frac{2^{i_0}}{\lambda}\right)\right]^{q'}dx\right)^{1/q'}dy.$$

On the one hand, $x\in E_k$ and $y\in B_{i,j}$ imply that $\theta r_{i,j}<|x-y|<5\theta r_{i,j}$, where $\theta:=2^k(3/2)^{(i-i_0)/n}$. Therefore, we have

$$\left(\int_{E_k}\frac{|\Omega(x-y)|^q}{|x-y|^{n+1/2}}\,dx\right)^{1/q}\leq\left(\int_{\theta r_{i,j}<|z|<5\theta r_{i,j}}\frac{|\Omega(z)|^q}{|z|^{n+1/2}}\,dz\right)^{1/q}$$

$$= \left(\int_{S^{n-1}} \int_{\theta r_{i,j}}^{5\theta r_{i,j}} \frac{|\Omega(z')|^q}{u^{n+1/2}} u^{n-1} du d\sigma(z') \right)^{1/q}$$
$$\sim (\theta r_{i,j})^{-1/2q}.$$

On the other hand, according to Lemmas 2.2 and 2.3 with $\varphi^{q'} \in \mathbb{A}_1$, it follows that, for any $\lambda \in (0, \infty)$,

$$\left(\int_{E_k} \frac{1}{|x-y|^{n+1/2}} \left[\varphi\left(x, \frac{2^{i_0}}{\lambda}\right) \right]^{q'} dx \right)^{1/q'}$$
$$\sim (\theta r_{i,j})^{-n/q'-1/2q'} \left\{ \int_{E_k} \left[\varphi\left(x, \frac{2^{i_0}}{\lambda}\right) \right]^{q'} dx \right\}^{1/q'}$$
$$\lesssim (\theta r_{i,j})^{-n/q'-1/2q'} \theta^{n/q'} (r_{i,j})^{-n/q} \varphi\left(B_{i,j}, \frac{2^{i_0}}{\lambda}\right).$$

If we plug the above two inequalities back into I_1, we obtain that, for any $\lambda \in (0, \infty)$,

$$\mathrm{I}_1 \lesssim 2^i (r_{i,j})^{1/2} \sum_{k=0}^{\infty} \int_{B_{i,j}} (\theta r_{i,j})^{-1/2q} (\theta r_{i,j})^{-n/q'-1/2q'} \theta^{n/q'} (r_{i,j})^{-n/q} \varphi\left(B_{i,j}, \frac{2^{i_0}}{\lambda}\right) dy$$
$$\sim 2^i \sum_{k=0}^{\infty} \theta^{-1/2} \varphi\left(B_{i,j}, \frac{2^{i_0}}{\lambda}\right) \sim 2^i \left(\frac{2}{3}\right)^{\frac{i-i_0}{2n}} \varphi\left(B_{i,j}, \frac{2^{i_0}}{\lambda}\right).$$

For I_2, it is apparent from $t > |x-x_{i,j}|+r_{i,j}$ that $B_{i,j} \subset \{y \in \mathbb{R}^n : |x-y| \le t\}$. From this, the vanishing moments of $b_{i,j}$ and Minkowski's inequality for integrals, it follows that, for any $\lambda \in (0, \infty)$,

$$\mathrm{I}_2 = \int_{(\widetilde{B_{i,j}})^\complement} \left(\int_{|x-x_{i,j}|+r_{i,j}}^{\infty} \left| \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} b_{i,j}(y)\, dy \right|^2 \frac{dt}{t^{2\rho+1}} \right)^{1/2} \varphi\left(x, \frac{2^{i_0}}{\lambda}\right) dx$$
$$\le \int_{(\widetilde{B_{i,j}})^\complement} \left[\int_{B_{i,j}} \left| \frac{\Omega(x-y)}{|x-y|^{n-\rho}} - \frac{\Omega(x-x_{i,j})}{|x-x_{i,j}|^{n-\rho}} \right| |b_{i,j}(y)| \left(\int_{|x-x_{i,j}|}^{\infty} \frac{dt}{t^{2\rho+1}} \right)^{1/2} dy \right]$$
$$\times \varphi\left(x, \frac{2^{i_0}}{\lambda}\right) dx$$
$$= C \int_{(\widetilde{B_{i,j}})^\complement} \left(\int_{B_{i,j}} \left| \frac{\Omega(x-y)}{|x-y|^{n-\rho}|x-x_{i,j}|^\rho} - \frac{\Omega(x-x_{i,j})}{|x-x_{i,j}|^n} \right| |b_{i,j}(y)|\, dy \right)$$
$$\times \varphi\left(x, \frac{2^{i_0}}{\lambda}\right) dx$$
$$\le C \int_{(\widetilde{B_{i,j}})^\complement} \left(\int_{B_{i,j}} \left| \frac{\Omega(x-y)}{|x-y|^{n-\rho}|x-x_{i,j}|^\rho} - \frac{\Omega(x-y)}{|x-y|^n} \right| |b_{i,j}(y)|\, dy \right)$$

$$\times \varphi\left(x,\, \frac{2^{i_0}}{\lambda}\right) dx$$
$$+ C \int_{(\widetilde{B_{i,\,j}})^\complement} \left(\int_{B_{i,\,j}} \left| \frac{\Omega(x-y)}{|x-y|^n} - \frac{\Omega(x-x_{i,\,j})}{|x-x_{i,\,j}|^n} \right| |b_{i,\,j}(y)|\, dy \right) \varphi\left(x,\, \frac{2^{i_0}}{\lambda}\right) dx$$
$$=: C(\mathrm{I}_{21} + \mathrm{I}_{22}).$$

On the one hand, using the mean value theorem again, we obtain that, for any $x \in (\widetilde{B_{i,\,j}})^\complement$ and $y \in B_{i,\,j}$,

$$\left| \frac{1}{|x-x_{i,\,j}|^\rho} - \frac{1}{|x-y|^\rho} \right| \sim \frac{|y-x_{i,\,j}|}{|x-y|^{\rho+1}} \lesssim \frac{|y-x_{i,\,j}|^{1/2}}{|x-y|^{\rho+1/2}} \lesssim \frac{(r_{i,\,j})^{1/2}}{|x-y|^{\rho+1/2}},$$

which, together with the same argument as that used in I_1, implies that, for any $\lambda \in (0,\, \infty)$,

$$\mathrm{I}_{21} \lesssim 2^i \int_{(\widetilde{B_{i,\,j}})^\complement} \left(\int_{B_{i,\,j}} \frac{|\Omega(x-y)|}{|x-y|^{n-\rho}} \left| \frac{1}{|x-x_{i,\,j}|^\rho} - \frac{1}{|x-y|^\rho} \right| dy \right) \varphi\left(x,\, \frac{2^{i_0}}{\lambda}\right) dx$$
$$\lesssim 2^i\, (r_{i,\,j})^{1/2} \int_{(\widetilde{B_{i,\,j}})^\complement} \left(\int_{B_{i,\,j}} \frac{|\Omega(x-y)|}{|x-y|^{n+1/2}}\, dy \right) \varphi\left(x,\, \frac{2^{i_0}}{\lambda}\right) dx$$
$$\lesssim 2^i \left(\frac{2}{3}\right)^{\frac{i-i_0}{2n}} \varphi\left(B_{i,\,j},\, \frac{2^{i_0}}{\lambda}\right).$$

On the other hand, the condition (2.4) gives that, for any $\lambda \in (0,\, \infty)$,

$$\mathrm{I}_{22} \lesssim 2^i \int_{|y-x_{i,\,j}|<r_{i,\,j}} \int_{|x-x_{i,\,j}|>(3/2)^{(i-i_0)/n} r_{i,\,j}} \left| \frac{\Omega(x-y)}{|x-y|^n} - \frac{\Omega(x-x_{i,\,j})}{|x-x_{i,\,j}|^n} \right|$$
$$\times \varphi\left(x,\, \frac{2^{i_0}}{\lambda}\right) dx dy$$
$$\sim 2^i \int_{|y|<r_{i,\,j}} \int_{|x|>(3/2)^{(i-i_0)/n} r_{i,\,j}} \left| \frac{\Omega(x-y)}{|x-y|^n} - \frac{\Omega(x)}{|x|^n} \right| \varphi\left(x + x_{i,\,j},\, \frac{2^{i_0}}{\lambda}\right) dx dy$$
$$\lesssim 2^i \int_{|y|<r_{i,\,j}} \left(\frac{2}{3}\right)^{\frac{i-i_0}{n}} \varphi\left(y + x_{i,\,j},\, \frac{2^{i_0}}{\lambda}\right) dy \lesssim 2^i \left(\frac{2}{3}\right)^{\frac{i-i_0}{2n}} \varphi\left(B_{i,\,j},\, \frac{2^{i_0}}{\lambda}\right).$$

Collecting the estimates of I_1, I_{21} and I_{22}, we obtain that, for any $\lambda \in (0,\, \infty)$,

$$\mathrm{I} \lesssim \mathrm{I}_1 + \mathrm{I}_{21} + \mathrm{I}_{22} \lesssim 2^i \left(\frac{2}{3}\right)^{\frac{i-i_0}{2n}} \varphi\left(B_{i,\,j},\, \frac{2^{i_0}}{\lambda}\right)$$

and hence

$$\varphi\left(\left\{x \in (A_{i_0})^\complement :\ |\mu_\Omega^\rho(F_2)(x)| > 2^{i_0}\right\},\, \frac{2^{i_0}}{\lambda}\right) \le 2^{-i_0} \sum_{i=i_0}^\infty \sum_{j\in\mathbb{Z}_+} \mathrm{I}$$
$$\lesssim 2^{-i_0} \sum_{i=i_0}^\infty \sum_{j\in\mathbb{Z}_+} 2^i \left(\frac{2}{3}\right)^{\frac{i-i_0}{2n}} \varphi\left(B_{i,\,j},\, \frac{2^{i_0}}{\lambda}\right) \lesssim \sum_{i=i_0}^\infty \sum_{j\in\mathbb{Z}_+} \left(\frac{2}{3}\right)^{\frac{i-i_0}{2n}} \varphi\left(B_{i,\,j},\, \frac{2^{i}}{\lambda}\right)$$

$$\lesssim \sup_{i\in\mathbb{Z}} \left\{ \sum_{j\in\mathbb{Z}_+} \varphi\left(B_{i,j},\, \frac{2^i}{\lambda}\right) \right\},$$

where we used the uniformly lower type 1 property of φ in the second "$\lesssim$". The proof is completed. □

Remark 3.5. We should point out that, if φ is a growth function of uniformly lower type 1 and of uniformly upper type 1, then $WH^\varphi = WH^1_{\varphi(\cdot\,,1)}$ and $WL^\varphi = WL^1_{\varphi(\cdot\,,1)}$. In fact, there exists a positive constant C such that, for any $x \in \mathbb{R}^n$ and $t \in (0,\,\infty)$,

$$C^{-1}\, t\varphi(x,\,1) = C^{-1}\, t\varphi(x,\, t/t) \le \varphi(x,\,t) \le C\, t\varphi(x,\,1),$$

which implies that

$$\sup_{t\in(0,\,\infty)} \varphi\left(\{|f|>t\},\, t\right) \sim \sup_{t\in(0,\,\infty)} \varphi\left(\{|f|>t\},\, 1\right) t.$$

Thus, we have $WL^\varphi = WL^1_{\varphi(\cdot\,,1)}$. Analogously, $WH^\varphi = WH^1_{\varphi(\cdot\,,1)}$.

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References

1. J. Álvarez and M. Milman, *H^p continuity properties of Calderón-Zygmund-type operators*, J. Math. Anal. Appl. **118** (1986), no. 1, 63–79.
2. L. Diening, P. A. Hästö and S. Roudenko, *Function spaces of variable smoothness and integrability*, J. Funct. Anal. **256** (2009), no. 6, 1731–1768.
3. X. T. Duong and T. D. Tran, *Musielak-Orlicz Hardy spaces associated to operators satisfying Davies-Gaffney estimates and bounded holomorphic functional calculus*, J. Math. Soc. Japan **68** (2016), no. 1, 1–30.
4. C. L. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), no. 3-4, 137–193.
5. R. Fefferman and F. Soria, *The space weak H^1*, Studia Math. **85** (1986), no. 1, 1–16.
6. L. Hörmander, *Estimates for translation invariant operators in L^p spaces*, Acta Math. **104** (1960), 93–140.
7. K.-P. Ho, *Intrinsic atomic and molecular decompositions of Hardy-Musielak-Orlicz spaces*, Banach J. Math. Anal. **10** (2016), no. 3, 565–591.
8. R. Jiang, D. Yang and Y. Zhou, *Orlicz-Hardy spaces associated with operators*, Sci. China Ser. A **52** (2009), no. 5, 1042–1080.
9. R. Jiang and D. Yang, *New Orlicz-Hardy spaces associated with divergence form elliptic operators*, J. Funct. Anal. **258** (2010), no. 4, 1167–1224.
10. R. Jiang and D. Yang, *Predual spaces of Banach completions of Orlicz-Hardy spaces associated with operators*, J. Fourier Anal. Appl. **17** (2011), no. 1, 1–35.
11. R. Jiang and D. Yang, *Orlicz-Hardy spaces associated with operators satisfying Davies-Gaffney estimates*, Commun. Contemp. Math. **13** (2011), no. 2, 331–373.
12. L. D. Ky, *New Hardy spaces of Musielak-Orlicz type and boundedness of sublinear operators*, Integral Equations Operator Theory **78** (2014), no. 1, 115–150.
13. H. Liu, *The weak H^p spaces on homogeneous groups*, Harmonic analysis (Tianjin, 1988), 113–118, Lecture Notes in Math. **Vol. 1494**, Springer, Berlin, 1991.
14. C.-C. Lin and Y.-C. Lin, *H^p_ω-L^p_ω boundedness of Marcinkiewicz integral*, Integral Equations Operator Theory **58** (2007), no. 1, 87–98.

15. Bo Li, M. Liao and Baode Li, *Boundedness of Marcinkiewicz integrals with rough kernels on Musielak-Orlicz Hardy spaces*, J. Inequal. Appl. **2017** (2017), 228.
16. Y. Liang, J. Huang and D. Yang, *New real-variable characterizations of Musielak-Orlicz Hardy spaces*, J. Math. Anal. Appl. **395** (2012), no. 1, 413–428.
17. Y. Liang, E. Nakai, D. Yang and J. Zhang, *Boundedness of intrinsic Littlewood-Paley functions on Musielak-Orlicz Morrey and Campanato spaces*, Banach J. Math. Anal. **8** (2014), no. 1, 221–268.
18. Y. Liang, D. Yang and R. Jiang, Weak Musielak-Orlicz Hardy spaces and applications. Math. Nachr. **289** (2016), no. 5-6, 634–677.
19. F.-Y. Maeda, Y. Mizuta, T. Ohno and T. Shimomura, Duality of non-homogeneous central Herz-Morrey-Musielak-Orlicz spaces. Potential Anal. **47** (2017), no. 4, 447–460.
20. Y. Mizuta, E. Nakai, T. Ohno and T. Shimomura, Maximal functions, Riesz potentials and Sobolev embeddings on Musielak-Orlicz-Morrey spaces of variable exponent in $\mathbb{R}^n$. Rev. Mat. Complut. **25** (2012), no. 2, 413–434.
21. T. Quek and D. Yang, Calderón-Zygmund-type operators on weighted weak Hardy spaces over $\mathbb{R}^n$. Acta Math. Sin. (Engl. Ser.) **16** (2000), no. 1, 141–160.
22. E. M. Stein, On the functions of Littlewood-Paley, Lusin, and Marcinkiewicz. Trans. Amer. Math. Soc. **88** (1958), 430–466.
23. X. Shi and Y. Jiang, Weighted boundedness of parametric Marcinkiewicz integral and higher order commutator. Anal. Theory Appl. **25** (2009), no. 1, 25–39.
24. J.-O. Strömberg and A. Torchinsky, Weighted Hardy spaces. Lecture Notes in Math. **Vol. 1381**, Springer-Verlag, Berlin, 1989.
25. D. Yang, Y. Liang and L. D. Ky, Real-Variable Theory of Musielak-Orlicz Hardy Spaces, Lecture Notes in Math. **Vol. 2182**, Springer-Verlag, Cham, 2017.

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