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## ON MONOTONICITY OF SOME COMBINATORIAL SEQUENCES

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ABSTRACT. We confirm Sun's conjecture that  $(\sqrt[n+1]{F_{n+1}}/\sqrt[n]{F_n})_{n \geq 4}$  is strictly decreasing to the limit 1, where  $(F_n)_{n \geq 0}$  is the Fibonacci sequence. We also prove that the sequence  $(\sqrt[n+1]{D_{n+1}}/\sqrt[n]{D_n})_{n \geq 3}$  is strictly decreasing with limit 1, where  $D_n$  is the  $n$ -th derangement number. For  $m$ -th order harmonic numbers  $H_n^{(m)} = \sum_{k=1}^n 1/k^m$  ( $n = 1, 2, 3, \dots$ ), we show that  $(\sqrt[n+1]{H_{n+1}^{(m)}}/\sqrt[n]{H_n^{(m)}})_{n \geq 3}$  is strictly increasing.

### 1. INTRODUCTION

A challenging conjecture of Firoozbakht states that

$$\sqrt[n]{p_n} > \sqrt[n+1]{p_{n+1}} \quad \text{for every } n = 1, 2, 3, \dots,$$

where  $p_n$  denotes the  $n$ -th prime. Note that  $\lim_{n \rightarrow \infty} \sqrt[n]{p_n} = 1$  by the Prime Number Theorem. In [4] the second author conjectured further that for any integer  $n > 4$  we have the inequality

$$\frac{\sqrt[n+1]{p_{n+1}}}{\sqrt[n]{p_n}} < 1 - \frac{\log \log n}{2n^2},$$

which has been verified for all  $n \leq 3.5 \times 10^6$ . Motivated by this and [3], Sun [4, Conj. 2.12] conjectured that the sequence  $(\sqrt[n+1]{S_{n+1}}/\sqrt[n]{S_n})_{n \geq 7}$  is strictly increasing, where  $S_n$  is the sum of the first  $n$  positive squarefree numbers. Moreover, he also posed many conjectures on monotonicity of sequences of the type  $(\sqrt[n+1]{a_{n+1}}/\sqrt[n]{a_n})_{n \geq N}$  with  $(a_n)_{n \geq 1}$  a familiar combinatorial sequence of positive integers.

Throughout this paper, we set  $\mathbb{N} = \{0, 1, 2, \dots\}$  and  $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$ .

Let  $A$  and  $B$  be integers with  $\Delta = A^2 - 4B \neq 0$ . The Lucas sequence  $u_n = u_n(A, B)$  ( $n \in \mathbb{N}$ ) is defined as follows:

$$u_0 = 0, \quad u_1 = 1, \quad \text{and } u_{n+1} = Au_n - Bu_{n-1} \quad \text{for } n = 1, 2, 3, \dots$$

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It is well known that  $u_n = (\alpha^n - \beta^n)/(\alpha - \beta)$  for all  $n \in \mathbb{N}$ , where

$$\alpha = \frac{A + \sqrt{\Delta}}{2} \quad \text{and} \quad \beta = \frac{A - \sqrt{\Delta}}{2}$$

are the two roots of the characteristic equation  $x^2 - Ax + B = 0$ . The sequence  $F_n = u_n(1, -1)$  ( $n \in \mathbb{N}$ ) is the famous Fibonacci sequence, see [1, p. 46] for combinatorial interpretations of Fibonacci numbers.

Our first result is as follows.

**Theorem 1.1.** *Let  $A > 0$  and  $B \neq 0$  be integers with  $\Delta = A^2 - 4B > 0$ , and set  $u_n = u_n(A, B)$  for  $n \in \mathbb{N}$ . Then there exists an integer  $N > 0$  such that the sequence  $(\sqrt[n+1]{u_{n+1}}/\sqrt[n]{u_n})_{n \geq N}$  is strictly decreasing with limit 1. In the case  $A = 1$  and  $B = -1$  we may take  $N = 4$ .*

*Remark 1.1.* Under the condition of Theorem 1.1, by [2, Lemma 4] we have  $u_n < u_{n+1}$  unless  $A = n = 1$ . Note that the second assertion in Theorem 1.1 confirms a conjecture of the second author [4, Conj. 3.1] on the Fibonacci sequence.

For  $n \in \mathbb{Z}^+$  the  $n$ th derangement number  $D_n$  denotes the number of permutations  $\sigma$  of  $\{1, \dots, n\}$  with  $\sigma(i) = i$  for no  $i = 1, \dots, k$ . It has the following explicit expression (cf. [1, p. 67]):

$$D_n = \sum_{k=0}^n (-1)^k \frac{n!}{k!}.$$

Our second theorem is the following result conjectured by the second author [4, Conj. 3.3].

**Theorem 1.2.** *The sequence  $(\sqrt[n+1]{D_{n+1}}/\sqrt[n]{D_n})_{n \geq 3}$  is strictly decreasing with limit 1.*

*Remark 1.2.* It follows from Theorem 1.2 that the sequence  $(\sqrt[n]{D_n})_{n \geq 2}$  is strictly increasing.

For each  $m \in \mathbb{Z}^+$  those  $H_n^{(m)} = \sum_{k=1}^n 1/k^m$  ( $n \in \mathbb{Z}^+$ ) are called harmonic numbers of order  $m$ . The usual harmonic numbers (of order 1) are those rational numbers  $H_n = H_n^{(1)}$  ( $n = 1, 2, 3, \dots$ ).

Our following theorem confirms Conjecture 2.16 of Sun [4].

**Theorem 1.3.** *For any positive integer  $m$ , the sequence  $(\sqrt[n+1]{H_{n+1}^{(m)}}/\sqrt[n]{H_n^{(m)}})_{n \geq 3}$  is strictly increasing.*

We will prove Theorems 1.1–1.3 in Sections 2–4 respectively. It seems that there is no simple form for the generating function  $\sum_{n=0}^{\infty} \sqrt[n]{a_n} x^n$  with  $a_n = u_n, D_n, H_n^{(m)}$ . Note also that the set of those sequences  $(a_n)_{n \geq 1}$  of positive numbers with  $(\sqrt[n+1]{a_{n+1}} / \sqrt[n]{a_n})_{n \geq 1}$  decreasing (or increasing) is closed under multiplication.

## 2. PROOF OF THEOREM 1.1

*Proof of Theorem 1.1.* Set

$$\alpha = \frac{A + \sqrt{\Delta}}{2}, \quad \beta = \frac{A - \sqrt{\Delta}}{2}, \quad \text{and} \quad \gamma = \frac{\beta}{\alpha} = \frac{A - \sqrt{\Delta}}{A + \sqrt{\Delta}}.$$

Then

$$\log u_n = \log \frac{\alpha^n (1 - \gamma^n)}{\alpha - \beta} = n \log \alpha + \log(1 - \gamma^n) - \log \sqrt{\Delta}$$

for any  $n \in \mathbb{Z}^+$ . Note that

$$\log \frac{\sqrt[n+1]{u_{n+1}}}{\sqrt[n]{u_n}} = \frac{\log u_{n+1}}{n+1} - \frac{\log u_n}{n} = \frac{\log(1 - \gamma^{n+1})}{n+1} - \frac{\log(1 - \gamma^n)}{n} + \frac{\log \sqrt{\Delta}}{n(n+1)}.$$

Since

$$\lim_{n \rightarrow \infty} \frac{\log(1 - \gamma^n)}{n} = \lim_{n \rightarrow \infty} \frac{-\gamma^n}{n} = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{1}{n(n+1)} = 0,$$

we deduce that

$$\lim_{n \rightarrow \infty} \log \frac{\sqrt[n+1]{u_{n+1}}}{\sqrt[n]{u_n}} = 0, \quad \text{i.e.,} \quad \lim_{n \rightarrow \infty} \frac{\sqrt[n+1]{u_{n+1}}}{\sqrt[n]{u_n}} = 1.$$

For any  $n \in \mathbb{Z}^+$ , clearly

$$\begin{aligned} \frac{\sqrt[n+1]{u_{n+1}}}{\sqrt[n]{u_n}} > \frac{\sqrt[n+2]{u_{n+2}}}{\sqrt[n+1]{u_{n+1}}} &\iff \frac{\log u_{n+1}}{n+1} - \frac{\log u_n}{n} > \frac{\log u_{n+2}}{n+2} - \frac{\log u_{n+1}}{n+1} \\ &\iff \Delta_n := \frac{2 \log u_{n+1}}{n+1} - \frac{\log u_n}{n} - \frac{\log u_{n+2}}{n+2} > 0. \end{aligned}$$

Observe that

$$\begin{aligned} \Delta_n &= 2 \log \alpha + \frac{2 \log(1 - \gamma^{n+1})}{n+1} - \frac{2 \log \sqrt{\Delta}}{n+1} \\ &\quad - \left( 2 \log \alpha + \frac{\log(1 - \gamma^n)}{n} + \frac{\log(1 - \gamma^{n+2})}{n+2} - \frac{\log \sqrt{\Delta}}{n} - \frac{\log \sqrt{\Delta}}{n+2} \right) \\ &= \frac{\log \Delta}{n(n+1)(n+2)} + \frac{2}{n+1} \log(1 - \gamma^{n+1}) - \frac{\log(1 - \gamma^n)}{n} - \frac{\log(1 - \gamma^{n+2})}{n+2}. \end{aligned}$$

The function  $f(x) = \log(1+x)$  on the interval  $(-1, +\infty)$  is concave since  $f''(x) = -1/(x+1)^2 < 0$ . Note that  $|\gamma| < 1$ . If  $-|\gamma| \leq x \leq 0$ , then  $t = -x/|\gamma| \in [0, 1]$  and hence

$$f(x) = f(t(-|\gamma|) + (1-t)0) \geq tf(-|\gamma|) + (1-t)f(0) = qx,$$

where  $q = -\log(1-|\gamma|)/|\gamma| > 0$ . Note also that  $\log(1+x) < x$  for  $x > 0$ . So we have

$$\begin{aligned} \log(1-\gamma^{n+1}) &\geq \log(1-|\gamma|^{n+1}) \geq -q|\gamma|^{n+1}, \\ \log(1-\gamma^n) &\leq \log(1+|\gamma|^n) < |\gamma|^n, \\ \log(1-\gamma^{n+2}) &\leq \log(1+|\gamma|^{n+2}) < |\gamma|^{n+2}. \end{aligned}$$

Therefore

$$\Delta_n > \frac{\log \Delta}{n(n+1)(n+2)} - |\gamma|^n \left( \frac{2q|\gamma|}{n+1} + \frac{1}{n} + \frac{|\gamma|^2}{n+2} \right)$$

and hence

$$\begin{aligned} n(n+1)(n+2)\Delta_n \\ > \log \Delta - |\gamma|^n (2q|\gamma|n(n+2) + (n+1)(n+2) + |\gamma|^2n(n+1)). \end{aligned} \quad (1)$$

Since  $\lim_{n \rightarrow \infty} n^2|\gamma|^n = 0$ , when  $\Delta > 1$  we have  $\Delta_n > 0$  for large  $n$ .

Now it remains to consider the case  $\Delta = 1$ . Clearly  $\gamma = (A-1)/(A+1) > 0$ . Recall that  $\log(1-x) < -x$  for  $x \in (0, 1)$ . As

$$\frac{d}{dx}(\log(1-x) + x + x^2) = -\frac{1}{1-x} + 1 + 2x = \frac{x(1-2x)}{1-x} > 0 \quad \text{for } x \in (0, 0.5),$$

we have  $\log(1-x) + x + x^2 > \log 1 + 0 + 0^2 = 0$  for  $x \in (0, 0.5)$ . If  $n$  is large enough, then  $\gamma^n < 0.5$  and hence

$$\Delta_n = \frac{2}{n+1} \log(1-\gamma^{n+1}) - \frac{\log(1-\gamma^n)}{n} - \frac{\log(1-\gamma^{n+2})}{n+2} > w_n,$$

where

$$w_n := \frac{2}{n+1}(-\gamma^{n+1} - \gamma^{2n+2}) + \frac{\gamma^n}{n} + \frac{\gamma^{n+2}}{n+2}.$$

Note that

$$\lim_{n \rightarrow \infty} \frac{nw_n}{\gamma^n} = -2\gamma + 1 + \gamma^2 = (1-\gamma)^2 > 0.$$

So, for sufficiently large  $n$  we have  $\Delta_n > w_n > 0$ .

Now we show that  $n \geq 4$  suffices in the case  $A = 1$  and  $B = -1$ . Note that  $\Delta = 5$  and  $\gamma \approx -0.382$ . The sequence  $(|\gamma|^n(n+1)(n+2))_{n \geq 1}$  is

decreasing since

$$|\gamma| \frac{(n+2)(n+3)}{(n+1)(n+2)} < \frac{1}{2} \left(1 + \frac{2}{n+1}\right) \leq 1$$

for  $n \geq 1$ . It follows that  $|\gamma|^n(n+1)(n+2) \leq \gamma^6 \times 7 \times 8 < 1/3$  for  $n \geq 6$ . In view of (1), if  $n \geq 6$  then

$$\begin{aligned} n(n+1)(n+2)\Delta_n &> \log 5 - |\gamma|^n(n+1)(n+2)(2q|\gamma| + 1 + |\gamma|^2) \\ &> \log 5 - \frac{1 + 1 + \gamma^2}{3} > \log 5 - 1 > 0. \end{aligned}$$

It is easy to verify that  $\Delta_4$  and  $\Delta_5$  are positive. So  $(\sqrt[n+1]{F_{n+1}}/\sqrt[n]{F_n})_{n \geq 4}$  is strictly decreasing.

In view of the above, we have completed the proof of Theorem 1.1.  $\square$

### 3. PROOF OF THEOREM 1.2

*Proof of Theorem 1.2.* Let  $n \geq 3$ . It is well known that  $|D_n - n!/e| \leq 1/2$  (cf. [1, p.67]). Applying the Intermediate Value Theorem in calculus, we obtain

$$\left| \log D_n - \log \left( \frac{n!}{e} \right) \right| \leq \left| D_n - \frac{n!}{e} \right| \leq 0.5.$$

Set  $R_0(n) = \log D_n - \log n!$ . Then  $|R_0(n)| \leq 1.5$ .

Since  $\lim_{n \rightarrow \infty} R_0(n)/n = 0$ , we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \left( \frac{\log D_{n+1}}{n+1} - \frac{\log D_n}{n} \right) &= \lim_{n \rightarrow \infty} \left( \frac{\log(n+1)!}{n+1} - \frac{\log n!}{n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{n \log(n+1) + n \log n! - (n+1) \log n!}{n(n+1)} \\ &= \lim_{n \rightarrow \infty} \frac{n \log n + n \log(1 + 1/n) - \log n!}{n(n+1)} \\ &= \lim_{n \rightarrow \infty} \frac{\log(n^n/n!)}{n(n+1)}. \end{aligned}$$

As  $n! \sim \sqrt{2\pi n}(n/e)^n$  (i.e.,  $\lim_{n \rightarrow \infty} n!/(\sqrt{2\pi n}(n/e)^n) = 1$ ) by Stirling's formula, we have  $\log(n^n/n!) \sim n$  and hence

$$\lim_{n \rightarrow \infty} \left( \frac{\log D_{n+1}}{n+1} - \frac{\log D_n}{n} \right) = 0.$$

Thus  $\lim_{n \rightarrow \infty} \sqrt[n+1]{D_{n+1}}/\sqrt[n]{D_n} = 1$ .

From the known identity  $D_n/n! = \sum_{k=0}^n (-1)^k/k!$ , we have the recurrence  $D_n = nD_{n-1} + (-1)^n$  for  $n > 1$ . Thus, if  $n \geq 3$  then

$$R_0(n) - R_0(n-1) = \log \frac{D_n}{n!} - \log \frac{D_{n-1}}{(n-1)!} = \log \frac{D_n}{nD_{n-1}} = \log \left( 1 + \frac{(-1)^n}{nD_{n-1}} \right).$$

Fix  $n \geq 4$ . If  $n$  is even, then

$$0 < R_0(n) - R_0(n-1) = \log \left( 1 + \frac{1}{nD_{n-1}} \right) < \frac{1}{nD_{n-1}} = \frac{1}{D_n - 1} \leq \frac{3}{D_n + 0.5}.$$

If  $n$  is odd, then

$$0 > R_0(n) - R_0(n-1) = \log \left( 1 - \frac{1}{nD_{n-1}} \right) > \frac{-2}{nD_{n-1}} = \frac{-2}{D_n + 1} \geq \frac{-3}{D_n + 0.5}$$

since  $\log(1-x) + 2x > 0$  for  $x \in (0, 0.5)$ . So

$$|R_0(n) - R_0(n-1)| < \frac{3}{D_n + 0.5} \leq \frac{3e}{n!}$$

and hence

$$\left| \frac{R_0(n-1) - R_0(n)}{n-1} \right| < \frac{3e}{n!(n-1)} \leq \frac{3e}{n(n-1)(n+1)}.$$

Similarly, we also have

$$\left| \frac{R_0(n+1) - R_0(n)}{n+1} \right| < \frac{3e}{n!(n+1)} \leq \frac{3e}{n(n-1)(n+1)}.$$

Therefore,

$$\begin{aligned} & \left| \frac{R_0(n+1)}{n+1} - \frac{2R_0(n)}{n} + \frac{R_0(n-1)}{n-1} - \frac{2R_0(n)}{n(n-1)(n+1)} \right| \\ &= \left| \frac{R_0(n+1) - R_0(n)}{n+1} + \frac{R_0(n-1) - R_0(n)}{n-1} \right| \leq \frac{6e}{n(n-1)(n+1)} \end{aligned}$$

and hence

$$\left| \frac{R_0(n+1)}{n+1} - \frac{2R_0(n)}{n} + \frac{R_0(n-1)}{n-1} \right| \leq \frac{2|R_0(n)| + 6e}{n(n-1)(n+1)} \leq \frac{6e+3}{n(n-1)(n+1)}.$$

Thus  $|R_1(n)| \leq 6e+3$ , where

$$R_1(n) := n(n-1)(n+1) \left( \frac{R_0(n+1)}{n+1} - \frac{2R_0(n)}{n} + \frac{R_0(n-1)}{n-1} \right).$$

Since

$$\begin{aligned}
\log((n-1)!) &= \sum_{k=1}^{n-1} \int_k^{k+1} (\log k) dx \\
&< \sum_{k=1}^{n-1} \int_k^{k+1} \log x dx = \int_1^n \log x dx = n \log n - n + 1 \\
&< \sum_{k=1}^{n-1} \int_k^{k+1} (\log(k+1)) dx = \log(n!),
\end{aligned}$$

we have

$$n \log n - n < \log(n!) = \log((n-1)!) + \log n < n \log n - n + \log n + 1$$

and so  $\log(n!) = n \log n - n + R_2(n)$  with  $|R_2(n)| < \log n + 1$ .

Observe that

$$\begin{aligned}
&\frac{\log D_{n+1}}{n+1} - \frac{2}{n} \log D_n + \frac{\log D_{n-1}}{n-1} \\
&= \frac{\log(n+1)!}{n+1} - \frac{2 \log n!}{n} + \frac{\log(n-1)!}{n-1} + \frac{R_1(n)}{(n-1)n(n+1)} \\
&= \frac{2 \log n!}{(n-1)n(n+1)} - \frac{\log n}{n-1} + \frac{\log(n+1)}{n+1} + \frac{R_1(n)}{(n-1)n(n+1)} \\
&= -\frac{2n}{(n-1)n(n+1)} + \frac{\log(n+1) - \log(n)}{n+1} + \frac{2R_2(n) + R_1(n)}{(n-1)n(n+1)} \\
&\leq -\frac{2n}{(n-1)n(n+1)} + \frac{n-1}{(n-1)n(n+1)} + \frac{2R_2(n) + R_1(n)}{(n-1)n(n+1)} \\
&= -\frac{n+1 - 2R_2(n) - R_1(n)}{(n-1)n(n+1)}.
\end{aligned}$$

If  $n \geq 27$ , then  $n+1 - 2R_2(n) - R_1(n) > n - 2 \log n - 1 - 6e - 3 > 0$ , and hence we get

$$\log \frac{\sqrt[n]{D_n}}{\sqrt[n-1]{D_{n-1}}} > \log \frac{\sqrt[n+1]{D_{n+1}}}{\sqrt[n]{D_n}}.$$

By a direct check via computer, the last inequality also holds for  $n = 4, \dots, 26$ . Therefore, the sequence  $(\sqrt[n+1]{D_{n+1}}/\sqrt[n]{D_n})_{n \geq 3}$  is strictly decreasing. This ends the proof.  $\square$

#### 4. PROOF OF THEOREM 1.3

**Lemma 4.1.** *For  $x > 0$  we have*

$$\log(1+x) > x - \frac{x^2}{2}. \quad (2)$$

*Proof.* As

$$\frac{d}{dx} \left( \log(1+x) - x + \frac{x^2}{2} \right) = \frac{x^2}{1+x},$$

we see that  $\log(1+x) - x + x^2/2 > \log 1 - 0 + 0^2/2 = 0$  for any  $x > 0$ .  $\square$

**Lemma 4.2.** *Let  $m, n \in \mathbb{Z}^+$  with  $n \geq 3$ . If  $m \geq 11$  or  $n \geq 30$ , then*

$$H_n^{(m)} \log H_n^{(m)} > 4 \left( \frac{2}{n+2} \right)^{m-1}. \quad (3)$$

*Proof.* Recall that  $H_n$  refers to  $H_n^{(1)}$ . If  $n \geq 30$ , then  $H_n \log H_n \geq H_{30} \log H_{30} > 4$  and hence (3) holds for  $m = 1$ .

Below we assume that  $m \geq 2$ . As  $n \geq 3$ , we have  $H_n^{(m)} \log H_n^{(m)} \geq H_3^{(m)} \log H_3^{(m)}$ . So it suffices to show that

$$\left( \frac{n+2}{2} \right)^{m-1} H_3^{(m)} \log H_3^{(m)} > 4 \quad (4)$$

whenever  $m \geq 11$  or  $n \geq 30$ . By Lemma 4.1,

$$\begin{aligned} \log H_3^{(m)} &= \log(1 + 2^{-m} + 3^{-m}) > 2^{-m} + 3^{-m} - \frac{(2^{-m} + 3^{-m})^2}{2} \\ &> 2^{-m} + 3^{-m} - \frac{(2^{1-m})^2}{2} = \frac{1}{2^m} + \frac{1}{3^m} - \frac{2}{4^m}. \end{aligned}$$

If  $m \geq 3$ , then  $(4/3)^m \geq (4/3)^3 > 2$  and hence  $\log H_3^{(m)} > 1/2^m$ . Note also that  $H_3^{(2)} \log H_3^{(2)} > 1/4$ . So we always have

$$H_3^{(m)} \log H_3^{(m)} > \frac{1}{2^m}.$$

If  $m \geq 11$ , then  $1.25^m \geq 1.25^{11} > 10$  and hence

$$\frac{1}{2^m} > \frac{4}{2.5^{m-1}} \geq \frac{4}{((n+2)/2)^{m-1}},$$

therefore (4) holds. When  $n \geq 30$ , we have

$$\frac{1}{2^m} \geq \frac{1}{2^{4m-6}} = \frac{4}{16^{m-1}} \geq 4 \left( \frac{2}{n+2} \right)^{m-1}$$

and hence (4) also holds.  $\square$

*Proof of Theorem 1.3.* Let  $m \geq 1$  and  $n \geq 3$ . Set

$$\Delta_n(m) := \log \frac{\sqrt[n+1]{H_{n+1}^{(m)}}}{\sqrt[n]{H_n^{(m)}}} - \log \frac{\sqrt[n+2]{H_{n+2}^{(m)}}}{\sqrt[n+1]{H_{n+1}^{(m)}}} = \frac{2}{n+1} \log H_{n+1}^{(m)} - \frac{\log H_n^{(m)}}{n} - \frac{\log H_{n+2}^{(m)}}{n+2}.$$

It suffices to show that  $\Delta_n(m) < 0$ . This can be easily verified by computer if  $m \in \{1, \dots, 10\}$  and  $n \in \{3, \dots, 29\}$ .



Below we assume that  $m \geq 11$  or  $n \geq 30$ . Recall (2) and the known fact that  $\log(1+x) < x$  for  $x > 0$ . We clearly have

$$\log \frac{H_{n+1}^{(m)}}{H_n^{(m)}} = \log \left( 1 + \frac{1}{(n+1)^m H_n^{(m)}} \right) < \frac{1}{(n+1)^m H_n^{(m)}}$$

and

$$\log \frac{H_{n+2}^{(m)}}{H_n^{(m)}} > \log \left( 1 + \frac{2}{(n+2)^m H_n^{(m)}} \right) > \frac{2}{(n+2)^m H_n^{(m)}} - \frac{2}{(n+2)^{2m} (H_n^{(m)})^2}.$$

It follows that

$$\begin{aligned} \Delta_n(m) &= \left( \frac{2}{n+1} - \frac{1}{n} - \frac{1}{n+2} \right) \log H_n^{(m)} \\ &\quad + \frac{2}{n+1} \log \frac{H_{n+1}^{(m)}}{H_n^{(m)}} - \frac{1}{n+2} \log \frac{H_{n+2}^{(m)}}{H_n^{(m)}} \\ &< \frac{-2 \log H_n^{(m)}}{n(n+1)(n+2)} + \frac{2}{(n+1)^{m+1} H_n^{(m)}} \\ &\quad - \frac{2}{(n+2)^{m+1} H_n^{(m)}} + \frac{2}{(n+2)^{2m+1} (H_n^{(m)})^2}. \end{aligned}$$

Since  $(n+2)^{m+1} = \sum_{k=0}^{m+1} \binom{m+1}{k} (n+1)^k$  by the binomial theorem, we obtain

$$\begin{aligned} \Delta_n(m) &\leq \frac{-2 \log H_n^{(m)}}{n(n+1)(n+2)} + \frac{2 \sum_{k=0}^m \binom{m+1}{k} (n+1)^k}{(n+1)^{m+1} (n+2)^{m+1} H_n^{(m)}} + \frac{2}{(n+2)^{m+2} H_n^{(m)}} \\ &< \frac{-2 \log H_n^{(m)}}{n(n+1)(n+2)} + \frac{2(n+1)^m \sum_{k=0}^m \binom{m+1}{k}}{(n+1)^{m+1} (n+2)^{m+1} H_n^{(m)}} + \frac{2}{(n+1)(n+2)^{m+1} H_n^{(m)}} \\ &= \frac{-2 \log H_n^{(m)}}{n(n+1)(n+2)} + \frac{2(2^{m+1} - 1) + 2}{(n+1)(n+2)^{m+1} H_n^{(m)}}. \end{aligned}$$

Thus

$$\begin{aligned} n(n+1)(n+2)\Delta_n(m) \frac{H_n^{(m)}}{2} &< -H_n^{(m)} \log H_n^{(m)} + \frac{2^{m+1}n}{(n+2)^m} \\ &< 4 \left( \frac{2}{n+2} \right)^{m-1} - H_n^{(m)} \log H_n^{(m)}. \end{aligned}$$

Applying (3) we find that  $\Delta_n(m) < 0$ . This completes the proof.  $\square$

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