Coupling by Change of Measure, Harnack Inequality and Hypercontractivity

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Abstract The coupling method is a powerful tool in analysis of stochastic processes. To make the coupling successful before a given time, it is essential that two marginal processes are constructed under different probability measures. We explain the main idea of establishing Harnack inequalities for Markov semigroups using these new type couplings, and apply the coupling and Harnack inequality to the study of hypercontractivity of Markov semigroups.

Keywords ■■■

1 Coupling Method for Harnack Inequality

In 1887, Carl Gustav Axel Harnack found out the following inequality: for an open domain $D \subset \mathbb{R}^2$ and a compact set $K \subset D$, there exists a constant $C(D, K)$ such that for any positive harmonic function u on D,

$$\sup_K u \le C(D, K) \inf_K u.$$

This inequality can be reformulated as follows: for any open domain D there exists a locally bounded positive function C on $D \times D$ such that

$$u(x) \le C(x, y)u(y), \quad x, y \in D$$

holds for all positive harmonic functions u on D. This type of inequality is called Harnack inequality and has been extended and applied to positive solutions of many other elliptic or parabolic PDEs.

In this part, we introduce the main idea of establishing Harnack inequalities for Markov semigroups using the coupling method. Let P_t be a Markov semigroup on

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a Polish space E. Let $\mathscr{B}_b^+(E)$ be the class of all non-negative bounded measurable functions on E. Given $t > 0$ and $x, y \in E$, we aim to compare $P_t f(x)$ and $P_t f(y)$ uniformly in $f \in \mathscr{B}_b^+(E)$.

To apply the coupling method, we assume that the semigroup P_t is associated to a strong Markov process. For fixed $x, y \in E$, we consider the processes $X^x(t)$, $X^y(t)$ on the same probability space starting from x and y respectively such that

$$P_t f(x) = \mathbb{E}[f(X^x(t))], \quad P_t f(y) = \mathbb{E}[f(X^y(t))], \quad t \geq 0, f \in \mathscr{B}_b^+(E). \tag{1}$$

Let $\tau = \inf\{t \geq 0 : X^x(t) = X^y(t)\}$ be the coupling time. By the strong Markov property, we may and do let $X^x(t) = X^y(t)$ for $t \geq \tau$. If $\mathbb{P}(\tau > t) = 0$ then $X^x(t) = X^y(t)$ $\mathbb{P}$-a.s., so that (1) gives

$$P_t f(x) = P_t f(y), \quad f \in \mathscr{B}_b^+(E).$$

This is, however, too strong to be true. Indeed, in general τ is an unbounded random variable such that $\mathbb{P}(\tau > t) > 0$ for $t > 0$. But if $\mathbb{P}(X^x(t) \neq X^y(t)) > 0$, then (1) does not provide any non-trivial comparison of $P_t f(x)$ and $P_t f(y)$ up to a constant independent of f, since, when $X^x(t) \neq X^y(t)$, a function f may be zero at $X^x(t)$ but arbitrarily large at $X^y(t)$. Therefore, to derive the Harnack inequality of P_t using coupling, it seems essential that $\tau \leq t$, which is however impossible as explained above. To avoid the contradiction, we will construct the coupling under different probability measures, which is called coupling by change of measure.

From now on, we fix a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$. We shall define the coupling by change of measure for stochastic processes. Let $\mathscr{L}(X)|_{\mathbb{P}}$ denote the law of a process $X(t)$ under the probability $\mathbb{P}$.

Definition 1.1 Let $X(t)$ and $Y(t)$ be two stochastic processes on E. A stochastic process $(\bar{X}(t), \bar{Y}(t))$ on $E \times E$ is called a coupling by change of measure for $X(t)$ and $Y(t)$ with changed measure $\mathbb{Q}$, if $\mathscr{L}(X)|_{\mathbb{P}} = \mathscr{L}(\bar{X})|_{\mathbb{P}}$ and $\mathbb{Q}$ is a probability measure on $(\Omega, \mathscr{F})$ such that $\mathscr{L}(Y)|_{\mathbb{P}} = \mathscr{L}(\bar{Y})|_{\mathbb{Q}}$. If, in particular, $\mathbb{Q} = \mathbb{P}$, we call $(\bar{X}(t), \bar{Y}(t))$ a coupling for $X(t)$ and $Y(t)$.

In applications, we assume that $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$. In this case, with a coupling by change of measure satisfying $\bar{X}(T) = \bar{Y}(T)$ $\mathbb{Q}$-a.s for a fixed $T > 0$, one may compare the distributions of $X(T)$ and $Y(T)$ using the density $R := \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ (we also denote $\mathbb{Q} = R\mathbb{P}$).

The following is a general result on Harnack type inequalities using coupling by change of measure.

Theorem 1.1 *Let P_t be the Markov semigroup and let $x, y \in E, T > 0$ be fixed. Suppose there is a coupling by change of measure $(\bar{X}(t), \bar{Y}(t))_{t \in [0,T]}$ with $\mathbb{Q} := R\mathbb{P}$ such that $\bar{X}(T) = \bar{Y}(T)$ $\mathbb{Q}$-a.s. Then for any $f \in \mathscr{B}^+(E)$,*

$$\begin{aligned} &(P_T f)^p(y) \leq \big\{P_T f^p(x)\big\}\big\{\mathbb{E}[R^{p/(p-1)}]\big\}^{p-1}, \quad p > 1, \\ &(P_T \log f)(y) \leq \log(P_T f)(x) + \mathbb{E}[R \log R]. \end{aligned}$$

Proof By the definition of coupling by change of measure, we have $P_T f(x) = \mathbb{E} f(\bar{X}(T)), \mathbb{E}[Rf(\bar{Y}(T))] = P_T f(y)$. Combining with $\bar{X}(T) = \bar{Y}(T)$ $\mathbb{Q}$-a.s. and using the Hölder inequality, we obtain

$$\begin{aligned}(P_T f)^p(y) &= \big\{\mathbb{E}[Rf(Y(T))]\big\}^p = \big\{\mathbb{E}[Rf(X(T))]\big\}^p \\ &\le \big\{\mathbb{E}[f^p(X(T))]\big\}\big\{\mathbb{E}R^{p/(p-1}\big\}^{p-1} = \big\{P_T f^p(x)\big\}\big\{\mathbb{E}[R^{p/(p-1)}]\big\}^{p-1}.\end{aligned}$$

Moreover, the Young inequality ([2, Lemma 2.4]) see implies

$$\begin{aligned}(P_T \log f)(y) &= \mathbb{E}[R \log f(Y(T))] = \mathbb{E}[R \log f(X(T))] \\ &\le \log \mathbb{E}[f(X(T))] + \mathbb{E}[R \log R] = \log(P_T f)(x) + \mathbb{E}[R \log R].\end{aligned}$$

□

The Harnack inequality with a power $p > 1$ was first found in [16] for diffusion semigroups on manifolds with curvature bounded below using gradient estimates, and was then extended in [1, 2, 18] to unbounded below curvatures using coupling by change of measure. The log-Harnack inequality was introduced in [14, 19] for semi-linear SPDEs and Neumann semigroups on manifolds respectively. Both inequalities have been intensively investigated and applied for many other models, see e.g. [9, 10, 18, 25] for non-linear SPDEs, [12–14, 28] for semi-linear SPDEs, [3, 5, 15, 27] for functional SDEs, [8, 22, 26] for degenerate SDEs, and [6, 24] for SDEs driven by Lévy and fractional noises. We refer to the survey paper [17] and the monograph [21] for more applications of coupling by change of measure and the above type Harnack inequalities.

In the next section, we introduce a general result on the hypercontractivity using coupling and Harnack inequality. Then we apply this result to degenerate SDEs and functional SPDEs in Sects. 3 and 4 respectively.

2 Hypercontractivity Using Coupling and Harnack Inequality

Let $(E, \mathscr{B}, \mu)$ be a probability space, and let P_t be a Markov semigroup on $\mathscr{B}_b(E)$ such that μ is P_t-invariant, i.e. $\mu(P_t f) = \mu(f)$ for $f \in L^1(\mu)$ and $t \ge 0$. P_t is called hypercontractive with respect to the invariant probability measure μ, if $\|P_t\|_{L^2(\mu)\to L^4(\mu)} = 1$ for large enough $t > 0$. By the interpolation theorem, one may replace the operator norm $\|\cdot\|_{L^2(\mu)\to L^4(\mu)}$ by $\|\cdot\|_{L^p(\mu)\to L^q(\mu)}$ for any $q > p > 1$. This property was found by Nelson [11] for the Ornstein-Uhlenbeck semigroup. In general, the hypercontractivity of P_t implies the exponential convergence in entropy, i.e.

$$\mathrm{Ent}_\mu(P_t f) \le c\mathrm{e}^{-\lambda t}\mathrm{Ent}_\mu(f), \quad t \ge 0,\ f \in \mathscr{B}_b^+(E)$$

holds for some constants $c, \lambda > 0$, where $\mathrm{Ent}_\mu(f) := \mu\big(f \log \frac{f}{\mu(f)}\big)$, see [23] and references therein. According to L. Gross (see e.g. [7]), the hypercontractivity of P_t follows from the log-Sobolev inequality

$$\mu(f^2 \log f^2) - \mu(f^2) \log \mu(f^2) \le C\mu(-f Lf), \quad f \in \mathbb{D}(L)$$

for some constant $C > 0$, where $(L, \mathbb{D}(L))$ is the generator of P_t in $L^2(\mu)$. When P_t is symmetric in $L^2(\mu)$, the hypercontractivity and the log-Sobolev inequality are equivalent. However, in the non-symmetric case, the log-Sobolev inequality is essentially stronger than the hypercontractivity, see Sects. 3 and 4 for hypercontractive semigroups for which the log-Sobolev inequality is not available.

We introduce below a general result on hypercontractivity using coupling and Harnack inequality. A process $(X(t), Y(t))$ on $E \times E$ is called a coupling of the Markov process with semigroup P_t, if

$$(P_t f)(X(0)) = \mathbb{E}\big[f(X_t)\big|X(0)\big], \quad (P_t f)(Y(0)) = \mathbb{E}\big[f(Y(t))\big|Y(0)\big], \quad f \in \mathscr{B}_b(E), t \ge 0.$$

Theorem 2.1 ([23]) *Assume that the following three conditions hold for some measurable functions $\rho : E \times E \to (0, \infty)$ and $\phi : [0, \infty) \to (0, \infty)$ with $\lim_{t\to\infty} \phi(t) = 0$:*

(i) There exist two constants $t_0, c_0 > 0$ such that

$$(P_{t_0} f(\xi))^2 \le (P_{t_0} f^2(\eta)) e^{c_0 \rho(\xi,\eta)^2}, \quad f \in \mathscr{B}_b(E), \xi, \eta \in E;$$

(ii) For any $(X(0), Y(0)) \in E \times E$, there exists a coupling $(X(t), Y(t))$ associated to P_t such that

$$\rho(X(t), Y(t)) \le \phi(t)\rho(X(0), Y(0)), \quad t \ge 0;$$

(iii) There exists $\varepsilon > 0$ such that $(\mu \times \mu)(e^{\varepsilon\rho^2}) < \infty$.

Then μ is the unique invariant probability measure and P_t is hypercontractive. Consequently, P_t is compact in $L^2(\mu)$ for large $t > 0$ and is exponentially convergent in entropy.

Proof (*Sketch*) The Harnack inequality implies that P_t has a density with respect to μ, so that besides the exponential convergence in entropy, the hypercontractivity also implies the compactness of P_t in $L^2(\mu)$ for large $t > 0$, see [23] and references therein for details.

According to [27, Proposition 3.1], (i) implies that μ is the unique invariant probability measure for P_{t_0}, and P_{t_0} has a density with respect to μ. It remains to prove $\|P_t\|^4_{L^2(\mu)\to L^4(\mu)} < 2$ for large enough $t > 0$, which implies the hypercontractivity according to [23, Proposition 2.2].

Let $f \in \mathscr{B}_b(E)$ with $\mu(f^2) \le 1$. By (i) and (ii) we have

$$\begin{aligned}(P_{t+t_0} f(\xi))^2 &\le \mathbb{E}[P_{t_0} f(X(t))]^2 \le \mathbb{E}\Big[\{P_{t_0} f^2(Y(t))\}\mathrm{e}^{c_0\rho(X(t),Y(t))^2}\Big]\\ &\le (P_{t_0+t} f^2(\eta))\mathrm{e}^{c_0\phi(t)^2\rho(\xi,\eta)^2}, \quad t \ge 0, (\xi,\eta) \in E \times E.\end{aligned}$$

Equivalently,

$$(P_{t_0+t} f(\xi))^2 \mathrm{e}^{-c_0\phi(t)^2\rho(\xi,\eta)^2} \le P_{t_0+t} f^2(\eta), \quad t \ge 0, (\xi,\eta) \in E \times E.$$

Integrating with respect to $\mu(\mathrm{d}\eta)$ gives

$$\begin{aligned}&(P_{t_0+t} f(\xi))^2 \int_E \mathrm{e}^{-c_0\phi(t)^2\rho(\xi,\eta)^2}\mu(\mathrm{d}\eta)\\ &\le \int_E P_{t_0+t} f^2(\eta)\mu(\mathrm{d}\eta) = \mu(f^2) \le 1, \quad t \ge 0, \xi \in E.\end{aligned}$$

Thus,

$$(P_{t_0+t} f(\xi))^4 \le \frac{1}{\big(\int_E \exp[-c_0\phi(t)^2\rho(\xi,\eta)^2]\mu(\mathrm{d}\eta)\big)^2}, \quad \mu(f^2) \le 1, t \ge 0, \xi \in E.$$

Then by Jensen's inequality, for $t \ge 0$

$$\begin{aligned}\sup_{\mu(f^2)\le 1} \int_E (P_{t+t_0} f(\xi))^4 \mu(\mathrm{d}\xi) &\le \int_E \frac{\mu(\mathrm{d}\xi)}{(\int_E \exp[-c_0\phi(t)^2\rho(\xi,\eta)^2]\mu(\mathrm{d}\eta))^2}\\ &\le \int_E \left(\int_E \mathrm{e}^{c_0\phi(t)^2\rho(\xi,\eta)^2}\mu(\mathrm{d}\eta)\right)^2 \mu(\mathrm{d}\xi) \le \int_{E\times E} \mathrm{e}^{2c_0\phi(t)^2\rho(\xi,\eta)^2}\mu(\mathrm{d}\xi)\mu(\mathrm{d}\eta).\end{aligned} \tag{2}$$

Since $\lim_{t\to\infty}\phi(t) = 0$, it follows from (iii) that

$$\lim_{t\to\infty} \int_{E\times E} \mathrm{e}^{2c_0\phi(t)^2\rho(\xi,\eta)^2}\mu(\mathrm{d}\xi)\mu(\mathrm{d}\eta) = 1.$$

Combining this with (2) we prove $\|P_t\|_{2\to 4}^4 < 2$ for large enough $t > 0$. □

3 Hypercontractivity for Degenerate SDEs

We only consider finite-dimensional stochastic Hamiltonian systems, see [23] for extensions to infinite-dimensions and typical examples.

Consider the following degenerate SDE for $(X(t), Y(t))$ on $\mathbb{R}^m \times \mathbb{R}^d$:

$$\begin{cases} \mathrm{d}X(t) = (AX(t) + BY(t))\,\mathrm{d}t, \\ \mathrm{d}Y(t) = Z(X(t), Y(t))\mathrm{d}t + \sigma \mathrm{d}W(t), \end{cases} \tag{3}$$

where $W(t)$ is a d-dimensional Brownian motion, and

(A1) A is an $m \times m$-matrix, B is a $d \times m$-matrix, σ is a $d \times d$-matrix, such that σ is invertible and $\mathrm{Rank}[B, AB, \cdots, A^{m-1}B] = m$.
(A2) $Z : \mathbb{R}^{m+d} \to \mathbb{R}^d$ is Lipschitz continuous.
(A3) There exist constants $r, \theta > 0$ and $r_0 \in (-\|B\|^{-1}, \|B\|^{-1})$ such that

$$\begin{aligned} &\big\langle r^2(x-\bar{x}) + rr_0B(y-\bar{y}), A(x-\bar{x}) + B(y-\bar{y})\big\rangle \\ &+ \big\langle Z(x,y) - Z(\bar{x},\bar{y}), y - \bar{y} + rr_0B^*(x-\bar{x})\big\rangle \\ &\le -\theta(|x-\bar{x}|^2 + |y-\bar{y}|^2), \quad (x,y),(\bar{x},\bar{y}) \in \mathbb{R}^{m+d}. \end{aligned}$$

Theorem 3.1 ([23]) *Assume* $(A1)$*,* $(A2)$ *and* $(A3)$*. Let* P_t *be the Markov semigroup associated with* (3)*. Then* P_t *has a unique invariant probability measure* μ *and it is hypercontractive. Consequently,* P_t *is compact in* $L^2(\mu)$ *for large* $t > 0$*, and is exponentially convergent in entropy.*

Proof (*Sketch*). Firstly, by $(A1)$ and $(A2)$ we may construct a coupling by change of measure such that Theorem 3.1 gives the following Harnack inequality: for any $t_0 > 0$,

$$(P_{t_0}f)^2(\xi) \le (P_{t_0}f^2(\eta))\mathrm{e}^{c_0|\xi-\eta|^2}, \quad f \in \mathscr{B}_b(\mathbb{R}^{m+d}), \xi, \eta \in \mathbb{R}^{m+d}$$

holds for some constant $c_0 > 0$.

Secondly, if $(A3)$ holds then we may find out two constants $c, \lambda > 0$ such that for any two solutions $(X(t), Y(t))$ and $(\mathtt{X}(\mathtt{t}), \mathtt{Y}(\mathtt{t}))$ of (3),

$$|X(t) - \mathtt{X}(\mathtt{t})|^2 + |\mathtt{Y}(\mathtt{t}) - \mathtt{Y}(\mathtt{t})|^2 \le \mathtt{c}\mathrm{e}^{-\lambda \mathtt{t}}(|\mathtt{X}(0) - \mathtt{X}(0)|^2 + |\mathtt{Y}(0) - \mathtt{Y}(0)|^2), \ \ \mathtt{t} \ge 0.$$

Finally, if $(A3)$ holds then the standard argument using a Lyapunov condition implies that P_t has an invariant probability measure μ such that $\mu(\mathrm{e}^{\varepsilon|\cdot|^2}) < \infty$ for some constant $\varepsilon > 0$.

Therefore, the proof is finished by Theorem 2.1. □

4 Hypercontractivity for Functional SPDEs

We will only consider non-degenerate functional semi-linear SPDEs, see [4] for results on degenerate functional SPDEs and specific examples.

Let $\mathbb{H}$ be a separable Hilbert space. For a fixed constant $r_0 > 0$, consider the path space $\mathscr{C} = C([-r_0, 0]; \mathbb{H})$ equipped with the uniform norm $\|f\|_\infty := \sup_{-r_0 \le \theta \le 0} |f(\theta)|$. For a map $h(\cdot) : [-r_0, \infty) \to \mathbb{H})$, we define its segment functional $h_\cdot : [0, \infty) \to \mathscr{C}$ by letting

$$h_t(\theta) = h(t + \theta), \quad \theta \in [-r_0, 0].$$

Consider the following SPDE on $\mathbb{H}$:

$$\mathrm{d}X(t) = \{AX(t) + b(X_t)\}\mathrm{d}t + \sigma \mathrm{d}W(t), \quad t > 0, \quad X_0 = \xi \in \mathscr{C}, \tag{4}$$

where $W(t)$ is a cylindrical Brownian motion on $\mathbb{H}$; is,

$$W(t) = \sum_{i=1}^{\infty} B_i(t) e_i, \quad t \ge 0$$

for an orthonormal basis $\{e_i\}_{i\ge1}$ on $\mathbb{H}$ and a sequence of independent one-dimensional Brownian motions $\{B_i(t)\}_{i\ge1}$. Moreover:

(H1) $(-A, \mathscr{D}(A))$ is a self-adjoint operator on $\mathbb{H}$ with discrete spectrum $0 < \lambda_1 \le \lambda_2 \le \cdots$ counting multiplicities such that $\lambda_i \uparrow \infty$, such that for some constant $\delta \in (0, 1)$,

$$\int_0^1 \|\mathrm{e}^{-t(-A)^{1-\delta}} \sigma\|_{HS}^2 \mathrm{d}t < \infty, \quad t > 0,$$

where $\|\cdot\|_{HS}$ stands for the Hilbert-Schmidt norm.

(H2) $b : \mathscr{C} \to \mathbb{H}$ is such that

$$|b(\xi) - b(\eta)| \le L\|\xi - \eta\|_\infty, \quad \xi, \eta \in \mathscr{C}$$

holds for some constant $L > 0$.

(H3) $(\sigma, \mathbb{D}(\sigma))$ is an invertible linear operator on $\mathbb{H}$, i.e. there exists bounded operator σ^{-1} such that $\sigma^{-1}\mathbb{H} \subset \mathbb{D}(\sigma)$ and $\sigma\sigma^{-1} = I$, the identity operator.

It is easy to see that **(H1)** and **(H2)** imply

$$\int_0^1 \|\mathrm{e}^{tA}\sigma\|_{HS}^{2(1+\varepsilon)} \mathrm{d}t < \infty$$

for some $\varepsilon > 0$. So, according to e.g. [21, Theorem 4.1.3], for any initial point $\xi \in \mathscr{C}$, the equation (4) has a unique continuous mild solution $(X^\xi(t))_{t\ge0}$. Let $\{X_t^\xi\}_{t\ge0}$ be the corresponding segment solution. Then the associated Markov semigroup is given by

$$P_t f(\xi) := \mathbb{E} f(X_t^\xi), \quad f \in \mathscr{B}_b(\mathscr{C}), \quad \xi \in \mathscr{C}.$$

Theorem 4.1 ([4]) *Let* **(H1)**–**(H3)** *hold. If* $\lambda := \sup_{s\in(0,\lambda_1]}\big(s - Le^{sr_0}\big) > 0$*, then* P_t *has a unique invariant probability measure and is hypercontractive. Consequently,* P_t *is compact in* $L^2(\mu)$ *for large* $t > 0$ *and is exponentially convergent in entropy.*

Proof (*Sketch*) By constructing a suitable coupling by change of measure in terms of **(H1)** and **(H2)**, we establish the following Harnack inequality according to Theorem 3.1: for any $t_0 > r_0$, there exists a constant $c_0 > 0$ such that (see [21, Theorem 4.2.4]):

$$\big(P_{t_0}f(\eta)\big)^2 \le (P_{t_0}f^2(\xi))\mathrm{e}^{c_0\|\xi-\eta\|_\infty^2}, \quad \xi,\eta\in\mathscr{C},\ f\in\mathscr{B}_b(\mathscr{C}).$$

Next, by **(H1)** and **(H2)** we have

$$\mathrm{e}^{\lambda_1 t}|X^\xi(t) - X^\eta(t)| \le |\xi(0)-\eta(0)| + L\int_0^t \mathrm{e}^{\lambda_1 s}\|X_s^\xi - X_s^\eta\|_\infty \mathrm{d}s.$$

By Gronwall's inequality this implies

$$\|X_t^\xi - X_t^\eta\|_\infty \le \mathrm{e}^{\lambda_1 r_0}\mathrm{e}^{-\lambda t}\|\xi-\eta\|_\infty, \quad t\ge 0,\ \ \xi,\eta\in\mathscr{C}.$$

According to Theorem 2.1, it remains to verify $\mu(\mathrm{e}^{\varepsilon\|\cdot\|_\infty^2}) < \infty$ for some constant $\varepsilon > 0$. This can be done by applying an infinite-dimensional Fernique inequality. □

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