Large Time Behavior and Convergence for the Camassa-Holm Equations with Fractional Laplacian Viscosity

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Abstract. In this paper, we consider the n-dimensional ($n = 2, 3$) Camassa-Holm equations with fractional Laplacian viscosity in the whole space. In contrast to the Camassa-Holm equations without any nonlocal effect, much less has been known on the large time behavior and convergences of solutions. Here we study first the large time behavior of solutions, then consider the relation between the equations under consideration and the imcompressible Navier-Stokes equations with fractional Laplacian viscosity (INSF). By applying the fractional Leibniz chain rule and the fractional Gagliardo-Nirenberg-Sobolev type estimates, the high and low frequency splitting method and the Fourier splitting method, we shall establish the large time non-uniform decays and algebraic rate decays of solutions. In the critical case $s = \frac{n}{4}$, the nonlocal version of Ladyzhenskaya's inequality along with the smallness of initial data in suitable Sobolev spaces are needed. In addition, by estimates for the fractional heat kernels, we prove that the solutions to the Camassa-Holm equations with nonlocal viscosity converge strongly as the filter parameter $\alpha \to 0$ to solutions of the equations (INSF).

Key Words. Camassa-Holm equations; Fractional Laplacian viscosity; Large time behavior; Convergence

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1 Introduction

In this article, we investigate the following Camassa-Holm equations with fractional Laplacian viscosity in $\mathbb{R}^n$ ($n = 2, 3$):

$$\begin{cases} \mathbf{v}_t + \mathbf{u} \cdot \nabla \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{u}^T + \nabla p = -\nu(-\Delta)^\beta \mathbf{v}, \ (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n, \\ \mathbf{u} - \alpha^2 \Delta \mathbf{u} = \mathbf{v}, \\ \operatorname{div} \mathbf{v} = \operatorname{div} \mathbf{u} = 0, \end{cases} \tag{1.1}$$

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together with the initial condition

$$\mathbf{v}(0,x)=\mathbf{v}_0(x),\ \mathbf{u}(0,x)=\mathbf{u}_0(x),\ x\in\mathbb{R}^n. \tag{1.2}$$

Here, $\mathbf{v}$, $\mathbf{u}$ denotes the fluid velocity field and the filtered fluid velocity, respectively, and p the scalar pressure. α is a length scale parameter representing the width of the filter, and $\nu>0$ is the viscosity coefficient which is fixed in our discussions. In particular, the divergence free condition div $\mathbf{v}=0$ indicates the imcompressibility of the fluid, $(-\Delta)^\beta$ denotes the fractional power of the Laplacian in $\mathbb{R}^n$, $\frac{n}{4}\le\beta<1$ and $n=2,3$. Recall that the Camassa-Holm equations with Laplacian viscosity (equations (1.1) with $\beta=1$) read

$$\begin{cases}\mathbf{v}_t+\mathbf{u}\cdot\nabla\mathbf{v}+\mathbf{v}\cdot\nabla\mathbf{u}^T+\nabla p=\nu\Delta\mathbf{v},\ (t,x)\in\mathbb{R}^+\times\mathbb{R}^n,\\ \mathbf{u}-\alpha^2\Delta\mathbf{u}=\mathbf{v},\\ \text{div }\mathbf{v}=0.\end{cases} \tag{1.3}$$

As it is well-known that the system (1.3) rose from work on shallow water equations [8]. Specifically, it was introduced in [26] as a natural mathematical generalization of the integrable inviscid one-dimensional Camassa-Holm equation discovered in [8] through a variational formulation and with a lagrangian averaging. It could be used as a closure model for the mean effects of subgrid excitations, and be also viewed as a filtered Navier-Stokes equations with the parameter α in the filter, which obeys a modified Kelvin circulation theorem along filtered velocities [26]. Numerical examples that seem to justify this intuition were reported in [10]. Formally, the system (1.3) reduces to the imcompressible Navier-Stokes equations as $\alpha\to 0$:

$$\begin{cases}\mathbf{v}_t+\mathbf{v}\cdot\nabla\mathbf{v}+\nabla p=\nu\Delta\mathbf{v},\\ \text{div }\mathbf{v}=0.\end{cases} \tag{1.4}$$

For the fractional Laplacian in the whole space, there are several ways to define it [5, 36, 42]. For example, for a function $f\in\mathscr{S}$, the integral fractional Laplacian $(-\Delta)^\beta$ at the point x can be defined as

$$\begin{aligned}I_\beta f(x)\ &\triangleq(-\Delta)^\beta f(x):=C_{n,\beta}\,\text{P.V.}\int_{\mathbb{R}^n}\tfrac{f(x)-f(\xi)}{|x-\xi|^{n+2\beta}}d\xi,\\ &:=C_{n,\beta}\,\text{P.V.}\int_{\mathbb{R}^n}\frac{f(x)-f(\xi)}{|x-\xi|^{n+2\beta}}d\xi\\ &:=C_{n,\beta}\lim_{\varepsilon\to0^+}\int_{|\xi|>\varepsilon}\frac{f(x+\xi)-f(x)}{|\xi|^{n+2\beta}}d\xi,\end{aligned} \tag{1.5}$$

or equivalently

$$I_\beta f(x)\triangleq(-\Delta)^\beta f(x):=\frac{C_{n,\beta}}{2}\int_{\mathbb{R}^n}\frac{2f(x)-f(x+y)-f(x-y)}{|y|^{n+2\beta}}dy, \tag{1.6}$$

where the parameter β is a real number with $0<\beta<1$, P.V. is a commonly used abbreviation for "in the principle value sense" (as defined by the latter equation), and $C_{n,\beta}$ is some normalization constant depending only on n and β given by

$$C_{n,\beta}=\left(\int_{\mathbb{R}^n}\frac{1-\cos(\zeta_1)}{|\zeta|^{n+2\beta}}d\zeta\right)^{-1}. \tag{1.7}$$

Before going further, we collect several definitions and basic facts concerning the fractional Sobolev spaces $W^{\beta,p}(\mathbb{R}^n)$ and $H^\beta(\mathbb{R}^n)$, as well as the fractional Laplacian [42].

Definition 1.1. *In the whole space, for $\beta \in (0,1)$, if $f \in \mathscr{S}(\mathbb{R}^n)$, let $\Lambda^\gamma = (-\Delta)^\beta$ with $\gamma = 2\beta$, and*

$$\widehat{\Lambda^{2\beta} f}(\xi) = \widehat{(-\Delta)^\beta f}(\xi) = |\xi|^{2\beta} \widehat{f}(\xi),$$

the domain of definition of the fractional Laplacian, $\mathcal{D}\left(\Lambda^\beta\right)$ is endowed with a natural norm $\|\cdot\|_{\mathcal{D}(\Lambda^\beta)}$ and is a Hilbert space. The norm of u in $\mathcal{D}\left(\Lambda^\beta\right)$ is defined by

$$\|u\|_{\mathcal{D}(\Lambda^\beta)} := \left\|\Lambda^\beta u\right\|_{L^2(\mathbb{R}^n)}. \tag{1.8}$$

It should be pointed out that in the whole space, if any function $\psi \in \mathscr{S}(\mathbb{R}^n)$, $\mathcal{D}\left(\Lambda^\beta\right)$ is equivalent to the fractional Sobolev space $\dot{H}^\beta(\mathbb{R}^n)$, defined as the completion of $C_0^\infty(\mathbb{R}^n)$ with the norm

$$\|\psi\|_{\dot{H}^\beta(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\xi|^{2\beta} \left|\widehat{\psi}\right|^2 d\xi\right)^{\frac{1}{2}} = \left\|(-\Delta)^{\frac{\beta}{2}}\psi\right\|_{L^2(\mathbb{R}^n)}. \tag{1.9}$$

On the other hand, the norm $\|u\|_{H^\beta(\mathbb{R}^n)}$ in the fractioal Laplacian Sobolev space $H^\beta(\mathbb{R}^n)$ is given by

$$\|\mathbf{u}\|^2_{H^\beta(\mathbb{R}^n)} := 2C(n,\beta)^{-1}\left\|\Lambda^\beta u\right\|^2_{L^2(\mathbb{R}^n)} + \|u\|^2_{L^2(\mathbb{R}^n)}. \tag{1.10}$$

In particular, the norm of $\mathcal{D}\left(\Lambda^2\right) = \mathcal{D}(-\Delta)$ is equivalent to the $H^2(\mathbb{R}^n)$ norm. □

Definition 1.2. *Let $\beta \in (0,1)$. For any $p \in [1,\infty)$, we define $W^{\beta,p}(\mathbb{R}^n)$ as follows*

$$W^{\beta,p}(\mathbb{R}^n) := \left\{u \in L^p(\mathbb{R}^n) : \ \frac{|u(x)-u(y)|}{|x-y|^{\frac{n}{p}+\beta}} \in L^p(\mathbb{R}^n \times \mathbb{R}^n)\right\}, \tag{1.11}$$

i.e., an intermediary Banach space between $L^p(\mathbb{R}^n)$ and $W^{1,p}(\mathbb{R}^n)$, endowed with the natural norm

$$\|u\|_{W^{\beta,p}(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |u|^p dx + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+\beta p}} dxdy\right)^{\frac{1}{p}}, \tag{1.12}$$

where the term

$$[u]_{W^{\beta,p}(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+\beta p}} dxdy\right)^{\frac{1}{p}} \tag{1.13}$$

is the so-called Gagliardo (semi) norm of u.

However, there is another case for $\beta \in (1,\infty)$ and β is not an integer. In this case, we write $\beta = m + m'$, where m is an integer and $m' \in (0,1)$. The space $W^{\beta,p}(\mathbb{R}^n)$ consists of those equivalence classes of functions $u \in W^{m,p}(\mathbb{R}^n)$ whose distributional derivatives $D^\alpha u$, with $|\alpha| = m$, belong to $W^{m',p}(\mathbb{R}^n)$, namely

$$W^{\beta,p}(\mathbb{R}^n) := \left\{u \in W^{m,p}(\mathbb{R}^n) : \ D^\alpha u \in W^{m',p}(\mathbb{R}^n) \ for \ any \ \alpha \ s.t. \ |\alpha| = m\right\},$$

and this is a Banach space with respect to the norm

$$\|u\|_{W^{\beta,p}(\mathbb{R}^n)} := \left(\|u\|^p_{W^{m,p}(\mathbb{R}^n)} + \sum_{|\alpha|=m} \left\|D^\alpha u\right\|^p_{W^{m',p}(\mathbb{R}^n)}\right)^{\frac{1}{p}}. \tag{1.14}$$

Clearly, if $\beta = m$ is an integer, the space $W^{\beta,p}(\mathbb{R}^n)$ coincides with the Sobolev space $W^{m,p}(\mathbb{R}^n)$.

Note that for any $\beta > 0$, the space $C_0^\infty(\mathbb{R}^n)$ of smooth functions with compact support is dense *in* $W^{\beta,p}(\mathbb{R}^n)$*, and* $W_0^{\beta,p}(\mathbb{R}^n) = W^{\beta,p}(\mathbb{R}^n)$*, where* $W_0^{\beta,p}(\mathbb{R}^n)$ *denotes the closure of* $C_0^\infty(\mathbb{R}^n)$ *in the space* $W^{\beta,p}(\mathbb{R}^n)$*.*

In particular, for $\beta \in (0,1)$ *and* $p = 2$*, the fractional Sobolev spaces* $W^{\beta,2}(\mathbb{R}^n)$ *and* $W_0^{\beta,2}(\mathbb{R}^n)$ *turn out to be Hilbert spaces, which are usually labeled by* $W^{\beta,2}(\mathbb{R}^n) = H^\beta(\mathbb{R}^n)$ *and* $W_0^{\beta,2}(\mathbb{R}^n) = H_0^\beta(\mathbb{R}^n)$*. That is,*

$$H^\beta(\mathbb{R}^n) := \left\{u \in L^2(\mathbb{R}^n) : \ \frac{|u(x) - u(y)|}{|x-y|^{\frac{n}{2}+\beta}} \in L^2(\mathbb{R}^n \times \mathbb{R}^n)\right\}, \tag{1.15}$$

i.e., an intermediary Hilbert space between $L^2(\mathbb{R}^n)$ *and* $H^1(\mathbb{R}^n)$*, endowed with the natural norm*

$$\|u\|_{H^\beta(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |u|^2 dx + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\beta}} dxdy\right)^{\frac{1}{2}}, \tag{1.16}$$

where the term

$$[u]_{H^\beta(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\beta}} dxdy\right)^{\frac{1}{2}} \tag{1.17}$$

is the so-called seminorm of u.

There is an alternative definition of the space $H^\beta(\mathbb{R}^n)$ *via the Fourier transform. For any real* $\beta \geq 0$*, we may define*

$$\widehat{H^\beta}(\mathbb{R}^n) := \left\{u \in L^2(\mathbb{R}^n) : \ \int_{\mathbb{R}^n} \left(1 + |\xi|^{2\beta}\right) |\mathcal{F}u(\xi)|^2 d\xi < \infty\right\}. \tag{1.18}$$

In the same manner, for $\beta < 0$ *there is an analogous definition for* $H^\beta(\mathbb{R}^n)$*:*

$$H^\beta(\mathbb{R}^n) := \left\{u \in \mathscr{S}'(\mathbb{R}^n) : \ \int_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^\beta |\mathcal{F}u(\xi)|^2 d\xi < \infty\right\}. \tag{1.19}$$

On the other hand, let $\beta \in (0,1)$ *and let* $(-\Delta)^\beta : \ \mathscr{S} \to L^2(\mathbb{R}^n)$ *be the fractional Laplacian operator defined by* (1.6)*. Then*

(1) For any $u \in \mathscr{S}$*,*

$$(-\Delta)^\beta u = \mathcal{F}^{-1}\left[|\xi|^{2\beta}(\mathcal{F}u)\right], \ \ \forall \xi \in \mathbb{R}^n. \tag{1.20}$$

(2) The fractional Sobolev space $H^\beta(\mathbb{R}^n)$ *defined in* (1.15) *coincides with* $\widehat{H^\beta}(\mathbb{R}^n)$ *defined in* (1.18)*. In particular, for any* $u \in H^\beta(\mathbb{R}^n)$

$$[u]^2_{H^\beta(\mathbb{R}^n)} = 2C(n,\beta)^{-1} \int_{\mathbb{R}^n} |\xi|^{2\beta} |\mathcal{F}u(\xi)|^2 \, d\xi, \tag{1.21}$$

where $C(n,\beta)$ *is defined by* (1.7)*.*

(3) For $u \in H^\beta(\mathbb{R}^n)$*,*

$$[u]^2_{H^\beta(\mathbb{R}^n)} = 2C(n,\beta)^{-1} \left\|(-\Delta)^{\frac{\beta}{2}} u\right\|^2_{L^2(\mathbb{R}^n)}, \tag{1.22}$$

where $C(n,\beta)$ *is defined by* (1.7)*.* □

Recently, a great attention has been paid to the study of nonlocal problems driven by fractional Laplacian type operators in the literature. Partially it is because of the fact that fractional Laplacian $(-\Delta)^\beta$ as a spatial integro-differential operator can be used to describe the spatial nonlocality and power law behaviors in various science and engineering problems. For example, it has been utilized to model energy dissipation of acoustic propagation in human tissue [7], turbulence diffusion [9], contaminant transport in ground water [44], non-local heat conduction [4, 12, 41], and electromagnetic fields on fractals [49].

For the system (1.3), the non-uniform decay and algebraic decay of solutions were considered in [3]. Concerning the convergence of solutions of (1.3) to that of the imcompressible Navier-Stokes equations (1.4), in [19, 22] it is shown that solutions of (1.3) approach to solutions of (1.4) weakly when the filter parameter α tends to zero. In a work of Bjorland and Schonbek [3], the convergence in strong norms are proved provided solutions of (1.4) are sufficiently regular. In a follow up work [2], Bjorland investigated further the relationship between solutions of the Navier-Stokes equations (1.4) and the Camassa-Holm equations (1.3) by computing the first and second order decay asymptotics for solutions with small initial data. Decay ratess for solutions of (1.4) have been studied by various authors earlier, see e.g., [6, 19, 22, 47, 48]. The asymptotic behavior of the 2-D vorticity equation for (1.4) has been investigated in [6, 19, 22]. For examples, in [6], Carpio studied the asymptotic behavior for the vorticity equation for (1.4) in two and three space dimensions; Gallay and Wayne in [19] calculated the asymptotics by applying invariant manifold technique to the semiflow governing the vorticity equation for (1.4). The large time behavior of the vorticity of two-dimensional viscous flow for (1.4) was established by Giga and Kambe in [22].

In contrast to those works on the Camassa-Holm equations (1.3), less has been known when equations contain nonlocal spatial fractional viscosity despite non-standard diffusions are very natural also for these problems. The Camassa-Holm equations with fractional Laplacian viscosity (1.1) is technically more challenging due to the vector integral expressions and nonlocal property.

The aim of this paper is twofold. We first want to establish the large time behavior and the non-uniform decay and algebraic decay of solutions to the nonlocal Camassa-Holm equations (1.1). Our second goal is to show the connections betweem the nonlocal equations under study and the incompressible Navier-Stokes equations with fractional Laplacian viscosity:

$$\begin{cases} \mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} + \nabla p = -\nu(-\Delta)^{\beta}\mathbf{v}, \\ \operatorname{div}\mathbf{v} = 0. \end{cases} \tag{1.23}$$

To achieve these results, we have to use some basic properties of the fractional Laplacian as formulated in [5, 11]. In particular, we need a fractional Leibniz chain rule and fractional Gagliardo-Nirenberg-Sobolev type estimates. To establish the large time behavior with non-uniform decays and algebraic decays of solutions to the nonlocal equations, we use the so-called ”high and low frequency splitting method” introduced in [40] and the Fourier splitting method as in [30, 31]. In the critical case $s = \frac{n}{4}$, we showed a nonlocal version of Ladyzhenskaya's inequality; with the smallness of initial data in Sobolev spaces we can show the desired global estimates. In addition, by sharp estimates on the fractional heat kernels [13] and on the Leray projector, we can prove the convergence of solutions of the nonlocal equations (1.1) to that of imcompressible viscous nonlocal Navier-Stokes equations (1.23) strongly as the filter parameter $\alpha \to 0$.

A couple remarks on the nonlocal Camassa-Holm equations (1.1) are in order.

Remark 1.3. Problems involving a fractional power of Laplacian have been appeared in many studies, see for examples [1, 5, 14, 15, 29, 38, 39, 50]. When the spatial dimension is one, in [25] the followings are proved:

- Global well-posedness and blow-up of solutions to the Camassa-Holm equations with fractional dissipation under the supercritical case: $\gamma \in \left[\frac{1}{2}, 1\right)$.

- The zero filter limit of the Camassa-Holm equation with fractional dissipation, as well as the possible blow-up of solutions in the sub-critical cases: $0 \le \gamma < \frac{1}{2}$. Here we study the related issues in both two and three dimensions.

Remark 1.4. As it is well-known for the Navier-Stokes equations, the scalling invariants played a crucial role. Adding a filter to a Camassa-Holm equation with a fractional viscousity does introduce a smoothing effect on the solution. But the filter-equations do not scale well with the original dynamical equations, and the resulting nonlinear terms do have a dependence on the scalings. The latter complicates the matter substantially and results in various difficulties in analysis.

The following notations will be used thoughout this paper.

Notations

$\mathscr{S}(\mathbb{R}^n)$ denotes the Schwartz calss. The i^{th} component of $\mathbf{v}\cdot\nabla\mathbf{u}^T$ is denoted by $\left(\mathbf{v}\cdot\nabla\mathbf{u}^T\right)_i = \sum_{j=1}^{n} v_j\partial_i u_j$. Let $\langle\mathbf{u},\mathbf{v}\rangle = \int_{\mathbb{R}^n}\mathbf{u}\cdot\mathbf{v}dx$ and $\Sigma = \left\{\phi\in C_0^\infty(\mathbb{R}^n)|\nabla\cdot\phi = 0\right\}$. $L_0^p(\mathbb{R}^n)$ denotes the closure of $C_0^\infty(\mathbb{R}^n)$ in the space $L^p(\mathbb{R}^n)$ and $H_0^m(\mathbb{R}^n)$ the completion of $C_0^\infty(\mathbb{R}^n)$ in the norm $\|\cdot\|_{H^m(\mathbb{R}^n)}$. We denote by $L^p(\mathbb{R}^n)$ the standard Lebesgue space, and $L^p_\sigma(\mathbb{R}^n)$ the completion of Σ in the norm $\|\cdot\|_{L^p(\mathbb{R}^n)}$. The completion of Σ under the $\mathcal{D}\left(\Lambda^\beta\right)(\mathbb{R}^n)$-norm is denoted by $\mathcal{D}_\sigma\left(\Lambda^\beta\right)(\mathbb{R}^n)$ and $\left(\mathcal{D}_\sigma\left(\Lambda^\beta\right)\right)'(\mathbb{R}^n)$ is its dual space. The completion of Σ under the $H^m(\mathbb{R}^n)$-norm will be denoted by $H^m_\sigma(\mathbb{R}^n)$ and $(H^m_\sigma(\mathbb{R}^n))'$ be the corresponding dual space. $\mathcal{F}(\phi)$ or $\hat{\phi}$ denotes the Fourier transform of a function ϕ, with $\mathcal{F}^{-1}(\phi)$ or $\check{\phi}$ the inverse Fourier transform. For $a \lesssim b$, we mean that there is a uniform constant C, which may be different on different lines, such that $a \le Cb$. For $\beta = 1$, $\mathcal{D}(-\Delta) = H^2(\mathbb{R}^n)\cap H_0^1(\mathbb{R}^n)$ and $\mathcal{D}(\Lambda)(\mathbb{R}^n) = \dot{H}_0^1(\mathbb{R}^n)$. Generally, the letter C will denote a generic constant. □

The rest of the paper is organized as follows: in Section 2 we collect some preliminaries. In Section 3 the non-uniform decay is established. Subsequently, in Section 4 we show the algebraic decay. In the last section (Section 5), we prove the convergence from the solution of (1.1)-(1.2) to the imcompressible Navier-Stokes equations with nonlocal viscosity (1.23).

2 Preliminaries

In this section, we collect several preliminary results.

Lemma 2.1. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be smooth divergence free functions with compact support. Then one has*

$$\begin{aligned}
&\mathbf{u}\cdot\nabla\mathbf{v} + \sum_{j=1}^{n} v_j\nabla u_j = -\mathbf{u}\times(\nabla\times\mathbf{v}) + \nabla(\mathbf{v}\cdot\mathbf{u}),\\
&\langle\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{u}\rangle + \left\langle\mathbf{v}\cdot\nabla\mathbf{u}^T,\mathbf{u}\right\rangle = 0,\\
&\langle\mathbf{u}\times(\nabla\times\mathbf{v}),\mathbf{u}\rangle = 0.
\end{aligned}$$

Proof. By direct calculation, it is easy to achieve these expected identities. □

Lemma 2.2. *For* $n = 2, 3$ *and* $0 < \beta < 1$, *let* $\mathbf{u}$ *and* $\mathbf{v}$ *be smooth divergence free functions with compact supports. Then if* $(\mathbf{v},\mathbf{u})$ *solves* (1.1)-(1.2) *, there holds*

$$\frac{1}{2}\frac{d}{dt}\left(\langle\mathbf{u},\mathbf{u}\rangle + \alpha^2\langle\nabla\mathbf{u},\nabla\mathbf{u}\rangle\right) + \nu\left(\left\langle\Lambda^\beta\mathbf{u},\Lambda^\beta\mathbf{u}\right\rangle + \alpha^2\left\langle\nabla\Lambda^\beta\mathbf{u},\nabla\Lambda^\beta\mathbf{u}\right\rangle\right) = 0, \tag{2.1}$$

and

$$\begin{aligned}
&\|\mathbf{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + \alpha^2\|\nabla\mathbf{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} \\
&\quad +2\nu\int_0^t \left\|\Lambda^\beta\mathbf{u}(\cdot,t)\right\|^2_{L^2(\mathbb{R}^n)} dt + 2\nu\alpha^2\int_0^t \left\|\nabla\Lambda^\beta\mathbf{u}(\cdot,t)\right\|^2_{L^2(\mathbb{R}^n)} dt \\
&\quad \le \|\mathbf{u}_0\|^2_{L^2(\mathbb{R}^n)} + \alpha^2\|\nabla\mathbf{u}_0\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned} \tag{2.2}$$

Proof. Thanks to div $\mathbf{v}$ = div $\mathbf{u} = 0$, making inner product with $\mathbf{u}$ on the both sides in the first equation in (1.1) gives

$$\langle \mathbf{v}_t, \mathbf{u}\rangle + \left\langle \mathbf{u}\cdot\nabla\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{u}^T, \mathbf{u}\right\rangle + \langle\nabla p, \mathbf{u}\rangle + \nu\left\langle(-\Delta)^\beta\mathbf{v}, \mathbf{u}\right\rangle = 0.$$

Note that Lemma 2.1, one deduces by integrating by parts

$$\langle \mathbf{v}_t, \mathbf{u}\rangle + \nu\left\langle(-\Delta)^\beta\mathbf{v}, \mathbf{u}\right\rangle = \langle \mathbf{v}_t, \mathbf{u}\rangle + \nu\left\langle\Lambda^\beta\mathbf{v}, \Lambda^\beta\mathbf{u}\right\rangle = 0.$$

This together with the second equation in (1.1) concludes that

$$\begin{aligned}
&\left\langle \mathbf{u}_t - \alpha^2\Delta\mathbf{u}_t, \mathbf{u}\right\rangle + \nu\left\langle(-\Delta)^\beta(\mathbf{u}-\alpha^2\Delta\mathbf{u}), \mathbf{u}\right\rangle \\
&= \frac{1}{2}\frac{d}{dt}\left(\langle\mathbf{u},\mathbf{u}\rangle + \alpha^2\langle\nabla\mathbf{u},\nabla\mathbf{u}\rangle\right) + \nu\left(\left\langle\Lambda^\beta\mathbf{u},\Lambda^\beta\mathbf{u}\right\rangle + \alpha^2\left\langle\nabla\Lambda^\beta\mathbf{u},\nabla\Lambda^\beta\mathbf{u}\right\rangle\right) \\
&= 0.
\end{aligned}$$

This is the equality (2.1). (2.2) follows by integrating both sides of (2.1) with respect to t. □

Before going further, we introduce the following notion of weak solutions to the Camassa-Holm equations with fractional Laplacian viscosity (1.1)-(1.2) in $\mathbb{R}^n$ ($n = 2,3$).

Definition 2.3. *Let* $\frac{n}{4} \le \beta < 1$ *with* $n = 2,3$*. A weak solution to* (1.1)-(1.2) *is a pair of functions* $(\mathbf{v},\mathbf{u})$ *such that*

$$\begin{aligned}
&\mathbf{v} \in \mathrm{L}^\infty\left([0,T];\mathrm{L}^2_\sigma(\mathbb{R}^n)\right) \cap \mathrm{L}^2\left([0,T];H^\beta_\sigma(\mathbb{R}^n)\right), \\
&\partial_t\mathbf{v} \in \mathrm{L}^2\left([0,T];\mathcal{B}'\right), \\
&\mathbf{u} \in \mathrm{L}^\infty\left([0,T];\mathrm{H}^2_\sigma(\mathbb{R}^n)\right) \cap \mathrm{L}^2\left([0,T];H^{2+\beta}_\sigma(\mathbb{R}^n)\right), \\
&\mathbf{v}(0,x) = \mathbf{v}_0(x).
\end{aligned}$$

Here, $\mathcal{B} = H^\beta_\sigma(\mathbb{R}^n)$*. In addition, for every* $\phi \in \mathrm{L}^2\left([0,T];H^1_\sigma(\mathbb{R}^n)\right)$ *with* $\phi(T) = 0$*, there holds*

$$\begin{aligned}
&-\int_0^T \langle\mathbf{v},\partial_t\phi\rangle\, ds + \int_0^T \langle\mathbf{u}\cdot\nabla\mathbf{v},\phi\rangle\, ds + \int_0^T \langle\phi\cdot\nabla\mathbf{u},\mathbf{v}\rangle\, ds + \nu\int_0^T \left\langle\Lambda^\beta\mathbf{v},\Lambda^\beta\phi\right\rangle ds \\
&\qquad = \langle\mathbf{v}_0,\phi(0)\rangle.
\end{aligned}$$

In particular, for $t \in [0,T]$ *there holds*

$$\langle\mathbf{u},\phi\rangle + \alpha^2\langle\nabla\mathbf{u},\nabla\phi\rangle = \langle\mathbf{v},\phi\rangle.$$ □

Let $\mathscr{S}(\mathbb{R}^n)$ be the Schwartz class. The nonlocal operator $(-\Delta)^\beta$ is defined for any $g \in \mathscr{S}(\mathbb{R}^n)$ through the Fourier transform: if $(-\Delta)^\beta g = h$, then

$$\widehat{h}(\xi) = |\xi|^{2\beta}\widehat{g}(\xi). \qquad (D-1)$$

It should be pointed out that if ψ and ϕ belong to the Schwartz class $\mathscr{S}(\mathbb{R}^n)$, (D-1) together with Plancherel's theorem yields

$$\int_{\mathbb{R}^n}(-\Delta)^\beta\psi\phi dx = \int_{\mathbb{R}^n}|\xi|^{2\beta}\widehat{\psi\phi}d\xi = \int_{\mathbb{R}^n}(-\Delta)^{\frac{\beta}{2}}\psi(-\Delta)^{\frac{\beta}{2}}\phi dx.$$

Thanks to Theorem 3.1 and Theorem 4.1 in [20] (see also [21]), using the energy method and a bootstrap argument, we obtain the following proposition concerning the existence, uniqueness and regularity of a weak solution to (1.1)-(1.2):

Proposition 2.4. *Let $\frac{n}{4} \le \beta < 1$ with $n = 2, 3$. Assume that*

(1) for $\dfrac{n}{4} < \beta < 1$, $\mathbf{v}_0 \in H^M_\sigma(\mathbb{R}^n)$, $M \ge 0$,

and

(2) for $\beta = \dfrac{n}{4}$, $\mathbf{v}_0 \in H^M_\sigma(\mathbb{R}^n)$, $M \ge 0$, and in addition, there exists an $\varepsilon^* = \varepsilon^*(\alpha, \nu, n)$ sufficiently small such that $\|\mathbf{v}_0\|_{H^M_0(\mathbb{R}^n)} \le \varepsilon^*$.

Then there exists a unique weak solution to (1.1)-(1.2) *in the sense of Definition 2.3. In addition, this solution satisfies the energy estimate* (2.2)*, and for all $m + 2k\beta \le M$ there holds*

$$\left\|\partial_t^k\nabla^m\mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)} + \nu\int_0^T \left\|\partial_t^k\nabla^m\Lambda^\beta\mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)} dt \le C\left(n, \beta, \alpha, \nu, \|\mathbf{v}_0\|_{\mathcal{A}}\right), \qquad (2.3)$$

where $\mathcal{A} = H^M_0(\mathbb{R}^n)$, m and k are both non-negative integers. □

By applying the Gagliardon-Nirenberg-Sobolev inequality to the bound (2.3) in Proposition 2.4, we achieve a corollary which describes the action of the filter.

Corollary 2.5. For $\frac{n}{4} \le \beta < 1$ with $n = 2, 3$, let $(\mathbf{v}, \mathbf{u})$ be the solution to the Cauchy problem (1.1)-(1.2) constructed in Proposition 2.4. Then the following estimates hold for all $m + 2k\beta \le M$:

$$\left\|\partial_t^k\nabla^m\mathbf{u}\right\|^2_{L^2(\mathbb{R}^n)} + 2\alpha^2\left\|\partial_t^k\nabla^{m+1}\mathbf{u}\right\|^2_{L^2(\mathbb{R}^n)} + \alpha^4\left\|\partial_t^k\nabla^{m+2}\mathbf{u}\right\|^2_{L^2(\mathbb{R}^n)} = \left\|\partial_t^k\nabla^m\mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)}, \qquad (2.4)$$

$$\left\|\partial_t^k\nabla^m\mathbf{u}\right\|^2_{L^n(\mathbb{R}^n)} + \left\|\partial_t^k\nabla^m\Lambda^\beta\mathbf{u}\right\|^2_{L^n(\mathbb{R}^n)} + \left\|\partial_t^k\nabla^{m+1}\mathbf{u}\right\|^2_{L^n(\mathbb{R}^n)} \lesssim \left\|\partial_t^k\nabla^m\mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)}, \qquad (2.5)$$

$$\left\|\partial_t^k\nabla^m\mathbf{u}\right\|^2_{L^n(\mathbb{R}^n)} + \nu\int_0^t \left\|\partial_t^k\nabla^m\Lambda^\beta\mathbf{u}\right\|^2_{L^n(\mathbb{R}^n)} ds \le C\left(n, \beta, \alpha, \nu, m, k, \|\mathbf{v}_0\|_{H^M_0(\mathbb{R}^n)}\right). \qquad (2.6)$$

Here, m and k are both non-negative integers.

□

We then claim a lemma concerning the Helmholtz equation $\mathbf{u} - \alpha^2\Delta\mathbf{u} = \mathbf{v}$.

Lemma 2.6. Let $n = 2,3$ and $\frac{n}{4} \leq \beta < 1$. Given $\mathbf{v} \in w^{\beta,p}(\mathbb{R}^n)$ with $1 < p < \infty$, there exists a weak solution $\mathbf{u} \in W^{2,p}(\mathbb{R}^n)$ to the Helmholtz equation $\mathbf{u} - \alpha^2 \Delta \mathbf{u} = \mathbf{v}$ such that the following estimates hold:

$$
\begin{aligned}
&\|\mathbf{u}\|_{L^p(\mathbb{R}^n)} \leq \|\mathbf{v}\|_{L^p(\mathbb{R}^n)},\\
&\|\mathbf{u}\|_{L^q(\mathbb{R}^n)} \leq \frac{C(n,p,q)}{\alpha^{2\gamma_1}} \|\mathbf{v}\|_{L^p(\mathbb{R}^n)},\\
&\|\nabla \mathbf{u}\|_{L^q(\mathbb{R}^n)} \leq \frac{C(n,p,q)}{\alpha^{1+2\gamma_2}} \|\mathbf{v}\|_{L^p(\mathbb{R}^n)},\\
&\|\Delta \mathbf{u}\|_{L^q(\mathbb{R}^n)} \leq \frac{C(n,p,q)}{\alpha^{2+2\gamma_3-\beta}} \left\|\Lambda^\beta \mathbf{v}\right\|_{L^p(\mathbb{R}^n)},
\end{aligned}
$$

where $w^{\beta,p}(\mathbb{R}^n)$ is defined by Definition 1.2, $\gamma_1 = \frac{n}{2}\left(\frac{1}{p} - \frac{1}{q}\right) < 1$, $\gamma_2 = \frac{n}{2}\left(\frac{1}{p} - \frac{1}{q}\right) < \frac{1}{2}$, $\gamma_3 = \frac{n}{2}\left(\frac{1}{p} - \frac{1}{q}\right) < \frac{\beta}{2}$. In particular, there holds that for $0 < \alpha < 1$:

$$
\begin{aligned}
&\|\mathbf{u}\|_{L^q(\mathbb{R}^n)} \leq \frac{C(n,p,q)}{\alpha^{1+\gamma_1}} \|\mathbf{v}\|_{L^p(\mathbb{R}^n)},\\
&\|\nabla \mathbf{u}\|_{L^q(\mathbb{R}^n)} \leq \frac{C(n,p,q)}{\alpha^{\frac{3}{2}+\gamma_2}} \|\mathbf{v}\|_{L^p(\mathbb{R}^n)},\\
&\|\Delta \mathbf{u}\|_{L^q(\mathbb{R}^n)} \leq \frac{C(n,p,q)}{\alpha^{2-\frac{\beta}{2}+\gamma_3}} \left\|\Lambda^\beta \mathbf{v}\right\|_{L^p(\mathbb{R}^n)}.
\end{aligned}
$$

In addition, if $n\left(\frac{2}{p} - 1\right) < \beta$, then the solution is unique.

Proof. Note that $1 - \alpha^2\Delta$ is a strictly positive, compact and self-adjoint operator, using standard elliptic theory and making suitable scaling on spatial variables, Sobolev embedding theorem and interpolation inequalities deduce the expected estimates. □

Lemma 2.7. For $\frac{n}{4} \leq \beta < 1$, $n = 2,3$, let

$$
E(t) \in C^1([0,\infty)),\ \psi \in C^1\left([0,\infty), C^1 \cap L^2(\mathbb{R}^n)\right),\ \widetilde{\psi} \in C^1([0,\infty), L^\infty(\mathbb{R}^n)).
$$

Solutions of (1.1)-(1.2) constructed in Proposition 2.4 admit the following genearlized energy inequalities:

$$
\begin{aligned}
&E(t)\|\psi(t) * \mathbf{v}(t)\|^2_{L^2(\mathbb{R}^n)}\\
&\quad = E(s)\|\psi(s) * \mathbf{v}(s)\|^2_{L^2(\mathbb{R}^n)} + \int_s^t E'(\tau)\|\psi(\tau) * \mathbf{v}(\tau)\|^2_{L^2(\mathbb{R}^n)}\, d\tau\\
&\qquad +2\int_s^t E(\tau)\left\langle \psi'(\tau) * \mathbf{v}(\tau), \psi(\tau) * \mathbf{v}(\tau)\right\rangle d\tau\\
&\qquad -2\nu \int_s^t E(\tau)\left\|\Lambda^\beta \psi(\tau) * \mathbf{v}(\tau)\right\|^2_{L^2(\mathbb{R}^n)} d\tau\\
&\qquad -2\int_s^t E(\tau)\langle \mathbf{u}\cdot\nabla\mathbf{v}, \psi(\tau) * \psi(\tau) * \mathbf{v}(\tau)\rangle\, d\tau\\
&\qquad -2\int_s^t E(\tau)\left\langle \mathbf{v}\cdot\nabla\mathbf{u}^T, \psi(\tau) * \psi(\tau) * \mathbf{v}(\tau)\right\rangle d\tau,
\end{aligned}
\tag{2.7}
$$

and

$$
\begin{aligned}
&E(t)\left\|\widetilde{\psi}(t)\hat{\mathbf{v}}(t)\right\|^2_{L^2(\mathbb{R}^n)}\\
&\quad= E(s)\left\|\widetilde{\psi}(s)\hat{\mathbf{v}}(s)\right\|^2_{L^2(\mathbb{R}^n)}+\int_s^t E'(\tau)\left\|\widetilde{\psi}(\tau)\hat{\mathbf{v}}(\tau)\right\|^2_{L^2(\mathbb{R}^n)}d\tau\\
&\quad+2\int_s^t E(\tau)\left\langle \widetilde{\psi}'(\tau)\hat{\mathbf{v}}(\tau),\widetilde{\psi}(\tau)\hat{\mathbf{v}}(\tau)\right\rangle d\tau\\
&\quad-2\nu\int_s^t E(\tau)\left\|\xi^\beta\widetilde{\psi}(\tau)\hat{\mathbf{v}}(\tau)\right\|^2_{L^2(\mathbb{R}^n)}d\tau\\
&\quad-2\int_s^t E(\tau)\left\langle \mathcal{F}(\mathbf{u}\cdot\nabla\mathbf{v})\widetilde{\psi}^2(\tau)\hat{\mathbf{v}}(\tau)\right\rangle d\tau\\
&\quad-2\int_s^t E(\tau)\left\langle \mathcal{F}(\mathbf{v}\cdot\nabla\mathbf{u}^T)\widetilde{\psi}^2(\tau)\hat{\mathbf{v}}(\tau)\right\rangle d\tau.
\end{aligned}
\tag{2.8}
$$

Proof. Multiplying the first equation in (1.1) by $E(t)\psi*\psi*\mathbf{v}$ and integrating in space variable x yields, after some integration by parts,

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\partial_t\mathbf{v}\cdot E(t)\psi*\psi*\mathbf{v}dx+\int_{\mathbb{R}^n}\mathbf{u}\cdot\nabla\mathbf{v}\cdot E(t)\psi*\psi*\mathbf{v}dx\\
&\quad+\int_{\mathbb{R}^n}\mathbf{v}\cdot\nabla\mathbf{u}^T\cdot E(t)\psi*\psi*\mathbf{v}dx+\int_{\mathbb{R}^n}\nabla p\cdot E(t)\psi*\psi*\mathbf{v}dx\\
&\quad+\nu\int_{\mathbb{R}^n}(-\Delta)^\beta\mathbf{v}\cdot E(t)\psi*\psi*\mathbf{v}dx=0.
\end{aligned}
\tag{2.9}
$$

Rearranging (2.9) gives rise to

$$
\begin{aligned}
&\frac{d}{dt}\left(E(t)\left\|\psi(t)*\mathbf{v}(t)\right\|^2_{L^2(\mathbb{R}^n)}\right)\\
&\quad=E'(t)\left\|\psi(t)*\mathbf{v}(t)\right\|^2_{L^2(\mathbb{R}^n)}+2E(t)\int_{\mathbb{R}^n}(\psi'(t)*\mathbf{v}(t))(\psi(t)*\mathbf{v}(t))\,dx\\
&\quad-2\nu E(t)\int_{\mathbb{R}^n}\left|\Lambda^\beta\psi(t)*\mathbf{v}(t)\right|^2dx-2E(t)\int_{\mathbb{R}^n}\mathbf{u}\cdot\nabla\mathbf{v}\psi(t)*\psi(t)*\mathbf{v}(t)dx\\
&\quad-2E(t)\int_{\mathbb{R}^n}\mathbf{v}\cdot\nabla\mathbf{u}^T\psi(t)*\psi(t)*\mathbf{v}(t)dx.
\end{aligned}
\tag{2.10}
$$

Integrating (2.10) over (s,t) concludes (2.7). To attain (2.8), note that div $\mathbf{v}=0$, making the Fourier transform on the both sides of the first equation in (1.1) with respect to the space variable x, then multiplying the resulting equation by $E(t)\widetilde{\psi}^2(t)\hat{\mathbf{v}}(t)$, one can deduce (2.8). □

Lemma 2.8 ([23, 24, 28])**.** Let $\Lambda^\beta=(-\Delta)^{\frac{\beta}{2}}$ be the standard Riesz potential of order $\beta\in\mathbb{R}$, $\beta>0$, $1<p,\ p_1,\ p_2,\ p_3,\ p_4<\infty$, and $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}+\frac{1}{p_4}$. Then the following bilinear estimate holds for all $f,g\in\mathscr{S}(\mathbb{R}^n)$:

$$
\left\|\Lambda^\beta(fg)\right\|_{L^p(\mathbb{R}^n)}\le C\left\|\Lambda^\beta f\right\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}+C\|f\|_{L^{p_3}(\mathbb{R}^n)}\left\|\Lambda^\beta g\right\|_{L^{p_4}(\mathbb{R}^n)}. \qquad \square
$$

Lemma 2.9 ([18])**.** Let $\Lambda^\beta=(-\Delta)^{\frac{\beta}{2}}$ be the standard Riesz potential of order $\beta\in\mathbb{R}$, $\beta_1,\beta_2\in[0,1]$, $\beta=\beta_1+\beta_2$, and $p,\ p_1,\ p_2\in(1,\infty)$ such that $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$. Then the following bilinear estimate holds for all $f,g\in\mathscr{S}(\mathbb{R}^n)$ with $n\ge1$:

$$
\left\|\Lambda^\beta(fg)-f\Lambda^\beta g-g\Lambda^\beta f\right\|_{L^p(\mathbb{R}^n)}\le C\left\|\Lambda^{\beta_1}f\right\|_{L^{p_1}(\mathbb{R}^n)}\left\|\Lambda^{\beta_2}g\right\|_{L^{p_2}(\mathbb{R}^n)}. \qquad \square
$$

We now give a nonlocal Sobolev type imbedding result.

Lemma 2.10. Let $0 < \beta < 1$ and $n = 2, 3$. Then the inclusion $H^{\beta}_{\sigma}(\mathbb{R}^n) \hookrightarrow L^2_{\sigma}(\mathbb{R}^n)$ is compact.

Proof. It is easy to check it by using standard functional analysis method (see also [42]). □

Lemma 2.11. For $\frac{n}{4} < \beta < 1$ with $n = 2, 3$, let $A = \frac{n}{2} + 1 - 2\beta$. Direct calculation gives

(I) $\beta \geq 1 - \beta,\ \dfrac{n-2}{2n} < \dfrac{2\beta - 1}{n} < \dfrac{1}{n},\ \dfrac{2\beta-1}{n} = \dfrac{1}{2} - \dfrac{A}{n},$
$\dfrac{n}{2} - 1 < A < 1 < \dfrac{n}{2} < 2$ for $n = 3$,
$\dfrac{n}{2} - 1 < A < 1 = \dfrac{n}{2} < 2$ for $n = 2$.

(II) Due to $\frac{n}{2\beta-1} = \frac{2n}{n-2A}$, for $\dfrac{n}{4} \leq \beta < 1$ with $n = 3$, and $\dfrac{n}{4} < \beta < 1$ with $n = 2$, we have the following fractional Sobolev-type continuous imbedding between $\dot{H}^{A}_{\sigma}(\mathbb{R}^n)$ and $L^{\frac{n}{2\beta-1}}(\mathbb{R}^n)$:

$$\dot{H}^{A}_{\sigma}(\mathbb{R}^n) \hookrightarrow L^{\frac{n}{2\beta-1}}(\mathbb{R}^n).$$
□

The following Lemma concerns the nonlocal version of the known estimates given in Ladyzhenskaya-Shkoller-Seregin [32, 33, 34, 35].

Lemma 2.12. *For $n = 2, 3$ and $u(x) \in H^1_0(\mathbb{R}^n)$, $\forall\, \varepsilon > 0$, the following estimates hold:*

$$\|u\|^2_{L^4(\mathbb{R}^n)} \leq \varepsilon \|\nabla u\|^2_{L^2(\mathbb{R}^n)} + \varepsilon^{-1}\|u\|^2_{L^2(\mathbb{R}^n)} \quad \textit{for } n = 2, \qquad (E-1)$$

$$\|u\|^2_{L^4(\mathbb{R}^n)} \leq 3^{-\frac{1}{4}} \sqrt{2\varepsilon} \|\nabla u\|^2_{L^2(\mathbb{R}^n)} + \sqrt{2}(3^{\frac{5}{2}}\varepsilon)^{-\frac{1}{6}}\|u\|^2_{L^2(\mathbb{R}^n)} \quad \textit{for } n = 3. \qquad (E-2)$$

The above inequalities (E-1) and (E-2) can be generalized to the following nonlocal version (fractional power Sobolev-type) estimates.

♡ *For $\dfrac{n}{4} < \beta < 1$ and $u \in H^{\beta}_0(\mathbb{R}^n)$, the following estimates hold:*

$$\|u\|^2_{L^4(\mathbb{R}^n)} \leq \varepsilon \left\|\Lambda^{\beta} u\right\|^2_{L^2(\mathbb{R}^n)} + \varepsilon^{-1}\|u\|^2_{L^2(\mathbb{R}^n)} \quad \textit{for } n = 2, \qquad (E-3)$$

$$\|u\|^2_{L^4(\mathbb{R}^n)} \leq C(\beta)\varepsilon \left\|\Lambda^{\beta} u\right\|^2_{L^2(\mathbb{R}^n)} + C(\varepsilon)\|u\|^2_{L^2(\mathbb{R}^n)} \quad \textit{for } n = 3. \qquad (E-4)$$

Here, ε, $C(\beta)$ and $C(\varepsilon)$ are constants; $C(\beta)$ depends only on spatial dimensions and β, and $C(\varepsilon) = O(\varepsilon^{-\frac{1}{3}})$.

♡ *For the critical case $s = \dfrac{n}{4}$ and $u \in H^{\frac{n}{4}}_0(\mathbb{R}^n)$, the following estimates hold:*

$$\|u\|^2_{L^4(\mathbb{R}^n)} \leq C_n \left(\left\|\Lambda^{\frac{n}{4}} u\right\|^2_{L^2(\mathbb{R}^n)} + \|u\|^2_{L^2(\mathbb{R}^n)} \right) \quad \textit{for } n = 2, 3. \qquad (E-5)$$

Here, C_n is a constant depending only on space dimensions n. □

3 Non-Uniform Decay

In this section we consider the non-uniform decay of the Cauchy problem for the Camassa-Holm equations (1.1)-(1.2) in $\mathbb{R}^n$ if the initial data is assumed only in $L^2(\mathbb{R}^n)$. In particular, if $\mathbf{v}_0 \in H^1_\sigma(\mathbb{R}^n)$ for $\frac{n}{4} \le \beta < 1$, then one deduces that the L^2-norm of the solution to (1.1)-(1.2) decays to zero as time t tends to infinity. Unfortunately, we can't determine the decay rate without more information on the initial data. We now formulate the non-uniform decay result as follows.

Theorem 3.1. For $\frac{n}{4} \le \beta < 1$, $n = 2, 3$, let $\mathbf{v}$ be the solution to the Cauchy problem (1.1)-(1.2) constructed in Proposition 2.4. Then

(I) If $\mathbf{v}_0 \in H^1_\sigma(\mathbb{R}^n)$, then $\lim\limits_{t\to\infty} \|\mathbf{v}(t)\|_{L^2(\mathbb{R}^n)} = 0$;

(II) If $\mathbf{v}_0 \in H^1_\sigma(\mathbb{R}^n)$, then $\lim\limits_{t\to\infty} \frac{1}{t}\int_0^t \|\mathbf{v}(\tau)\|_{L^2(\mathbb{R}^n)} d\tau = 0$;

(III) If $\mathbf{v}_0 \in H^1_\sigma(\mathbb{R}^n)$ for $\frac{n}{4} < \beta < 1$, and if $\mathbf{v}_0 \in H^1_\sigma(\mathbb{R}^n)$ for $\beta = \frac{n}{4}$ with $\|\mathbf{v}_0\|_{L^2(\mathbb{R}^n)} \le \varepsilon^*$ for an $\varepsilon^* = \varepsilon^*(\alpha, \nu, n)$ sufficiently small, then there exists no function $G(t,s) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ admitting the following two properties simultaneously:

$$(1)\ \ \|\mathbf{v}(t)\|_{L^2(\mathbb{R}^n)} \le G\left(t, \|\mathbf{v}_0\|_{L^2(\mathbb{R}^n)}\right), \text{ and } \ (2)\ \text{ for all } s,\ \lim_{t\to\infty} G(t,s) = 0.$$

Proof. We shall follow the idea introduced in [40, 43]. The idea is to split the energy into low and high frequency parts firsty used in [40], to use a cut-off function and the generalized energy inequalities, and then to show that both the high and low frequency terms approach zero.

We first show (I).

Due to $\|\mathbf{v}(t)\|_{L^2(\mathbb{R}^n)} = \|\hat{\mathbf{v}}(t)\|_{L^2(\mathbb{R}^n)}$, it suffices to show that $\lim\limits_{t\to\infty} \|\hat{\mathbf{v}}(t)\|_{L^2(\mathbb{R}^n)} = 0$. Splitting the energy into low and high frequency parts gives rise to

$$\|\hat{\mathbf{v}}\|_{L^2(\mathbb{R}^n)} \le \|\phi\hat{\mathbf{v}}\|_{L^2(\mathbb{R}^n)} + \|(1-\phi)\hat{\mathbf{v}}\|_{L^2(\mathbb{R}^n)}, \tag{3.1}$$

where $\phi = e^{-\nu|\xi|^{2\beta}}$. In the following, we shall divide the proof into two steps.

Step 1. Estimate the low frequency part of the energy $\|\hat{\mathbf{v}}\|_{L^2(\mathbb{R}^n)}$.

Fix t temporarily, then make the choice of $E = 1$ and $\psi(\tau) = \mathscr{F}^{-1}\left[e^{-\nu|\xi|^{2\beta}(t+1-\tau)}\right]$ in (2.7). Note that ψ and $\mathscr{F}(\psi)$ are rapidly decreasing functions for $\tau < t+1$, the relation $\hat{\psi}'(\tau) = \nu|\xi|^{2\beta}\hat{\psi}$ assures that the third and fourth terms on the right hand side of (2.7) add to zero. By Plancherel's theorem, it follows from (2.7) and $\phi = e^{-\nu|\xi|^{2\beta}} = \hat{\psi}(t)$ that

$$\begin{aligned} \|\phi\hat{\mathbf{v}}(t)\|^2_{L^2(\mathbb{R}^n)} \le &\left\|e^{-\nu|\xi|^{2\beta}(t-s)}\phi\hat{\mathbf{v}}(s)\right\|^2_{L^2(\mathbb{R}^n)} \\ &+2\int_s^t \left|\left\langle \check{\phi}^2 * \left(\mathbf{u}\cdot\nabla\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{u}^T\right), e^{-2\nu(-\Delta)^\beta(t-\tau)}\mathbf{v}(\tau)\right\rangle\right| d\tau. \end{aligned} \tag{3.2}$$

Due to Lemma 2.7, by the aid of Hölder's inequality, Young's inequality and Gagliardo-Nirenberg-Sobolev inequality, we have for $\frac{n}{4} \leq \beta < 1$

$$\begin{aligned}
&\left|\left\langle \breve{\phi}^2 * \mathbf{u}\cdot\nabla\mathbf{v}, e^{-2\nu(-\Delta)^\beta(t-\tau)}\mathbf{v}(\tau)\right\rangle\right| \\
&\quad\leq \left\|\breve{\phi}^2 * \mathbf{u}\cdot\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \left\|e^{-2\nu(-\Delta)^\beta(t-\tau)}\mathbf{v}(\tau)\right\|_{L^2(\mathbb{R}^n)} \\
&\quad\leq \left\|\breve{\phi}^2 * \mathbf{u}\right\|_{L^\infty(\mathbb{R}^n)} \|\nabla\mathbf{v}\|_{L^2(\mathbb{R}^n)} \|\mathbf{v}\|_{L^2(\mathbb{R}^n)} \\
&\quad\leq \left\|\breve{\phi}^2\right\|_{L^{\frac{2n}{n+2\beta}}(\mathbb{R}^n)} \|\mathbf{u}\|_{L^{\frac{2n}{n-2\beta}}(\mathbb{R}^n)} \|\nabla\mathbf{v}\|_{L^2(\mathbb{R}^n)} \|\mathbf{v}\|_{L^2(\mathbb{R}^n)} \\
&\quad\leq C(\phi) \|\mathbf{v}\|_{L^2(\mathbb{R}^n)} \left\|\Lambda^\beta\mathbf{u}\right\|_{L^2(\mathbb{R}^n)} \|\nabla\mathbf{v}\|_{L^2(\mathbb{R}^n)}.
\end{aligned} \tag{3.3}$$

In the same manner, one deduces

$$\begin{aligned}
&\left|\left\langle \breve{\phi}^2 * \mathbf{v}\cdot\nabla\mathbf{u}^T, e^{-2\nu(-\Delta)^\beta(t-\tau)}\mathbf{v}(\tau)\right\rangle\right| \\
&\quad\leq C(\phi)\|\mathbf{v}\|_{L^2(\mathbb{R}^n)} \left\|\Lambda^\beta\mathbf{u}\right\|_{L^2(\mathbb{R}^n)} \|\nabla\mathbf{v}\|_{L^2(\mathbb{R}^n)}.
\end{aligned} \tag{3.4}$$

Thanks to the triangle inequality, Hölder's inequality, Proposition 2.4, (2.2) and (3.2), one achieves

$$\begin{aligned}
\|\phi\hat{\mathbf{v}}(t)\|^2_{L^2(\mathbb{R}^n)} \lesssim{}& \left\|e^{-\nu|\xi|^{2\beta}(t-s)}\phi\hat{\mathbf{v}}(s)\right\|^2_{L^2(\mathbb{R}^n)} \\
&+2C(\phi)C\left(\|\mathbf{v}_0\|_{L^2(\mathbb{R}^n)}\right)\left(\int_s^t \|\Lambda^\beta\mathbf{u}\|^2_{L^2(\mathbb{R}^n)}d\tau\right)^{\frac{1}{2}}\left(\int_s^t \|\nabla\mathbf{v}\|^2_{L^2(\mathbb{R}^n)}d\tau\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.5}$$

Since

$$\lim_{t\to\infty} \left\|e^{-\nu|\xi|^{2\beta}(t-s)}\phi\hat{\mathbf{v}}(s)\right\|^2_{L^2(\mathbb{R}^n)} = 0,$$

letting $t \to \infty$ gives rise to

$$\begin{aligned}
&\limsup_{t\to\infty} \|\phi\hat{\mathbf{v}}(t)\|^2_{L^2(\mathbb{R}^n)} \\
&\quad\leq C(\phi)\left(\|\mathbf{v}_0\|_{L^2(\mathbb{R}^n)}\right)\left(\int_s^\infty \left\|\Lambda^\beta\mathbf{u}\right\|^2_{L^2(\mathbb{R}^n)} d\tau\right)^{\frac{1}{2}}\left(\int_s^\infty \|\nabla\mathbf{v}\|^2_{L^2(\mathbb{R}^n)}d\tau\right)^{\frac{1}{2}}.
\end{aligned}$$

Recall Proposition 2.4, for $\mathbf{v}_0 \in H^1_\sigma(\mathbb{R}^n)$ with $\frac{n}{4} \leq \beta < 1$, there holds

$$\begin{cases}
\|\mathbf{v}\|^2_{L^2(\mathbb{R}^n)} + \nu\displaystyle\int_0^T \|\Lambda^\beta\mathbf{v}\|^2_{L^2(\mathbb{R}^n)}dt \leq C\left(n,\beta,\alpha,\nu,\|\mathbf{v}_0\|_{\mathcal{A}_1}\right), \\
\|\nabla\mathbf{v}\|^2_{L^2(\mathbb{R}^n)} + \nu\displaystyle\int_0^T \|\nabla\Lambda^\beta\mathbf{v}\|^2_{L^2(\mathbb{R}^n)}dt \leq C\left(n,\beta,\alpha,\nu,\|\mathbf{v}_0\|_{\mathcal{A}_2}\right).
\end{cases} \tag{3.6}$$

Here, $\mathcal{A}_1 = L^2_0(\mathbb{R}^n)$, and $\mathcal{A}_2 = H^1_0(\mathbb{R}^n)$.
By interpolation inequality and Young's inequality, we have

$$\|\nabla\mathbf{v}\|^2_{L^2(\mathbb{R}^n)}dt \leq \|\Lambda^\beta\mathbf{v}\|^{2\beta}_{L^2(\mathbb{R}^n)}\|\nabla\Lambda^\beta\mathbf{v}\|^{2-2\beta}_{L^2(\mathbb{R}^n)}. \tag{3.6 − a}$$

Hence, note that Hölder's inequality and Cauchy-Schwartz inequality, (3.6) yields that

$$\int_0^T \|\nabla\mathbf{v}\|^2_{L^2(\mathbb{R}^n)}dt \leq \left(\int_0^T \|\Lambda^\beta\mathbf{v}\|^2_{L^2(\mathbb{R}^n)}dt\right)^\beta\left(\int_0^T \|\nabla\Lambda^\beta\mathbf{v}\|^{1-\beta}_{L^2(\mathbb{R}^n)}dt\right)^{\frac{1}{2}}. \tag{3.6 − b}$$

This together with (2.2) and (2.3) implies that $\left\|\Lambda^\beta \mathbf{u}\right\|^2_{L^2(\mathbb{R}^n)}$ and $\|\nabla \mathbf{v}\|^2_{L^2(\mathbb{R}^n)}$ are both integrable on the positive real line. Letting $s \to \infty$ then gives

$$\limsup_{t\to\infty} \|\phi \hat{\mathbf{v}}(t)\|^2_{L^2(\mathbb{R}^n)} = 0. \tag{3.7}$$

Step 2 We now estimate the high-frequency part of the energy $\|\mathbf{v}(t)\|_{L^2(\mathbb{R}^n)}$.

Put $\widetilde{\psi} = 1 - e^{-\nu|\xi|^{2\beta}} = 1 - \phi$ in (2.8). Let $B_G(t) = \{\xi : \ |\xi| \le G(t)\}$, where $G(t)$ will be determined later. Note that $\langle \mathbf{u}\cdot\nabla\mathbf{v}, \mathbf{v}\rangle = 0$, replacing $\widetilde{\psi}^2$ by $1-\widetilde{\psi}^2$ in the fourth term on the right hand side of (2.8) yields

$$\begin{aligned}
&E(t)\left\|\widetilde{\psi}(t)\hat{\mathbf{v}}(t)\right\|^2_{L^2(\mathbb{R}^n)}\\
&\quad\le E(s)\left\|\widetilde{\psi}(s)\hat{\mathbf{v}}(s)\right\|^2_{L^2(\mathbb{R}^n)} + \int_s^t E'(\tau)\int_{B_G(\tau)}\left|\widetilde{\psi}(\tau)\hat{\mathbf{v}}(\tau)\right|^2 d\xi d\tau\\
&\qquad + \int_s^t \left(E'(\tau) - 2\nu E(\tau)G^{2\beta}(\tau)\right)\int_{B^c_G(\tau)}\left|\widetilde{\psi}(\tau)\hat{\mathbf{v}}(\tau)\right|^2 d\xi d\tau\\
&\qquad +2\int_s^t E(\tau)\left|\left\langle \mathcal{F}\left(\mathbf{u}\cdot\nabla\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{u}^T\right), \left(1-\widetilde{\psi}^2(\tau)\right)\hat{\mathbf{v}}(\tau)\right\rangle\right| d\tau\\
&\qquad +2\int_s^t E(\tau)\left|\left\langle \mathcal{F}\left(\mathbf{v}\cdot\nabla\mathbf{u}^T\right), \hat{\mathbf{v}}(\tau)\right\rangle\right| d\tau.
\end{aligned}\tag{3.8}$$

Since

$$\begin{aligned}
&\psi(\tau) = \mathcal{F}^{-1}\left[e^{-\nu|\xi|^{2\beta}(t+1-\tau)}\right],\ \ \phi = e^{-\nu|\xi|^{2\beta}} = \hat{\psi}(t),\\
&\widetilde{\psi} = 1-\phi,\ \ \widetilde{\psi}(\tau) = \mathcal{F}^{-1}\left[1 - e^{-\nu|\xi|^{2\beta}(t+1-\tau)}\right],
\end{aligned}$$

$\mathcal{F}(\phi) = 1 - \widetilde{\psi}^2$ and $\phi = 1 - \widetilde{\psi}$ are rapidly decreasing functions, applying Hölder's inequality, the Plancherel's Theorem, Young's inequality and Gagliardo-Nirenberg-Sobolev inequality, we obtain

$$\begin{aligned}
&\left|\left\langle \mathcal{F}\left(\mathbf{u}\cdot\nabla\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{u}^T\right), \left(1-\widetilde{\psi}^2(\tau)\right)\hat{\mathbf{v}}(\tau)\right\rangle\right|\\
&\quad = \left|\left\langle \left(1-\widetilde{\psi}^2(\tau)\right)\mathcal{F}\left(\mathbf{u}\cdot\nabla\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{u}^T\right), \hat{\mathbf{v}}(\tau)\right\rangle\right|\\
&\quad \le \left\|1-\widetilde{\psi}^2(\tau)\right\|_{L^2(\mathbb{R}^n)}\|\hat{\mathbf{v}}(\tau)\|_{L^2(\mathbb{R}^n)}\\
&\qquad\quad \cdot\left(\|\mathcal{F}\left(\mathbf{u}\cdot\nabla\mathbf{v}\right)\|_{L^\infty(\mathbb{R}^n)} + \left\|\mathcal{F}\left(\mathbf{v}\cdot\nabla\mathbf{u}^T\right)\right\|_{L^\infty(\mathbb{R}^n)}\right)\\
&\quad \le C(\phi)\left(\|\hat{\mathbf{u}} * \xi\hat{\mathbf{v}}\|_{L^\infty(\mathbb{R}^n)} + \|\hat{\mathbf{v}} * \xi\hat{\mathbf{u}}\|_{L^\infty(\mathbb{R}^n)}\right)\|\mathbf{v}\|_{L^2(\mathbb{R}^n)}\\
&\quad \le C(\phi)\|\mathbf{v}\|_{L^2(\mathbb{R}^n)}\left(\|\hat{\mathbf{u}} * \xi\hat{\mathbf{v}}\|_{L^\infty(\mathbb{R}^n)} + \|\hat{\mathbf{v}} * \xi\hat{\mathbf{u}}\|_{L^\infty(\mathbb{R}^n)}\right)\\
&\quad \le C(\phi)\|\mathbf{v}\|_{L^2(\mathbb{R}^n)}\left(\|\mathbf{u}\|_{L^2(\mathbb{R}^n)}\|\nabla\mathbf{v}\|_{L^2(\mathbb{R}^n)} + \|\mathbf{v}\|_{L^2(\mathbb{R}^n)}\|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^n)}\right).
\end{aligned}\tag{3.9}$$

In the same manner, using interpolation inequality one concludes

$$
\begin{aligned}
\left|\left\langle \mathcal{F}\left(\mathbf{v}\cdot\nabla\mathbf{u}^T\right),\hat{\mathbf{v}}(\tau)\right\rangle\right| &\le \left\|\mathbf{v}\cdot\nabla\mathbf{u}^T\right\|_{L^2(\mathbb{R}^n)}\|\mathbf{v}\|_{L^2(\mathbb{R}^n)}\\
&\le C\|\mathbf{v}\|_{L^{\frac{2n}{n-2\beta}}(\mathbb{R}^n)}\|\nabla\mathbf{u}\|_{L^{\frac{n}{\beta}}(\mathbb{R}^n)}\|\mathbf{v}\|_{L^2(\mathbb{R}^n)}\\
&\le C\left\|\Lambda^\beta\mathbf{u}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{2}-\beta}\nabla\mathbf{u}\right\|_{L^2(\mathbb{R}^n)}\|\mathbf{v}\|_{L^2(\mathbb{R}^n)}\\
&\le C\left\|\Lambda^\beta\mathbf{u}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^\beta\mathbf{u}\right\|_{L^2(\mathbb{R}^n)}^{2\beta-\frac{n}{2}}\left\|\Lambda^\beta\nabla\mathbf{u}\right\|_{L^2(\mathbb{R}^n)}^{\frac{n}{2}+1-2\beta}\|\mathbf{v}\|_{L^2(\mathbb{R}^n)}\\
&\le C\left\|\Lambda^\beta\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2\|\mathbf{v}\|_{L^2(\mathbb{R}^n)}.
\end{aligned}
\tag{3.10}
$$

Choosing $E(t)=(1+t)^k$, and $G^{2\beta}(t)=\dfrac{k}{2\nu(1+t)}$ in (3.8) such that $E'-2\nu EG^{2\beta}=0$, then taking $k>0$ sufficiently large yields

$$
\begin{aligned}
&\|(1-\phi)\hat{\mathbf{v}}(\tau)\|_{L^2(\mathbb{R}^n)}^2\\
&\quad\le\frac{(1+s)^k}{(1+t)^k}\|(1-\phi)\hat{\mathbf{v}}(s)\|_{L^2(\mathbb{R}^n)}^2\\
&\qquad+\int_s^t\frac{k(1+\tau)^{k-1}}{(1+t)^k}\int_{B_G(\tau)}|(1-\phi)\hat{\mathbf{v}}(\tau)|^2\,d\xi d\tau\\
&\qquad+C\|\mathbf{v}_0\|_{L^2(\mathbb{R}^n)}\int_s^t\frac{(1+\tau)^k}{(1+t)^k}\cdot\Big(\|\mathbf{u}\|_{L^2(\mathbb{R}^n)}\|\nabla\mathbf{v}\|_{L^2(\mathbb{R}^n)}\\
&\qquad\qquad+\|\mathbf{v}\|_{L^2(\mathbb{R}^n)}\|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^n)}+\left\|\Lambda^\beta\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2\Big)\,d\tau.
\end{aligned}
$$

For $\xi\in B_G(t)$ and t sufficiently large, there holds that $\widetilde{\psi}=|1-\phi|\le\nu|\xi|^{2\beta}$. In particular, $|1-\phi|^2\le\dfrac{k^2}{4(1+t)^2}$. Thus the second term on the right hand side of the above inequality is bounded as follows:

$$
\begin{aligned}
&\int_s^t\frac{k^3(1+\tau)^{-3}}{4}\int_{B_G(\tau)}|\hat{\mathbf{v}}(\tau)|^2d\xi d\tau\\
&\quad\le\int_s^t\frac{k^3(1+\tau)^{-3}}{4}\|\mathbf{v}(\tau)\|_{L^2(\mathbb{R}^n)}^2d\tau\\
&\quad\le\frac{k^3}{4}\|\mathbf{v}_0\|_{L^2(\mathbb{R}^n)}^2\int_s^t(1+\tau)^{-3}d\tau\\
&\quad\le\frac{k^3}{8}\|\mathbf{v}_0\|_{L^2(\mathbb{R}^n)}^2(1+s)^{-2}.
\end{aligned}
$$

Letting $t\to\infty$ gives rise to

$$
\begin{aligned}
&\limsup_{t\to\infty}\|(1-\phi)\hat{\mathbf{v}}(\tau)\|_{L^2(\mathbb{R}^n)}^2\\
&\quad\le\frac{k^3}{8}\|\mathbf{v}_0\|_{L^2(\mathbb{R}^n)}^2(1+s)^{-2}\\
&\qquad+C\|\mathbf{v}_0\|_{L^2(\mathbb{R}^n)}\left(\int_s^\infty\|\nabla\mathbf{v}\|_{L^2(\mathbb{R}^n)}^2d\tau+\int_s^\infty\left\|\Lambda^\beta\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2d\tau+\int_s^\infty\|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^n)}^2d\tau\right).
\end{aligned}
\tag{3.11}
$$

Recall Proposition 2.4, for $\mathbf{v}_0 \in H^1_\sigma(\mathbb{R}^n)$ with $\frac{n}{4} \leq \beta < 1$, there holds

$$\|\mathbf{v}\|^2_{L^2(\mathbb{R}^n)} + \nu \int_0^T \left\|\Lambda^\beta \mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)} dt \leq C\left(n, \beta, \alpha, \nu, \|\mathbf{v}_0\|_{H^1_0(\mathbb{R}^n)}\right). \tag{3.12}$$

Thanks to (2.2), (3.6) and (3.12), by interpolation inequality, one deduces that $\left\|\Lambda^\beta \mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)}$, $\|\nabla \mathbf{v}\|^2_{L^2(\mathbb{R}^n)}$ and $\|\nabla \mathbf{u}\|^2_{L^2(\mathbb{R}^n)}$ are all integrable on the real line. Letting $s \to \infty$ gives

$$\limsup_{t\to\infty} \|(1-\phi)\hat{\mathbf{v}}(\tau)\|^2_{L^2(\mathbb{R}^n)} = 0.$$

By the aid of the Plancherel's theorem, combining this with (3.1) and (3.7) finishes the proof of (I).

We next show (II).

According to (I), given an $\epsilon > 0$ we can choose s large enough such that $\|\mathbf{v}\|_{L^2(\mathbb{R}^n)} \leq \varepsilon$ for $\tau > s$. Thus there holds

$$\begin{aligned}
&\frac{1}{t}\int_0^t \|\mathbf{v}(\tau)\|_{L^2(\mathbb{R}^n)} d\tau \\
&\quad = \frac{1}{t}\int_0^s \|\mathbf{v}(\tau)\|_{L^2(\mathbb{R}^n)} d\tau + \frac{1}{t}\int_s^t \|\mathbf{v}(\tau)\|_{L^2(\mathbb{R}^n)} d\tau \\
&\quad \leq \frac{1}{t}\int_0^s \|\mathbf{v}(\tau)\|_{L^2(\mathbb{R}^n)} d\tau + \varepsilon\frac{t-s}{t}.
\end{aligned} \tag{3.13}$$

Since ε can be chosen arbitrarily, letting $t \to \infty$ finishes the proof of (II).

We are now in the position to show (III).

Let $\mathbf{u}_0(x)$ be any smooth function with compact support, and $\mathbf{u}^\varepsilon_0(x) = \varepsilon^{\frac{n}{2}}\mathbf{u}_0(\varepsilon x)$. In addition, let $\mathbf{v}^\varepsilon_0 = \mathbf{u}^\varepsilon_0 - \alpha^2\Delta\mathbf{u}^\varepsilon_0$ and $\mathbf{v}^\varepsilon$ be the solution of (1.1)-(1.2) given by Proposition 2.4 corresponding to the initial data $\mathbf{v}_0$. For any $\varepsilon > 0$, a straightforward computation shows that

$$\left\|\mathbf{u}^\varepsilon_0\right\|_{L^2(\mathbb{R}^n)} = \|\mathbf{u}_0\|_{L^2(\mathbb{R}^n)}, \quad \left\|\nabla^m \mathbf{u}^\varepsilon_0\right\|_{L^2(\mathbb{R}^n)} = \varepsilon^m \|\nabla \mathbf{u}_0\|_{L^2(\mathbb{R}^n)}, \tag{3.14}$$

$$\begin{aligned}
\left\|\mathbf{v}^\varepsilon_0\right\|^2_{L^2(\mathbb{R}^n)} &= \left\|\mathbf{u}^\varepsilon_0\right\|^2_{L^2(\mathbb{R}^n)} + \alpha^2\left\|\nabla\mathbf{u}^\varepsilon_0\right\|^2_{L^2(\mathbb{R}^n)} + \alpha^4\left\|\Delta\mathbf{u}^\varepsilon_0\right\|^2_{L^2(\mathbb{R}^n)} \\
&= \|\mathbf{u}_0\|^2_{L^2(\mathbb{R}^n)} + \alpha^2\varepsilon^2\|\nabla\mathbf{u}_0\|^2_{L^2(\mathbb{R}^n)} + \alpha^4\varepsilon^4\|\Delta\mathbf{u}_0\|^2_{L^2(\mathbb{R}^n)},
\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}
\left\|\nabla\mathbf{v}^\varepsilon_0\right\|^2_{L^2(\mathbb{R}^n)} &= \left\|\nabla\mathbf{u}^\varepsilon_0\right\|^2_{L^2(\mathbb{R}^n)} + 2\alpha^2\left\|\nabla^2\mathbf{u}^\varepsilon_0\right\|^2_{L^2(\mathbb{R}^n)} + \alpha^4\left\|\nabla\Delta\mathbf{u}^\varepsilon_0\right\|^2_{L^2(\mathbb{R}^n)} \\
&= \varepsilon^2\left\|\nabla^2\mathbf{u}_0\right\|^2_{L^2(\mathbb{R}^n)} + 2\alpha^2\varepsilon^4\|\Delta\mathbf{u}_0\|^2_{L^2(\mathbb{R}^n)} + \alpha^4\varepsilon^6\|\nabla\Delta\mathbf{u}_0\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned} \tag{3.16}$$

It follows from (3.15), (3.16) and Corollary 2.5 that there exists a constant $C = C\left(\|\mathbf{u}_0\|^2_{H^3_\sigma(\mathbb{R}^n)}\right)$ such that for all $\varepsilon > 0$,

$$\left\|\mathbf{v}^\varepsilon_0\right\|^2_{L^2(\mathbb{R}^n)} \leq C, \quad \left\|\nabla\mathbf{v}^\varepsilon_0\right\|^2_{L^2(\mathbb{R}^n)} \leq C\varepsilon^2. \tag{3.17}$$

We then claim

$$\frac{d}{dt}\left(\left\|\mathbf{u}^\varepsilon\right\|^2_{L^2(\mathbb{R}^n)} + \alpha^2\left\|\nabla\mathbf{u}^\varepsilon\right\|^2_{L^2(\mathbb{R}^n)}\right) \geq -C\varepsilon^2, \tag{3.18}$$

which is equivalent to

$$\begin{aligned}
&\|\mathbf{u}^\varepsilon\|^2_{L^2(\mathbb{R}^n)} + \alpha^2 \|\nabla \mathbf{u}^\varepsilon\|^2_{L^2(\mathbb{R}^n)} \\
&\quad \geq \left\|\mathbf{u}^\varepsilon_0\right\|^2_{L^2(\mathbb{R}^n)} + \alpha^2 \left\|\nabla \mathbf{u}^\varepsilon_0\right\|^2_{L^2(\mathbb{R}^n)} - C\varepsilon^2 t \\
&\quad = \|\mathbf{u}_0\|^2_{L^2(\mathbb{R}^n)} + \alpha^2\varepsilon^2 \|\nabla \mathbf{u}_0\|^2_{L^2(\mathbb{R}^n)} - C\varepsilon^2 t \\
&\quad \geq \|\mathbf{u}_0\|^2_{L^2(\mathbb{R}^n)} - C\varepsilon^2 t.
\end{aligned} \tag{3.19}$$

Thanks to (3.18) and (3.19), we conclude that there is not a function $G(t, s)$ continuous and approaching zero in t for each fixed s, such that

$$\|\mathbf{v}\|_{L^2(\mathbb{R}^n)} \leq G\left(t, \|\mathbf{u}_0\|_{L^2(\mathbb{R}^n)}\right). \tag{3.20}$$

Otherwise, if there were such a function, then at some t_0 it would admit the bound

$$G\left(t_0, \|\mathbf{u}_0\|_{L^2(\mathbb{R}^n)}\right) \leq \frac{1}{2}\|\mathbf{u}_0\|_{L^2(\mathbb{R}^n)}. \tag{3.21}$$

Choosing ε sufficiently small in (3.19), in particular, $\varepsilon^{2\beta} \leq \dfrac{\|\mathbf{u}_0\|^2_{L^2(\mathbb{R}^n)}}{4Ct_0}$, one deduces that

$$G\left(t_0, \|\mathbf{u}_0\|^2_{L^2(\mathbb{R}^n)}\right) \geq \|\mathbf{v}\|^2_{L^2(\mathbb{R}^n)} \geq \frac{3}{4}\|\mathbf{u}_0\|^2_{L^2(\mathbb{R}^n)}.$$

This is contradictory to (3.21).

Once we have shown (3.18) or (3.19), the proof of (III) will be finished. The proof of (3.18) will be given in Appendix A.

4 Algebraic Decay

Motivated by these works concerning the algebraic decay of the imcompressible Navier-Stokes equations [27, 37], in this section we shall establish the algebraic decay estimate for the solutions of the Cauchy problem (1.1)-(1.2). From Section 3, we have known that there is no uniform rate of decay for solutions with data exclusively in $H^1_\sigma(\mathbb{R}^n)$ for $\frac{n}{4} \leq \beta < 1$. However, we claim here that there is a uniform rate of decay depending on $H^1_\sigma(\mathbb{R}^n)$ and $L^1(\mathbb{R}^n)$ norms of the initial data for $\frac{n}{4} \leq \beta < 1$. We first in this section establish the decay rate for the filtered velocity $\mathbf{u}$ by applying the Fourier splitting argument introduced in [30, 31] to the natural energy relation (2.2). This decay rate is then applied with an inductive argument to achieve deacy rates for the unfiltered velocity $\mathbf{v}$ and all of its derivatives. It should be pointed out that the Fourier splitting method was originally applied to parabolic conservation laws in [45], and later to Navier-Stokes equations in [46].

The algebraic decay result is the following.

Theorem 4.1. For $\frac{n}{4} \leq \beta < 1$, $n = 2, 3$, let $\mathbf{v}$ be the solution of the Camassa-Holm equations with fractional Laplacian viscosity (1.1)-(1.2) constructed in Proposition 2.4. Then we have

(I) If $\mathbf{v}_0 \in H^1_\sigma(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ for $\frac{n}{4} < \beta < 1$, and $\mathbf{v}_0 \in H^1_\sigma(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ for $\beta = \frac{n}{4}$ with an additional assumption that there exists an $\varepsilon^* = \varepsilon^*(\alpha, \nu, n)$ sufficiently small such that $\|\mathbf{v}_0\|_{H_0^1(\mathbb{R}^n)} \lesssim \varepsilon^*$, then the solution satisfies the "energy" decay rate

$$\int_{\mathbb{R}^n} \mathbf{v}\cdot\mathbf{u}dx = \|\mathbf{u}\|^2_{L^2(\mathbb{R}^n)} + \alpha^2\|\nabla\mathbf{u}\|^2_{L^2(\mathbb{R}^n)} \le C(1+t)^{-\frac{n}{2\beta}}.$$

(II) Under the condition of (I), the solution satisfies the decay rate

$$\|\nabla\mathbf{v}\|^2_{L^2(\mathbb{R}^n)} \le C(1+t)^{-\frac{1}{\beta}-\frac{n}{2\beta}}.$$

(III) Under the condition of (I), then

(III-1) $|\mathcal{F}(\mathbf{v})| \le C, \quad |\mathcal{F}(\mathbf{u})| \le C,$

(III-2) $\|\mathbf{v}\|^2_{L^2(\mathbb{R}^n)} \le C(1+t)^{-\frac{n}{2\beta}}.$

(IV) Let $\|\nabla^m w_0\|_{L^2(\mathbb{R}^n)} < \infty$. Given an energy inequality of the form

$$\frac{1}{2}\frac{d}{dt}\left\|\nabla^m w\right\|^2_{L^2(\mathbb{R}^n)} + \nu\left\|\Lambda^\beta\nabla^m w\right\|^2_{L^2(\mathbb{R}^n)} \le C(1+t)^\gamma, \tag{4.1}$$

and the bound $|\hat{w}(\xi,t)| \le C(1+t)^\eta$ which holds for $|\xi|^{2\beta} < \frac{b}{\nu(1+t)}$, we then achieve

$$\left\|\nabla^m w\right\|^2_{L^2(\mathbb{R}^n)} \le C\left[(1+t)^{-\frac{m}{\beta}-\frac{n}{2\beta}+2\eta} + (1+t)^{\gamma+1}\right]. \tag{4.2}$$

(V) Let $\mathbf{v}_0 \in H^1_\sigma(\mathbb{R}^n)\cap L^1(\mathbb{R}^n)$. For $P \ge 1$, if for all $p < P$ and $m = 0, 1$,

$$\left\|\partial_t^p\nabla^m\mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)} \le C(1+t)^{-2p-\frac{m}{\beta}-\frac{n}{2\beta}},$$

then for $|\xi|^{2\beta} \le \frac{b}{\nu(1+t)}$, there holds

$$\left|\partial_t^P\hat{\mathbf{v}}(\xi)\right| \le C(1+t)^{-P}.$$

(VI) If $\mathbf{v}_0 \in H^K_\sigma(\mathbb{R}^n)\cap L^1(\mathbb{R}^n)$ for $\dfrac{n}{4} < \beta < 1$, and $\mathbf{v}_0 \in H^K_\sigma(\mathbb{R}^n)\cap L^1(\mathbb{R}^n)$ for $\beta = \dfrac{n}{4}$ with an additional assumption that there exists an $\varepsilon^{**} = \varepsilon^{**}(\alpha,\nu,n)$ sufficiently small such that $\|\mathbf{v}_0\|_{H_0^K(\mathbb{R}^n)} \lesssim \varepsilon^{**}$, then

(VI-1) For all $m \le K$, the solution satisfies the following decay

$$\left\|\nabla^m\mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)} \le C(1+t)^{-\frac{m}{\beta}-\frac{n}{2\beta}}.$$

(VI-2) For all $m + 2p\beta \le K$, the solution satisfies the decay estimate

$$\left\|\partial_t^p\nabla^m\mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)} \le C(1+t)^{-2p-\frac{m}{\beta}-\frac{n}{2\beta}}.$$

Here, m, p and P are all non-negative integers in (IV), (V) and (VI), the constant C in (I)-(VI) depends only on the initial data, the dimension of space, and the constants in (1.1), which may be different on different lines.

We shall apply the Fourier splitting method and the bootstrap argument to show Theorem 4.1. Before going further, we first establish an estimate on $\|\hat{\mathbf{v}}\|_{L^\infty(\mathbb{R}^n)}$.

Lemma 4.2. Let $\mathbf{v}$ be the solution of (1.1)-(1.2) constructed in Proposition 2.4 corresponding to $\mathbf{v}_0 \in \left(L^2_\sigma \cap L^1\right)(\mathbb{R}^n)$. Then

$$|\mathcal{F}(\mathbf{v})| \leq C\left[1+\left(\int_0^t \|\mathbf{u}(s)\|^2_{L^2(\mathbb{R}^n)}ds\right)^{\frac{1}{2}}\left(\int_0^t \|\nabla\mathbf{v}(s)\|^2_{L^2(\mathbb{R}^n)}ds\right)^{\frac{1}{2}}\right]. \tag{4.3}$$

Here, C depends only on the initial data, the dimension of space and the constants in (1.1), but not on α.

Proof. Note that

$$\sum_{i=1}^n \nabla(u_i v_i) = \sum_{i=1}^n u_i\nabla v_i + \sum_{i=1}^n v_i \nabla u_i = \mathbf{u}\cdot\nabla\mathbf{v}^T + \mathbf{v}\cdot\nabla\mathbf{u}^T, \tag{4.4}$$

taking the Fourier transform with respect to x for the first equation in (1.1) yields

$$\hat{\mathbf{v}}_t = -\mathcal{F}\left(\mathbf{u}\cdot\nabla\mathbf{v} - \mathbf{u}\cdot\nabla\mathbf{v}^T\right) - \mathcal{F}\left[\nabla\left(\sum_{i=1}^n u_i v_i + p\right)\right] - \nu|\xi|^{2\beta}\hat{\mathbf{v}}.$$

A straightforward computation shows that

$$\mathcal{F}(\mathbf{v}) = e^{-\nu t|\xi|^{2\beta}}\mathcal{F}(\mathbf{v}_0) + \int_0^t e^{-\nu(t-s)|\xi|^{2\beta}}\psi(\xi,s)ds, \tag{4.5}$$

where

$$\psi(\xi,t) = -\xi\cdot\mathcal{F}\left(p+\sum_{i=1}^n u_i v_i\right) - \mathcal{F}\left(\mathbf{u}\cdot\nabla\mathbf{v} - \mathbf{u}\cdot\nabla\mathbf{v}^T\right). \tag{4.6}$$

We first deal with the term $\psi(\xi,t)$.

Thanks to $\|\mathcal{F}(\phi)\|_{L^\infty(\mathbb{R}^n)} \leq \|\phi\|_{L^1(\mathbb{R}^n)}$ and Young's inequality, one deduces that

$$|\mathcal{F}(\mathbf{u}\cdot\nabla\mathbf{v})| \leq \|\mathbf{u}\cdot\nabla\mathbf{v}\|_{L^1(\mathbb{R}^n)} \leq C\|\mathbf{u}\|_{L^2(\mathbb{R}^n)}\|\nabla\mathbf{v}\|_{L^2(\mathbb{R}^n)}. \tag{4.7}$$

In the same manner, one achieves

$$\left|\mathcal{F}\left(\mathbf{u}\cdot\nabla\mathbf{v}^T\right)\right| \leq \|\mathbf{u}\|_{L^2(\mathbb{R}^n)}\|\nabla\mathbf{v}\|_{L^2(\mathbb{R}^n)}. \tag{4.8}$$

On the other hand, taking the divergence for both sides of the first equation in (1.1) leads to

$$-\Delta\left(p+\sum_{i=1}^n u_i v_i\right) = \operatorname{div}\left(\mathbf{u}\cdot\nabla\mathbf{v} - \mathbf{u}\cdot\nabla\mathbf{v}^T\right). \tag{4.9}$$

Combining (4.6) with (4.7), (4.8) and (4.9) yields that

$$|\psi(\xi,t)| \leq C\|\mathbf{u}\|_{L^2(\mathbb{R}^n)}\|\nabla\mathbf{v}\|_{L^2(\mathbb{R}^n)}.$$

Taking the supremum over ξ for (4.4) and applying Cauchy-Schwarz inequality, one obtains

$$|\mathcal{F}(\mathbf{v})| \leq |\mathcal{F}(\mathbf{v}_0)| + C\left(\int_0^t \|\mathbf{u}(s)\|^2_{L^2(\mathbb{R}^n)}ds\right)^{\frac{1}{2}}\left(\int_0^t \|\nabla\mathbf{v}(s)\|^2_{L^2(\mathbb{R}^n)}ds\right)^{\frac{1}{2}}.$$

In view of $\|\mathcal{F}(\mathbf{v}_0)\|_{L^\infty(\mathbb{R}^n)} \leq \|\mathbf{v}_0\|_{L^1(\mathbb{R}^n)}$, the above inequality deduces the desired estimate (4.3). □

In the following, we start the proof of Theorem 4.1.

Proof of Theorem 4.1.

We first show (I).

Note that the assumption of (I), by Proposition 2.4, one obtains

$$\int_0^t \left\|\nabla\Lambda^\beta \mathbf{v}(s)\right\|^2_{L^2(\mathbb{R}^n)} ds \leq C,$$

and

$$\int_0^t \left\|\Lambda^\beta \mathbf{v}(s)\right\|^2_{L^2(\mathbb{R}^n)} ds \leq C,$$

which imply by interpolation inequality that

$$\int_0^t \|\nabla\mathbf{v}(s)\|^2_{L^2(\mathbb{R}^n)} ds \leq C,$$

where C depends only on n, β, α, ν and $\|\mathbf{v}_0\|_{H^1_0(\mathbb{R}^n)}$. Lemma 4.2 then gives rise to

$$\begin{aligned} |\hat{\mathbf{v}}| &\leq C\left[1 + \left(\int_0^t \|\mathbf{u}(s)\|^2_{L^2(\mathbb{R}^n)} ds\right)^{\frac{1}{2}} \left(\int_0^t \|\nabla\mathbf{v}(s)\|^2_{L^2(\mathbb{R}^n)} ds\right)^{\frac{1}{2}}\right] \\ &\leq C\left[1 + \left(\int_0^t \|\mathbf{u}(s)\|^2_{L^2(\mathbb{R}^n)} ds\right)^{\frac{1}{2}}\right]. \end{aligned} \tag{4.10}$$

Thanks to the Plancherel's theorem, the energy equality (2.1) is equivalent to

$$\frac{d}{dt}\int_{\mathbb{R}^n} \left(1 + \alpha^2|\xi|^2\right) \hat{\mathbf{u}}^2 d\xi + 2\nu \int_{\mathbb{R}^n} |\xi|^{2\beta} \left(1 + \alpha^2|\xi|^2\right) \hat{\mathbf{u}}^2 d\xi = 0.$$

Let $B(\rho)$ be the ball of radius ρ with $\rho^{2\beta} = \dfrac{\frac{n}{2\beta} + 1}{2\nu(1+t)}$. Put

$$E^2 = \hat{\mathbf{u}} \cdot \hat{\mathbf{v}} = \left(1 + \alpha^2|\xi|^2\right) \hat{\mathbf{u}}^2. \tag{4.11}$$

Then

$$\frac{d}{dt}\int_{\mathbb{R}^n} E^2 d\xi + 2\nu\rho^{2\beta} \int_{B^c(\rho)} E^2 d\xi \leq 0,$$

or

$$\frac{d}{dt}\int_{\mathbb{R}^n} E^2 d\xi + 2\nu\rho^{2\beta} \int_{\mathbb{R}^n} E^2 d\xi \leq 2\nu\rho^{2\beta} \int_{B(\rho)} E^2 d\xi. \tag{4.12}$$

The equation $\mathbf{u} - \alpha^2\Delta\mathbf{u} = \mathbf{v}$ implies that $\hat{\mathbf{u}} = \dfrac{\hat{\mathbf{v}}}{1 + \alpha^2|\xi|^2}$. This together with (4.10) and (4.11) yields that

$$\left\|E^2\right\|_{L^\infty(\mathbb{R}^n)} \leq \frac{\left\|\hat{\mathbf{v}}^2\right\|_{L^\infty(\mathbb{R}^n)}}{1 + \alpha^2|\xi|^2} \leq C\left[1 + \int_0^t \|u(s)\|^2_{L^2(\mathbb{R}^n)} ds\right].$$

With this, (4.12) then leads to

$$\frac{d}{dt}\int_{\mathbb{R}^n} E^2 d\xi + 2\nu\rho^{2\beta} \int_{\mathbb{R}^n} E^2 d\xi \leq 2\nu\rho^{2\beta+n} \left[1 + \int_0^t \|\mathbf{u}(s)\|^2_{L^2(\mathbb{R}^n)} ds\right],$$

which yields a differential inequality by using the integrating factor $f = (1+t)^{\frac{n}{2\beta}+1}$:

$$\frac{d}{dt}\left((1+t)^{\frac{n}{2\beta}+1}\int_{\mathbb{R}^n} E^2 d\xi\right) \le 2\nu\rho^{2\beta+n}(1+t)^{\frac{n}{2\beta}+1}\left[1+\int_0^t \|\mathbf{u}(s)\|^2_{L^2(\mathbb{R}^n)} ds\right].$$

Integrating this differential inequality in time t from 0 to r gives rise to

$$\begin{aligned}&(1+r)^{\frac{n}{2\beta}+1}\int_{\mathbb{R}^n} E^2(\xi,r)d\xi \\ &\quad\le \int_{\mathbb{R}^n} E^2(\xi,0)d\xi + \frac{\left(\frac{n}{2\beta}+1\right)^{\frac{n}{2\beta}+1}}{(2\nu)^{\frac{n}{2\beta}}}\int_0^r\left(1+\int_0^t \|\mathbf{u}(s)\|^2_{L^2(\mathbb{R}^n)}ds\right)dt.\end{aligned}\tag{4.13}$$

By the Tonelli theorem, a simple calculation shows that

$$\begin{aligned}&\int_0^r (r-s)\|\mathbf{u}(s)\|^2_{L^2(\mathbb{R}^n)}ds \\ &\quad= r\int_0^r \|\mathbf{u}(s)\|^2_{L^2(\mathbb{R}^n)}ds - s\int_0^s \|\mathbf{u}(t)\|^2_{L^2(\mathbb{R}^n)}dt\Big|_0^r + \int_0^r\int_0^s \|\mathbf{u}(t)\|^2_{L^2(\mathbb{R}^n)}dtds \\ &\quad\ge \int_0^r\int_0^t \|\mathbf{u}(s)\|^2_{L^2(\mathbb{R}^n)}dsdt.\end{aligned}$$

Furthermore, it is a simple exercise to obtain the following estimate

$$\begin{aligned}\int_0^r\left(1+\int_0^t \|\mathbf{u}(s)\|^2_{L^2(\mathbb{R}^n)}ds\right)dt &\le r+\int_0^r\int_0^t \|\mathbf{u}(s)\|^2_{L^2(\mathbb{R}^n)}dsdt \\ &\le r+\int_0^r (r-s)\|\mathbf{u}(s)\|^2_{L^2(\mathbb{R}^n)}ds.\end{aligned}$$

Due to $\frac{n}{4} \le \beta < 1$, $\hat{\mathbf{u}}^2 = \frac{E^2}{1+\alpha^2|\xi|^2} \le E^2$ by (4.11) and $\|\mathbf{u}(s)\|^2_{L^2(\mathbb{R}^n)} \le \int_{\mathbb{R}^n} E^2 d\xi$, if follows from (4.13) that

$$(1+r)^{\frac{n}{2\beta}+1}\int_{\mathbb{R}^n} E^2(\xi,r)d\xi \le C(1+r) + C\int_0^r (r-s)\int_{\mathbb{R}^n} E^2(\xi,s)d\xi ds.\tag{4.14}$$

Let $\phi(r) = (1+r)^{\frac{n}{2\beta}+1}\int_{\mathbb{R}^n} E^2(\xi,r)d\xi$. (4.14) has the following equivalent form:

$$\phi \le C(1+r) + C\int_0^r \phi(s)(r-s)(1+s)^{-\frac{n}{2\beta}-1}ds.$$

The Gronwall inequality implies that

$$(1+r)^{\frac{n}{2\beta}+1}\int_{\mathbb{R}^n} E^2(\xi,r)d\xi \le C(1+r)exp\left(C\int_0^r (r-s)(1+s)^{-\frac{n}{2\beta}-1}ds\right).\tag{4.15}$$

Thanks to the fact that $\frac{n}{2} < \frac{n}{2\beta} \le 2$ for $\frac{n}{4} \le \beta < 1$ and $n = 2, 3$, the integral $\int_0^r (r-s)(1+s)^{-\frac{n}{2\beta}-1}ds$ is bounded independent of r. Applying the Plancherel's theorem finishes the proof of (I).

We next prove (II).

Recalling that $\Delta\mathbf{v}$ is divergence free, thanks to the identity (4.4) and Hölder's inequality, multiplying the first equation in (1.1) by $\Delta\mathbf{v}$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{v}\|^2_{L^2(\mathbb{R}^n)}+\nu\left\|\Lambda^\beta\nabla\mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)} &\lesssim \langle\mathbf{u}\cdot\nabla\mathbf{v},\Delta\mathbf{v}\rangle + C\left\langle\mathbf{u}\cdot\nabla\mathbf{v}^T,\Delta\mathbf{v}\right\rangle\\ &\lesssim \left\|\Lambda^{1-\beta}(\mathbf{u}\cdot\nabla\mathbf{v})\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^\beta\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}.\end{aligned}\tag{4.16}$$

There are two cases to consider for estimating the term $\left\|\Lambda^{1-\beta}(\mathbf{u}\cdot\nabla\mathbf{v})\right\|_{L^2(\mathbb{R}^n)}$.

Case **(I)** $\frac{n}{4}<\beta<1$ for $n=2,3$;

Case **(II)** $\beta=\frac{n}{4}$ for $n=2,3$.

♡ We first deal with **Case (I)** $\frac{n}{4}<\beta<1$ for $n=2,3$. The following auxiliary computations will be needed for **Case (I)**.

$$\begin{cases}\frac{n}{\beta}=\frac{2n}{n-2(n-2\beta)/2},\ \ \frac{1}{2}=\frac{n-2}{2}<\frac{n-2\beta}{2}<\frac{3}{4} & \text{for } n=3,\\ 0=\frac{n-2}{2}<\frac{n-2\beta}{2}<\frac{1}{2} & \text{for } n=2,\\ B=\frac{n-2\beta}{2}+1-\beta=\frac{n}{2}+1-2\beta,\quad \frac{n}{2}-1<B<1 & \text{for } n=2,3.\end{cases}\tag{4.17}$$

A straightforward computation shows that

$$\begin{aligned}&\left\|\Lambda^{1-\beta}(\mathbf{u}\cdot\nabla\mathbf{v})\right\|_{L^2(\mathbb{R}^n)}\\ &\le\left\|\Lambda^{1-\beta}(\mathbf{u}\cdot\nabla\mathbf{v})-\Lambda^{1-\beta}\mathbf{u}\cdot\nabla\mathbf{v}-\mathbf{u}\Lambda^{1-\beta}\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\\ &\quad+\left\|\Lambda^{1-\beta}\mathbf{u}\cdot\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}+\left\|\mathbf{u}\Lambda^{1-\beta}\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}.\end{aligned}\tag{4.18}$$

Note that interpolation inequality and Young's inequality, in view of Lemma 2.9, Lemma 2.10, Lemma 2.11 and (4.17), for $\frac{1}{2}=\frac{1}{n/\beta}+\frac{1}{2n/(n-2\beta)}$ and $0<1-\beta<\beta<1$, the first term on the right hand side of (4.18) can be bounded as follows:

$$\begin{aligned}&\left\|\Lambda^{1-\beta}(\mathbf{u}\cdot\nabla\mathbf{v})-\mathbf{u}\Lambda^{1-\beta}\nabla\mathbf{v}-\Lambda^{1-\beta}\mathbf{u}\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\\ &\le C\left\|\Lambda^{1-\beta}\mathbf{u}\right\|_{L^{\frac{n}{\beta}}(\mathbb{R}^n)}\|\nabla\mathbf{v}\|_{L^{\frac{2n}{n-2\beta}}(\mathbb{R}^n)}\\ &\le C\left\|\Lambda^{\frac{n}{2}+1-2\beta}\mathbf{u}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^\beta\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\\ &\le C\|\mathbf{u}\|^{2\beta-\frac{n}{2}}_{L^2(\mathbb{R}^n)}\|\nabla\mathbf{u}\|^{\frac{n}{2}+1-2\beta}_{L^2(\mathbb{R}^n)}\left\|\Lambda^\beta\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\\ &\le C\left(\|\mathbf{u}\|_{L^2(\mathbb{R}^n)}+\|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^n)}\right)\left\|\Lambda^\beta\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}.\end{aligned}\tag{4.19}$$

Here, we have used interpolation inequality in the last line. Due to Lemma 2.11, a similar estimate to (4.19) holds for the second term on the right hand side of (4.18)

$$\begin{aligned}\left\|\Lambda^{1-\beta}\mathbf{u}\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}&\le C\left\|\Lambda^{1-\beta}\mathbf{u}\right\|_{L^{\frac{n}{\beta}}(\mathbb{R}^n)}\|\nabla\mathbf{v}\|_{L^{\frac{2n}{n-2\beta}}(\mathbb{R}^n)}\\ &\le C\left(\|\mathbf{u}\|_{L^2(\mathbb{R}^n)}+\|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^n)}\right)\left\|\Lambda^\beta\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}.\end{aligned}\tag{4.20}$$

In a same manner, recall (4.17) again, we deduce the estimate for the third term on the right hand side of (4.18)

$$\begin{aligned}
&\left\|\mathbf{u}\Lambda^{1-\beta}\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\quad\leq C\left\|\mathbf{u}\right\|_{L^{\frac{n}{2\beta-1}}(\mathbb{R}^n)}\left\|\Lambda^{1-\beta}\nabla\mathbf{v}\right\|_{L^{\frac{2n}{n-2(2\beta-1)}}(\mathbb{R}^n)} \\
&\quad\leq C\left\|\mathbf{u}\right\|_{L^{\frac{2n}{n-2A}}(\mathbb{R}^n)}\left\|\Lambda^{\beta}\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\quad\leq C\left\|\Lambda^{A}\mathbf{u}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\beta}\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\quad\leq C\left(\|\mathbf{u}\|_{L^2(\mathbb{R}^n)}+\|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^n)}\right)\left\|\Lambda^{\beta}\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)},
\end{aligned} \tag{4.21}$$

where $A=\frac{n}{2}+1-2\beta$ is given by Lemma 2.11. Note that (I) of this theorem, combining (4.16) with (4.17),(4.18), (4.19), (4.20) and (4.21) yields that

$$\frac{d}{dt}\|\nabla\mathbf{v}\|_{L^2(\mathbb{R}^n)}^2+2\nu\left\|\Lambda^{\beta}\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2\leq C(1+t)^{-\frac{n}{4\beta}}\left\|\Lambda^{\beta}\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2. \tag{4.22}$$

♡ we next consider Case **(II)** $\beta=\dfrac{n}{4}$ for $n=2,3$.

In this case, $\left\|\Lambda^{1-\frac{n}{4}}(\mathbf{u}\cdot\nabla\mathbf{v})\right\|_{L^2(\mathbb{R}^n)}$ can be bounded as follows:

$$\begin{aligned}
&\left\|\Lambda^{1-\frac{n}{4}}(\mathbf{u}\cdot\nabla\mathbf{v})\right\|_{L^2(\mathbb{R}^n)} \\
&\quad\leq\left\|\Lambda^{1-\frac{n}{4}}(\mathbf{u}\cdot\nabla\mathbf{v})-\Lambda^{1-\frac{n}{4}}\mathbf{u}\cdot\nabla\mathbf{v}-\mathbf{u}\Lambda^{1-\frac{n}{4}}\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\qquad+\left\|\Lambda^{1-\frac{n}{4}}\mathbf{u}\cdot\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}+\left\|\mathbf{u}\Lambda^{1-\frac{n}{4}}\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}.
\end{aligned} \tag{4.23}$$

We first bound the first term on the right hand side of estimate (4.23). By Lemma 2.9, one attains that for $0<\beta_1<1-\dfrac{n}{4}$,

$$\begin{aligned}
&\left\|\Lambda^{1-\frac{n}{4}}(\mathbf{u}\cdot\nabla\mathbf{v})-\mathbf{u}\Lambda^{1-\frac{n}{4}}\nabla\mathbf{v}-\Lambda^{1-\frac{n}{4}}\mathbf{u}\cdot\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\quad\leq C\left\|\Lambda^{1-\frac{n}{4}-\beta_1}\mathbf{u}\right\|_{L^{\frac{2n}{n-2(n/4+\beta_1)}}(\mathbb{R}^n)}\left\|\Lambda^{\beta_1}\nabla\mathbf{v}\right\|_{L^{\frac{2n}{n-2(n/4-\beta_1)}}(\mathbb{R}^n)} \\
&\quad\leq C\left\|\nabla\mathbf{u}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{4}}\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\quad\lesssim\left\|\mathbf{v}_0\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{4}}\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)},
\end{aligned} \tag{4.24}$$

where we have used the fact that $\beta_1\in\left(0,\dfrac{1}{2}\right)$, $1-\dfrac{n}{4}-\beta_1\in\left(0,\dfrac{1}{2}\right)$, $\dfrac{1}{2}=\dfrac{1}{p_1}+\dfrac{1}{p_2}$ with $p_1,p_2\in(1,\infty)$, $p_1=\dfrac{2n}{n-2(n/4+\beta_1)}$, $p_2=\dfrac{2n}{n-2(n/4-\beta_1)}$. Thanks to Lemma 2.12, Agmon's inequality, Young's inequality and the interpolation inequality, note that $0<1-\dfrac{n}{4}<\dfrac{n}{4}$, $n=\frac{2n}{n-2(n/2-1)}$, (2.4) and the assumption of (II) for $\beta=\dfrac{n}{4}$, the second and the third terms on the right hand side of (4.23) enjoy the similar estimates to (4.24):

For $n = 2$, we have

$$\begin{aligned}
&\left\|\Lambda^{1-\frac{n}{4}}\mathbf{u}\cdot\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} + \left\|\mathbf{u}\Lambda^{1-\frac{n}{4}}\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\quad\le C\left\|\Lambda^{1-\frac{n}{4}}\mathbf{u}\right\|_{L^4(\mathbb{R}^n)}\|\nabla\mathbf{v}\|_{L^4(\mathbb{R}^n)} + C\|\mathbf{u}\|_{L^\infty(\mathbb{R}^n)}\left\|\Lambda^{1-\frac{n}{4}}\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\quad\le C\left\|\Lambda^{1-\frac{n}{4}+\frac{n}{4}}\mathbf{u}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{4}}\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} + \|\mathbf{u}\|^{\frac{1}{2}}_{H^1(\mathbb{R}^n)}\|\mathbf{u}\|^{\frac{1}{2}}_{H^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{4}}\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\quad\lesssim \|\mathbf{v}\|_{L^2_0(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{4}}\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}.
\end{aligned} \tag{4.25}$$

For $n = 3$, we also achieve

$$\begin{aligned}
&\left\|\Lambda^{1-\frac{n}{4}}\mathbf{u}\cdot\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} + \left\|\mathbf{u}\Lambda^{1-\frac{n}{4}}\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\quad\le C\left\|\Lambda^{1-\frac{n}{4}}\mathbf{u}\right\|_{L^4(\mathbb{R}^n)}\|\nabla\mathbf{v}\|_{L^4(\mathbb{R}^n)} + C\|\mathbf{u}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)}\left\|\Lambda^{1-\frac{n}{4}}\nabla\mathbf{v}\right\|_{L^n(\mathbb{R}^n)} \\
&\quad\le C\|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{4}}\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} + C\|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{4}}\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\quad\lesssim \|\mathbf{v}\|_{L^2_0(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{4}}\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}.
\end{aligned} \tag{4.25a}$$

In the above estimates, we have used the fact that $\|\mathbf{v}\|_{L^2(\mathbb{R}^n)} \le C\left(\|\mathbf{v}_0\|_{L^2(\mathbb{R}^n)}\right)$. Combining (4.16) with (4.18), (4.23), (4.24) and (4.25) then yields that

$$\frac{d}{dt}\|\nabla\mathbf{v}\|^2_{L^2(\mathbb{R}^n)} + 2\nu\left\|\Lambda^{\frac{n}{4}}\nabla\mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)} \le C(1+t)^{-\frac{n}{4\beta}}\left\|\Lambda^{\frac{n}{4}}\nabla\mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)}. \tag{4.26}$$

Therefore, from the above arguments of Case (I) and Case (II), for any $\frac{n}{4} \le \beta < 1$ with $n = 2, 3$, choosing t large enough such that $C(1+t)^{-\frac{n}{4\beta}} < \nu$, one deduces from (4.22) and (4.26) that

$$\frac{d}{dt}\|\nabla\mathbf{v}\|^2_{L^2(\mathbb{R}^n)} + \nu\left\|\Lambda^{\beta}\nabla\mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)} \le 0. \tag{4.27}$$

In the following, we continue our proof by applying the Fourier splitting method as used in the proof of (I) of this theorem.

Let $B(\rho)$ be the ball of radius ρ, where $\rho^{2\beta} = \dfrac{\frac{n}{2\beta} + \frac{1}{\beta} + 1}{\nu(1+t)}$. Thanks to the Plancherel's theorem, it follows from (4.27) that

$$\frac{d}{dt}\|\xi\hat{\mathbf{v}}\|^2_{L^2(\mathbb{R}^n)} + \nu\rho^{2\beta}\int_{B^C(\rho)}|\xi\hat{\mathbf{v}}|^2\,d\xi \le 0,$$

which gives rise to

$$\frac{d}{dt}\|\xi\hat{\mathbf{v}}\|^2_{L^2(\mathbb{R}^n)} + \nu\rho^{2\beta}\|\xi\hat{\mathbf{v}}\|^2_{L^2(\mathbb{R}^n)} \le \nu\rho^{2\beta+2}\int_{B(\rho)}|\hat{\mathbf{v}}|^2\,d\xi. \tag{4.28}$$

On the other hand, we obtain by Lemma 4.2 and (I) of this theorem

$$|\hat{\mathbf{v}}|^2 \le C\left[1 + \int_0^t (1+s)^{-\frac{n}{2\beta}}ds \cdot \int_0^t \|\nabla\mathbf{v}\|^2_{L^2(\mathbb{R}^n)}ds\right].$$

With this bound and (4.28), we arrive at

$$\frac{d}{dt}\|\xi\hat{\mathbf{v}}\|^2_{L^2(\mathbb{R}^n)} + \nu\rho^{2\beta}\|\xi\hat{\mathbf{v}}\|^2_{L^2(\mathbb{R}^n)}$$

$$\leq C\nu\rho^{2\beta+2+n}\left[1+\left(\int_0^t (1+s)^{-\frac{n}{2\beta}}ds\right)\left(\int_0^t \|\nabla\mathbf{v}\|^2_{L^2(\mathbb{R}^n)}ds\right)\right].$$

Let $K(t) = (1+t)^{\frac{n}{2\beta}+\frac{1}{\beta}+1}$. Taking $K(t)$ as an integrating factor, we then claim

$$\frac{d}{dt}\left((1+t)^{\frac{n}{2\beta}+\frac{1}{\beta}+1}\|\xi\hat{\mathbf{v}}\|^2_{L^2(\mathbb{R}^n)}\right)$$
$$\leq C\left[1+\left(\int_0^t (1+s)^{-\frac{n}{2\beta}}ds\right)\left(\int_0^t \|\nabla\mathbf{v}\|^2_{L^2(\mathbb{R}^n)}ds\right)\right].$$

Thanks to the Tonelli theorem and the Plancherel's theorem, we obtain by applying (I) of this theorem again and integrating in time from 0 to r

$$(1+r)^{\frac{n}{2\beta}+\frac{1}{\beta}+1}\|\nabla\mathbf{v}\|^2_{L^2(\mathbb{R}^n)}$$
$$\leq C(1+r)$$
$$+\int_0^r\left(\int_0^t (1+s)^{-\frac{n}{2\beta}}ds\right)\cdot\left(\int_0^t \frac{(1+s)^{\frac{n}{2\beta}+\frac{1}{\beta}+1}}{(1+s)^{\frac{n}{2\beta}+\frac{1}{\beta}+1}}\|\nabla\mathbf{v}(s)\|^2_{L^2(\mathbb{R}^n)}ds\right)dt.$$

The Gronwall inequality then implies that

$$(1+r)^{\frac{n}{2\beta}+\frac{1}{\beta}+1}\|\nabla\mathbf{v}\|^2_{L^2(\mathbb{R}^n)} \leq C(1+r)e^A,$$

where

$$A = \int_0^r\left(\int_0^t (1+s)^{-\frac{n}{2\beta}}ds\right)\left(\int_0^t (1+s)^{-\frac{n}{2\beta}-\frac{1}{\beta}-1}ds\right)dt.$$

Note that the term A is bounded independent of r for $n = 2, 3$, we then obtain

$$\|\nabla\mathbf{v}(r)\|^2_{L^2(\mathbb{R}^n)} \leq C(1+r)^{-\frac{n}{2\beta}-\frac{1}{\beta}}.$$

This finishes the proof of (II).

We then prove (III-1).

Due to Lemma 2.10, Lemma 4.2 with (I) and (II) of this theorem, we have $|\mathcal{F}(\mathbf{v})| \leq C$. Note that the Helmholtz equation $\mathbf{u} - \alpha^2\Delta\mathbf{u} = \mathbf{v}$, simple computation gives $|\mathcal{F}(\mathbf{u})| \leq |\mathcal{F}(\mathbf{v})|$ yields the conclusion of (III-1).

We next show (III-2).

From (I) of this theorem, we have shown that

$$\|\mathbf{u}\|^2_{L^2(\mathbb{R}^n)} + \alpha^2\|\nabla\mathbf{u}\|^2_{L^2(\mathbb{R}^n)} \leq C(1+t)^{-\frac{n}{2\beta}}. \tag{4.29}$$

Differentiating the Helmholtz equation $\mathbf{u} - \alpha^2\Delta\mathbf{u} = \mathbf{v}$ and squaring the resulting equation yields, after some integration by parts,

$$\|\nabla\mathbf{u}\|^2_{L^2(\mathbb{R}^n)} + 2\alpha^2\left\|\nabla^2\mathbf{u}\right\|^2_{L^2(\mathbb{R}^n)} + \alpha^4\left\|\nabla^3\mathbf{u}\right\|^2_{L^2(\mathbb{R}^n)} = \|\nabla\mathbf{v}\|^2_{L^2(\mathbb{R}^n)}.$$

Combining this with (II) of this theorem gives rise to

$$\left\|\nabla^2\mathbf{u}\right\|^2_{L^2(\mathbb{R}^n)} \leq C(1+t)^{-\frac{n}{2\beta}-\frac{1}{\beta}} \leq C(1+t)^{-\frac{n}{2\beta}}.$$

This together with (4.29) deduces

$$\|\mathbf{v}\|_{L^2(\mathbb{R}^n)}^2 \leq \|\mathbf{u}\|_{L^2(\mathbb{R}^n)}^2 + 2\alpha^2 \|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^n)}^2 + \alpha^4 \|\Delta\mathbf{u}\|_{L^2(\mathbb{R}^n)}^2 \leq C(1+t)^{-\frac{n}{2\beta}}.$$

This ends the proof of (III-2).

In the following, we begin to show (IV).

We will adopt the Fourier splitting argument again. Let $B(\rho)$ be the ball of radius ρ. Thanks to the Plancherel's theorem, breaking up the left hand side of the integral (4.1) deduces that

$$\frac{1}{2}\frac{d}{dt}\left\|\xi^m\hat{w}\right\|_{L^2(\mathbb{R}^n)}^2 + \nu\rho^{2\beta}\left\|\xi^m\hat{w}\right\|_{L^2(\mathbb{R}^n)}^2 \leq \nu\rho^{2\beta+2m}\int_{B(\rho)}|\hat{w}|^2 d\xi + C(1+t)^{\gamma}. \tag{4.30}$$

Let $\rho^{2\beta} = \frac{b}{\nu(1+t)}$ for some large b. Note that the assumption for the bound on $\hat{w}$, making direct calculation for the right hand side of (4.30) gives

$$\frac{d}{dt}\left[(1+t)^b\|\xi^m\hat{w}\|_{L^2(\mathbb{R}^n)}^2\right] \leq C\left[(1+t)^{-\frac{m}{\beta}-1+b+2\eta-\frac{n}{2\beta}} + (1+t)^{\gamma+b}\right]. \tag{4.31}$$

Integrating both sides of (4.31) with respect to time t, and then applying the plancherel's theorem once again, we arrive at the conclusion of (IV).

We next prove (V).

Note that the chain rule

$$\frac{d}{dt}\int_0^t f(t,s)ds = f(t,t) + \int_0^t \frac{\partial f(t,s)}{\partial t}ds,$$

one deduces from (4.5) and (4.6) that

$$\begin{aligned}\partial_t^P\mathcal{F}(\mathbf{v}) = (-\nu)^P|\xi|^{2\beta P}e^{-\nu t|\xi|^{2\beta}}\mathcal{F}(\mathbf{v}_0)\\ + \sum_{p=0}^{P-1}\left(-\nu|\xi|^{2\beta}\right)^{P-1-p}\partial_t^p\psi(\xi,t)\\ + \int_0^t\left(-\nu|\xi|^{2\beta}\right)^P e^{-\nu(t-s)|\xi|^{2\beta}}\psi(\xi,s)ds.\end{aligned}$$

With this expression, to achieve (V), the key ingredient is to first bound $\partial_t^p\psi(\xi,t)$, with $\psi(\xi,t)$ defined by (4.6). Applying an argument similar to the proof of Lemma 4.2, one obtains

$$\begin{aligned}\partial_t^p\psi(\xi,t) &= -\partial_t^p\mathcal{F}(\mathbf{u}\cdot\nabla\mathbf{v}) + \partial_t^p\mathcal{F}\left(\mathbf{u}\cdot\nabla\mathbf{v}^T\right) + \partial_t^p\left[-\xi\cdot\mathcal{F}\left(p+\sum_{i=1}^n u_iv_i\right)\right]\\ &\triangleq \partial_t^pA + \partial_t^pB + \partial_t^pC.\end{aligned}$$

Thanks to div $\mathbf{u} = 0$, by the assumptions of (V), ∂_t^pA can be bounded as follows:

$$\begin{aligned}|\partial_t^pA| &= \left|\partial_t^p\mathcal{F}\left(\sum_{j=1}^n u_j\partial_j\mathbf{v}\right)\right| = \left|\partial_t^p\mathcal{F}\left(\sum_{j=1}^n \partial_j(u_j\mathbf{v})\right)\right| = \left|\partial_t^p\sum_{j=1}^n\xi_j\mathcal{F}(u_j\mathbf{v})\right|\\ &\leq \sum_{l=0}^p C|\xi|\left\|\partial_t^l\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\partial_t^{p-l}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\end{aligned}$$

$$\leq C(1+t)^{-\frac{1}{2\beta}}(1+t)^{-l-\frac{n}{4\beta}}(1+t)^{-(p-l)-\frac{n}{4\beta}}$$
$$\leq C(1+t)^{-p-\frac{n}{2\beta}-\frac{1}{2\beta}}.$$

In the same manner, one attains the bound for $\partial_t^p B$:

$$\begin{aligned}\left|\partial_t^p B\right| &= \left|\partial_t^p \sum_{j=1}^{n} \mathcal{F}\left(u_j \nabla v_j\right)\right| \\ &\leq \sum_{l=0}^{p} C|\xi| \left\|\partial_t^l \mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \left\|\partial_t^{p-l}\nabla \mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\ &\leq C(1+t)^{-l-\frac{n}{4\beta}}(1+t)^{-(p-l)-\frac{1}{2\beta}-\frac{n}{4\beta}} \\ &\leq C(1+t)^{-p-\frac{n}{2\beta}-\frac{1}{2\beta}},\end{aligned}$$

Due to (4.9), putting together the above estimates for $\partial_t^p A$ and $\partial_t^p B$, we get

$$\begin{aligned}\left|\partial_t^p C\right| = \left|\partial_t^p \xi \mathcal{F}\left(p + \sum_{i=1}^{n} u_i v_i\right)\right| &\leq \left|\partial_t^p A\right| + \left|\partial_t^p B\right| \\ &\leq C(1+t)^{-p-\frac{n}{2\beta}-\frac{1}{2\beta}}.\end{aligned}$$

In view of (4.6), the bound $|\hat{\mathbf{v}}| \leq C$ by (III-1) of this theorem, $|\xi|^{2\beta} \leq \dfrac{b}{\nu(1+t)}$ and $1 - \dfrac{n}{2\beta} - \dfrac{1}{2\beta} < 0$, we finish the proof of (V).

For readers' convenience, the proof of (VI-1) will be given in Appendix B and (VI-2) in Appendix C.

So far, we finish the proof of Theorem 4.1. □

5 Convergence to the NSE with nonlocal viscosity

We observe that for $\alpha = 0$, the system (1.1) formally reduces to the incompressible Navier-Stokes equations with fractional Laplacian viscosity

$$\begin{cases} \mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} + \nabla p = -\nu(-\Delta)^{\beta}\mathbf{v} \\ \operatorname{div} \mathbf{v} = 0. \end{cases} \tag{5.1}$$

By means of the fractional heat kernel estimates [13] and Leray projection, we figure out the relation between the nonlocal system (1.1) and (5.1). In particular, we investigate the convergence of the solution of (1.1) to that of (5.1) strongly as the filter parameter $\alpha \to 0$, and relate the limit to . To achieve this, we need first exploring how a solution $\mathbf{u}$ of the Helmholtz equation

$$\mathbf{u} - \alpha^2 \Delta \mathbf{u} = \mathbf{v} \tag{5.2}$$

approaches $\mathbf{v}$ as α tends to zero. In [16, 17], the authors clarified that how the solutions of the Camassa-Holm equations (1.3) approach that of the corresponding imcompressible Navier-Stokes equations (1.4) weakly as the filter parameter α tends to zero. In [3], the authors established how solutions to the viscous Camassa-Holm equations (1.3) approach that to (1.4) strongly as $\alpha \to 0$ when the solution to (1.4) is known to be regular enough. Here, we expect to establish a similar result for the nonlocal Camassa-Holm equations (1.1) to that for (1.3) mentioned as above . Precisely, we hope to make sure how solutions of (1.1) tend to that of (5.1) strongly as $\alpha \to 0$ when the solutions to (5.1) are to be sufficiently regular. To attain this goal, we must establish some a priori estimates on the solutions of (1.1) which are independent of α, but on regions of time where a solution to the nonlocal Navier-Stokes equations (5.1) is known to be regular by the functional analytic argument.

The object of this section is to prove the following convergence theorem for (1.1):

Theorem 5.1. For $\frac{n}{4} \le \beta < 1$, $n = 2, 3$, let $\{\alpha_i\}$ be a sequence of filter coefficients tending to zero, and let $\mathbf{v}_{\alpha_i}$ be the solutions of (1.1) constructed in Proposition 2.4 corresponding to the initial data $\mathbf{w}_0 \in H^{\beta}_{\sigma}(\mathbb{R}^n)$. Let $\mathbf{w}$ be the solution of (5.1) with the same initial data $\mathbf{w}_0$. In any time interval $[0, T]$, where a solution to (5.1) is known to be sufficiently regular, if there exists a bound

$$\sup_{\alpha_i} \sup_{t\in[0,T]} \left(\left\|\mathbf{v}_{\alpha_i}\right\|_{L^l(\mathbb{R}^n)} + \left\|\Lambda^{\beta}\mathbf{v}_{\alpha_i}\right\|_{L^l(\mathbb{R}^n)} \right) \le C,$$

which is independent of α, then $\mathbf{v}_\alpha$ approaches $\mathbf{w}$ strongly in $L^\infty([0, T], L^q(\mathbb{R}^n))$ as $\alpha \to 0$, where $q = \dfrac{2s}{s-2}$, $s = \dfrac{ln}{n - l\beta}$ and $l > \dfrac{n}{3\beta - 1}$.

Before proving this theorem, we first make some preliminary remarks and preparations.

Remark 5.2. By a similar proof to that for the Camassa-Holm equations without any fractional viscosity term (1.3), we deduce that a solution $\mathbf{u}$ of (5.2) approaches $\mathbf{v}$ weakly as the filter parameter α tends to zero. That is, fix $\mathbf{v} \in L^p(\mathbb{R}^n)$, let $\{\alpha_i\}$ be a sequence of filter coefficients tending to zero, for each α_i there is a weak solution $\mathbf{u}_{\alpha_i} \in W^{1,p}(\mathbb{R}^n)$ of (5.2) such that

$$\mathbf{u}_{\alpha_i} \rightharpoonup \mathbf{v} \text{ weakly in } L^p(\mathbb{R}^n) \text{ as } \alpha_i \to 0. \qquad \square$$

Due to Remark 5.2, we claim a stronger result if $\mathbf{v}$ is sufficiently differentiable.

Proposition 5.3. For $\dfrac{n}{4} \le \beta < 1$, $n = 2, 3$, let $\mathbf{v} \in W^{\beta,p}(\mathbb{R}^n)$ and $\mathbf{u}$ be the solution of (5.2). Then for $\alpha \in (0, 1)$, there holds

$$\|\mathbf{u} - \mathbf{v}\|_{L^q(\mathbb{R}^n)} \le C(n, p, q)\alpha^{\frac{\beta}{2}-\gamma} \left\|\Lambda^{\beta}\mathbf{v}\right\|_{L^p(\mathbb{R}^n)} \text{ for } \gamma = \frac{n}{2}\left(\frac{1}{p} - \frac{1}{q}\right) < \frac{\beta}{2}.$$

In particular, if $\{\alpha_i\}$ is a sequence tending to zero, and $\mathbf{u}_{\alpha_i}$ are solutions of (5.2), then for $\dfrac{1}{p} - \dfrac{1}{q} < \dfrac{\beta}{n}$,

$$\mathbf{u}_{\alpha_i} \to \mathbf{v} \text{ strongly in } L^q(\mathbb{R}^n) \text{ as } \alpha_i \to 0.$$

Here, $W^{\beta,p}(\mathbb{R}^n)$ is defined by Definition 1.2.

Proof. If $\mathbf{u}$ and $\mathbf{v}$ satisfy (5.2), then a preliminary calculation gives rise to

$$\|\mathbf{u} - \mathbf{v}\|_{L^q(\mathbb{R}^n)} \le \alpha^2 \|\Delta\mathbf{u}\|_{L^q(\mathbb{R}^n)}. \tag{5.3}$$

Since (5.2) is linear, the derivatives of the corresponding functions obey the relation

$$\Lambda^{\beta}\mathbf{u} - \alpha^2\Lambda^{\beta}\Delta\mathbf{u} = \Lambda^{\beta}\mathbf{v}. \tag{5.4}$$

Applying Lemma 2.6 to (5.4) with $\gamma = \gamma_3 = \dfrac{n}{2}\left(\dfrac{1}{p} - \dfrac{1}{q}\right) < \dfrac{\beta}{2}$, we arrive at the following:

$$\|\Delta\mathbf{u}\|_{L^q(\mathbb{R}^n)} \le \left\|\Lambda^{\beta}\Delta\mathbf{u}\right\|_{L^p(\mathbb{R}^n)} \le \frac{C(n, p, q)}{\alpha^{2-\frac{\beta}{2}+\gamma}} \left\|\Lambda^{\beta}\mathbf{v}\right\|_{L^p(\mathbb{R}^n)}.$$

This together with (5.3) yields that

$$\|\mathbf{u}-\mathbf{v}\|_{L^q(\mathbb{R}^n)} \le \alpha^{\frac{\beta}{2}-\gamma}\left\|\Lambda^\beta \mathbf{v}\right\|_{L^p(\mathbb{R}^n)}. \tag{5.5}$$

Note that $\gamma < \dfrac{\beta}{2}$, replacing α with α_i in (5.5) and then letting $\alpha_i \to 0$, we immediately deduce the second statement that $u_{\alpha_i} \to v$ strongly in $L^q(\mathbb{R}^n)$ for $\dfrac{1}{p}-\dfrac{1}{q}<\dfrac{\beta}{n}$. □

We now mention some estimates concerning the fundamental solution of the linear nonlocal operator $\partial_t + (-\Delta)^{\frac{\gamma_0}{2}}$ in [13], which are key to the proof of Theorem 5.1.

Lemma 5.4 ([13])**.** Let $\gamma_0 \in (1,2]$. Define $G_{\gamma_0}(t,x)$ by its Fourier transform $\widehat{G}_{\gamma_0}(t,\xi) = e^{-t|\xi|^{\gamma_0}}$ for $t>0$. Then $G_{\gamma_0}(t,x)$ is the fundamental solution of the linear operator: $\partial_t + (-\Delta)^{\frac{\gamma_0}{2}}$. In addition, it enjoys the scaling property:

$$G_{\gamma_0}(t,x) = t^{-\frac{n}{\gamma_0}} G_{\gamma_0}\left(1, t^{-\frac{1}{\gamma_0}}x\right).$$ □

Lemma 5.5 ([13])**.** For $\gamma_0 \in (0,\infty)$ and $p\in[1,\infty]$, let $k \ge 0$ be an integer and $\varepsilon \in (0,1]$. Then for some constant $C = C(n,\gamma_0,\varepsilon)$, there holds that

$$\left\|D_x^k \Lambda^\alpha G_{\gamma_0}(t,\cdot)\right\|_{L_x^p(\mathbb{R}^n)} \le C^{k+1} k^{\frac{k}{\gamma_0}} t^{-\frac{k+\alpha}{\gamma_0}-\frac{n}{\gamma_0}\left(1-\frac{1}{p}\right)}$$

for any α satisfying

$$\begin{cases} \varepsilon - 1 \le \alpha \le 1 & \text{if } k \ge 1 \\ \varepsilon \le \alpha \le 1 \ \text{ or } \ \alpha = 0 & \text{if } k = 0. \end{cases}$$

Here, the constant C can be taken to be independent of p. □

In addition, the following auxiliary lemma will be needed for the proof of Theorem 5.1.

Lemma 5.6. For $\dfrac{n}{4} \le \beta < 1$, $n = 2,3$, let $\dfrac{1}{p}+\dfrac{1}{2}=\dfrac{1}{q}+1$, $q=\dfrac{2s}{s-2}$, $s=\dfrac{ln}{n-l\beta}$ and $l > \dfrac{n}{3\beta-1}$. It follows that

$$1 \le p < \begin{cases} \frac{3}{2} & \text{for } n = 3, \\ 2 & \text{for } n = 2. \end{cases}$$

Proof. Direct calculation gives $p = \dfrac{ln}{ln-n+l\beta}$. Thanks to $l > \dfrac{n}{3\beta-1}$, $\dfrac{n}{4} \le \beta < 1$ and $n=2,3$, it follows that $1 \le p < \dfrac{n}{n-1}$. This completes the proof of this lemma. □

With the previous preparations, we begin to show Theorem 5.1.

Proof of Theorem 5.1.

We will work in a time interval with known regularity of the solutions to the Camassa-Holm equations with fractional Laplacian viscosity (1.1) and the imcompressible Navier-Stokes equations with fractional Laplacian viscosity (5.1). Hence these are the unique solutions. Note that (1.1) and (5.1), if $\mathbb{P}$ is the Leray projector onto the divergence free subspace of $L^2(\mathbb{R}^n)$ and $\phi(t)$ is the fractional power heat kernel $\phi(t) = e^{-(-\Delta)^\beta t}$, then

$$\mathbf{w}(t) = \phi(t) * \mathbf{w}_0 - \int_0^t \phi(t-s) * \mathbb{P}\left[\mathbf{w}\cdot\nabla\mathbf{w}\right](s)ds, \tag{5.6}$$

$$\mathbf{v}(t) = \phi(t) * \mathbf{w}_0 - \int_0^t \phi(t-s) * \mathbb{P}\left[\mathbf{u}\cdot\nabla\mathbf{v} - \mathbf{u}\cdot\nabla\mathbf{v}^T\right](s)ds. \tag{5.7}$$

Thanks to (4.4), a straightforward computation shows that

$$\begin{aligned}\mathbf{w}(t) - \mathbf{v}(t) &= -\int_0^t \phi(t-s) * \mathbb{P}\left[(\mathbf{w}-\mathbf{u})\cdot\nabla\mathbf{w} + \mathbf{u}\cdot\nabla(\mathbf{w}-\mathbf{v})\right](s)ds \\ &\quad - \int_0^t \phi(t-s) * \mathbb{P}\left[\sum_{j=1}^n u_j\nabla(v_j - w_j) + \sum_{j=1}^n (u_j - w_j)\nabla w_j\right](s)ds.\end{aligned} \tag{5.8}$$

Note that the definition of the Leray projector, and the fact that the projector commutes with derivative for smooth functions in the entire space, using Young's inequality and Gagliardo-Nirenberg-Sobolev inequality, one deduces the following estimate for the first term of the integrand in (5.8):

$$\begin{aligned}&\|\phi(t-s) * \mathbb{P}\left[(\mathbf{w}-\mathbf{u})\cdot\nabla\mathbf{w}\right](s)\|_{L^q(\mathbb{R}^n)} \\ &\quad\le \|\nabla\phi(t-s) * \mathbb{P}\left[(\mathbf{w}-\mathbf{u})\cdot\mathbf{w}\right](s)\|_{L^q(\mathbb{R}^n)} \\ &\quad\le \|\nabla\phi(t-s)\|_{L^p(\mathbb{R}^n)} \|(\mathbf{w}-\mathbf{u})\cdot\mathbf{w}\|_{L^2(\mathbb{R}^n)} \\ &\quad\le \|\nabla\phi(t-s)\|_{L^p(\mathbb{R}^n)} \|\mathbf{w}\|_{L^{\frac{ln}{n-l\beta}}(\mathbb{R}^n)} \|\mathbf{w}-\mathbf{u}\|_{L^q(\mathbb{R}^n)} \\ &\quad\le \|\nabla\phi(t-s)\|_{L^p(\mathbb{R}^n)} \left\|\Lambda^\beta\mathbf{w}\right\|_{L^l(\mathbb{R}^n)} \|\mathbf{w}-\mathbf{u}\|_{L^q(\mathbb{R}^n)}.\end{aligned} \tag{5.9}$$

Here and hereafter, $\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{2}$, $\frac{1}{2} = \frac{1}{q} + \frac{n-l\beta}{ln}$ and $1 - \frac{1}{p} = \frac{1}{2} - \frac{1}{q} = \frac{n-l\beta}{ln} < \frac{2\beta-1}{n}$ for $l > \frac{n}{3\beta-1}$.

Due to Proposition 5.3 with $\gamma = \frac{n}{2}\left(\frac{1}{2}-\frac{1}{q}\right) < \frac{\beta}{2}$, (5.9) can be bounded as follows:

$$\begin{aligned}&\|\phi(t-s) * \mathbb{P}\left[(\mathbf{w}-\mathbf{u})\cdot\nabla\mathbf{w}\right](s)\|_{L^q(\mathbb{R}^n)} \\ &\quad\le \|\nabla\phi(t-s)\|_{L^p(\mathbb{R}^n)} \left\|\Lambda^\beta\mathbf{w}\right\|_{L^l(\mathbb{R}^n)} \\ &\qquad\cdot\left(\|\mathbf{w}-\mathbf{v}\|_{L^q(\mathbb{R}^n)} + C\alpha^{\frac{\beta}{2}-\gamma}\left\|\Lambda^\beta\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\right).\end{aligned} \tag{5.10}$$

Making a similar derivation to (5.10) for the second term of (5.8), one achieves

$$\begin{aligned}&\|\phi(t-s) * \mathbb{P}\left[\mathbf{u}\cdot\nabla(\mathbf{w}-\mathbf{v})\right](s)\|_{L^q(\mathbb{R}^n)} \\ &\quad= \|\nabla\phi(t-s) * \mathbb{P}\left[\mathbf{u}\cdot(\mathbf{w}-\mathbf{v})\right](s)\|_{L^q(\mathbb{R}^n)} \\ &\quad\le \|\nabla\phi(t-s)\|_{L^p(\mathbb{R}^n)} \|\mathbf{u}\cdot(\mathbf{w}-\mathbf{v})\|_{L^2(\mathbb{R}^n)} \\ &\quad\le \|\nabla\phi(t-s)\|_{L^p(\mathbb{R}^n)} \|\mathbf{u}\|_{L^{\frac{ln}{n-l\beta}}(\mathbb{R}^n)} \|\mathbf{w}-\mathbf{v}\|_{L^q(\mathbb{R}^n)} \\ &\quad\le \|\nabla\phi(t-s)\|_{L^p(\mathbb{R}^n)} \left\|\Lambda^\beta\mathbf{u}\right\|_{L^l(\mathbb{R}^n)} \|\mathbf{w}-\mathbf{v}\|_{L^q(\mathbb{R}^n)}.\end{aligned} \tag{5.11}$$

In the same manner, one can deduce the following estimate for the third term of (5.8):

$$\begin{aligned}
&\left\|\phi(t-s)*\mathbb{P}\left[\sum_{j=1}^{n} u_j\nabla(v_j-w_j)\right]\right\|_{L^q(\mathbb{R}^n)}\\
&\quad\leq\left\|\nabla\phi(t-s)*\mathbb{P}\left[\sum_{j=1}^{n} u_j(v_j-w_j)\right]\right\|_{L^q(\mathbb{R}^n)}\\
&\qquad+\left\|\phi(t-s)*\mathbb{P}\left[\sum_{j=1}^{n} \nabla u_j(v_j-w_j)\right]\right\|_{L^q(\mathbb{R}^n)}\\
&\quad\leq\|\nabla\phi(t-s)\|_{L^p(\mathbb{R}^n)}\sum_{j=1}^{n}\left\|u_j\right\|_{L^{\frac{ln}{n-l\beta}}(\mathbb{R}^n)}\left\|v_j-w_j\right\|_{L^q(\mathbb{R}^n)}\\
&\qquad+\|\phi(t-s)\|_{L^p(\mathbb{R}^n)}\sum_{j=1}^{n}\left\|\nabla u_j\right\|_{L^{\frac{ln}{n-l\beta}}(\mathbb{R}^n)}\left\|w_j-v_j\right\|_{L^q(\mathbb{R}^n)}\\
&\quad\leq\|\nabla\phi(t-s)\|_{L^p(\mathbb{R}^n)}\sum_{j=1}^{n}\left\|\Lambda^\beta u_j\right\|_{L^l(\mathbb{R}^n)}\left\|v_j-w_j\right\|_{L^q(\mathbb{R}^n)}\\
&\qquad+\|\phi(t-s)\|_{L^p(\mathbb{R}^n)}\sum_{j=1}^{n}\left\|\Lambda^\beta\nabla u_j\right\|_{L^l(\mathbb{R}^n)}\left\|w_j-v_j\right\|_{L^q(\mathbb{R}^n)}.
\end{aligned}\tag{5.12}$$

The fourth term of (5.8) can also be bounded as

$$\begin{aligned}
&\left\|\phi(t-s)*\mathbb{P}\left[\sum_{j=1}^{n}(v_j-w_j)\nabla w_j\right]\right\|_{L^q(\mathbb{R}^n)}\\
&\quad\leq\left\|\nabla\phi(t-s)*\mathbb{P}\left[\sum_{j=1}^{n}(u_j-w_j)w_j\right]\right\|_{L^q(\mathbb{R}^n)}\\
&\qquad+\left\|\phi(t-s)*\mathbb{P}\left[\sum_{j=1}^{n}\nabla(u_j-w_j)w_j\right]\right\|_{L^q(\mathbb{R}^n)}\\
&\quad\leq\left\|\nabla\phi(t-s)*\mathbb{P}\left[\sum_{j=1}^{n}(u_j-w_j)w_j\right]\right\|_{L^q(\mathbb{R}^n)}\\
&\quad\leq\|\nabla\phi(t-s)\|_{L^p(\mathbb{R}^n)}\left\|\sum_{j=1}^{n}(u_j-w_j)w_j\right\|_{L^2(\mathbb{R}^n)}\\
&\quad\leq\|\nabla\phi(t-s)\|_{L^p(\mathbb{R}^n)}\|\mathbf{u}-\mathbf{w}\|_{L^q(\mathbb{R}^n)}\|\mathbf{w}\|_{L^{\frac{ln}{n-l\beta}}(\mathbb{R}^n)}\\
&\quad\leq\|\nabla\phi(t-s)\|_{L^p(\mathbb{R}^n)}\|\mathbf{u}-\mathbf{w}\|_{L^q(\mathbb{R}^n)}\left\|\Lambda^\beta\mathbf{w}\right\|_{L^l(\mathbb{R}^n)}\\
&\quad\leq\|\nabla\phi(t-s)\|_{L^p(\mathbb{R}^n)}\left\|\Lambda^\beta\mathbf{w}\right\|_{L^l(\mathbb{R}^n)}\\
&\qquad\quad\cdot\left(\|\mathbf{w}-\mathbf{v}\|_{L^q(\mathbb{R}^n)}+C\alpha^{\frac{\beta}{2}-\gamma}\left\|\Lambda^\beta\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\right).
\end{aligned}\tag{5.13}$$

Putting together estimates (5.8), (5.9), (5.10), (5.11), (5.12) and (5.13), we conclude

$$\begin{aligned}\|\mathbf{v}-\mathbf{w}\|_{L^q(\mathbb{R}^n)} \le{}& C\alpha^{\frac{\beta}{2}-\gamma}\int_0^t \left\|\Lambda^\beta \mathbf{v}\right\|_{L^2(\mathbb{R}^n)} ds \\ &+C\sup_{t\in[0,T]}\left(\left\|\Lambda^\beta \mathbf{w}\right\|_{L^l(\mathbb{R}^n)}+\left\|\Lambda^\beta \mathbf{u}\right\|_{L^l(\mathbb{R}^n)}+\left\|\Lambda^\beta \nabla\mathbf{u}\right\|_{L^l(\mathbb{R}^n)}\right) \\ &\cdot\int_0^t \left(\|\phi(t-s)\|_{L^p(\mathbb{R}^n)}+\|\nabla\phi(t-s)\|_{L^p(\mathbb{R}^n)}\right)\|\mathbf{v}-\mathbf{w}\|_{L^q(\mathbb{R}^n)}(s)ds.\end{aligned} \tag{5.14}$$

Finally, thanks to Lemma 5.4, Lemma 5.5 and Lemma 5.6, we deduce that for $\phi(t)=e^{-(-\Delta)^\beta t}$

$$\|\phi(t-s)\|_{L^p(\mathbb{R}^n)} \le \frac{1}{(t-s)^{\delta_1}},\quad \delta_1=\left(1-\frac{1}{p}\right)\frac{n}{2\beta}, \tag{5.15}$$

$$\|\nabla\phi(t-s)\|_{L^p(\mathbb{R}^n)} \le \frac{1}{(t-s)^{\delta_2}},\quad \delta_2=\frac{1}{2\beta}+\left(1-\frac{1}{p}\right)\frac{n}{2\beta}. \tag{5.16}$$

As a consequence, for $\gamma=\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)<\frac{\beta}{2}$, we infer from (5.14), (5.15) and (5.16) that

$$\|\mathbf{v}-\mathbf{w}\|_{L^q(\mathbb{R}^n)} \le C\alpha^{\frac{\beta}{2}-\gamma}+B\int_0^t \frac{1}{(t-s)^\delta}\|\mathbf{v}-\mathbf{w}\|_{L^q(\mathbb{R}^n)}(s)ds, \tag{5.17}$$

where

$$\begin{cases} A=C\int_0^t \left\|\Lambda^\beta \mathbf{v}\right\|_{L^2(\mathbb{R}^n)} ds, \\ B=C\sup_{t\in[0,T]}\left(\left\|\Lambda^\beta \mathbf{w}\right\|_{L^l(\mathbb{R}^n)}+\left\|\Lambda^\beta \mathbf{u}\right\|_{L^l(\mathbb{R}^n)}+\left\|\Lambda^\beta \nabla\mathbf{u}\right\|_{L^l(\mathbb{R}^n)}\right), \\ \delta=\max\{\delta_1,\delta_2\}=\frac{1}{2\beta}+\left(1-\frac{1}{p}\right)\frac{n}{2\beta}<\frac{1}{2\beta}+\frac{2\beta-1}{n}\frac{n}{2\beta}<1. \end{cases} \tag{5.18}$$

Here, we have used some known facts:

$$l>\frac{n}{3\beta-1},\ \frac{1}{q}+1=\frac{1}{p}+\frac{1}{2},\ \frac{1}{2}=\frac{1}{q}+\frac{n-l\beta}{ln},\ 1-\frac{1}{p}=\frac{1}{2}-\frac{1}{q}=\frac{n-l\beta}{ln}<\frac{2\beta-1}{n}.$$

Note that (5.17) and (5.18), the Gronwall inequality then implies that

$$\|\mathbf{v}-\mathbf{w}\|_{L^q(\mathbb{R}^n)} \le A\alpha^{\frac{\beta}{2}-\gamma}\exp\left(\int_0^t \frac{B}{(t-s)^\delta}ds\right), \tag{5.19}$$

where $\int_0^t (t-s)^{-\delta}ds=\left.\frac{(t-s)^{-\delta+1}}{-\delta+1}\right|_0^t=-\frac{t^{-\delta+1}}{-\delta+1}$, which is finite for $\delta<1$ and $t\in[0,T]$. Hence, for $t\in[0,T]$, letting $\alpha\to 0$ deduces that $\mathbf{v}\to\mathbf{w}$ strongly in $L^q(\mathbb{R}^n)$.

This finishes the proof of Theorem 5.1. □

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Appendix A

We are now in the position to show (3.18). Note that $\mathbf{v}^\varepsilon$ is a solution of (1.1)-(1.2), multiplying the first equation for $\mathbf{v}^\varepsilon$ in (1.1) by $\Delta\mathbf{v}^\varepsilon$, then integrating by parts yields

$$\frac{1}{2}\frac{d}{dt}\left\|\nabla\mathbf{v}^\varepsilon\right\|_{L^2(\mathbb{R}^n)}^2+\nu\left\|\Lambda^\beta\nabla\mathbf{v}^\varepsilon\right\|_{L^2(\mathbb{R}^n)}^2=\langle\mathbf{u}^\varepsilon\cdot\nabla\mathbf{v}^\varepsilon,\Delta\mathbf{v}^\varepsilon\rangle+\left\langle\mathbf{v}^\varepsilon\cdot\nabla\mathbf{u}^{\varepsilon T},\Delta\mathbf{v}^\varepsilon\right\rangle. \qquad (A-1)$$

We then deal with the two terms on the right hand side of (A-1) through two cases:

Case (I) $\frac{n}{4}<\beta<1$ for $n=2,3$;

Case (II) $\beta=\frac{n}{4}$ for $n=2,3$.

We first consider Case **(I)** $\frac{n}{4}<\beta<1$ for $n=2,3$.

In this case, notice that $\langle\mathbf{u}^\varepsilon\cdot\nabla\mathbf{v}^\varepsilon,\mathbf{v}^\varepsilon\rangle=0$, $\frac{\beta}{n}=\frac{1}{2}-\frac{n/2-\beta}{n}$, $\frac{n}{2}-1<\frac{n}{2}-\beta<\frac{n}{4}$, $\frac{n}{2}-1<\frac{n}{2}-2\beta+1<1$, Hölder's inequality, Sobolev inequality and Cauchy-Schwartz inequality yield that

$$\begin{aligned}
\left|\langle\mathbf{u}^\varepsilon\cdot\nabla\mathbf{v}^\varepsilon,\Delta\mathbf{v}^\varepsilon\rangle\right|&=\left|\langle\nabla\mathbf{u}^\varepsilon\cdot\nabla\mathbf{v}^\varepsilon,\nabla\mathbf{v}^\varepsilon\rangle\right|\\
&\le C\left\|\nabla\mathbf{v}^\varepsilon\right\|_{L^2(\mathbb{R}^n)}\left\|\nabla\mathbf{u}^\varepsilon\nabla v^\varepsilon\right\|_{L^2(\mathbb{R}^n)}\\
&\le C\left\|\nabla\mathbf{v}^\varepsilon\right\|_{L^2(\mathbb{R}^n)}\left\|\nabla\mathbf{v}^\varepsilon\right\|_{L^{\frac{2n}{n-2\beta}}(\mathbb{R}^n)}\left\|\nabla\mathbf{u}^\varepsilon\right\|_{L^{\frac{n}{\beta}}(\mathbb{R}^n)}\\
&\le C\left\|\nabla\mathbf{v}^\varepsilon\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^\beta\nabla\mathbf{v}^\varepsilon\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{2}-\beta}\nabla\mathbf{u}^\varepsilon\right\|_{L^2(\mathbb{R}^n)} \qquad (A-2)\\
&\le C\left\|\nabla\mathbf{v}^\varepsilon\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^\beta\nabla\mathbf{v}^\varepsilon\right\|_{L^2(\mathbb{R}^n)}\left\|\nabla\mathbf{u}^\varepsilon\right\|_{L^2(\mathbb{R}^n)}^{1+\beta-\frac{n}{2}}\left\|\Delta\mathbf{u}^\varepsilon\right\|_{L^2(\mathbb{R}^n)}^{\frac{n}{2}-\beta}\\
&\le C\left\|\nabla\mathbf{v}^\varepsilon\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^\beta\nabla\mathbf{v}^\varepsilon\right\|_{L^2(\mathbb{R}^n)}\left\|\mathbf{v}^\varepsilon\right\|_{L^2(\mathbb{R}^n)}\\
&\le\frac{\nu}{4}\left\|\Lambda^\beta\nabla\mathbf{v}^\varepsilon\right\|_{L^2(\mathbb{R}^n)}^2+C\left\|\mathbf{v}^\varepsilon\right\|_{L^2(\mathbb{R}^n)}^2\left\|\nabla\mathbf{v}^\varepsilon\right\|_{L^2(\mathbb{R}^n)}^2.
\end{aligned}$$

On the other hand, Lemma 2.8 and Lemma 2.11 ensure that

$$\begin{aligned}
&\left|\left\langle\mathbf{v}^\varepsilon\cdot\nabla\mathbf{u}^{\varepsilon T},\Delta\mathbf{v}^\varepsilon\right\rangle\right|\\
&\le\left|\left\langle\Lambda^{1-\beta}\left(\mathbf{v}^\varepsilon\cdot\nabla\mathbf{u}^{\varepsilon T}\right),\Lambda^\beta\nabla\mathbf{v}^\varepsilon\right\rangle\right|\\
&\le\left\|\Lambda^{1-\beta}\left(\mathbf{v}^\varepsilon\cdot\nabla\mathbf{u}^{\varepsilon T}\right)\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^\beta\nabla\mathbf{v}^\varepsilon\right\|_{L^2(\mathbb{R}^n)}\\
&\le C\Bigg(\left\|\Lambda^{1-\beta}\mathbf{v}^\varepsilon\right\|_{L^{\frac{2n}{n-4\beta+2}}(\mathbb{R}^n)}\left\|\nabla\mathbf{u}^\varepsilon\right\|_{L^{\frac{2n}{4\beta-2}}(\mathbb{R}^n)}\\
&\qquad+\left\|\mathbf{v}^\varepsilon\right\|_{L^{\frac{2n}{n-2\beta}}(\mathbb{R}^n)}\left\|\nabla\mathbf{u}^\varepsilon\right\|_{L^{\frac{n}{\beta}}(\mathbb{R}^n)}\Bigg)\cdot\left\|\Lambda^\beta\nabla\mathbf{v}^\varepsilon\right\|_{L^2(\mathbb{R}^n)} \qquad (A-3)\\
&\le\frac{\nu}{4}\left\|\Lambda^\beta\nabla\mathbf{v}^\varepsilon\right\|_{L^2(\mathbb{R}^n)}^2+C\left\|\Lambda^\beta\mathbf{v}^\varepsilon\right\|_{L^2(\mathbb{R}^n)}^2\left\|\Lambda^{\frac{n}{2}-2\beta+1}\nabla\mathbf{u}^\varepsilon\right\|_{L^2(\mathbb{R}^n)}^2\\
&\le\frac{\nu}{4}\left\|\Lambda^\beta\nabla\mathbf{v}^\varepsilon\right\|_{L^2(\mathbb{R}^n)}^2+C\left\|\Lambda^\beta\mathbf{v}^\varepsilon\right\|_{L^2(\mathbb{R}^n)}^2\|\mathbf{v}^\varepsilon\|_{L^2(\mathbb{R}^n)}^2\\
&\le\frac{\nu}{4}\left\|\Lambda^\beta\nabla\mathbf{v}^\varepsilon\right\|_{L^2(\mathbb{R}^n)}^2+C\left(\left\|\mathbf{v}^\varepsilon\right\|_{L^2(\mathbb{R}^n)}+\left\|\nabla\mathbf{v}^\varepsilon\right\|_{L^2(\mathbb{R}^n)}\right)^2\|\mathbf{v}^\varepsilon\|_{L^2(\mathbb{R}^n)}^2.
\end{aligned}$$

Combining (A-1) with (A-2) and (A-3) gives rise to

$$\frac{d}{dt}\left\|\nabla\mathbf{v}^{\varepsilon}\right\|^2_{L^2(\mathbb{R}^n)}+\frac{\nu}{2}\left\|\Lambda^{\beta}\nabla\mathbf{v}^{\varepsilon}\right\|^2_{L^2(\mathbb{R}^n)}\leq C\left\|\nabla\mathbf{v}^{\varepsilon}\right\|^2_{L^2(\mathbb{R}^n)}\left\|\mathbf{v}^{\varepsilon}\right\|^2_{L^2(\mathbb{R}^n)}. \qquad (A-4)$$

We next consider Case **(II)** $\quad \beta=\dfrac{n}{4}$ for $n=2,3$.

In this case, thanks to $\langle\mathbf{u}^{\varepsilon}\cdot\nabla\mathbf{v}^{\varepsilon},\mathbf{v}^{\varepsilon}\rangle=0$ and $\frac{1}{4}=\frac{1}{2}-\frac{\beta}{n}$, note that Lemma 2.12, applying Hölder's inequality and Gagliardo-Nirenberg-Sobolev inequality imply

$$\begin{aligned}|\langle\mathbf{u}^{\varepsilon}\cdot\nabla\mathbf{v}^{\varepsilon},\Delta\mathbf{v}^{\varepsilon}\rangle|&\leq|\langle\nabla\mathbf{u}^{\varepsilon}\cdot\nabla\mathbf{v}^{\varepsilon},\nabla\mathbf{v}^{\varepsilon}\rangle|\\&\leq\|\nabla\mathbf{u}^{\varepsilon}\|_{L^2(\mathbb{R}^n)}\|\nabla\mathbf{v}^{\varepsilon}\|^2_{L^4(\mathbb{R}^n)}\\&\leq C\|\nabla\mathbf{u}^{\varepsilon}\|_{L^2(\mathbb{R}^n)}\left(\|\nabla\mathbf{v}^{\varepsilon}\|^2_{L^2(\mathbb{R}^n)}+\left\|\Lambda^{\frac{n}{4}}\nabla\mathbf{v}^{\varepsilon}\right\|^2_{L^2(\mathbb{R}^n)}\right),\end{aligned} \qquad (A-5)$$

and

$$\begin{aligned}&\left|\left\langle\mathbf{v}^{\varepsilon}\cdot\nabla\mathbf{u}^{\varepsilon T},\Delta\mathbf{v}^{\varepsilon}\right\rangle\right|\\&\leq\|\nabla\mathbf{v}^{\varepsilon}\|^2_{L^4(\mathbb{R}^n)}\|\nabla\mathbf{u}^{\varepsilon}\|_{L^2(\mathbb{R}^n)}+\|\Delta\mathbf{u}^{\varepsilon}\|_{L^2(\mathbb{R}^n)}\|\mathbf{v}^{\varepsilon}\nabla v^{\varepsilon}\|_{L^2(\mathbb{R}^n)}\\&\leq\|\nabla\mathbf{v}^{\varepsilon}\|^2_{L^4(\mathbb{R}^n)}\|\nabla\mathbf{u}^{\varepsilon}\|_{L^2(\mathbb{R}^n)}+\|\Delta\mathbf{u}^{\varepsilon}\|_{L^2(\mathbb{R}^n)}\|\mathbf{v}^{\varepsilon}\|_{L^4(\mathbb{R}^n)}\|\nabla v^{\varepsilon}\|_{L^4(\mathbb{R}^n)}\\&\leq C\|\nabla\mathbf{u}^{\varepsilon}\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{4}}\nabla\mathbf{v}^{\varepsilon}\right\|^2_{L^2(\mathbb{R}^n)}+\|\Delta\mathbf{u}^{\varepsilon}\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{4}}\mathbf{v}^{\varepsilon}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{4}}\nabla\mathbf{v}^{\varepsilon}\right\|_{L^2(\mathbb{R}^n)}\\&\leq C\|\nabla\mathbf{u}^{\varepsilon}\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{4}}\nabla\mathbf{v}^{\varepsilon}\right\|^2_{L^2(\mathbb{R}^n)}+\|\Delta\mathbf{u}^{\varepsilon}\|_{L^2(\mathbb{R}^n)}\left(\left\|\Lambda^{\frac{n}{4}}\mathbf{v}^{\varepsilon}\right\|^2_{L^2(\mathbb{R}^n)}+\left\|\Lambda^{\frac{n}{4}}\nabla\mathbf{v}^{\varepsilon}\right\|^2_{L^2(\mathbb{R}^n)}\right)\\&\lesssim\left(\|\nabla\mathbf{u}^{\varepsilon}\|_{L^2(\mathbb{R}^n)}+\|\Delta\mathbf{u}^{\varepsilon}\|_{L^2(\mathbb{R}^n)}\right)\left\|\Lambda^{\frac{n}{4}}\nabla\mathbf{v}^{\varepsilon}\right\|^2_{L^2(\mathbb{R}^n)}+\|\Delta\mathbf{u}^{\varepsilon}\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{4}}\mathbf{v}^{\varepsilon}\right\|^2_{L^2(\mathbb{R}^n)}\\&\lesssim\left(\|\nabla\mathbf{u}^{\varepsilon}\|_{L^2(\mathbb{R}^n)}+\|\Delta\mathbf{u}^{\varepsilon}\|_{L^2(\mathbb{R}^n)}\right)\left\|\Lambda^{\frac{n}{4}}\nabla\mathbf{v}^{\varepsilon}\right\|^2_{L^2(\mathbb{R}^n)}\\&\quad+\|\Delta\mathbf{u}^{\varepsilon}\|_{L^2(\mathbb{R}^n)}\left(\|\mathbf{v}^{\varepsilon}\|^2_{L^2(\mathbb{R}^n)}+\|\nabla\mathbf{v}^{\varepsilon}\|^2_{L^2(\mathbb{R}^n)}\right).\end{aligned} \qquad (A-6)$$

By Proposition 2.4, the assumptions in (III) of this theorem, once we choose $\|\nabla\mathbf{u}^{\varepsilon}\|_{L^2(\mathbb{R}^n)}+\|\Delta\mathbf{u}^{\varepsilon}\|_{L^2(\mathbb{R}^n)}\lesssim\|\mathbf{v}^{\varepsilon}\|_{L^2(\mathbb{R}^n)}\lesssim\|\mathbf{v}^{\varepsilon}_0\|_{L^2(\mathbb{R}^n)}\lesssim\|\mathbf{v}_0\|_{L^2(\mathbb{R}^n)}\leq\varepsilon^*\leq\frac{\nu}{4}$, (A-1), (A-5) and (A-6) yield

$$\frac{d}{dt}\|\nabla\mathbf{v}^{\varepsilon}\|^2_{L^2(\mathbb{R}^n)}+\frac{\nu}{2}\left\|\Lambda^{\frac{n}{4}}\nabla\mathbf{v}^{\varepsilon}\right\|^2_{L^2(\mathbb{R}^n)}\lesssim\|\mathbf{v}^{\varepsilon}\|_{L^2(\mathbb{R}^n)}\|\nabla\mathbf{v}^{\varepsilon}\|^2_{L^2(\mathbb{R}^n)}. \qquad (A-7)$$

Using (A-4) and (A-7), Gronwall's lemma yields that for $\dfrac{n}{4}\leq\beta<1$ with $n=2,3$

$$\|\nabla\mathbf{v}^{\varepsilon}\|^2_{L^2(\mathbb{R}^n)}\leq\|\nabla\mathbf{v}^{\varepsilon}_0\|^2_{L^2(\mathbb{R}^n)}e^{C\|\mathbf{v}^{\varepsilon}_0\|^2_{L^2(\mathbb{R}^n)}}\leq C\varepsilon^2. \qquad (A-8)$$

This gives

$$\left\|\Lambda^{\beta}\mathbf{u}^{\varepsilon}\right\|^2_{L^2(\mathbb{R}^n)}+\alpha^2\left\|\Lambda^{\beta}\nabla\mathbf{u}^{\varepsilon}\right\|^2_{L^2(\mathbb{R}^n)}\leq\left\|\nabla\mathbf{u}^{\varepsilon}\right\|^2_{L^2(\mathbb{R}^n)}+\alpha^2\left\|\Delta\mathbf{u}^{\varepsilon}\right\|^2_{L^2(\mathbb{R}^n)}\leq\left\|\nabla\mathbf{v}^{\varepsilon}\right\|^2_{L^2(\mathbb{R}^n)}\leq C\varepsilon^2. \qquad (A-9)$$

It follows from (2.1) and (A-9) that

$$\frac{d}{dt}\left(\|\mathbf{u}^{\varepsilon}\|^2_{L^2(\mathbb{R}^n)}+\alpha^2\|\nabla\mathbf{u}^{\varepsilon}\|^2_{L^2(\mathbb{R}^n)}\right)\geq-C\varepsilon^2.$$

This is the estimate (3.18). So far, we finish the proof of Theorem 3.1. □

Appendix B

We shall prove (VI-1) of Theorem 4.1 here by using inductive argument. Due to the regularity of solutions (Proposition 2.4), we present the proof only formally. It should be pointed out that the key point of the proof is to establish an inequality in a form satisfing the conclusion in (IV) of this theorem. To achieve this, we shall divide the proof into the following three steps.

Step 1. For $m = 0, 1$, the inequality holds by (III-2) and (II), respectively. That is,

$$\|\mathbf{v}\|^2_{L^2(\mathbb{R}^n)} \le C(1+t)^{-\frac{n}{2\beta}}, \ \ \|\nabla\mathbf{v}\|^2_{L^2(\mathbb{R}^n)} \le C(1+t)^{-\frac{1}{\beta}-\frac{n}{2\beta}}.$$

Step 2. We now assume (inductive assumption) that the decay

$$\|\nabla^m\mathbf{v}\|^2_{L^2(\mathbb{R}^n)} \le C(1+t)^{-\frac{m}{\beta}-\frac{n}{2\beta}} \qquad (B-1)$$

holds for all $m < M$. Here, m and M are both non-negative integers.

Step 3. We will verify that the inequality (B-1) is true for $m = M$.

Multiplying the first equation in (1.1) by $\Delta^M\mathbf{v}$, and then integrating by parts the resulting equation gives rise to

$$\begin{aligned} &\tfrac{d}{dt}\left\|\nabla^M\mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)} + 2\nu\left\|\Lambda^\beta\nabla^M\mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)} \\ &\quad\le \left|\left\langle \mathbf{u}\cdot\nabla\mathbf{v}, \Delta^M\mathbf{v}\right\rangle\right| + \left|\left\langle \mathbf{v}\cdot\nabla\mathbf{u}^T, \Delta^M\mathbf{v}\right\rangle\right| \\ &\quad\triangleq\ I_M + J_M. \end{aligned} \qquad (B-2)$$

To bound I_M and J_M, there are two cases to consider.

Case **(I)** $\dfrac{n}{4} < \beta < 1$ for $n = 2, 3$;

Case **(II)** $\beta = \dfrac{n}{4}$ for $n = 2, 3$.

We first consider Case **(I)** $\dfrac{n}{4} < \beta < 1$ for $n = 2, 3$.

In this case, it is easy to check that $\dfrac{n}{2} - \beta < \beta$ and $\frac{n}{\beta} = \frac{2n}{n-2(n/2-\beta)}$. Recall that (B-2) and $\langle \mathbf{u}\cdot\nabla\mathbf{v}, \mathbf{v}\rangle = 0$, thanks to Cauchy's inequality, Hölder's inequality and Gagliardo-Nirenberg-Sobolev inequality, one deduces that

$$
\begin{aligned}
I_M &= \left|\left\langle \mathbf{u}\cdot\nabla\mathbf{v}, \Delta^M \mathbf{v}\right\rangle\right| \\
&= \left|\sum_{m=1}^{M}\binom{M}{m}\left\langle \nabla^m\mathbf{u}\cdot\nabla\nabla^{M-m}\mathbf{v}, \nabla^M\mathbf{v}\right\rangle\right| \\
&\leq C\sum_{m=1}^{M}\left\|\nabla^{M+1-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\nabla^m\mathbf{u}\cdot\nabla^M\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\leq C\sum_{m=1}^{M}\left\|\nabla^{M+1-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\nabla^m\mathbf{u}\right\|_{L^{\frac{n}{\beta}}(\mathbb{R}^n)}\left\|\nabla^M\mathbf{v}\right\|_{L^{\frac{2n}{n-2\beta}}(\mathbb{R}^n)} \\
&\leq C\sum_{m=1}^{M}\left\|\nabla^{M+1-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^\beta\nabla^m\mathbf{u}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^\beta\nabla^M\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \qquad (B-3)\\
&\leq C\sum_{m=1}^{M}\left\|\nabla^{M+1-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{2}-\beta}\nabla^m\mathbf{u}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^\beta\nabla^M\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\leq C\sum_{m=1}^{M}\left\|\nabla^{M+1-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^\beta\nabla^{m-1}\mathbf{u}\right\|_{L^2(\mathbb{R}^n)}^{2\beta-\frac{n}{2}}\left\|\Lambda^\beta\nabla^m\mathbf{u}\right\|_{L^2(\mathbb{R}^n)}^{\frac{n}{2}+1-2\beta}\left\|\Lambda^\beta\nabla^M\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\leq C\sum_{m=1}^{M}\left\|\nabla^{M+1-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^\beta\nabla^{m-1}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^\beta\nabla^M\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\leq C\sum_{m=1}^{M}\left\|\nabla^{M+1-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2\left\|\Lambda^\beta\nabla^{m-1}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2+\frac{\nu}{2}\left\|\Lambda^\beta\nabla^M\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
J_M &= \left|\left\langle \mathbf{v}\cdot\nabla\mathbf{u}^T, \Delta^M\mathbf{v}\right\rangle\right| \\
&= \sum_{m=0}^{M}\binom{M}{m}\left|\left\langle \nabla^M\mathbf{v}\cdot\nabla\nabla^m\mathbf{u}, \nabla^{M-m}\mathbf{v}\right\rangle\right| \qquad (B-4)\\
&\leq C\left|\left\langle \nabla^M\mathbf{v}\cdot\nabla\mathbf{u}, \nabla^M\mathbf{v}\right\rangle\right| + C\sum_{m=1}^{M}\left|\left\langle \nabla^M\mathbf{v}\cdot\nabla^{m+1}\mathbf{u}, \nabla^{M-m}\mathbf{v}\right\rangle\right|.
\end{aligned}
$$

By Lemma 2.12, a similar computation to (B-3) shows that

$$
\begin{aligned}
&\left|\left\langle \nabla^M\mathbf{v}\cdot\nabla\mathbf{u}, \nabla^M\mathbf{v}\right\rangle\right| \\
&\quad\leq \left\|\nabla^M\mathbf{v}\right\|_{L^4(\mathbb{R}^n)}^2\|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^n)} \\
&\quad\leq \left(\left\|\nabla^M\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^{1-\frac{n}{4\beta}}\left\|\nabla^M\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^{\frac{n}{4\beta}}\right)\|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^n)} \qquad (B-5)\\
&\quad\leq \left(C(\varepsilon)\left\|\nabla^M\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2+\varepsilon\left\|\Lambda^\beta\nabla^M\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2\right)\|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^n)},
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{m=1}^{M}\left|\left\langle\nabla^{M}\mathbf{v}\cdot\nabla^{m+1}\mathbf{u},\nabla^{M-m}\mathbf{v}\right\rangle\right| \\
&\qquad\le\sum_{m=1}^{M}\left\|\nabla^{M-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\nabla^{m+1}\mathbf{u}\cdot\nabla^{M}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\qquad\le\sum_{m=1}^{M}\left\|\nabla^{M-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\nabla^{m+1}\mathbf{u}\right\|_{L^{\frac{n}{\beta}}(\mathbb{R}^n)}\left\|\nabla^{M}\mathbf{v}\right\|_{L^{\frac{2n}{n-2\beta}}(\mathbb{R}^n)} \\
&\qquad\le\sum_{m=1}^{M}\left\|\nabla^{M-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{2}-\beta}\nabla^{m+1}\mathbf{u}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\beta}\nabla^{M}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \qquad (B-6)\\
&\qquad\le\sum_{m=1}^{M}\left\|\nabla^{M-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\beta}\nabla^{m-1}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\beta}\nabla^{M}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\qquad\le C\sum_{m=1}^{M}\left\|\nabla^{M-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2\left\|\Lambda^{\beta}\nabla^{m-1}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2+\frac{\nu}{4}\left\|\Lambda^{\beta}\nabla^{M}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2.
\end{aligned}
$$

We have used the relation $\mathbf{u}-\alpha^2\Delta\mathbf{u}=\mathbf{v}$ in the estimates (B-3) and (B-6). Choosing $\varepsilon\le\dfrac{\nu}{4\|\nabla\mathbf{u}_0\|_{L^2(\mathbb{R}^n)}}$, if follows from (B-2), (B-3), (B-4), (B-5) and (B-6) that

$$
\begin{aligned}
&\frac{d}{dt}\left\|\nabla^{M}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2+\nu\left\|\Lambda^{\beta}\nabla^{M}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad\le C\left\|\nabla^{M}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2\left\|\Lambda^{\beta}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2+C(\varepsilon)\left\|\nabla^{M}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2\|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^n)}^2 \qquad (B-7)\\
&\qquad+\left\|\nabla^{M-1}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2\left\|\Lambda^{\beta}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2+C\sum_{m=2}^{M}\left\|\nabla^{M-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2\left\|\Lambda^{\beta}\nabla^{m-1}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2.
\end{aligned}
$$

We next consider Case **(II)** $\quad\beta=\dfrac{n}{4}$ for $n=2,3$.

In this case, thanks to Lemma 2.12, note that (B-2) and $\langle\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{v}\rangle=0$, we deduce the following two estimates:

$$
\begin{aligned}
I_M&=\left|\left\langle\mathbf{u}\cdot\nabla\mathbf{v},\Delta^{M}\mathbf{v}\right\rangle\right| \\
&=\left|\sum_{m=1}^{M}\begin{pmatrix}M\\ m\end{pmatrix}\left\langle\nabla^{m}\mathbf{u}\cdot\nabla\nabla^{M-m}\mathbf{v},\nabla^{M}\mathbf{v}\right\rangle\right| \\
&\le C\sum_{m=1}^{M}\left\|\nabla^{M+1-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\nabla^{m}\mathbf{u}\cdot\nabla^{M}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\le C\sum_{m=1}^{M}\left\|\nabla^{M+1-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\nabla^{m}\mathbf{u}\right\|_{L^4(\mathbb{R}^n)}\left\|\nabla^{M}\mathbf{v}\right\|_{L^4(\mathbb{R}^n)} \qquad (B-8)\\
&\le C\sum_{m=1}^{M}\left\|\nabla^{M+1-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{4}}\nabla^{m}\mathbf{u}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{4}}\nabla^{M}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\le C\sum_{m=1}^{M}\left\|\nabla^{M+1-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{4}}\nabla^{m-1}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{4}}\nabla^{M}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}.
\end{aligned}
$$

and

$$
\begin{aligned}
J_M &= \left|\left\langle \mathbf{v}\cdot\nabla\mathbf{u}^T, \Delta^M\mathbf{v}\right\rangle\right| \\
&= \sum_{m=0}^{M}\begin{pmatrix} M \\ m \end{pmatrix}\left|\left\langle \nabla^M\mathbf{v}\cdot\nabla\nabla^m\mathbf{u}, \nabla^{M-m}\mathbf{v}\right\rangle\right| \\
&\leq C\sum_{m=0}^{M}\left\|\nabla^{M-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\nabla^{m+1}\mathbf{u}\cdot\nabla^M\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\leq C\sum_{m=0}^{M}\left\|\nabla^{M-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\nabla^{m+1}\mathbf{u}\right\|_{L^4(\mathbb{R}^n)}\left\|\nabla^M\mathbf{v}\right\|_{L^4(\mathbb{R}^n)} \\
&\leq C\sum_{m=0}^{M}\left\|\nabla^{M-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{4}}\nabla^{m+1}\mathbf{u}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{4}}\nabla^M\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\leq C\sum_{m=0}^{M}\left\|\nabla^{M-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{4}}\nabla^{m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{4}}\nabla^M\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}.
\end{aligned} \tag{B-9}
$$

Combining (B-8) with (B-9) yields that

$$
\begin{aligned}
I_M + J_M &\leq C\sum_{m=0}^{M}\left\|\nabla^{M-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{4}}\nabla^{m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{4}}\nabla^M\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\leq C\left\|\nabla^M\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{4}}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{4}}\nabla^M\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\quad + C\sum_{m=1}^{M-1}\left\|\nabla^{M-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{4}}\nabla^{m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{4}}\nabla^M\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\quad + C\left\|\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{4}}\nabla^M\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\leq C\left\|\nabla^M\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^{1-\frac{n}{4}}\left\|\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^{\frac{n}{4}}\left\|\Lambda^{\frac{n}{4}}\nabla^M\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\quad + C\sum_{m=1}^{M-1}\left\|\nabla^{M-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\nabla^{m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^{1-\frac{n}{4}}\left\|\nabla^{m+1}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^{\frac{n}{4}}\left\|\Lambda^{\frac{n}{4}}\nabla^M\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\quad + C\left\|\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{4}}\nabla^M\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\lesssim \left\|\nabla^M\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2\left\|\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^{2-\frac{n}{2}}\left\|\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^{\frac{n}{2}} \\
&\quad + \sum_{m=1}^{M-1}\left(\left\|\nabla^{M-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 + \left\|\nabla^{m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^{(1-\frac{n}{4})\times 2}\left\|\nabla^{m+1}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^{\frac{n}{4}\times 2}\right) \\
&\quad + \left\|\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{4}}\nabla^M\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 + \frac{\nu}{4}\left\|\Lambda^{\frac{n}{4}}\nabla^M\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2.
\end{aligned} \tag{B-10}
$$

This together with (B-2) and the smallness assumption of the initial data for $\beta = \frac{n}{4}$ in (VI) with $M \leq K$ ensures that

$$
\begin{aligned}
&\frac{d}{dt}\left\|\nabla^M \mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)} + \nu\left\|\Lambda^{\frac{n}{4}}\nabla^M \mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)} \\
&\quad\lesssim \left\|\nabla^M \mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{4}}\mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)} + \sum_{m=1}^{M-1}\left\|\nabla^{M-m}\mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{4}}\nabla^m \mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)} \\
&\quad\lesssim \left\|\nabla^M \mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)}\|\mathbf{v}\|^{2-\frac{n}{2}}_{L^2(\mathbb{R}^n)}\|\nabla\mathbf{v}\|^{\frac{n}{2}}_{L^2(\mathbb{R}^n)} + \sum_{m=1}^{M-1}\left\|\nabla^{M-m}\mathbf{v}\right\|^{2-\frac{n}{2}}_{L^2(\mathbb{R}^n)}\left\|\nabla^{m+1}\mathbf{v}\right\|^{\frac{n}{2}}_{L^2(\mathbb{R}^n)} \\
&\quad\lesssim \left(\|\mathbf{v}\|^2_{L^2(\mathbb{R}^n)} + \|\nabla\mathbf{v}\|^2_{L^2(\mathbb{R}^n)}\right)\left\|\nabla^M \mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)} \\
&\qquad + \sum_{m=1}^{M-2}\left(\left\|\nabla^m \mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)} + \left\|\nabla^{m+1}\mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)}\right)\left\|\nabla^{M-m}\mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)} \qquad (B-11)\\
&\qquad + \left(\left\|\nabla^{M-1}\mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)} + \left\|\nabla^M \mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)}\right)\|\nabla\mathbf{v}\|^2_{L^2(\mathbb{R}^n)} \\
&\quad\lesssim \left((1+t)^{-\frac{n}{2\beta}} + (1+t)^{-\frac{1}{\beta}-\frac{n}{2\beta}}\right)\left\|\nabla^M \mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)} \\
&\qquad + \sum_{m=1}^{M-2}\left((1+t)^{-\frac{m}{\beta}-\frac{n}{2\beta}} + (1+t)^{-\frac{m+1}{\beta}-\frac{n}{2\beta}}\right)(1+t)^{-\frac{M-m}{\beta}-\frac{n}{2\beta}} \\
&\qquad + (1+t)^{-\frac{M-1}{\beta}-\frac{n}{2\beta}}(1+t)^{-\frac{1}{\beta}-\frac{n}{2\beta}} + (1+t)^{-\frac{1}{\beta}-\frac{n}{2\beta}}\left\|\nabla^M \mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)} \\
&\quad\lesssim (1+t)^{-\frac{n}{2\beta}}\left\|\nabla^M \mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)} + (1+t)^{-\frac{M}{\beta}-\frac{n}{\beta}}.
\end{aligned}
$$

Note that $\left\|\nabla^M \mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)} \leq C$ by Proposition 2.4, (I), (II) of this theorem and the inductive assumption (B-1), applying interpolation inequality and a bootstrap argument, it follows from (B-7) and (B-11) that for $\frac{n}{4} \leq \beta < 1$ with $n = 2, 3$,

$$
\begin{aligned}
&\frac{d}{dt}\left\|\nabla^M \mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)} + \nu\left\|\Lambda^{\beta}\nabla^M \mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)} \\
&\quad\leq C(1+t)^{-\frac{n}{2\beta}}\left\|\nabla^M \mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)} + C(1+t)^{-\frac{M}{\beta}-\frac{n}{\beta}} \\
&\quad\leq C(1+t)^{-\frac{n}{2\beta}}\left\|\nabla^M \mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)} + C(1+t)^{-\frac{M}{\beta}-\frac{n}{2\beta}-1}(1+t)^{-\frac{n}{2\beta}+\frac{1}{\beta}} \qquad (B-12)\\
&\quad\leq C(1+t)^{-\frac{M}{\beta}-\frac{n}{2\beta}-1}.
\end{aligned}
$$

Here, we applied the fact that for $n = 2, 3, -1 + \frac{1}{\beta} > 0$ and $-\frac{n}{2\beta} + \frac{1}{\beta} \leq 0$. Since $|\mathcal{F}(\mathbf{v})| \leq C$ by (III-1) of this theorem, applying (IV) of this theorem to estimate (B-12) gives rise to

$$\left\|\nabla^M \mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)} \leq C(1+t)^{-\frac{M}{\beta}-\frac{n}{2\beta}}.$$

Combinig Step 1 with Step 2 and Step 3 finishes the proof of (VI-1).

Appendix C

We show (VI-2) of Theorem 4.1 here. We will adopt an inductive argument as above. The inductive assumption is as follows.

For $p \le \frac{K}{2\beta}$, the decay rate

$$\left\|\partial_t^p \nabla^m \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \le C(1+t)^{-2p-\frac{m}{\beta}-\frac{n}{2\beta}} \tag{$C-1$}$$

holds for all $p < P$ and m such that $2p\beta + m \le K$. Here, p, P and m are all non-negative integers.

In the following, based on the inductive assumption (C-1), we divided the proof into four steps. In Step 1, we show that for $|\xi|^{2\beta} \le \dfrac{b}{\nu(1+t)}$, $\left\|\partial_t^p \hat{\mathbf{v}}(\xi)\right\| \le C(1+t)^{-P}$. In the second step, we verify that the decay rate (C-1) holds for $p = P$ and $m = 0$ by an inductive argument on p. We will check the decay rate (C-1) holds for any $m > 0$ by another inductive argument on m in the third step. In the fourth step, we conclude the expected result by a bootstrap argument.

We begin to show (VI-2) step by step in detail.

Step 1 We show for $|\xi|^{2\beta} \le \dfrac{b}{\nu(1+t)}$, $\left\|\partial_t^p \hat{\mathbf{v}}(\xi)\right\| \le C(1+t)^{-P}$.

By (C-1) we get for all $p < P$ and $m = 0, 1$,

$$\left\|\partial_t^p \nabla^m \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \le C(1+t)^{-2p-\frac{m}{\beta}-\frac{n}{2\beta}}. \tag{$C-2$}$$

By the aid of (V) of theorem 4.1, (C-2) implies that for $|\xi|^{2\beta} \le \dfrac{b}{\nu(1+t)}$,

$$\left\|\partial_t^P \hat{\mathbf{v}}(\xi)\right\|_{L^2(\mathbb{R}^n)}^2 \le C(1+t)^{-P}. \tag{$C-3$}$$

Step 2 We now show that the decay rate (C-1) holds for $p = P$ and $m = 0$ by an inductive argument on p.

Note that $\mathbf{v} \cdot \nabla \mathbf{u}^T = \nabla(\mathbf{u}\mathbf{v}) - \mathbf{u} \cdot \nabla \mathbf{v}^T$ by (4.4) and div $\mathbf{v} = 0$, choosing P and M such that $M + 2P \le K$, then applying ∂_t^P to the first equation in (1.1), multiplying the resulting equation by $\partial_t^P \Delta^M \mathbf{v}$ and integrating in space variable x yields, after some integration by parts,

$$\begin{aligned}
&\frac{d}{dt}\left\|\partial_t^P \nabla^M \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 + 2\nu\left\|\partial_t^P \nabla^M \Lambda^\beta \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad \le \left|\left\langle \partial_t^P(\mathbf{u}\cdot\nabla\mathbf{v}), \partial_t^P \Delta^M \mathbf{v}\right\rangle\right| + \left|\left\langle \partial_t^P\left(\mathbf{v}\cdot\nabla\mathbf{u}^T\right), \partial_t^P \Delta^M \mathbf{v}\right\rangle\right| \\
&\quad = \left|\left\langle \partial_t^P(\mathbf{u}\cdot\nabla\mathbf{v}), \partial_t^P \Delta^M \mathbf{v}\right\rangle\right| + \left|\left\langle \partial_t^P\left(\mathbf{u}\cdot\nabla\mathbf{v}^T\right), \partial_t^P \Delta^M \mathbf{v}\right\rangle\right| \\
&\quad \triangleq I_{M,P} + J_{M,P}.
\end{aligned} \tag{$C-4$}$$

In the following, we deal with the two terms on the right hand side of (C-4) by considering two cases:

Case **(1)** $\dfrac{n}{4} < \beta < 1$ for $n = 2, 3$;

Case **(2)** $\beta = \dfrac{n}{4}$ for $n = 2, 3$.

♡ We first consider Case **(1)** $\dfrac{n}{4} < \beta < 1$ for $n = 2, 3$.

In this case, a straightforward computation shows that

$$\begin{cases} \dfrac{n}{\beta} = \dfrac{2n}{n-2\cdot\frac{n-2\beta}{2}}, \quad \dfrac{1}{2} = \dfrac{n-2}{2} < \dfrac{n-2\beta}{2} < \dfrac{3}{4} & \text{for } n = 3, \\ 0 = \dfrac{n-2}{2} < \dfrac{n-2\beta}{2} < \dfrac{1}{2} & \text{for } n = 2, \\ B = \dfrac{n-2\beta}{2} + 1 - \beta = \dfrac{n}{2} + 1 - 2\beta, & \\ \dfrac{n}{2} - 1 < B < 1, & \text{for } n = 2, 3. \end{cases} \tag{$C-5$}$$

Note that (C-4), one attains

$$\begin{aligned} I_{M,P} &= \left|\left\langle \partial_t^P(\mathbf{u}\cdot\nabla\mathbf{v}), \partial_t^P\Delta^M\mathbf{v}\right\rangle\right| \\ &= \sum_{p=0}^{P}\sum_{m=0}^{M-1}\binom{P}{p}\binom{M-1}{m}\left|\left\langle \Lambda^{1-\beta}\left(\partial_t^p\nabla^m\mathbf{u}\cdot\partial_t^{P-p}\nabla^{M-m}\mathbf{v}\right), \partial_t^P\nabla^M\Lambda^\beta\mathbf{v}\right\rangle\right| \\ &\le \sum_{p=0}^{P}\sum_{m=0}^{M-1}\binom{P}{p}\binom{M-1}{m}\left\|\Lambda^{1-\beta}\left(\partial_t^p\nabla^m\mathbf{u}\cdot\partial_t^{P-p}\nabla^{M-m}\mathbf{v}\right)\right\|_{L^2(\mathbb{R}^n)} \\ &\qquad \cdot\left\|\partial_t^P\nabla^M\Lambda^\beta\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}. \end{aligned} \tag{$C-6$}$$

Thanks to higher order fractional Leibniz's rule [18], $\left\|\Lambda^{1-\beta}\left(\partial_t^p\nabla^m\mathbf{u}\cdot\partial_t^{P-p}\nabla^{M-m}\mathbf{v}\right)\right\|_{L^2(\mathbb{R}^n)}$ can be bounded as follows:

$$\begin{aligned} &\left\|\Lambda^{1-\beta}\left(\partial_t^p\nabla^m\mathbf{u}\cdot\partial_t^{P-p}\nabla^{M-m}\mathbf{v}\right)\right\|_{L^2(\mathbb{R}^n)} \\ &\le \Big\|\Lambda^{1-\beta}\left(\partial_t^p\nabla^m\mathbf{u}\cdot\partial_t^{P-p}\nabla^{M-m}\mathbf{v}\right) \\ &\qquad -\partial_t^p\nabla^m\mathbf{u}\cdot\partial_t^{P-p}\nabla^{M-m}\Lambda^{1-\beta}\mathbf{v} \\ &\qquad -\partial_t^p\Lambda^{1-\beta}\nabla^m\mathbf{u}\cdot\partial_t^{P-p}\nabla^{M-m}\mathbf{v}\Big\|_{L^2(\mathbb{R}^n)} \\ &\quad +\left\|\partial_t^p\nabla^m\mathbf{u}\cdot\partial_t^{P-p}\nabla^{M-m}\Lambda^{1-\beta}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\ &\quad +\left\|\partial_t^p\Lambda^{1-\beta}\nabla^m\mathbf{u}\cdot\partial_t^{P-p}\nabla^{M-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}. \end{aligned} \tag{$C-7$}$$

Due to Lemma 2.10, (I) of Lemma 2.11 and (C-5), for $\frac{1}{2} = \frac{1}{n/\beta} + \frac{1}{2n/(n-2\beta)}$ and $0 < 1-\beta < \beta < 1$, the first term on the right hand side of (C-7) can be bounded by

$$\begin{aligned} &\Big\|\Lambda^{1-\beta}\left(\partial_t^p\nabla^m\mathbf{u}\cdot\partial_t^{P-p}\nabla^{M-m}\mathbf{v}\right) \\ &\qquad -\partial_t^p\nabla^m\mathbf{u}\cdot\partial_t^{P-p}\nabla^{M-m}\Lambda^{1-\beta}\mathbf{v} \\ &\qquad -\partial_t^p\Lambda^{1-\beta}\nabla^m\mathbf{u}\cdot\partial_t^{P-p}\nabla^{M-m}\mathbf{v}\Big\|_{L^2(\mathbb{R}^n)} \\ &\le C\left\|\partial_t^p\Lambda^{1-\beta}\nabla^m\mathbf{u}\right\|_{L^{\frac{n}{\beta}}(\mathbb{R}^n)}\left\|\partial_t^{P-p}\nabla^{M-m}\mathbf{v}\right\|_{L^{\frac{2n}{n-2\beta}}(\mathbb{R}^n)} \\ &\le C\left\|\partial_t^p\Lambda^{\frac{n}{2}+1-2\beta}\nabla^m\mathbf{u}\right\|_{L^2(\mathbb{R}^n)}\left\|\partial_t^{P-p}\nabla^{M-m}\Lambda^\beta\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\ &\le C\left\|\partial_t^p\nabla^m\mathbf{u}\right\|_{L^2(\mathbb{R}^n)}^{2\beta-\frac{n}{2}}\left\|\partial_t^p\nabla^{m+1}\mathbf{u}\right\|_{L^2(\mathbb{R}^n)}^{\frac{n}{2}+1-2\beta}\left\|\partial_t^{P-p}\nabla^{M-m}\Lambda^\beta\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\ &\le C\left\|\partial_t^p\nabla^{m-1}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\partial_t^{P-p}\nabla^{M-m}\Lambda^\beta\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}. \end{aligned} \tag{$C-8$}$$

On the other hand, note that Lemma 2.8, Lemma 2.9 and Lemma 2.11, the second and the third terms on the right hand side of (C-7) can be bounded by

$$\begin{aligned}
&\left\|\partial_t^p \nabla^m \mathbf{u} \cdot \partial_t^{P-p} \nabla^{M-m} \Lambda^{1-\beta} \mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\quad \leq C \left\|\partial_t^p \nabla^m \mathbf{u}\right\|_{L^{\frac{n}{2\beta-1}}(\mathbb{R}^n)} \left\|\partial_t^{P-p} \nabla^{M-m} \Lambda^{1-\beta} \mathbf{v}\right\|_{L^{\frac{2n}{n-2(2\beta-1)}}(\mathbb{R}^n)} \\
&\quad \leq C \left\|\partial_t^p \nabla^m \mathbf{u}\right\|_{L^{\frac{2n}{n-2\left(\frac{n}{2}+1-2\beta\right)}}(\mathbb{R}^n)} \left\|\partial_t^{P-p} \nabla^{M-m} \Lambda^{\beta} \mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\quad \leq C \left\|\partial_t^p \nabla^m \Lambda^{\frac{n}{2}+1-2\beta} \mathbf{u}\right\|_{L^2(\mathbb{R}^n)} \left\|\partial_t^{P-p} \nabla^{M-m} \Lambda^{\beta} \mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\quad \leq C \left\|\partial_t^p \nabla^{m-1} \mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \left\|\partial_t^{P-p} \nabla^{M-m} \Lambda^{\beta} \mathbf{v}\right\|_{L^2(\mathbb{R}^n)},
\end{aligned} \tag{C-9}$$

and

$$\begin{aligned}
&\left\|\partial_t^p \Lambda^{1-\beta} \nabla^m \mathbf{u} \cdot \partial_t^{P-p} \nabla^{M-m} \mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\quad \leq C \left\|\partial_t^p \Lambda^{1-\beta} \nabla^m \mathbf{u}\right\|_{L^{\frac{n}{\beta}}(\mathbb{R}^n)} \left\|\partial_t^{P-p} \nabla^{M-m} \mathbf{v}\right\|_{L^{\frac{2n}{n-2\beta}}(\mathbb{R}^n)} \\
&\quad \leq C \left\|\partial_t^p \Lambda^{\frac{n}{2}+1-2\beta} \nabla^m \mathbf{u}\right\|_{L^{\frac{n}{\beta}}(\mathbb{R}^n)} \left\|\partial_t^{P-p} \nabla^{M-m} \Lambda^{\beta} \mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\quad \leq C \left\|\partial_t^p \nabla^{m-1} \mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \left\|\partial_t^{P-p} \nabla^{M-m} \Lambda^{\beta} \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}.
\end{aligned} \tag{C-10}$$

Hence, combining (C-6) with (C-7), (C-8), (C-9) and (C-10) gives rise to

$$\begin{aligned}
I_{M,P} &= \left|\left\langle \partial_t^P (\mathbf{u} \cdot \nabla \mathbf{v}), \partial_t^P \Delta^M \mathbf{v} \right\rangle\right| \\
&\leq C \sum_{p=0}^{P} \sum_{m=0}^{M-1} \left\|\partial_t^p \nabla^{m+1} \mathbf{u}\right\|_{L^2(\mathbb{R}^n)} \left\|\partial_t^{P-p} \nabla^{M-m} \Lambda^{\beta} \mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\qquad\qquad \cdot \left\|\partial_t^P \nabla^M \Lambda^{\beta} \mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\leq C \sum_{p=0}^{P} \left\|\partial_t^p \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \left\|\partial_t^{P-p} \nabla^M \Lambda^{\beta} \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad + C \sum_{p=0}^{P} \left\|\partial_t^p \nabla \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \left\|\partial_t^{P-p} \nabla^{M-1} \Lambda^{\beta} \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad + C \sum_{p=0}^{P} \sum_{m=2}^{M-1} \left\|\partial_t^p \nabla^{m-1} \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \left\|\partial_t^{P-p} \nabla^{M-m} \Lambda^{\beta} \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad + \frac{\nu}{4} \left\|\partial_t^P \nabla^M \Lambda^{\beta} \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2.
\end{aligned} \tag{C-11}$$

Similar estimates to that used in the estimates for $I_{M,P}$ are valid for $J_{M,P}$ in (C-4):

$$\begin{aligned}
J_{M,P} &= \left|\left\langle \partial_t^P\left(\mathbf{u}\cdot\nabla\mathbf{v}^T\right), \partial_t^P\Delta^M\mathbf{v}\right\rangle\right| \\
&\leq C\sum_{p=0}^{P}\left\|\partial_t^p\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2\left\|\partial_t^{P-p}\nabla^M\Lambda^\beta\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad +C\sum_{p=0}^{P}\left\|\partial_t^p\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2\left\|\partial_t^{P-p}\nabla^{M-1}\Lambda^\beta\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad +C\sum_{p=0}^{P}\sum_{m=2}^{M-1}\left\|\partial_t^p\nabla^{m-1}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2\left\|\partial_t^{P-p}\nabla^{M-m}\Lambda^\beta\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad +\frac{\nu}{4}\left\|\partial_t^P\nabla^M\Lambda^\beta\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 .
\end{aligned} \tag{$C-12$}$$

Substituting the above estimates (C-11) and (C-12) into (C-4) leads to the following estimate under the case $\frac{n}{4} < \beta < 1$ with $n = 2, 3$:

$$\begin{aligned}
&\frac{d}{dt}\left\|\partial_t^P\nabla^M\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 + \nu\left\|\partial_t^P\nabla^M\Lambda^\beta\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\leq C\sum_{p=0}^{P}\left\|\partial_t^p\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2\left\|\partial_t^{P-p}\nabla^M\Lambda^\beta\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad +C\sum_{p=0}^{P}\left\|\partial_t^p\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2\left\|\partial_t^{P-p}\nabla^{M-1}\Lambda^\beta\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad +C\sum_{p=0}^{P}\sum_{m=2}^{M-1}\left\|\partial_t^p\nabla^{m-1}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2\left\|\partial_t^{P-p}\nabla^{M-m}\Lambda^\beta\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 .
\end{aligned} \tag{$C-13$}$$

♡ We now tackle (C-4) under the Case **(2)** $\beta = \frac{n}{4}$ for $n = 2, 3$.

In this case, it is easy to check that $1 - \frac{n}{4} \leq \frac{n}{4}$. Recall (C-4), we shall estimate $I_{M,P}$ and $J_{M,P}$, respectively. We first handle $I_{M,P}$.

By a similar proof to that for case (1), one deduces the following:

$$\begin{aligned}
I_{M,P} &= \left|\left\langle \partial_t^P(\mathbf{u}\cdot\nabla\mathbf{v}), \partial_t^P\Delta^M\mathbf{v}\right\rangle\right| \\
&= \sum_{p=0}^{P}\sum_{m=0}^{M-1}\begin{pmatrix} P \\ p \end{pmatrix}\begin{pmatrix} M-1 \\ m \end{pmatrix}\left\langle\Lambda^{1-\frac{n}{4}}\left(\partial_t^p\nabla^m\mathbf{u}\cdot\partial_t^{P-p}\nabla^{M-m}\mathbf{v}\right), \partial_t^P\nabla^M\Lambda^{\frac{n}{4}}\mathbf{v}\right\rangle \\
&\leq \sum_{p=0}^{P}\sum_{m=0}^{M-1}\begin{pmatrix} P \\ p \end{pmatrix}\begin{pmatrix} M-1 \\ m \end{pmatrix}\left\|\Lambda^{1-\frac{n}{4}}\left(\partial_t^p\nabla^m\mathbf{u}\cdot\partial_t^{P-p}\nabla^{M-m}\mathbf{v}\right)\right\|_{L^2(\mathbb{R}^n)} \\
&\qquad\qquad\qquad\qquad\cdot\left\|\partial_t^P\nabla^M\Lambda^{\frac{n}{4}}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} .
\end{aligned} \tag{$C-14$}$$

However, direct calculation gives

$$\begin{aligned}
&\left\|\Lambda^{1-\frac{n}{4}}\left(\partial_t^p\nabla^m\mathbf{u}\cdot\partial_t^{P-p}\nabla^{M-m}\mathbf{v}\right)\right\|_{L^2(\mathbb{R}^n)}\\
&\le\Big\|\Lambda^{1-\frac{n}{4}}\left(\partial_t^p\nabla^m\mathbf{u}\cdot\partial_t^{P-p}\nabla^{M-m}\mathbf{v}\right)\\
&\quad-\partial_t^p\nabla^m\mathbf{u}\cdot\partial_t^{P-p}\nabla^{M-m}\Lambda^{1-\frac{n}{4}}\mathbf{v}\\
&\quad-\partial_t^p\Lambda^{1-\frac{n}{4}}\nabla^m\mathbf{u}\cdot\partial_t^{P-p}\nabla^{M-m}\mathbf{v}\Big\|_{L^2(\mathbb{R}^n)}\\
&\quad+\left\|\partial_t^p\nabla^m\mathbf{u}\cdot\partial_t^{P-p}\nabla^{M-m}\Lambda^{1-\frac{n}{4}}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\\
&\quad+\left\|\partial_t^p\Lambda^{1-\frac{n}{4}}\nabla^m\mathbf{u}\cdot\partial_t^{P-p}\nabla^{M-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}.
\end{aligned}\tag{$C-15$}$$

Due to Lemma 2.9 and Lemma 2.11, one deduces that for $0<\beta_1<1-\dfrac{n}{4}$, the first term on the right hand side of (C-15) can be estimated by

$$\begin{aligned}
&\Big\|\Lambda^{1-\frac{n}{4}}\left(\partial_t^p\nabla^m\mathbf{u}\cdot\partial_t^{P-p}\nabla^{M-m}\mathbf{v}\right)\\
&\quad-\partial_t^p\nabla^m\mathbf{u}\cdot\partial_t^{P-p}\nabla^{M-m}\Lambda^{1-\frac{n}{4}}\mathbf{v}\\
&\quad-\partial_t^p\Lambda^{1-\frac{n}{4}}\nabla^m\mathbf{u}\cdot\partial_t^{P-p}\nabla^{M-m}\mathbf{v}\Big\|_{L^2(\mathbb{R}^n)}\\
&\le C\left\|\partial_t^p\Lambda^{1-\frac{n}{4}-\beta_1}\nabla^m\mathbf{u}\right\|_{L^{\frac{2n}{n-2(n/4+\beta_1)}}(\mathbb{R}^n)}\left\|\partial_t^{P-p}\nabla^{M-m}\Lambda^{\beta_1}\mathbf{v}\right\|_{L^{\frac{2n}{n-2(n/4-\beta_1)}}(\mathbb{R}^n)}\\
&\le C\left\|\partial_t^p\nabla^m\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\partial_t^{P-p}\nabla^{M-m}\Lambda^{\frac{n}{4}}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)},
\end{aligned}\tag{$C-16$}$$

where $\beta_1\in\left(0,\dfrac{1}{2}\right)$, $1-\dfrac{n}{4}-\beta_1\in\left(0,\dfrac{1}{2}\right)$, $\dfrac{1}{2}=\dfrac{1}{p_1}+\dfrac{1}{p_2}$ with $p_1,p_2\in(1,\infty)$, $p_1=\dfrac{2n}{n-2(n/4+\beta_1)}$, $p_2=\dfrac{2n}{n-2(n/4-\beta_1)}$.

Let us turn to estimate the second and the third terms on the right hand side of (C-15). Thanks to Agmon's inequality, interpolation inequality, Lemma 2.12 and the assumption of (VI) for $\beta=\dfrac{n}{4}$, we have

$$\begin{aligned}
&\left\|\partial_t^p\nabla^m\mathbf{u}\cdot\partial_t^{P-p}\nabla^{M-m}\Lambda^{1-\frac{n}{4}}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\\
&\quad+\left\|\partial_t^p\Lambda^{1-\frac{n}{4}}\nabla^m\mathbf{u}\cdot\partial_t^{P-p}\nabla^{M-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\\
&\le C\left\|\partial_t^p\nabla^m\mathbf{u}\right\|_{L^\infty(\mathbb{R}^n)}\left\|\partial_t^{P-p}\nabla^{M-m}\Lambda^{1-\frac{n}{4}}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\\
&\quad+\left\|\partial_t^p\Lambda^{1-\frac{n}{4}}\nabla^m\mathbf{u}\right\|_{L^4(\mathbb{R}^n)}\left\|\partial_t^{P-p}\nabla^{M-m}\mathbf{v}\right\|_{L^4(\mathbb{R}^n)}\\
&\le C\left\|\partial_t^p\nabla^m\mathbf{u}\right\|_{H^1(\mathbb{R}^n)}^{\frac{1}{2}}\left\|\partial_t^p\nabla^m\mathbf{u}\right\|_{H^2(\mathbb{R}^n)}^{\frac{1}{2}}\left\|\partial_t^{P-p}\nabla^{M-m}\Lambda^{\frac{n}{4}}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\\
&\quad+C\left\|\partial_t^p\Lambda^{1-\frac{n}{4}+\frac{n}{4}}\nabla^m\mathbf{u}\right\|_{L^2(\mathbb{R}^n)}\left\|\partial_t^{P-p}\Lambda^{\frac{n}{4}}\nabla^{M-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\\
&\le C\left\|\partial_t^p\nabla^m\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\partial_t^{P-p}\nabla^{M-m}\Lambda^{\frac{n}{4}}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}.
\end{aligned}\tag{$C-17$}$$

This together with (C-14), (C-15) and (C-16) gives rise to

$$
\begin{aligned}
I_{M,P} &= \left|\left\langle \partial_t^P(\mathbf{u}\cdot\nabla\mathbf{v}), \partial_t^P\Delta^M\mathbf{v}\right\rangle\right| \\
&\le C\sum_{p=0}^{P}\sum_{m=0}^{M-1}\left\|\partial_t^p\nabla^m\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\partial_t^{P-p}\Lambda^{\frac{n}{4}}\nabla^{M-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\partial_t^{P-p}\Lambda^{\frac{n}{4}}\nabla^{M}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\le C\sum_{p=0}^{P}\left\|\partial_t^p\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2\left\|\partial_t^{P-p}\Lambda^{\frac{n}{4}}\nabla^{M}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad +C\sum_{p=0}^{P}\left\|\partial_t^p\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2\left\|\partial_t^{P-p}\Lambda^{\frac{n}{4}}\nabla^{M-1}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad +C\sum_{p=0}^{P}\sum_{m=2}^{M-1}\left\|\partial_t^p\nabla^m\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2\left\|\partial_t^{P-p}\Lambda^{\frac{n}{4}}\nabla^{M-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad +\frac{\nu}{4}\left\|\partial_t^P\Lambda^{\frac{n}{4}}\nabla^{M}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 .
\end{aligned} \tag{C-18}
$$

Here we have used Cauchy-Schwarz inequality and Young's inequality in the third inequality.

In the same manner, one may deduce the following estimate for $J_{M,P}$ in (C-4):

$$
\begin{aligned}
J_{M,P} &= \left|\left\langle \partial_t^P\left(\mathbf{u}\cdot\nabla\mathbf{v}^T\right), \partial_t^P\Delta^M\mathbf{v}\right\rangle\right| \\
&\le C\sum_{p=0}^{P}\left\|\partial_t^p\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2\left\|\partial_t^{P-p}\Lambda^{\frac{n}{4}}\nabla^{M}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad +C\sum_{p=0}^{P}\left\|\partial_t^p\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2\left\|\partial_t^{P-p}\Lambda^{\frac{n}{4}}\nabla^{M-1}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad +C\sum_{p=0}^{P}\sum_{m=2}^{M-1}\left\|\partial_t^p\nabla^m\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2\left\|\partial_t^{P-p}\Lambda^{\frac{n}{4}}\nabla^{M-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad +\frac{\nu}{4}\left\|\partial_t^P\Lambda^{\frac{n}{4}}\nabla^{M}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 .
\end{aligned} \tag{C-19}
$$

Therefore, under the case of $\beta = \dfrac{n}{4}$ with $n = 2, 3$, substituting (C-18) and (C-19) into (C-4) yields that

$$
\begin{aligned}
&\frac{d}{dt}\left\|\partial_t^P\nabla^M\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 + \nu\left\|\partial_t^P\Lambda^{\frac{n}{4}}\nabla^{M}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\le C\sum_{p=0}^{P}\left\|\partial_t^p\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2\left\|\partial_t^{P-p}\Lambda^{\frac{n}{4}}\nabla^{M}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad +C\sum_{p=0}^{P}\left\|\partial_t^p\nabla\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2\left\|\partial_t^{P-p}\Lambda^{\frac{n}{4}}\nabla^{M-1}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad +C\sum_{p=0}^{P}\sum_{m=2}^{M-1}\left\|\partial_t^p\nabla^m\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2\left\|\partial_t^{P-p}\Lambda^{\frac{n}{4}}\nabla^{M-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 .
\end{aligned} \tag{C-20}
$$

With (C-4), (C-13) and (C-20), for $\frac{n}{4} \leq \beta < 1$ with $n = 2, 3$, we always have

$$\begin{aligned}
&\frac{d}{dt}\left\|\partial_t^P \nabla^M \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 + \nu \left\|\partial_t^P \Lambda^\beta \nabla^M \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad \leq C \sum_{p=0}^{P} \left\|\partial_t^p \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \left\|\partial_t^{P-p} \Lambda^\beta \nabla^M \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad + C \sum_{p=0}^{P} \left\|\partial_t^p \nabla \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \left\|\partial_t^{P-p} \Lambda^\beta \nabla^{M-1} \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad + C \sum_{p=0}^{P} \sum_{m=2}^{M} \left\|\partial_t^p \nabla^m \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \left\|\partial_t^{P-p} \Lambda^\beta \nabla^{M-m} \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 .
\end{aligned} \tag{$C-21$}$$

Note that the inductive assumption (C-1), one deduces that for $p = P$ and $m = 0$

$$\begin{aligned}
&\frac{d}{dt}\left\|\partial_t^P \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 + \nu \left\|\partial_t^P \Lambda^\beta \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad \leq C(1+t)^{-\frac{n}{2\beta}} \left\|\partial_t^P \Lambda^\beta \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 + C(1+t)^{-2p-\frac{n}{2\beta}}(1+t)^{-2(P-p)-\frac{\beta}{\beta}-\frac{n}{2\beta}} \\
&\quad \leq C(1+t)^{-\frac{n}{2\beta}} \left\|\partial_t^P \Lambda^\beta \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 + C(1+t)^{-2P-1-\frac{n}{\beta}}.
\end{aligned}$$

Takeing t large enough such that $C(1+t)^{-\frac{n}{2\beta}} \leq \frac{\nu}{2}$, the above inequality then implies that

$$\frac{d}{dt}\left\|\partial_t^P \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 + \frac{\nu}{2} \left\|\partial_t^P \Lambda^\beta \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \leq C(1+t)^{-2P-1-\frac{n}{\beta}}. \tag{$C-22$}$$

This together with (IV) of this theorem ensures that (C-1) and (C-3) hold for $p = P$ and $m = 0$.

So far we have shown that for $m = 0$, and $\forall P \leq \frac{K}{2}$, there holds

$$\frac{d}{dt}\left\|\partial_t^P \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 + \frac{\nu}{2} \left\|\partial_t^P \Lambda^\beta \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \leq C(1+t)^{-2P-1-\frac{n}{\beta}}.$$

This deduces by Gronwall's inequality that

$$\frac{d}{dt}\left\|\partial_t^P \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \leq C(1+t)^{-2P-\frac{n}{\beta}}. \tag{$C-23$}$$

Step 3 We show that the decay rate (C-1) holds for any $m \leq M+1$ for $p < P$, and $m < M$ for $p = P$. That is,

$$\left\|\partial_t^p \nabla^m \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \leq C(1+t)^{-2p-\frac{m}{\beta}-\frac{n}{2\beta}}. \tag{$C-24$}$$

The base case is (C-23) where (C-24) holds for $p = P$ and $m = 0$. In the following, based on the inductive assumption (C-24), we will show that the decay rate (C-24) holds for $m = M$ and $p = P$.

Recall (I) and (II) of this theorem, applying the inductive assumption (C-24) to (C-21), one deduces that

$$\begin{aligned}
&\frac{d}{dt}\left\|\partial_t^P \nabla^M \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 + \nu \left\|\partial_t^P \Lambda^\beta \nabla^M \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad \leq C(1+t)^{-\frac{n}{2\beta}} \left\|\partial_t^P \Lambda^\beta \nabla^M \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad + C(1+t)^{-\frac{1}{\beta}-\frac{n}{2\beta}} \left\|\partial_t^P \Lambda^\beta \nabla^{M-1} \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad + C(1+t)^{-2P-\frac{M}{\beta}-\frac{n}{\beta}-1}.
\end{aligned} \tag{$C-25$}$$

Taking t large enough such that $C(1+t)^{-\frac{n}{2\beta}} \leq \frac{\nu}{2}$, thanks to (C-3), using (IV) once again deduces that

$$\left\|\partial_t^P \nabla^M \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \leq C(1+t)^{-2P-\frac{M}{\beta}-\frac{n}{\beta}} \leq C(1+t)^{-2P-\frac{M}{\beta}-\frac{n}{2\beta}}. \qquad (C-26)$$

This implies that the inductive assumption (C-24) holds for $m = M$ and $p = P$. By another bootstrap argument, we obtain for all $m + 2p\beta \leq K$, the following optimal decay holds:

$$\left\|\partial_t^p \nabla^m \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \leq C(1+t)^{-2p-\frac{m}{\beta}-\frac{n}{2\beta}}.$$

This completes the proof of (VI-2).

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