Collapse for the Generalized Three-Dimensional Nonlocal Nonlinear Schrödinger Equations *

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Abstract: Finite time collapse of the solution to the generalized three-dimensional nonlocal nonlinear Schrödinger equations is studied. This result is achieved through establishing some a priori estimates for the nonlocal terms and introducing a type of virial identities.

Key words: Nonlocal Nonlinear Schrödinger Equations, Collapse, Virial identity

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1. Introduction

In the present paper, we study the finite time collapse of the solution to the generalized nonlocal nonlinear Schrödinger equations in $\mathbb{R}^3$:

$$\begin{aligned}
&i\partial_t E_1 + \Delta E_1 + \left(|E_1|^2 + ||E_2|^2 + |E_3|^2\right) E_1 \\
&\quad -E_2\mathcal{F}^{-1}\left\{\tfrac{\eta}{|\xi|^2+\delta}\left[\xi_1\xi_3\mathcal{F}(E_2\overline{E_3} - \overline{E_2}E_3) - (\xi_1^2+\xi_2^2)\mathcal{F}(E_1\overline{E_2} - \overline{E_1}E_2)\right.\right. \\
&\quad \left.\left. +\xi_2\xi_3\mathcal{F}(\overline{E_1}E_3 - E_1\overline{E_3})\right]\right\} + E_3\mathcal{F}^{-1}\left\{\tfrac{\eta}{|\xi|^2+\delta}\left[\xi_2\xi_3\mathcal{F}(E_1\overline{E_2} - \overline{E_1}E_2)\right.\right. \\
&\quad \left.\left. -\ (\xi_1^2+\xi_3^2)\mathcal{F}(\overline{E_1}E_3 - E_1\overline{E_3}) + \xi_1\xi_2\mathcal{F}(E_2\overline{E_3} - \overline{E_2}E_3)\right]\right\} = 0,
\end{aligned} \tag{1.1}$$

$$\begin{aligned}
&i\partial_t E_2 + \Delta E_2 + \left(|E_1|^2 + |E_2|^2 + |E_3|^2\right) E_2 \\
&\quad -E_3\mathcal{F}^{-1}\left\{\tfrac{\eta}{|\xi|^2+\delta}\left[\xi_1\xi_2\mathcal{F}(\overline{E_1}E_3 - E_1\overline{E_3}) - (\xi_2^2+\xi_3^2)\mathcal{F}(E_2\overline{E_3} - \overline{E_2}E_3)\right.\right. \\
&\quad \left.\left. +\xi_1\xi_3\mathcal{F}(E_1\overline{E_2} - \overline{E_1}E_2)\right]\right\} + E_1\mathcal{F}^{-1}\left\{\tfrac{\eta}{|\xi|^2+\delta}\left[\xi_1\xi_3\mathcal{F}(E_2\overline{E_3} - \overline{E_2}E_3)\right.\right. \\
&\quad \left.\left. -(\xi_1^2+\xi_2^2)\mathcal{F}(E_1\overline{E_2} - \overline{E_1}E_2) + \xi_2\xi_3\mathcal{F}(\overline{E_1}E_3 - E_1\overline{E_3})\right]\right\} = 0,
\end{aligned} \tag{1.2}$$

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$$
\begin{aligned}
&i\partial_t E_3 + \Delta E_3 + \left(|E_1|^2 + ||E_2|^2 + |E_3|^2\right) E_3 \\
&\quad -E_1\mathcal{F}^{-1}\left\{\tfrac{\eta}{|\xi|^2+\delta}\left[\xi_2\xi_3\mathcal{F}(E_1\overline{E_2} - \overline{E_1}E_2) - (\xi_1^2+\xi_3^2)\mathcal{F}(\overline{E_1}E_3 - E_1\overline{E_3})\right.\right. \\
&\quad \left.\left.+\xi_1\xi_2\mathcal{F}(E_2\overline{E_3} - \overline{E_2}E_3)\right]\right\} + E_2\mathcal{F}^{-1}\left\{\tfrac{\eta}{|\xi|^2+\delta}\left[\xi_1\xi_2\mathcal{F}(\overline{E_1}E_3 - E_1\overline{E_3})\right.\right. \\
&\quad \left.\left. -\ (\xi_2^2+\xi_3^2)\mathcal{F}(E_2\overline{E_3} - \overline{E_2}E_3) + \xi_1\xi_3\mathcal{F}(E_1\overline{E_2} - \overline{E_1}E_2)\right]\right\} = 0,
\end{aligned}
\tag{1.3}
$$

along with the initial data

$$
E_1(0,x) = E_{10}(x), E_2(0,x) = E_{20}(x), E_3(0,x) = E_{30}(x). \tag{1.4}
$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and the Fourier inverse transform, respectively (see[10, 11, 12, 13]), $\eta > 0$, $\delta \geq 0$, $(E_1, E_2, E_3)(t,x)$ are complex vector-valued functions from $\mathbb{R}^+ \times \mathbb{R}^3$ into $\mathbb{C}^3$.

Equations (1.1)-(1.3) arise in the infinite ion acoustic speed limit of the self-generated magnetic field in a cold plasma, (E_1, E_2, E_3) denote the slowly varying complex amplitudes of the high-frequency electric field [3, 9, 19, 20].

In the case of the nonlinear Schrödinger equation without any nonlocal term

$$
i\varphi_t + \theta\Delta\varphi = \alpha|\varphi|^{p-1}\varphi, \quad \varphi(0,x) = \varphi_0(x), \tag{$S-1$}
$$

where $\varphi = \varphi(t,x) : R^+ \times \mathbb{R}^3 \to \mathbb{C}$, θ, α are real parameters and $p > 1$. Equation (S-1) may describe the propagation of a narrow electromagnetic beam through a nonlinear medium or electromagnetic (Langmuir) wave in a plasma (see[1]). For (S-1) there have been many investigations. Ginibre and Velo [6] studied the local and global existence of the solutions in the energy class. Glassey [7], Ozawa and Tsutsumi [15, 16] established the finite time collapse properties of the solutions to the Cauchy problem (S-1). In the study of the Cauchy problem (1.1)-(1.4), we still concentrate our attentions on the finite time collapse property of the solutions. To our best knowledge, for the nonlocal nonlinear Schrödinger equations, there have been few researches on the collapse property except in our previous papers [4, 5], we studied the similar problem for a simplified version for the nonlocal nonlinear Schrödinger equations, in which $\mathbf{E} = (E_1, E_2, 0)$, $\xi = (\xi_1, \xi_2, 0)$. The main difficulty for searching for the finite time collapse property here is to deal with the nonlocal terms in the equations (1.1)-(1.3). Like the nonlinear Schrödinger equations without any nonlocal term, we must set up the local wellposedness of the Cauchy problem (1.1)-(1.4), derive the conservation laws of mass and energy, as well as establish some suitable virial type identities.

2 Preliminaries

In this section, we first establish the local wellposedness for the Cauchy problem (1.1)-(1.4), and then conclude the conservation identities of mass and energy.

2.1 Local Wellposedness

The Cauchy problem (1.1)-(1.4) can written as the integral equations:

$$E_i(t) = U(t)E_{i0} + i\int_0^t U(t-t^{'})\left[\left(|E_1|^2+|E_2|^2+|E_3|^2\right)E_i + K_i(E_1,E_2,E_3)\right](t^{'})dt^{'}, \tag{2.1}$$

where $i = 1,2,3$, $U(t) = e^{it\triangle}$ is the unitary semigroup generated by the free Schrödinger Equation $iE_t + \Delta E = 0$ in $H^s(\mathbb{R}^3)(with\ s \in \mathbb{R})$, $K_i(E_1,E_2,E_3)$ have the following expressions:

$$\begin{aligned}
&K_1(E_1,E_2,E_3)\\
&\quad = -E_2\mathcal{F}^{-1}\left\{\tfrac{\eta}{|\xi|^2+\delta}\left[\xi_1\xi_3\mathcal{F}(E_2\overline{E_3}-\overline{E_2}E_3)-(\xi_1^2+\xi_2^2)\mathcal{F}(E_1\overline{E_2}-\overline{E_1}E_2)\right.\right.\\
&\qquad \left.\left.+\xi_2\xi_3\mathcal{F}(\overline{E_1}E_3-E_1\overline{E_3})\right]\right\} + E_3\mathcal{F}^{-1}\left\{\tfrac{\eta}{|\xi|^2+\delta}\left[\xi_2\xi_3\mathcal{F}(E_1\overline{E_2}-\overline{E_1}E_2)\right.\right.\\
&\qquad \left.\left.-\ (\xi_1^2+\xi_3^2)\mathcal{F}(\overline{E_1}E_3-E_1\overline{E_3})+\xi_1\xi_2\mathcal{F}(E_2\overline{E_3}-\overline{E_2}E_3)\right]\right\},
\end{aligned} \tag{K-1}$$

$$\begin{aligned}
&K_2(E_1,E_2,E_3)\\
&\quad = -E_3\mathcal{F}^{-1}\left\{\tfrac{\eta}{|\xi|^2+\delta}\left[\xi_1\xi_2\mathcal{F}(\overline{E_1}E_3-E_1\overline{E_3})-(\xi_2^2+\xi_3^2)\mathcal{F}(E_2\overline{E_3}-\overline{E_2}E_3)\right.\right.\\
&\qquad \left.\left.+\xi_1\xi_3\mathcal{F}(E_1\overline{E_2}-\overline{E_1}E_2)\right]\right\} + E_1\mathcal{F}^{-1}\left\{\tfrac{\eta}{|\xi|^2+\delta}\left[\xi_1\xi_3\mathcal{F}(E_2\overline{E_3}-\overline{E_2}E_3)\right.\right.\\
&\qquad \left.\left.-(\xi_1^2+\xi_2^2)\mathcal{F}(E_1\overline{E_2}-\overline{E_1}E_2)+\xi_2\xi_3\mathcal{F}(\overline{E_1}E_3-E_1\overline{E_3})\right]\right\},
\end{aligned} \tag{K-2}$$

$$\begin{aligned}
&K_3(E_1,E_2,E_3)\\
&\quad = -E_1\mathcal{F}^{-1}\left\{\tfrac{\eta}{|\xi|^2+\delta}\left[\xi_2\xi_3\mathcal{F}(E_1\overline{E_2}-\overline{E_1}E_2)-(\xi_1^2+\xi_3^2)\mathcal{F}(\overline{E_1}E_3-E_1\overline{E_3})\right.\right.\\
&\qquad \left.\left.+\xi_1\xi_2\mathcal{F}(E_2\overline{E_3}-\overline{E_2}E_3)\right]\right\} + E_2\mathcal{F}^{-1}\left\{\tfrac{\eta}{|\xi|^2+\delta}\left[\xi_1\xi_2\mathcal{F}(\overline{E_1}E_3-E_1\overline{E_3})\right.\right.\\
&\qquad \left.\left.-\ (\xi_2^2+\xi_3^2)\mathcal{F}(E_2\overline{E_3}-\overline{E_2}E_3)+\xi_1\xi_3\mathcal{F}(E_1\overline{E_2}-\overline{E_1}E_2)\right]\right\}.
\end{aligned} \tag{K-3}$$

Let (E_1^j,E_2^j,E_3^j) $(j=1,2)$ solve the cauchy problem (1.1)-(1.4). We mainly here calculate these terms $K_i(E_1^1,E_2^1,E_3^1)-K_i(E_1^2,E_2^2,E_3^2)$ $(i=1,2,3)$. We first make a calculation as follows.

$$\begin{aligned}
&K_1(E_1^1,E_2^1,E_3^1)-K_1(E_1^2,E_2^2,E_3^2)\\
&\quad = -E_2^1\mathcal{F}^{-1}\left\{\tfrac{\eta}{|\xi|^2+\delta}\left[\xi_1\xi_3\mathcal{F}(E_2^1\overline{E_3^1}-\overline{E_2^1}E_3^1)-(\xi_1^2+\xi_2^2)\mathcal{F}(E_1^1\overline{E_2^1}-\overline{E_1^1}E_2^1)\right.\right.\\
&\qquad \left.\left.+\xi_2\xi_3\mathcal{F}(\overline{E_1^1}E_3^1-E_1^1\overline{E_3^1})\right]\right\} + E_3^1\mathcal{F}^{-1}\left\{\tfrac{\eta}{|\xi|^2+\delta}\left[\xi_2\xi_3\mathcal{F}(E_1^1\overline{E_2^1}-\overline{E_1^1}E_2^1)\right.\right.\\
&\qquad \left.\left.-\ (\xi_1^2+\xi_3^2)\mathcal{F}(\overline{E_1^1}E_3^1-E_1^1\overline{E_3^1})+\xi_1\xi_2\mathcal{F}(E_2^1\overline{E_3^1}-\overline{E_2^1}E_3^1)\right]\right\}\\
&\quad -\left\{-E_2^2\mathcal{F}^{-1}\left\{\tfrac{\eta}{|\xi|^2+\delta}\left[\xi_1\xi_3\mathcal{F}(E_2^2\overline{E_3^2}-\overline{E_2^2}E_3^2)-(\xi_1^2+\xi_2^2)\mathcal{F}(E_1^2\overline{E_2^2}-\overline{E_1^2}E_2^2)\right.\right.\right.\\
&\qquad \left.\left.+\xi_2\xi_3\mathcal{F}(\overline{E_1^2}E_3^2-E_1^2\overline{E_3^2})\right]\right\} + E_3^2\mathcal{F}^{-1}\left\{\tfrac{\eta}{|\xi|^2+\delta}\left[\xi_2\xi_3\mathcal{F}(E_1^2\overline{E_2^2}-\overline{E_1^2}E_2^2)\right.\right.\\
&\qquad \left.\left.\left.-\ (\xi_1^2+\xi_3^2)\mathcal{F}(\overline{E_1^2}E_3^2-E_1^2\overline{E_3^2})+\xi_1\xi_2\mathcal{F}(E_2^2\overline{E_3^2}-\overline{E_2^2}E_3^2)\right]\right\}\right\}.
\end{aligned} \tag{2.2}$$

Now we can check

$$
\begin{aligned}
&-E_2^1\mathcal{F}^{-1}\left[\tfrac{\eta\xi_1\xi_3}{|\xi|^2+\delta}\mathcal{F}(E_2^1\overline{E_3^1}-\overline{E_2^1}E_3^1)\right]-\left\{-E_2^2\mathcal{F}^{-1}\left[\tfrac{\eta\xi_1\xi_3}{|\xi|^2+\delta}\mathcal{F}(E_2^2\overline{E_3^2}-\overline{E_2^2}E_3^2)\right]\right\}\\
&\quad=(E_2^1-E_2^2)\mathcal{F}^{-1}\left[\tfrac{\eta\xi_1\xi_3}{|\xi|^2+\delta}\mathcal{F}(E_2^1\overline{E_3^1}-\overline{E_2^1}E_3^1)\right]+E_2^2\mathcal{F}^{-1}\left\{\tfrac{\eta\xi_1\xi_3}{|\xi|^2+\delta}\mathcal{F}\left[(E_2^2-E_2^1)\overline{E_3^2}\right.\right.\\
&\qquad\left.\left.+\,E_2^1(\overline{E_3^2}-\overline{E_3^1})+(\overline{E_2^1}-\overline{E_2^2})E_3^1+\overline{E_2^2}(E_3^1-E_3^2)\right]\right\}.
\end{aligned}
\tag{2.3}
$$

Making the similar estimates to (2.3), we can establish the similar forms for the other terms of $K_i(E_1^1,E_2^1,E_3^1)-K_i(E_1^2,E_2^2,E_3^2)$, $(i=1,2,3)$.

Since by $\eta>0$ and $\delta\geq 0$, $\left|\frac{\eta\xi_i\xi_k}{|\xi|^2+\delta}\right|\leq\eta$, using the contraction mapping principle, we obtain that the solutions of the integral equations (2.1) are local wellposedness in $H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)$ by a standard argument (see, for example, [6, 8, 11, 12, 13, 14, 17]). This result goes as follows.

Proposition 2.1 Let $\eta>0$ and $\delta\geq 0$, $(E_{10},E_{20},E_{30})\in H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)$. Then the integral equation (2.1) has a unique solution $(E_1,E_2,E_3)\in X^1_{4,loc}([0,T))\times X^1_{4,loc}([0,T))\times X^1_{4,loc}([0,T))$ for some positive time $T=T(E_{10},E_{20},E_{30})$, and for any $0\leq T_1<T_2<T$, the mapping

$$(E_{10},E_{20},E_{30})\left(\in H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)\right)$$

$$\mapsto(E_1,E_2,E_3)(t)\left(\in X^1_{4,loc}([0,T))\times X^1_{4,loc}([0,T))\times X^1_{4,loc}([0,T))\right)$$

is continuous. Moreover, there holds either $T=+\infty$, or $T<+\infty$ and

$$\lim_{t\to T}\left(\|E_1\|_{H^1(\mathbb{R}^3)}+\|E_2\|_{H^1(\mathbb{R}^3)}+\|E_3\|_{H^1(\mathbb{R}^3)}\right)=+\infty.$$

Here, for any interval $I\subset\mathbb{R}$, $0\leq\frac{2}{q}=3(\frac{1}{2}-\frac{1}{\theta})<1$, $s\in\mathbb{R}$,

$$
\begin{aligned}
&X^s_\theta(I)=(C\cap L^\infty)(I;H^s(\mathbb{R}^3))\cap L^q(I;H^s_\theta(\mathbb{R}^3)),\\
&X^s_{\theta,loc}(I)=\{u;\,u\in X^s_\theta(J),\quad\forall\,J\subset\subset I\},\\
&H^s_\theta(\mathbb{R}^3)=J_s(L^\theta(\mathbb{R}^3)),\quad \mathrm{J}_s=(I-\Delta)^{-\frac{s}{2}}.
\end{aligned}
$$

□

As a direct consequence of Proposition 2.1, using the ideas in papers [2, 6, 8, 10, 14, 17](see also Chapter 12:Theorem 4.1 in [11]) we obtain the following local wellposedness theory for the Cauchy problem (1.1)-(1.4).

Proposition 2.2 The Cauchy problem (1.1)-(1.4), for $\eta>0$, $\delta\geq 0$ and $(E_{10},E_{20},E_{30})\in H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)$, has a unique solution

$$(E_1,E_2,E_3)\in C\left([0,T);H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)\right)$$

for some $T\in(0,+\infty)$ with $T=+\infty$ or $T<+\infty$ and

$$\lim_{t\to T}\left(\|E_1\|_{H^1(\mathbb{R}^3)}+\|E_2\|_{H^1(\mathbb{R}^3)}+\|E_3\|_{H^1(\mathbb{R}^3)}\right)=+\infty.$$

2.2 Conservation Laws of the Mass and of the Energy

According to the structure of the equations (1.1)-(1.4), we can establish the conservation laws of the total mass and of the total energy.

Lemma 2.1 Let (E_1, E_2, E_3) is a smooth solution of the Cauchy Problem $(1.1)-(1.4)$. Then the total mass and total energy are conserved:

$$\int_{\mathbb{R}^3}(|E_1|^2+|E_2|^2+|E_3|^2)dx = \int_{\mathbb{R}^3}(|E_{10}|^2+|E_{20}|^2+|E_{30}|^2)dx, \tag{2.4}$$

$$\begin{aligned}
&\mathcal{H}(E_1, E_2, E_3)\\
&\quad= \textstyle\int_{\mathbb{R}^3}(|\nabla E_1|^2+|\nabla E_2|^2+|\nabla E_3|^2)dx - \frac{1}{2}\int_{\mathbb{R}^3}(|E_1|^4+|E_2|^4+|E_3|^4)dx\\
&\quad - \textstyle\int_{\mathbb{R}^3}(|E_1|^2|E_2|^2+|E_1|^2|E_3|^2+|E_2|^2|E_3|^2)dx\\
&\quad -\tfrac{1}{2}\textstyle\int_{\mathbb{R}^3}\frac{\eta(\xi_1^2+\xi_2^2)}{|\xi|^2+\delta}|\mathcal{F}(E_1\overline{E_2}-\overline{E_1}E_2)|^2d\xi\\
&\quad -\tfrac{1}{2}\textstyle\int_{\mathbb{R}^3}\frac{\eta(\xi_1^2+\xi_3^2)}{|\xi|^2+\delta}|\mathcal{F}(\overline{E_1}E_3-E_1\overline{E_3})|^2d\xi\\
&\quad -\tfrac{1}{2}\textstyle\int_{\mathbb{R}^3}\frac{\eta(\xi_2^2+\xi_3^2)}{|\xi|^2+\delta}|\mathcal{F}(E_2\overline{E_3}-\overline{E_2}E_3)|^2d\xi\\
&\quad +Re\textstyle\int_{\mathbb{R}^3}\frac{\eta\xi_1\xi_2}{|\xi|^2+\delta}\mathcal{F}(E_2\overline{E_3}-\overline{E_2}E_3)\overline{\mathcal{F}(\overline{E_1}E_3-E_1\overline{E_3})}d\xi\\
&\quad +Re\textstyle\int_{\mathbb{R}^3}\frac{\eta\xi_1\xi_3}{|\xi|^2+\delta}\mathcal{F}(E_1\overline{E_2}-\overline{E_1}E_2)\overline{\mathcal{F}(E_2\overline{E_3}-\overline{E_2}E_3)}d\xi\\
&\quad +Re\textstyle\int_{\mathbb{R}^3}\frac{\eta\xi_2\xi_3}{|\xi|^2+\delta}\mathcal{F}(\overline{E_1}E_3-E_1\overline{E_3})\overline{\mathcal{F}(E_1\overline{E_2}-\overline{E_1}E_2)}d\xi\\
&\quad = \mathcal{H}(E_{10}, E_{20}, E_{30}).
\end{aligned} \tag{2.5}$$

Proof. Multiplying (1.1) by $\overline{E_1}$, (1.2) by $\overline{E_2}$ and (1.3) by $\overline{E_3}$, taking the imaginary part and then integrating with respect to the space variable $x \in \mathbb{R}^3$, we get

$$\begin{aligned}
&Im\textstyle\int_{\mathbb{R}^3}(i\partial_t E_1\overline{E_1}+i\partial_t E_2\overline{E_2}+i\partial_t E_3\overline{E_3})dx\\
&\quad +Im\textstyle\int_{\mathbb{R}^3}(\Delta E_1\overline{E_1}+\Delta E_2\overline{E_2}+\Delta E_3\overline{E_3})dx\\
&\quad +Im\textstyle\int_{\mathbb{R}^3}(|E_1|^2+|E_2|^2+|E_3|^2)(E_1\overline{E_1}+E_2\overline{E_2}+E_3\overline{E_3})dx\\
&\quad -Im\textstyle\int_{\mathbb{R}^3}\overline{E_1}A_1dx+Im\int_{\mathbb{R}^3}\overline{E_1}A_2dx-Im\int_{\mathbb{R}^3}\overline{E_2}B_1dx\\
&\quad +Im\textstyle\int_{\mathbb{R}^3}\overline{E_2}B_2dx-Im\int_{\mathbb{R}^3}\overline{E_3}C_1dx+Im\int_{\mathbb{R}^3}\overline{E_3}C_2dx=0,
\end{aligned} \tag{2.6}$$

where

$$\begin{aligned}
A_1 = E_2\mathcal{F}^{-1}\Big\{\tfrac{\eta}{|\xi|^2+\delta}\big[&\xi_1\xi_3\mathcal{F}(E_2\overline{E_3}-\overline{E_2}E_3)-(\xi_1^2+\xi_2^2)\mathcal{F}(E_1\overline{E_2}-\overline{E_1}E_2)\\
&+\xi_2\xi_3\mathcal{F}(\overline{E_1}E_3-E_1\overline{E_3})\big]\Big\},
\end{aligned} \tag{2.6 - 1}$$

$$\begin{aligned}
A_2 = E_3\mathcal{F}^{-1}\Big\{\tfrac{\eta}{|\xi|^2+\delta}\big[&\xi_2\xi_3\mathcal{F}(E_1\overline{E_2}-\overline{E_1}E_2)-(\xi_1^2+\xi_3^2)\mathcal{F}(\overline{E_1}E_3-E_1\overline{E_3})\\
&+\xi_1\xi_2\mathcal{F}(E_2\overline{E_3}-\overline{E_2}E_3)\big]\Big\},
\end{aligned} \tag{2.6 - 2}$$

$$\begin{aligned}
B_1 = E_3\mathcal{F}^{-1}\Big\{\tfrac{\eta}{|\xi|^2+\delta}\big[&\xi_1\xi_2\mathcal{F}(\overline{E_1}E_3-E_1\overline{E_3})-(\xi_2^2+\xi_3^2)\mathcal{F}(E_2\overline{E_3}-\overline{E_2}E_3)\\
&+\xi_1\xi_3\mathcal{F}(E_1\overline{E_2}-\overline{E_1}E_2)\big]\Big\},
\end{aligned} \tag{2.6 - 3}$$

$$\begin{aligned}
B_2 = E_1\mathcal{F}^{-1}\Big\{\tfrac{\eta}{|\xi|^2+\delta}\big[&\xi_1\xi_3\mathcal{F}(E_2\overline{E_3}-\overline{E_2}E_3)-(\xi_1^2+\xi_2^2)\mathcal{F}(E_1\overline{E_2}-\overline{E_1}E_2)\\
&+\xi_2\xi_3\mathcal{F}(\overline{E_1}E_3-E_1\overline{E_3})\big]\Big\},
\end{aligned} \tag{2.6 - 4}$$

$$C_1 = E_1\mathcal{F}^{-1}\left\{\tfrac{\eta}{|\xi|^2+\delta}\left[\xi_2\xi_3\mathcal{F}(E_1\overline{E_2}-\overline{E_1}E_2)-(\xi_1^2+\xi_3^2)\mathcal{F}(\overline{E_1}E_3-E_1\overline{E_3})\right.\right.$$
$$\left.\left.+\xi_1\xi_2\mathcal{F}(E_2\overline{E_3}-\overline{E_2}E_3)\right]\right\}, \tag{2.6 − 5}$$

$$C_2 = E_2\mathcal{F}^{-1}\left\{\tfrac{\eta}{|\xi|^2+\delta}\left[\xi_1\xi_2\mathcal{F}(\overline{E_1}E_3-E_1\overline{E_3})-(\xi_2^2+\xi_3^2)\mathcal{F}(E_2\overline{E_3}-\overline{E_2}E_3)\right.\right.$$
$$\left.\left.+\xi_1\xi_3\mathcal{F}(E_1\overline{E_2}-\overline{E_1}E_2)\right]\right\}. \tag{2.6 − 6}$$

By a direct calculation and rearranging these terms in (2.6), one has

$$Im\int_{\mathbb{R}^3} i\partial_t E_j\overline{E_j}dx = Re\int_{\mathbb{R}^3}\partial_t E_j\overline{E_j}dx = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|E_j|^2dx, \tag{2.7}$$

$$\begin{aligned}
&-Im\textstyle\int_{\mathbb{R}^3}\overline{E_1}E_2\mathcal{F}^{-1}\left[\frac{\eta\xi_1\xi_3}{|\xi|^2+\delta}\mathcal{F}(E_2\overline{E_3}-\overline{E_2}E_3)\right]dx\\
&\quad+Im\textstyle\int_{\mathbb{R}^3}E_1\overline{E_2}\mathcal{F}^{-1}\left[\frac{\eta\xi_1\xi_3}{|\xi|^2+\delta}\mathcal{F}(E_2\overline{E_3}-\overline{E_2}E_3)\right]dx\\
&\quad= Im\textstyle\int_{\mathbb{R}^3}\overline{E_1}E_2\mathcal{F}^{-1}\left[\frac{\eta\xi_1\xi_3}{|\xi|^2+\delta}\mathcal{F}(\overline{E_2}E_3-E_2\overline{E_3})\right]dx\\
&\quad\quad+Im\textstyle\int_{\mathbb{R}^3}E_1\overline{E_2}\mathcal{F}^{-1}\left[\frac{\eta\xi_1\xi_3}{|\xi|^2+\delta}\mathcal{F}(E_2\overline{E_3}-\overline{E_2}E_3)\right]dx\\
&\quad= Im\left\{2Re\textstyle\int_{\mathbb{R}^3}\overline{E_1}E_2\mathcal{F}^{-1}\left[\frac{\eta\xi_1\xi_3}{|\xi|^2+\delta}\mathcal{F}(\overline{E_2}E_3-E_2\overline{E_3})\right]\right\}=0.
\end{aligned} \tag{2.8}$$

Similarly, we can make some estimates on these term in (2.6) as (2.8). Thus we can conclude the mass identity (2.4). We next show (2.5).

Multiplying (1.1) by $2\partial_t\overline{E_1}$, (1.2) by $2\partial_t\overline{E_2}$ and (1.3) by $2\partial_t\overline{E_3}$, taking the real part and then integrating with respect to the space variable $x\in\mathbb{R}^3$, we get

$$\begin{aligned}
&2Re\textstyle\int_{\mathbb{R}^3}(i\partial_tE_1\partial_t\overline{E_1}+i\partial_tE_2\partial_t\overline{E_2}+i\partial_tE_3\partial_t\overline{E_3})dx\\
&\quad=2Re\textstyle\int_{\mathbb{R}^3}\left[(-\Delta E_1)\partial_t\overline{E_1}+(-\Delta E_2)\partial_t\overline{E_2}+(-\Delta E_3)\partial_t\overline{E_3}\ \right]dx\\
&\quad-2Re\textstyle\int_{\mathbb{R}^3}(|E_1|^2+|E_2|^2+|E_3|^2)(E_1\partial_t\overline{E_1}+E_2\partial_t\overline{E_2}+E_3\partial_t\overline{E_3})dx\\
&\quad+2Re\textstyle\int_{\mathbb{R}^3}\partial_t\overline{E_1}A_1dx-2Re\int_{R^3}\partial_t\overline{E_1}A_2dx+2Re\int_{R^3}\partial_t\overline{E_2}B_1dx\\
&\quad-2Re\textstyle\int_{\mathbb{R}^3}\partial_t\overline{E_2}B_2dx+2Re\int_{R^3}\partial_t\overline{E_3}C_1dx-2Re\int_{R^3}\partial_t\overline{E_3}C_2dx,
\end{aligned} \tag{2.9}$$

where $A_1, A_2, B_1, B_2, C_1, C_2$ are the same as (2.6-1)-(2.6-6). We now estimate (2.9) term by term.

$$2Re\int_{\mathbb{R}^3}(i\partial_tE_1\partial_t\overline{E_1}+i\partial_tE_2\partial_t\overline{E_2}+i\partial_tE_3\partial_t\overline{E_3})dx=0, \tag{2.10}$$

$$\begin{aligned}
&2Re\int_{\mathbb{R}^3}\left[(-\triangle E_1)\partial_t\overline{E_1}+(-\triangle E_2)\partial_t\overline{E_2}+(-\triangle E_3)\partial_t\overline{E_3}\ \right]dx\\
&\quad=\frac{d}{dt}\int_{\mathbb{R}^3}(|\nabla E_1|^2+|\nabla E_2|^2+|\nabla E_3|^2)dx,
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
&2Re\int_{\mathbb{R}^3}(|E_1|^2+|E_2|^2+|E_3|^2)(E_1\partial_t\overline{E_1}+E_2\partial_t\overline{E_2}+E_3\partial_t\overline{E_3})dx\\
&=\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}(|E_1|^4+|E_2|^4+|E_3|^4)dx+\frac{d}{dt}\int_{\mathbb{R}^3}(|E_1|^2|E_2|^2+|E_1|^2|E_3|^2+|E_2|^2|E_3|^2)dx,
\end{aligned} \tag{2.12}$$

$$
\begin{aligned}
&2Re\int_{\mathbb{R}^3}\partial_t\overline{E_1}E_2\mathcal{F}^{-1}\left[\tfrac{\eta\xi_1\xi_3}{|\xi|^2+\delta}\mathcal{F}(E_2\overline{E_3}-\overline{E_2}E_3)\right]dx\\
&\quad+2Re\int_{\mathbb{R}^3}\partial_t\overline{E_2}E_3\mathcal{F}^{-1}\left[\tfrac{\eta\xi_1\xi_3}{|\xi|^2+\delta}\mathcal{F}(E_1\overline{E_2}-\overline{E_1}E_2)\right]dx\\
&\quad-2Re\int_{\mathbb{R}^3}\partial_t\overline{E_2}E_1\mathcal{F}^{-1}\left[\tfrac{\eta\xi_1\xi_3}{|\xi|^2+\delta}\mathcal{F}(E_2\overline{E_3}-\overline{E_2}E_3)\right]dx\\
&\quad-2Re\int_{\mathbb{R}^3}\partial_t\overline{E_3}E_2\mathcal{F}^{-1}\left[\tfrac{\eta\xi_1\xi_3}{|\xi|^2+\delta}\mathcal{F}(E_1\overline{E_2}-\overline{E_1}E_2)\right]dx\\
&=\tfrac{1}{2}\tfrac{d}{dt}\int_{\mathbb{R}^3}\tfrac{\eta\xi_1\xi_3}{|\xi|^2+\delta}\mathcal{F}(E_2\overline{E_3}-\overline{E_2}E_3)\overline{\mathcal{F}(E_1\overline{E_2}-\overline{E_1}E_2)}d\xi\\
&\quad+\tfrac{1}{2}\tfrac{d}{dt}\int_{\mathbb{R}^3}\tfrac{\eta\xi_1\xi_3}{|\xi|^2+\delta}\overline{\mathcal{F}(E_2\overline{E_3}-\overline{E_2}E_3)}\mathcal{F}(E_1\overline{E_2}-\overline{E_1}E_2)d\xi\\
&=\tfrac{d}{dt}Re\int_{\mathbb{R}^3}\tfrac{\eta\xi_1\xi_3}{|\xi|^2+\delta}\mathcal{F}(E_2\overline{E_3}-\overline{E_2}E_3)\overline{\mathcal{F}(E_1\overline{E_2}-\overline{E_1}E_2)}d\xi.
\end{aligned}\tag{2.13}
$$

Making the similar estimates to (2.13) for the rest terms in (2.9) and employing (2.10)-(2.13), we can conclude energy identity (2.5).

The proof of Lemma 2.1 is complete. □

3 Finite Time Collapse

In this section, we will discuss the blow-up result of the solution to the Cauchy problem (1.1)-(1.4).

3.1 Main collapse result

Let

$$
\Sigma := \left\{(E_1,E_2,E_3) : (|x|E_1,|x|E_2,|x|E_3)\in L^2(\mathbb{R}^3)\times L^2(\mathbb{R}^3)\times L^2(\mathbb{R}^3)\right\}
$$

$$
\cap H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3).
$$

The main results of this paper can be stated as follows:

Theorem 3.1 Let $\eta>0$ and $\delta\geq 0$. Assume that

(1) $(E_{10},E_{20},E_{30})\in H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)$ and

$$
(E_1,E_2,E_3)\in C\left([0,T);H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)\right)
$$

is a solution to the Cauchy problem $(1.1)-(1.4)$;

(2) $(|x|E_{10},|x|E_{20},|x|E_{30})\in L^2(\mathbb{R}^3)\times L^2(\mathbb{R}^3)\times L^2(\mathbb{R}^3)$ and one of the following three conditions holds:

(i) $\mathcal{H}(E_{10},E_{20},E_{30})<0$;

(ii) $\mathcal{H}(E_{10},E_{20},E_{30})=0$ and $\sum\limits_{i=1}^{3} Im\int_{\mathbb{R}^3}(x\nabla E_{i0})\overline{E_{i0}}dx<0$;

(iii) $\mathcal{H}(E_{10},E_{20},E_{30})>0$,

$$
\sum_{i=1}^{3} Im\int_{\mathbb{R}^3}(x\nabla E_{i0})\overline{E_{i0}}dx \leq -\left[2\mathcal{H}(E_{10},E_{20},E_{30})\int_{\mathbb{R}^3}|x|^2\sum_{i=1}^{3}|E_{i0}|^2dx\right]^{\frac{1}{2}}.
$$

Then there exists $0 < T < +\infty$ such that

$$\lim_{t\to T}\left(\|E_1\|_{H^1(\mathbb{R}^3)} + \|E_2\|_{H^1(\mathbb{R}^3)} + \|E_3\|_{H^1(\mathbb{R}^3)}\right) = +\infty.$$

Furthermore, if the solutions $(E_1(t,x), E_2(t,x), E_3(t,x))$ of the Cauchy problem (1.1)-(1.4) is radially symmetric, we then obtain another finite time collapse result.

Theorem 3.2 Let $\eta > 0, \delta \geq 0$ and (E_1, E_2, E_3) be a classical and radially symmetric solution to the Cauchy Problem (1.1)-(1.4) with $(E_{10}, E_{20}, E_{30}) \in \Sigma$. Assume that

(I) $\mathcal{H}(E_{10}, E_{20}, E_{30}) \leq 0$ and $\sum\limits_{i=1}^{3} Im \int_{\mathbb{R}^3} r\overline{E_{i0}}\partial_r E_{i0}dx < 0.$

Then there exists a finite time T such that

$$\lim_{t\to T}\left(\|\nabla E_1\|^2_{L^2(\mathbb{R}^3)} + \|\nabla E_2\|^2_{L^2(\mathbb{R}^3)} + \|\nabla E_3\|^2_{L^2(\mathbb{R}^3)}\right) = +\infty.$$

Remark 3.1 For $(E_{10}, E_{20}, E_{30}) \in H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)$ and $(|x|E_{10}, |x|E_{20}, |x|E_{30}) \in L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$, there always exists initial data (E_{10}, E_{20}, E_{30}) such that

(i) $\mathcal{H}(E_{10}, E_{20}, E_{30}) < 0;$

or

(ii) $\mathcal{H}(E_{10}, E_{20}, E_{30}) = 0$ and $\sum\limits_{i=1}^{3} Im \int_{\mathbb{R}^3}(x\nabla E_{i0})\overline{E_{i0}}dx < 0;$

or

(iii) $\mathcal{H}(E_{10}, E_{20}, E_{30}) > 0,$

$$\sum_{i=1}^{3} Im \int_{\mathbb{R}^3}(x\nabla E_{i0})\overline{E_{i0}}dx \leq -\left[2\mathcal{H}(E_{10}, E_{20}, E_{30})\int_{\mathbb{R}^3}|x|^2\sum_{i=1}^{3}|E_{i0}|^2dx\right]^{\frac{1}{2}}.$$

In fact, let $E_{i0}(x) = \varphi_i(x)$ $(i = 1, 2, 3)$, we can always find some $\varphi_i(x) \neq 0$ and an $\varepsilon > 0$ such that

$$\varphi_i(x) \in H^1(\mathbb{R}^3), x\varphi_i(x) \in L^2(\mathbb{R}^3), \mathcal{H}(\varphi_1(x), \varphi_2(x), \varphi_3(x)) = 0,$$
$$\sum_{i=1}^{3} Im \int_{\mathbb{R}^3}(x\nabla\varphi_i(x))\overline{\varphi_i(x)}dx \leq -\varepsilon < 0,$$

where

$$\varepsilon \geq \frac{\left[2\mathcal{H}(\zeta\varphi_1(x), \zeta\varphi_2(x), \zeta\varphi_3(x))\sum\limits_{i=1}^{3}\int_{\mathbb{R}^3}|x|^2|\varphi_i(x)|^2dx\right]^{\frac{1}{2}}}{\zeta} \text{ for a fixed } \zeta \in (0,1). \quad (*)$$

Thus (ii) holds for such $E_{i0}(x)$ $(i = 1, 2, 3)$.

Next, let $\psi_i(x) = \lambda\varphi_i(x)$ with $\lambda > 1$, we get by (2.5)

$$\mathcal{H}(\psi_1(x), \psi_2(x), \psi_3(x)) = \mathcal{H}(\lambda\varphi_1(x), \lambda\varphi_2(x), \lambda\varphi_3(x)) < 0.$$

If we choose $E_{i0}(x) = \lambda\varphi_i(x)$ $(i = 1, 2, 3)$ with $\lambda > 1$, then $\mathcal{H}(E_{10}(x), E_{20}(x), E_{30}(x)) < 0$ and hence condition (i) is not empty.

Similarly, let $\phi_i(x) = \zeta\varphi_i(x)$ for the same ζ in (*), we obtain by (2.5)

$$\mathcal{H}(\phi_1(x), \phi_2(x), \phi_3(x)) = \mathcal{H}(\zeta\varphi_1(x), \zeta\varphi_2(x), \zeta\varphi_3(x)) > 0.$$

In this case we may choose $E_{i0}(x)=\zeta\varphi_i(x)$ $(i=1,2,3)$, then there holds

$$\begin{aligned}
&\mathcal{H}(E_{10},E_{20},E_{30})>0,\\
&\sum_{i=1}^{3} Im\int_{\mathbb{R}^3}(x\nabla E_{i0}(x))\overline{E_{i0}(x)}dx=\sum_{i=1}^{3} Im\int_{\mathbb{R}^3}(x\nabla\zeta\varphi_i(x))\overline{\zeta\varphi_i(x)}dx\\
&\quad\le-\zeta^2\varepsilon\le-\zeta^2\frac{\left[2\mathcal{H}(\zeta\varphi_1(x),\zeta\varphi_2(x),\zeta\varphi_3(x))\sum_{i=1}^{3}\int_{\mathbb{R}^3}|x|^2|\varphi_i(x)|^2dx\right]^{\frac{1}{2}}}{\zeta}\\
&\quad=-\left[2\mathcal{H}(E_{10}(x),E_{20}(x),E_{30}(x))\sum_{i=1}^{3}\int_{\mathbb{R}^3}|x|^2|E_{i0}(x)|^2dx\right]^{\frac{1}{2}}.
\end{aligned}$$

Hence such $E_{i0}(x)$ $(i=1,2,3)$ meets condition (iii).

Similar discussion then yields the existence of the initial data which meets (I) and (II) in Theorem 3.2. □

3.2 Key Ingredients

Before proving Theorem 3.1 and Theorem 3.2, we first give a key lemma and a proposition which concerns about the related virial identities, this kind of identity was found independently by Zakharov and Shabat in [21].

Lemma 3.1[18] Let f be a scalar-valued function. If $|x|f$ and ∇f belong to $L^2(\mathbb{R}^3)$, then f is in $L^2(\mathbb{R}^3)$ and satisfies

$$\int_{\mathbb{R}^3}|f|^2dx\le\frac{2}{3}\left(\int_{\mathbb{R}^3}|\nabla f|^2dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3}|x|^2|f|^2dx\right)^{\frac{1}{2}}.$$

The following proposition establishes the key virial identities.

Propositin 3.1 Let $(E_{10},E_{20},E_{30})\in\Sigma$ and $(E_1,E_2,E_3)\in C([0,T);\Sigma)$ be a solution to the Cauchy Problem (1.1)-(1.4) on $[0,T)$. Putting

$$J(t)=:\int_{\mathbb{R}^3}|x|^2(|E_1|^2+|E_2|^2+|E_3|^2)dx,\tag{3.1}$$

one gets

$$\frac{dJ(t)}{dt}=4Im\int_{\mathbb{R}^3}\left[(x\nabla E_1)\overline{E_1}+(x\nabla E_2)\overline{E_2}+(x\nabla E_3)\overline{E_3}\right]dx,\tag{3.2}$$

$$\begin{aligned}
\frac{d^2J(t)}{dt^2}&=8\int_{\mathbb{R}^3}\left(|\nabla E_1|^2+|\nabla E_2|^2+|\nabla E_3|^2\right)dx-6\int_{\mathbb{R}^3}\left(|E_1|^4+|E_2|^4+|E_3|^4\right)dx\\
&\quad-12\int_{\mathbb{R}^3}\left(|E_1|^2|E_2|^2+|E_1|^2|E_3|^2+|E_2|^2|E_3|^2\right)dx-6A+12B+C\\
&=8\mathcal{H}(E_{10},E_{20},E_{30})-2\int_{\mathbb{R}^3}\left(|E_1|^4+|E_2|^4+|E_3|^4\right)dx\\
&\quad-4\int_{\mathbb{R}^3}\left(|E_1|^2|E_2|^2+|E_1|^2|E_3|^2+|E_2|^2|E_3|^2\right)dx-2A+4B+C\\
&=-4\int_{\mathbb{R}^3}(|\nabla E_1|^2+|\nabla E_2|^2+|\nabla E_3|^2)dx+12\mathcal{H}(E_{10},E_{20},E_{30})+C,
\end{aligned}\tag{3.3}$$

where

$$\begin{aligned}
A&=\int_{\mathbb{R}^3}\frac{\eta(\xi_1^2+\xi_2^2)}{|\xi|^2+\delta}|\mathcal{F}(E_1\overline{E_2}-\overline{E_1}E_2)|^2d\xi\\
&\quad+\int_{\mathbb{R}^3}\frac{\eta(\xi_1^2+\xi_3^2)}{|\xi|^2+\delta}|\mathcal{F}(\overline{E_1}E_3-E_1\overline{E_3})|^2d\xi\\
&\quad+\int_{\mathbb{R}^3}\frac{\eta(\xi_2^2+\xi_3^2)}{|\xi|^2+\delta}|\mathcal{F}(E_2\overline{E_3}-\overline{E_2}E_3)|^2d\xi,
\end{aligned}\tag{3.3-1}$$

$$
\begin{aligned}
B = & Re\int_{\mathbb{R}^3}\tfrac{\eta\xi_1\xi_2}{|\xi|^2+\delta}\mathcal{F}(E_2\overline{E_3}-\overline{E_2}E_3)\overline{\mathcal{F}(\overline{E_1}E_3-E_1\overline{E_3})}d\xi\\
& +Re\int_{\mathbb{R}^3}\tfrac{\eta\xi_1\xi_3}{|\xi|^2+\delta}\mathcal{F}(E_1\overline{E_2}-\overline{E_1}E_2)\overline{\mathcal{F}(E_2\overline{E_3}-\overline{E_2}E_3)}d\xi\\
& +Re\int_{\mathbb{R}^3}\tfrac{\eta\xi_2\xi_3}{|\xi|^2+\delta}\overline{\mathcal{F}(E_1\overline{E_2}-\overline{E_1}E_2)}\mathcal{F}(\overline{E_1}E_3-E_1\overline{E_3})d\xi,
\end{aligned}
\tag{3.3-2}
$$

$$
\begin{aligned}
C = & 4\delta\int_{\mathbb{R}^3}\tfrac{\eta(\xi_1^2+\xi_2^2)}{(|\xi|^2+\delta)^2}|\mathcal{F}(E_1\overline{E_2}-\overline{E_1}E_2)|^2d\xi\\
& +4\delta\int_{\mathbb{R}^3}\tfrac{\eta(\xi_1^2+\xi_3^2)}{(|\xi|^2+\delta)^2}|\mathcal{F}(\overline{E_1}E_3-E_1\overline{E_3})|^2d\xi\\
& +4\delta\int_{\mathbb{R}^3}\tfrac{\eta(\xi_2^2+\xi_3^2)}{(|\xi|^2+\delta)^2}|\mathcal{F}(E_2\overline{E_3}-\overline{E_2}E_3)|^2d\xi\\
& -8\delta Re\int_{\mathbb{R}^3}\tfrac{\eta\xi_1\xi_2}{(|\xi|^2+\delta)^2}\mathcal{F}(E_2\overline{E_3}-\overline{E_2}E_3)\overline{\mathcal{F}(\overline{E_1}E_3-E_1\overline{E_3})}d\xi\\
& -8\delta Re\int_{\mathbb{R}^3}\tfrac{\eta\xi_1\xi_3}{(|\xi|^2+\delta)^2}\mathcal{F}(E_1\overline{E_2}-\overline{E_1}E_2)\overline{\mathcal{F}(E_2\overline{E_3}-\overline{E_2}E_3)}d\xi\\
& -8\delta Re\int_{\mathbb{R}^3}\tfrac{\eta\xi_2\xi_3}{(|\xi|^2+\delta)^2}\overline{\mathcal{F}(E_1\overline{E_2}-\overline{E_1}E_2)}\mathcal{F}(\overline{E_1}E_3-E_1\overline{E_3})d\xi.
\end{aligned}
\tag{3.3-3}
$$

Proof of Proposition3.1. Since $(E_{10}, E_{20}, E_{30}) \in \Sigma$, one has $(E_1, E_2, E_3) \in \Sigma$ by Ginibre-Velo [6], where $(E_1, E_2, E_3) \in C([0,T);\Sigma)$ is a solution to the Cauchy problem (1.1)-(1.4). In view of (1.1)-(1.3), (3.1) yields that

$$
\begin{aligned}
\frac{dJ(t)}{dt} & = \int_{\mathbb{R}^3}|x|^2\left[(\partial_t E_1\overline{E_1}+E_1\partial_t\overline{E_1})+(\partial_t E_2\overline{E_2}+E_2\partial_t\overline{E_2})+(\partial_t E_3\overline{E_3}+E_3\partial_t\overline{E_3})\right]dx\\
& = 2Im\int_{\mathbb{R}^3}|x|^2\left(i\partial_t E_1\overline{E_1}+i\partial_t E_2\overline{E_2}+i\partial_t E_3\overline{E_3}\right)dx\\
& = 2Im\int_{\mathbb{R}^3}|x|^2(-\Delta E_1\overline{E_1}-\Delta E_2\overline{E_2}-\Delta E_3\overline{E_3})dx\\
& \quad -2Im\int_{\mathbb{R}^3}|x|^2\left(|E_1|^2+||E_2|^2+|E_3|^2\right)(E_1\overline{E_1}+E_2\overline{E_2}+E_3\overline{E_3})dx\\
& \quad +2Im\int_{\mathbb{R}^3}|x|^2(\overline{E_1}A_1-\overline{E_1}A_2+\overline{E_2}B_1-\overline{E_2}B_2+\overline{E_3}C_1-\overline{E_3}C_2)dx,
\end{aligned}
\tag{3.4}
$$

where $A_1, A_2, B_1, B_2, C_1, C_2$ have the same forms in (2.6-1)-(2.6-6). We further calculate (3.4) term by term.

$$
\begin{aligned}
& Im\int_{\mathbb{R}^3}|x|^2(-\Delta E_1\overline{E_1}-\Delta E_2\overline{E_2}-\Delta E_3\overline{E_3})dx\\
& \quad = 2Im\int_{\mathbb{R}^3}\left(x\nabla E_1\overline{E_1}+x\nabla E_2\overline{E_2}+x\nabla E_3\overline{E_3}\right)dx,
\end{aligned}
\tag{3.5}
$$

$$
Im\int_{\mathbb{R}^3}|x|^2\left(|E_1|^2+||E_2|^2+|E_3|^2\right)(E_1\overline{E_1}+E_2\overline{E_2}+E_3\overline{E_3})dx = 0,
\tag{3.6}
$$

$$
\begin{aligned}
& Im\int_{\mathbb{R}^3}|x|^2\overline{E_1}E_2\mathcal{F}^{-1}\left[\tfrac{\eta\xi_1\xi_3}{|\xi|^2+\delta}\mathcal{F}(E_2\overline{E_3}-\overline{E_2}E_3)\right]dx\\
& \quad +Im\int_{\mathbb{R}^3}|x|^2\overline{E_2}E_3\mathcal{F}^{-1}\left[\tfrac{\eta\xi_1\xi_3}{|\xi|^2+\delta}\mathcal{F}(E_1\overline{E_2}-\overline{E_1}E_2)\right]dx\\
& \quad -Im\int_{\mathbb{R}^3}|x|^2E_1\overline{E_2}\mathcal{F}^{-1}\left[\tfrac{\eta\xi_1\xi_3}{|\xi|^2+\delta}\mathcal{F}(E_2\overline{E_3}-\overline{E_2}E_3)\right]dx\\
& \quad -Im\int_{\mathbb{R}^3}|x|^2E_2\overline{E_3}\mathcal{F}^{-1}\left[\tfrac{\eta\xi_1\xi_3}{|\xi|^2+\delta}\mathcal{F}(E_1\overline{E_2}-\overline{E_1}E_2)\right]dx\\
& = Im\int_{\mathbb{R}^3}|x|^2\overline{E_1}E_2\mathcal{F}^{-1}\left[\tfrac{\eta\xi_1\xi_3}{|\xi|^2+\delta}\mathcal{F}(E_2\overline{E_3}-\overline{E_2}E_3)\right]dx\\
& \quad +Im\int_{\mathbb{R}^3}|x|^2E_1\overline{E_2}\mathcal{F}^{-1}\left[\tfrac{\eta\xi_1\xi_3}{|\xi|^2+\delta}\mathcal{F}(\overline{E_2}E_3-E_2\overline{E_3})\right]dx\\
& \quad +Im\int_{\mathbb{R}^3}|x|^2\overline{E_2}E_3\mathcal{F}^{-1}\left[\tfrac{\eta\xi_1\xi_3}{|\xi|^2+\delta}\mathcal{F}(E_1\overline{E_2}-\overline{E_1}E_2)\right]dx\\
& \quad +Im\int_{\mathbb{R}^3}|x|^2E_2\overline{E_3}\mathcal{F}^{-1}\left[\tfrac{\eta\xi_1\xi_3}{|\xi|^2+\delta}\mathcal{F}(\overline{E_1}E_2-E_1\overline{E_2})\right]dx\\
& = 2Im\left\{Re\int_{\mathbb{R}^3}|x|^2\overline{E_1}E_2\mathcal{F}^{-1}\left[\tfrac{\eta\xi_1\xi_3}{|\xi|^2+\delta}\mathcal{F}(E_2\overline{E_3}-\overline{E_2}E_3)\right]dx\right\}\\
& \quad +2Im\left\{Re\int_{\mathbb{R}^3}|x|^2\overline{E_2}E_3\mathcal{F}^{-1}\left[\tfrac{\eta\xi_1\xi_3}{|\xi|^2+\delta}\mathcal{F}(E_1\overline{E_2}-\overline{E_1}E_2)\right]dx\right\}\\
& = 0.
\end{aligned}
\tag{3.7}
$$

Making the same estimate as (3.7) to the rest terms in $Im\int_{\mathbb{R}^3}|x|^2(\overline{E_1}A_1-\overline{E_1}A_2+\overline{E_2}B_1-\overline{E_2}B_2+\overline{E_3}C_1-\overline{E_3}C_2)dx$, we can obtain

$$Im\int_{\mathbb{R}^3}|x|^2(\overline{E_1}A_1-\overline{E_1}A_2+\overline{E_2}B_1-\overline{E_2}B_2+\overline{E_3}C_1-\overline{E_3}C_2)dx=0. \tag{3.8}$$

(3.4)-(3.8) thus yield

$$\frac{dJ(t)}{dt}=4Im\int_{\mathbb{R}^3}\left[(x\nabla E_1)\overline{E_1}+(x\nabla E_2)\overline{E_2}+(x\nabla E_3)\overline{E_3}\right]dx. \tag{3.9}$$

Differentiating (3.9) with respect to t , after a careful computation and proper groupings, we get

$$\begin{aligned}\frac{d^2J(t)}{dt^2} &= 4Im\int_{\mathbb{R}^3}x\left[(\nabla E_1)\partial_t\overline{E_1}+\nabla(\partial_tE_1)\overline{E_1}\right]dx\\
&\quad+4Im\int_{\mathbb{R}^3}x\left[(\nabla E_2)\partial_t\overline{E_2}+\nabla(\partial_tE_2)\overline{E_2}\right]dx\\
&\quad+4Im\int_{\mathbb{R}^3}x\left[(\nabla E_3)\partial_t\overline{E_3}+\nabla(\partial_tE_3)\overline{E_3}\right]dx\\
&= -8Im\int_{\mathbb{R}^3}\left(x\partial_tE_1\nabla\overline{E_1}+x\partial_tE_2\nabla\overline{E_2}+x\partial_tE_3\nabla\overline{E_3}\right)dx\\
&\quad-12Im\int_{\mathbb{R}^3}\left(\overline{E_1}\partial_tE_1+\overline{E_2}\partial_tE_2+\overline{E_3}\partial_tE_3\right)dx\\
&= 8Re\int_{\mathbb{R}^3}\left(xi\partial_tE_1\nabla\overline{E_1}+xi\partial_tE_2\nabla\overline{E_2}+xi\partial_tE_3\nabla\overline{E_3}\right)dx\\
&\quad+12Re\int_{\mathbb{R}^3}\left(\overline{E_1}i\partial_tE_1+\overline{E_2}i\partial_tE_2+\overline{E_3}i\partial_tE_3\right)dx\\
&= 8Re\int_{\mathbb{R}^3}x\left(-\Delta E_1\nabla\overline{E_1}-\Delta E_2\nabla\overline{E_2}-\Delta E_3\nabla\overline{E_3}\right)dx\\
&\quad-8Re\int_{\mathbb{R}^3}x\left(|E_1|^2+||E_2|^2+|E_3|^2\right)\left(E_1\nabla\overline{E_1}+E_2\nabla\overline{E_2}+E_3\nabla\overline{E_3}\right)dx\\
&\quad+8Re\int_{\mathbb{R}^3}\left[x\nabla\overline{E_1}(A_1-A_2)+x\nabla\overline{E_2}(B_1-B_2)+x\nabla\overline{E_3}(C_1-C_2)\right]dx\\
&\quad+12Re\int_{\mathbb{R}^3}\left(-\Delta E_1\overline{E_1}-\Delta E_2\overline{E_2}-\Delta E_3\overline{E_3}\right)dx\\
&\quad-12Re\int_{\mathbb{R}^3}\left(|E_1|^2+||E_2|^2+|E_3|^2\right)\left(E_1\overline{E_1}+E_2\overline{E_2}+E_3\overline{E_3}\right)dx\\
&\quad+12Re\int_{\mathbb{R}^3}\left[\overline{E_1}(A_1-A_2)+\overline{E_2}(B_1-B_2)+\overline{E_3}(C_1-C_2)\right]dx.\end{aligned} \tag{3.10}$$

Here, A_1,A_2,B_1,B_2,C_1,C_2 have the same forms in (2.6-1)-(2.6-6). Using integration by parts, the parseval identiey and the properties of Fourier transforms, we attain the following estimates.

$$\begin{aligned}&Re\int_{\mathbb{R}^3}x\left(-\Delta E_1\nabla\overline{E_1}-\Delta E_2\nabla\overline{E_2}-\Delta E_3\nabla\overline{E_3}\right)dx\\
&\quad=-\frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla E_1|^2+|\nabla E_2|^2+|\nabla E_3|^2\right)dx,\end{aligned} \tag{3.11}$$

$$\begin{aligned}&-Re\int_{\mathbb{R}^3}x\left(|E_1|^2+||E_2|^2+|E_3|^2\right)\left(E_1\nabla\overline{E_1}+E_2\nabla\overline{E_2}+E_3\nabla\overline{E_3}\right)dx\\
&\quad=\frac{3}{4}\int_{\mathbb{R}^3}\left(|E_1|^4+|E_2|^4+|E_3|^4\right)dx\\
&\qquad+\frac{3}{2}\int_{\mathbb{R}^3}\left(|E_1|^2|E_2|^2+||E_1|^2|E_3|^2+|E_2|^2|E_3|^2\right)dx,\end{aligned} \tag{3.12}$$

$$\begin{aligned}&Re\int_{\mathbb{R}^3}\left(-\Delta E_1\nabla\overline{E_1}-\Delta E_2\nabla\overline{E_2}-\Delta E_3\nabla\overline{E_3}\right)dx\\
&\quad=\int_{\mathbb{R}^3}\left(|\nabla E_1|^2+|\nabla E_2|^2+|\nabla E_3|^2\right)dx,\end{aligned} \tag{3.13}$$

$$\begin{aligned}&-Re\int_{\mathbb{R}^3}\left(|E_1|^2+||E_2|^2+|E_3|^2\right)\left(E_1\overline{E_1}+E_2\overline{E_2}+E_3\overline{E_3}\right)dx\\
&\quad=-\int_{\mathbb{R}^3}\left(|E_1|^4+|E_2|^4+|E_3|^4\right)dx\\
&\qquad-2\int_{\mathbb{R}^3}\left(|E_1|^2|E_2|^2+||E_1|^2|E_3|^2+|E_2|^2|E_3|^2\right)dx,\end{aligned} \tag{3.14}$$

$$
\begin{aligned}
&8Re\int_{\mathbb{R}^3}\left[x\nabla\overline{E_1}(A_1-A_2)+x\nabla\overline{E_2}(B_1-B_2)+x\nabla\overline{E_3}(C_1-C_2)\right]dx\\
&\quad+12Re\int_{\mathbb{R}^3}\left[\overline{E_1}(A_1-A_2)+\overline{E_2}(B_1-B_2)+\overline{E_3}(C_1-C_2)\right]dx\\
&=-6A+12B+C
\end{aligned}
\tag{3.15}
$$

where A, B, C are the same as (3.3-1)-(3.3-3). By these estimates as above, using (2.5) we complete the proof of Proposition 3.1. □

Remark 3.2 According to Ginibre-Velo [6], if $E_{i0} \in H^1(\mathbb{R}^3)$ and $xE_{i0} \in L^2(\mathbb{R}^3)$ $(i = 1, 2, 3)$, we can always obtain that the solution E_i $(i = 1, 2, 3)$ of the Cauchy problem (1.1)-(1.4) satisfies $E_i \in H^1(\mathbb{R}^3)$ and $xE_i \in L^2(\mathbb{R}^3)$, which together with Proposition 2.1 and Proposition 2.2 guarantees that the virial identity (3.3) holds for the class of local solutions. □

We now begin to prove Theorem 3.1 and Theorem 3.2.

Proof of Theorem 3.1 We show this theorem by contradiction. Assume that the maximal existence time T of the solution to the Cauchy Problem (1.1)-(1.4) is infinity. Thanks to Young's inequality, Proposition 3.1 implies

$$
\frac{d^2J(t)}{dt^2} \le 8\mathcal{H}(E_{10}, E_{20}, E_{30}). \tag{3.15}
$$

Integrating (3.15) with respect to t, one has

$$
J(t) \le 8\mathcal{H}(E_{10}, E_{20}, E_{30})t^2 + J'(0)t + J(0). \tag{3.16}
$$

Under the hypothesis (i),(ii) or (iii), (3.2) and (3.16) yield that if $(E_1, E_2, E_3) \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$, then there is a $T^* < \infty$ such that

$$
\lim_{t\to T^*}\int_{\mathbb{R}^3}|x|^2(|E_1|^2+|E_2|^2+|E_3|^2)dx = 0. \tag{3.17}
$$

On the other hand, by Lemma 3.1 and (2.4) one gets

$$
\begin{aligned}
&\int_{\mathbb{R}^3}(|E_{10}|^2+|E_{20}|^2+|E_{30}|^2)dx = \int_{\mathbb{R}^3}(|E_1|^2+|E_2|^2+|E_3|^2)dx\\
&\quad\le \tfrac{2}{3}\left(\int_{\mathbb{R}^3}|\nabla E_1|^2dx\right)^{\frac12}\left(\int_{\mathbb{R}^3}|x|^2|E_1|^2dx\right)^{\frac12}+\tfrac{2}{3}\left(\int_{\mathbb{R}^3}|\nabla E_2|^2dx\right)^{\frac12}\\
&\qquad\times\left(\int_{\mathbb{R}^3}|x|^2|E_2|^2dx\right)^{\frac12}+\tfrac{2}{3}\left(\int_{\mathbb{R}^3}|\nabla E_3|^2dx\right)^{\frac12}\left(\int_{\mathbb{R}^3}|x|^2|E_3|^2dx\right)^{\frac12}\\
&\quad\le 2\left[\int_{\mathbb{R}^3}\left(|\nabla E_1|^2+|\nabla E_2|^2+|\nabla E_3|^2\right)dx\right]^{\frac12}\left[\int_{\mathbb{R}^3}|x|^2\left(|E_1|^2+|E_2|^2+|E_3|^2\right)dx\right]^{\frac12},
\end{aligned}
$$

which together with (3.17) thus concludes that

$$
\lim_{t\to T^*}\int_{\mathbb{R}^3}(|\nabla E_1|^2+|\nabla E_2|^2+|\nabla E_3|^2)dx = +\infty. \tag{3.18}
$$

(3.18) then implies that the maximal existence time $T_{max} \le T^*$ and hence is a contradiction.

The proof of Proposition 3.1 is complete. □

We next prove Theorem 3.2.

Proof of Theorem 3.2

Let

$$P(t) = -Im\int_{\mathbb{R}^3} r\overline{E_1}\partial_r E_1 dx - Im\int_{\mathbb{R}^3} r\overline{E_2}\partial_r E_2 dx - Im\int_{\mathbb{R}^3} r\overline{E_3}\partial_r E_3 dx. \tag{3.19}$$

From assumption (II) in Theorem 3.2 and (3.3), it follows that $P(0) > 0$ by Proposition 3.1, and

$$\frac{dF(t)}{dt} = \int_{\mathbb{R}^3}(|\nabla E_1|^2 + |\nabla E_2|^2 + |\nabla E_3|^2)dx - 3\mathcal{H}(E_{10}, E_{20}, E_{30}) - \frac{1}{4}C, \tag{3.20}$$

where C is the same as (3.3-3). By $\eta > 0$, $\delta \geq 0$ and assumption (I) in Theorem 3.2, we get

$$\frac{dP(t)}{dt} \geq \int_{\mathbb{R}^3}(|\nabla E_1|^2 + |\nabla E_2|^2 + |\nabla E_3|^2)dx. \tag{3.21}$$

Since $P(0) > 0$, (3.19) implies that $P(t) > 0$ whenever (E_1, E_2, E_3) exists. On the other hand, from (3.21) it follows that

$$\frac{d}{dt}\int_{\mathbb{R}^3} r^2(|E_1|^2 + |E_2|^2 + |E_3|^2)dx = -4P(t) < 0, \tag{3.22}$$

which yields that

$$\int_{\mathbb{R}^3} r^2(|E_1|^2+|E_2|^2+|E_3|^2)dx \leq \int_{\mathbb{R}^3} r^2(|E_{10}|^2+|E_{20}|^2+|E_{30}|^2)dx = C_0^2 < +\infty, \tag{3.23}$$

$$|P(t)| = P(t) \leq C_*\left(\| \nabla E_1 \|^2_{L^2(\mathbb{R}^3)} + \| \nabla E_2 \|^2_{L^2(\mathbb{R}^3)} + \| \nabla E_3 \|^2_{L^2(\mathbb{R}^3)}\right)^{\frac{1}{2}}. \tag{3.24}$$

(3.21) and (3.24) imply that

$$\frac{dP(t)}{dt} \geq \frac{P^2(t)}{C_*^2} \quad with \quad P(0) > 0. \tag{3.25}$$

(3.25) reads that

$$P(t) \geq \frac{C_*^2(P(0)}{C_*^2 - P(0)t}, \quad t \in \left[0, \frac{C_*^2}{P(0)}\right).$$

Hence we obtain

$$\lim_{t\to T}\left(\| \nabla E_1 \|^2_{L^2(\mathbb{R}^3)} + \| \nabla E_2 \|^2_{L^2(\mathbb{R}^3)} + \| \nabla E_3 \|^2_{L^2(\mathbb{R}^3)}\right) = +\infty,$$

for some $T \leq T_0 < +\infty$, $\quad T_0 = \frac{C_*^2}{P(0)}$.

So far, the proof of Theorem 3.2 is complete. □

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