Mild Solutions and Harnack Inequality for Functional SPDEs with Dini Drift *

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Abstract

The existence and uniqueness of the mild solution for a class of functional SPDEs with multiplicative noise and a locally Dini continuous drift are proved. In addition, under a reasonable condition the solution is non-explosive. Moreover, Harnack inequalities are derived for the associated semigroup under certain global conditions, which is new even in the case without delay.

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1 Introduction

Recently, using Zvonkin type transformation and gradient estimate, Wang [11] has proved the existence and uniqueness of the mild solution for a class of SPDEs with multiplicative noise and a locally Dini continuous drift. Following this, Wang and Huang [5] extend the results to a class of functional SPDEs, where the drift without delay is assumed to be Dini continuous, and the delayed drift is Lipschitzian in some square integrable space. In this paper, we try to investigate the existence, uniqueness and non-explosion for functional equations in which the drift without delay is assumed to be Dini continuous and the drift

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with delay is Lipschitzian w.r.t. some uniform norm (finite delay) or weighted uniform norm (infinite delay). Moreover, Harnack inequalities are also established in the case of finite delay.

Let $(\mathbb{H}, \langle , \rangle, |\cdot|)$ and $\left(\bar{\mathbb{H}}, \langle , \rangle_{\bar{\mathbb{H}}}, |\cdot|_{\bar{\mathbb{H}}}\right)$ be two separable Hilbert spaces. Let $\mathscr{L}\left(\bar{\mathbb{H}};\mathbb{H}\right)$ ($\mathscr{L}_{\rm HS}\left(\bar{\mathbb{H}};\mathbb{H}\right)$) be the space of bounded linear operators (Hilbert-Schmidt operators) from $\bar{\mathbb{H}}$ to $\mathbb{H}$ with operator norm $\|\cdot\|$ (Hilbert-Schmidt norm $\|\cdot\|_{\rm HS}$). For any $r \in [0,\infty]$, let

$$\mathscr{C} = \left\{ \xi \;\middle|\; \xi \in C(-\infty,0]\cap[-r,0];\mathbb{H}),\ \|\xi\|_\infty := \sup_{s\in(-\infty,0]\cap[-r,0]} (\mathrm{e}^s 1_{r=\infty} + 1_{r<\infty})|\xi(s)| < \infty \right\}.$$

For any $f \in C((-\infty,\infty)\cap[-r,\infty);\mathbb{H})$, $t \geq 0$, let $f_t(s) = f(t+s), s \in (-\infty,0]\cap[-r,0]$. Then $f_t \in \mathscr{C}$. $\{f_t\}_{t\geq 0}$ is called the segment process of f.

Let $W = (W(t))_{t\geq 0}$ be a cylindrical Brownian motion on $\bar{\mathbb{H}}$ with respect to a complete filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$. More precisely, $W(\cdot) = \sum_{n=1}^\infty \bar{W}^n(\cdot)\bar{e}_n$ for a sequence of independent one dimensional standard Brownian motions $\left\{\bar{W}^n(\cdot)\right\}_{n\geq 1}$ with respect to $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$, where $\{\bar{e}_n\}_{n\geq 1}$ is an orthonormal basis on $\bar{\mathbb{H}}$.

Consider the following functional SPDE on $\mathbb{H}$:

$$\mathrm{d}X(t) = AX(t)\mathrm{d}t + b(t,X(t))\mathrm{d}t + B(t,X_t)\mathrm{d}t + Q(t,X(t))\mathrm{d}W(t), \quad X_0 = \xi \in \mathscr{C}, \tag{1.1}$$

where $(A, \mathscr{D}(A))$ is a negative definite self-adjoint operator on $\mathbb{H}$, $B : [0,\infty)\times\mathscr{C} \to \mathbb{H}$ and $b : [0,\infty)\times\mathbb{H} \to \mathbb{H}$ are measurable and locally bounded (i.e. bounded on bounded sets), and $Q : [0,\infty)\times\mathbb{H} \to \mathscr{L}\left(\bar{\mathbb{H}};\mathbb{H}\right)$ is measurable. Let A, B and Q satisfy the following two assumptions:

- **(a1)** $(-A)^{\varepsilon-1}$ is of trace class for some $\varepsilon \in (0,1)$; i.e. $\sum_{n=1}^\infty \lambda_n^{\varepsilon-1} < \infty$ for $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ being all eigenvalues of $-A$ counting multiplicities. The eigenbasis of $-A$ on $\mathbb{H}$ corresponding to the eigenvalues $\{\lambda_i\}_{i=1}^\infty$ is $\{e_i\}_{i=1}^\infty$.

- **(a2)** (i) $Q \in C([0,\infty)\times\mathbb{H};\mathscr{L}(\bar{\mathbb{H}};\mathbb{H}))$ and for every $t \geq 0$, $Q(t,\cdot) \in C^2(\mathbb{H};\mathscr{L}(\bar{\mathbb{H}};\mathbb{H}))$. $(QQ^*)(t,x)$ is invertible for all $(t,x) \in [0,\infty)\times\mathbb{H}$. Moreover,

$$\sum_{j=0}^{2} \left\|[\nabla^j Q(t,\cdot)](x)\right\| + \left\|(QQ^*)^{-1}(t,x)\right\|$$

is locally bounded in $(t,x) \in [0,\infty)\times\mathbb{H}$. Furthermore, for any $(t,x) \in [0,\infty)\times\mathbb{H}$,

$$\lim_{n\to\infty} \|Q(t,x) - Q(t,\pi_n x)\|_{\rm HS}^2 := \lim_{n\to\infty} \sum_{k\geq 1} |[Q(t,x) - Q(t,\pi_n x)]\bar{e}_k|^2 = 0, \tag{1.2}$$

where π_n is the orthogonal projection map from $\mathbb{H}$ to $\mathbb{H}_n := \mathrm{span}\{e_1,\cdots,e_n\}$.

(ii) $B \in C([0,\infty)\times\mathscr{C};\mathbb{H})$, and there exists an increasing function $C_B : [0,\infty) \to [0,\infty)$ such that for any $n \geq 1$,

$$|B(t,\xi) - B(t,\eta)| \leq C_B(n)\|\xi-\eta\|_\infty, \quad t \in [0,n], \xi,\eta \in \mathscr{C}, \|\xi\|_\infty \vee \|\eta\|_\infty \leq n.$$

To describe the singularity of b, we introduce

$$\mathscr{D} = \left\{\phi : [0,\infty) \to [0,\infty) \text{ is increasing}, \phi^2 \text{ is concave}, \int_0^1 \frac{\phi(s)}{s}\mathrm{d}s < \infty\right\}.$$

(**a4**) For any $n \geq 1$, there exits $\phi_n \in \mathscr{D}$ such that

$$|b(t,x) - b(t,y)| \leq \phi_n(|x-y|), \quad t \in [0,n], x, y \in \mathbb{H}, |x| \vee |y| \leq n. \tag{1.3}$$

Remark 1.1. *The condition $\int_0^1 \frac{\phi(s)}{s}\mathrm{d}s < \infty$ is well known as the Dini condition, due to the notion of Dini continuity. One can check that the class $\mathscr{D}$ contains $\phi(s) := \frac{K}{\log^{1+\delta}(c+s^{-1})}$ for constants $K, \delta > 0$ and large enough $c \geq \mathrm{e}$ such that ϕ^2 is concave.*

When $r = 0$, (**a1**)-(**a3**) imply the existence and uniqueness of the mild solution to (1.1) by [11, Theorem 1.1 (1)]. However, when $r > 0$, due to some technical reasons (see Remark 3.2 for more details), extra condition on the singular drift besides (**a1**)-(**a3**) is needed to obtain the pathwise uniqueness. Precisely,

(**a4**) For any $n \geq 1$, there exits $a_n \in \mathscr{A}_A$ such that

$$\sup_{t\in[0,n], x\in\mathbb{H}, |x|\leq n} |a_n(-A)b(t,x)| < \infty, \tag{1.4}$$

where

$$\mathscr{A}_A = \left\{a \in \mathscr{B}((0,\infty);(0,\infty)), \int_0^1 \sup_{i\geq 1} \frac{\lambda_i \mathrm{e}^{-\lambda_i s}}{a(\lambda_i)}\mathrm{d}s < \infty\right\}$$

with $\mathscr{B}((0,\infty);(0,\infty))$ denoting all the Borel-measurable functions from $(0,\infty)$ to $(0,\infty)$.

By (**a3**) and (**a4**), we mean that when $t, |x| \leq n$, $b(t,x)$ takes value in a smaller space $\mathbb{H}_{a_n}$ instead of $\mathbb{H}$, but it is still locally Dini continuous from $\mathbb{H}$ to $\mathbb{H}$.

Remark 1.2. *We note that the values $\{a(\lambda_i)\}_{i=1}^\infty$ determine the integration in the definition of $\mathscr{A}_A$, and this means that $\mathscr{A}_A$ depends on A. However, $\mathscr{A}_A$ contains a subset which is independent of A. Indeed, by the definition of $\mathscr{A}_A$, if $\{a(\lambda_i)\}_{i=1}^\infty$ has a bounded subsequence $\{a(\lambda_{i_k})\}_{k\geq 1}$, then there exists a constant $c > 0$ such that*

$$\int_0^1 \sup_{i\geq 1} \frac{\lambda_i \mathrm{e}^{-\lambda_i s}}{a(\lambda_i)}\mathrm{d}s \geq \int_0^1 \frac{\sup_{k\geq 1} \lambda_{i_k} \mathrm{e}^{-\lambda_{i_k} s}}{\sup_{k\geq 1} a(\lambda_{i_k})}\mathrm{d}s \geq \frac{c}{\sup_{k\geq 1} a(\lambda_{i_k})} \int_0^1 \frac{1}{s}\mathrm{d}s = \infty.$$

This means $\lim_{i\to\infty} a(\lambda_i) = \infty$ for any $a \in \mathscr{A}_A$. So we can impose some monotonicity conditions on a, and introduce the following $\mathscr{A}' \subset \mathscr{A}_A$, containing enough functions as well, in which the condition is much easier to check than that in $\mathscr{A}_A$. Letting

$$\mathscr{A}' = \left\{a \in \mathscr{B}((0,\infty);(0,\infty)), a \text{ and } \frac{x}{a(x)} \text{ are non-decreasing}, \int_1^\infty \frac{1}{sa(s)}\mathrm{d}s < \infty\right\},$$

we claim $\mathscr{A}' \subset \mathscr{A}_A$.

Proof. For any $a \in \mathscr{A}'$, $s \in (0,1)$, we have

$$\sup_{x \geq \frac{1}{s}} \frac{x}{a(x)} \mathrm{e}^{-xs} \leq \sup_{x \geq \frac{1}{s}} \frac{x}{a(\frac{1}{s})} \mathrm{e}^{-xs} \leq \frac{\frac{1}{s}}{a(\frac{1}{s})} \mathrm{e}^{-1} \leq \frac{\frac{1}{s}}{a(\frac{1}{s})}.$$

On the other hand,

$$\sup_{1 \wedge \lambda_1 \leq x < \frac{1}{s}} \frac{x}{a(x)} \mathrm{e}^{-xs} \leq \sup_{1 \wedge \lambda_1 \leq x < \frac{1}{s}} \frac{x}{a(x)} \leq \frac{\frac{1}{s}}{a(\frac{1}{s})}.$$

So

$$\int_0^1 \sup_{i \geq 1} \frac{\lambda_i \mathrm{e}^{-\lambda_i s}}{a(\lambda_i)} \mathrm{d}s \leq \int_0^1 \sup_{x \in [1 \wedge \lambda_1, \infty)} \frac{x}{a(x)} \mathrm{e}^{-xs} \mathrm{d}s \leq \int_0^1 \frac{\frac{1}{s}}{a(\frac{1}{s})} \mathrm{d}s = \int_1^\infty \frac{1}{sa(s)} \mathrm{d}s < \infty.$$

This means $a \in \mathscr{A}_A$, i.e. $\mathscr{A}' \subset \mathscr{A}_A$. □

Finally, we give some functions which belong to $\mathscr{A}_A$*.*

(1) $a(x) := x^\delta$ *for any* $\delta \in (0,1]$*;*

(2) $a(x) := \log^{1+\delta}(c+x)$ *for* $\delta > 0$ *and* $c \geq \mathrm{e}^{1+\delta}$*;*

(3) $a(x) = x^\delta(\sin x + 2)$, $\delta \in (0,1]$.

(1) *and* **(2)** *are in* $\mathscr{A}'$. *As to* **(3)**, *we only need to notice the fact that if* $a_1 \in \mathscr{A}_A$, *then* $a \in \mathscr{B}((0,\infty);(0,\infty))$ *satisfying* $a(x) \geq a_1(x)$, $x \geq R_0$ *for some constant* $R_0 > 0$ *is also in* $\mathscr{A}_A$. *It is clear that* $a(x) \geq x^\delta$, *and* **(1)** *implies that* $a \in \mathscr{A}_A$.

For simplicity, let $\mathscr{A}_1 = \mathscr{A}_A \cup \{a : a \equiv 1\}$. For any $a \in \mathscr{A}_1$, let $\mathbb{H}_a = \{x \in \mathbb{H}, |a(-A)x| < \infty\}$ equipped the norm $\|x\|_a := |a(-A)x|$, $x \in \mathbb{H}_a$. Then $(\mathbb{H}_a, \|\cdot\|_a)$ is a Banach space and $\mathbb{H}_1 = \mathbb{H}$.

In general, the mild solution (if exists) to (1.1) can be explosive, so we consider mild solutions with life time.

Definition 1.1. A continuous $\mathbb{H}$-valued process $(X(t))_{t \in [-r,\zeta) \cap (-\infty,\zeta)}$ is called a mild solution to (1.1) with life time ζ, if the segment process X_t is $\mathscr{F}_t$-measurable, $\zeta > 0$ is a stopping time such that $\mathbb{P}$-a.s $\limsup_{t \uparrow \zeta} |X(t)| = \infty$ holds on $\{\zeta < \infty\}$, and $\mathbb{P}$-a.s

$$\begin{aligned} X(t) = \mathrm{e}^{A(t \vee 0)} X(t \wedge 0) &+ \int_0^{t \vee 0} \mathrm{e}^{A(t-s)} (b(s, X(s)) + B(s, X_s)) \mathrm{d}s \\ &+ \int_0^{t \vee 0} \mathrm{e}^{A(t-s)} Q(s, X(s)) \mathrm{d}W(s), \quad t \in [-r,\zeta) \cap (-\infty,\zeta). \end{aligned}$$

The following lemma is a crucial tool in the proof of our results, see [2, Proposition 7.9].

Lemma 1.1. *Let $\{S(t)\}_{t\geq 0}$ be a C_0-contractive semigroup on $\mathbb{H}$. Assume there exist $\alpha \in \left(0, \frac{1}{2}\right)$ and $s > 0$ such that*

$$\int_0^s t^{-2\alpha} \|S(t)\|_{\mathrm{HS}}^2 \mathrm{d}t < \infty. \tag{1.5}$$

Then for every $q \in \left(1, \frac{1}{2\alpha}\right)$, $T > 0$, there exists $c_q > 0$ such that for any $\mathscr{L}\left(\bar{\mathbb{H}}; \mathbb{H}\right)$-valued predictable process Φ, there exists a continuous version of $\int_0^{\cdot} S(\cdot - s)\Phi(s)\mathrm{d}W(s)$ such that

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t S(t-s)\Phi(s)\mathrm{d}W(s)\right|^{2q}\right] &\leq c_q\left[\int_0^T t^{-2\alpha}\|S(t)\|_{\mathrm{HS}}^2\mathrm{d}t\right]^q \\ &\quad \times \mathbb{E}\left[\int_0^T \|\Phi(t)\|^{2q}\mathrm{d}t\right].\end{aligned} \tag{1.6}$$

Remark 1.4 (**a1**) implies that (1.5) holds for $\alpha = \frac{\varepsilon}{2}$:

$$\begin{aligned}\int_0^s t^{-2\alpha}\|S(t)\|_{\mathrm{HS}}^2\mathrm{d}t &= \sum_{i=1}^{\infty}\int_0^s t^{-2\alpha}\mathrm{e}^{-2\lambda_i t}\mathrm{d}t \\ &\leq \sum_{i=1}^{\infty}\lambda_i^{2\alpha-1}\int_0^{\infty}u^{-2\alpha}\mathrm{e}^{-2u}\mathrm{d}u < \infty.\end{aligned}$$

2 Main results

Via Yamada-Watanabe principle and Zvonkin type transformation, we obtain the first main result on existence, uniqueness and non-explosion of the mild solution.

Theorem 2.1. *Assume* (**a1**)-(**a4**).

(1) *The equation* (1.1) *has a unique mild solution $(X(t))_{t\in[-r,\zeta)\cap(-\infty,\zeta)}$ with life time ζ.*

(2) *Let $\|Q(t)\|_\infty := \sup_{x\in\mathbb{H}}\|Q(t,x)\|$ be locally bounded in $t \geq 0$. If there exist two positive functions $\Phi, h : [0,\infty)\times[0,\infty) \to (0,\infty)$ increasing in each variable such that $\int_1^\infty \frac{\mathrm{d}s}{\Phi_t(s)} = \infty$ for any $t \geq 0$ and*

$$\langle B(t,\xi+\eta) + b(t,(\xi+\eta)(0)), \xi(0)\rangle \leq \Phi_t\left(\|\xi\|_\infty^2\right) + h_t(\|\eta\|_\infty), \quad \xi,\eta\in\mathscr{C}, t\geq 0, \tag{2.1}$$

then the mild solution is non-explosive.

To apply Zvonkin type transformation, we in fact need some global conditions. However, Theorem 2.1 can be proved by localization. For simplicity, we introduce some notations firstly. For any $a \in \mathscr{A}_1$ and $\mathbb{H}_a$-valued function f on $[0,T]\times\mathbb{H}$, let

$$\|f\|_{T,\infty,a} = \sup_{t\in[0,T],x\in\mathbb{H}}|a(-A)f(t,x)|.$$

Similarly, for a $\mathscr{L}(\mathbb{H},\mathbb{H}_a)$-valued or $\mathbb{H}_a$-valued function f on $[0,T]\times\mathbb{H}$, let

$$\|f\|_{T,\infty,a}=\sup_{t\in[0,T],x\in\mathbb{H}}\|a(-A)f(t,x)\|.$$

If $a=1$, we write $\|\cdot\|_{T,\infty,a}$ as $\|\cdot\|_{T,\infty}$. Moreover, for any $\mathbb{H}$-valued map f on $[0,T]\times\mathscr{C}$, let

$$\|f\|_{T,\infty}=\sup_{t\in[0,T],\xi\in\mathscr{C}}|f(t,\xi)|.$$

Then we introduce that

($\mathbf{a2}'$) (i') Q satisfies ($\mathbf{a2}$) (i), and there exists a positive increasing function $C_Q:[0,\infty)\to(0,\infty)$ such that

$$\sum_{j=0}^{2}\left\|\nabla^j Q\right\|_{T,\infty}+\left\|(QQ^*)^{-1}\right\|_{T,\infty}<C_Q(T),\quad T\geq 0.$$

(ii') For any $t\geq 0$, $\|B\|_{t,\infty}<\infty$. B satisfies ($\mathbf{a2}$) (ii), and there exists an increasing function $C_B':[0,\infty)\to[0,\infty)$ such that for any $n\geq 1$,

$$|B(t,\xi)-B(t,\eta)|\leq C_B'(n)\|\xi-\eta\|_\infty,\quad t\in[0,n],\xi,\eta\in\mathscr{C}.$$

($\mathbf{a3}'$) For any $T>0$, there exists $\phi\in\mathscr{D}$ such that

$$|b(t,x)-b(t,y)|\leq\phi(|x-y|),\quad t\in[0,T],x,y\in\mathbb{H}. \tag{2.2}$$

($\mathbf{a4}'$) For any $T>0$, there exists $a\in\mathscr{A}_A$ such that

$$\|b\|_{T,\infty,a}<\infty. \tag{2.3}$$

According to Theorem 2.1, under ($\mathbf{a1}$), ($\mathbf{a2}'$)-($\mathbf{a4}'$), the unique mild solution X_t^ξ of (1.1) is non-explosive. The associated Markov semigroup P_t of X_t^ξ is defined as

$$P_tf(\xi)=\mathbb{E}f(X_t^\xi),\quad f\in\mathscr{B}_b(\mathscr{C}),t\geq 0,\xi\in\mathscr{C},$$

where $\mathscr{B}_b(\mathscr{C})$ is the set of all bounded measurable functions on $\mathscr{C}$. The next main result is about Harnack inequalities. Here, we only study Harnack inequalities in the case of $r<\infty$. In fact, Harnack inequalities do not hold generally if $r=\infty$, see Remark 2.1 for details. On the other hand, according to [13, Theorem 1.4.1], the log-Harnack inequality implies the strong Feller property, while for $T\leq r$, we have $X_T(s)=X(T+s)=\xi(T+s),\ s\in[-r,-T]$ which is deterministic. Thus P_T is strong Feller only if $T>r$. So the restriction on $r<\infty$ is essential for the study.

To obtain Harnack inequalities, we need the following stronger conditions ($\mathbf{a3}''$) and ($\mathbf{a4}''$) instead of ($\mathbf{a3}'$) and ($\mathbf{a4}'$) to ensure (5.33) in Lemma 5.3:

(a3$''$) For any $T > 0$, there exists $\phi \in \mathscr{D}$ such that

$$\left|(-A)^{\frac{1-\varepsilon}{2}}[b(t,x) - b(t,y)]\right| \le \phi(|x-y|), \quad t \in [0,T], x, y \in \mathbb{H}, \tag{2.4}$$

where ε is in **(a1)**.

(a4$''$) For any $T > 0$,

$$\left\|(-A)^{\frac{1}{2}}b\right\|_{T,\infty} < \infty. \tag{2.5}$$

Then we have

Theorem 2.2. *Assume* (**a1**), (**a2$'$**), (**a3$''$**) *and* (**a4$''$**). *If for any $T > 0$, there exists a constant $C(T) > 0$ such that*

$$\|Q(t,x) - Q(t,y)\|_{\mathrm{HS}}^2 \le C(T)|x-y|^2, \quad t \in [0,T], x, y \in \mathbb{H}. \tag{2.6}$$

Then for every $T > r$ and positive function $f \in \mathscr{B}_b(\mathscr{C})$,

(1) the log*-Harnack inequality holds, i.e.*

$$P_T \log f(\eta) \le \log P_T f(\xi) + H(T,\xi,\eta), \;\; \xi, \eta \in \mathscr{C} \tag{2.7}$$

with

$$H(T,\xi,\eta) = C\left(\frac{|\xi(0) - \eta(0)|^2}{T-r} + \|\xi - \eta\|_\infty^2\right)$$

for some constant $C > 0$.

(2) There exists $K > 0$ such that for any $p > (1+K)^2$, the Harnack inequality with power

$$P_T f(\eta) \le (P_T f^p(\xi))^{\frac{1}{p}} \exp \Psi_p(T;\xi,\eta), \quad \xi, \eta \in \mathscr{C} \tag{2.8}$$

holds, where

$$\Psi_p(T;\xi,\eta) = C(p)\left\{1 + \frac{|\xi(0) - \eta(0)|^2}{T-r} + \|\xi - \eta\|_\infty^2\right\}$$

for a decreasing function $C : ((1+K)^2, \infty) \to (0,\infty)$.

Remark 2.1. *If $r = \infty$, the Harnack inequality does not hold for P_t for any $t > 0$. In fact, fix $x^0 \in \mathbb{H}$, $t > 0$, and let $g(x) = 1_{\{x^0\}}(x), x \in \mathbb{H}$. Then*

$$f_t(\eta) := g(\eta(-t)), \;\; \eta \in \mathscr{C}$$

is in $\mathscr{B}_b(\mathscr{C})$, but $P_t f_t(\xi) = \mathbb{E} f_t(X_t^\xi) = g(\xi(0))$. It is clear that the Harnack inequality does not hold for f_t.

The remainder of the paper is organized as follows: in Section 3, we prove the pathwise uniqueness; in Section 4, combining Section 3 with a truncating argument, we prove Theorem 2.1; in Section 5, we investigate Harnack inequalities for the semigroup by finite-dimensional approximations; in Section 6, we give [13, Theorem 4.3.1 and Theorem 4.3.2] in detail.

3 Pathwise uniqueness

In this section, we transform (1.1) to a regular equation and then investigate its pathwise uniqueness, which is equivalent to that of (1.1). We start from the following SPDE

$$
\mathrm{d}Z_{s,t}^x = AZ_{s,t}^x\mathrm{d}t + Q(t, Z_{s,t}^x)\mathrm{d}W(t), \quad Z_{s,s}^x = x, t \geq s \geq 0. \tag{3.1}
$$

Under (**a1**) and (**a2**$'$) with $B = 0$, (3.1) has a unique mild solution $\{Z_{s,t}^x\}_{t\geq s}$. Let $P_{s,t}^0$ be the associated Markov semigroup.

We first consider modified gradient estimates for $P_{s,t}^0$, which will be used to study the regularity of the solution to the equation (3.6). Before moving on, we shall recall two gradient estimates for $P_{s,t}^0 f$ with any bounded Borel measurable function f on $\mathbb{H}$. For η, $x \in \mathbb{H}$, $0 \leq s < t \leq T$, there are, see [11, (2.12),(2.16)],

$$
\left|\nabla P_{s,t}^0 f(x)\right|^2 \leq \frac{c}{t-s} P_{s,t}^0 |f|^2(x), \quad \|\nabla^2 P_{s,t}^0 f(x)\|^2 \leq \frac{c}{(t-s)^2} P_{s,t}^0 |f|^2(x). \tag{3.2}
$$

Next, for $f \in \mathscr{B}_b(\mathbb{H}, \mathbb{H})$, by (3.2),

$$
P_{s,t}^0 f(x) := \sum_{i=1}^{\infty} \left(P_{s,t}^0 \langle f, e_i \rangle(x)\right) e_i, \ \nabla_\eta P_{s,t}^0 f(x) := \sum_{i=1}^{\infty} \left(\nabla_\eta P_{s,t}^0 \langle f, e_i \rangle(x)\right) e_i
$$

are well defined. Then, for any $a \in \mathscr{A}_1$ and $f \in \mathscr{B}_b(\mathbb{H}, \mathbb{H}_a)$, it holds that

$$
\begin{aligned}
\sum_{i=1}^{\infty} a(\lambda_i)^2 \left(\nabla_\eta P_{s,t}^0 \langle f, e_i \rangle(x)\right)^2 &= \sum_{i=1}^{\infty} \left(\nabla_\eta P_{s,t}^0 \langle a(-A)f, e_i \rangle(x)\right)^2 \\
&\leq \frac{c}{t-s} P_{s,t}^0 |a(-A)f|^2(x)|\eta|^2 < \infty.
\end{aligned}
$$

Thus $\nabla_\eta P_{s,t}^0 f(x)$ belongs to the domain of $a(-A)$ and

$$
\begin{aligned}
a(-A)\nabla_\eta P_{s,t}^0 f(x) &= \sum_{i=1}^{\infty} a(\lambda_i) \nabla_\eta P_{s,t}^0 \langle f, e_i \rangle(x) e_i \\
&= \sum_{i=1}^{\infty} \nabla_\eta P_{s,t}^0 \langle a(-A)f, e_i \rangle(x) e_i \\
&= \nabla_\eta P_{s,t}^0 (a(-A)f)(x).
\end{aligned}
$$

We can define $\nabla^2 P_{s,t} f(x)$ in a similar way. Then (3.2) and the similar argument as above imply that

$$
a(-A)\nabla_{\eta'} \nabla_\eta P_{s,t}^0 f(x) = \nabla_{\eta'} \nabla_\eta P_{s,t}^0 (a(-A)f)(x). \tag{3.3}
$$

In a word,

$$
a(-A)\nabla^k P_{s,t}^0 f = \nabla^k P_{s,t}^0 (a(-A)f), \ f \in \mathscr{B}_b(\mathbb{H}, \mathbb{H}_a), 0 \leq s < t \leq T, k = 0, 1, 2. \tag{3.4}
$$

Next, as in [11], to obtain the pathwise uniqueness of (1.1), formally, we need to study a parabolic equation with terminal condition:

$$\partial_t u(\cdot,x)(t) = -L_t u(t,\cdot)(x) - b(t,x) + \lambda u(t,x),\ u(T,x)=0,\ t\in[0,T], \tag{3.5}$$

where

$$L_t = \frac{1}{2}\sum_{i,j}\langle Q(t,\cdot)Q^*(t,\cdot)e_i, e_j\rangle \nabla_{e_i}\nabla_{e_j} + \nabla_{A\cdot} + \nabla_{b(t,\cdot)}.$$

The precise meaning of (3.5) is the following integral equation, which will be solved by the fixed-point theorem on a suitable Banach space which depends on the function a.

$$u(s,x) = \int_s^T \mathrm{e}^{-\lambda(t-s)} P^0_{s,t}(\nabla_{b(t,\cdot)}u(t,\cdot) + b(t,\cdot))(x)\mathrm{d}t, \quad s\in[0,T]. \tag{3.6}$$

The following Lemma 3.1 is a modified version of [11, Lemma 2.3]. Set

$$\theta(t,x) = x + u(t,x).$$

Formally, taking (3.5) into account, $\partial_t\theta(\cdot,x)(t) = Ax - L_t\theta(t,\cdot)(x)$ with $\theta(T,x)=x$. Formally again, Itô's formula implies that

$$\begin{aligned}\mathrm{d}\theta(t,X(t)) &= A\theta(t,X(t))\mathrm{d}t + (\lambda - A)u(t,X(t))\mathrm{d}t\\ &\quad + \nabla\theta(t,\cdot)(X(t))\left\{Q(t,X(t))\mathrm{d}W(t) + B(t,X_t)\mathrm{d}t\right\},\end{aligned} \tag{3.7}$$

which is similar to [11, (2.1)] without singular term. Thus we can also obtain a regular representation of (1.1) in the following Lemma 3.2, which is the precise meaning of the equation (3.7).

Lemma 3.1. *Assume* (**a1**), (**a2**$'$) *with* $B=0$, *and* (2.3) *with* $a\in\mathscr{A}_1$. *Let* $T>0$ *be fixed. Then there exists a constant* $\lambda(T)>0$ *such that the following assertions hold.*

(1) For any $\lambda\ge\lambda(T)$, *the equation* (3.6) *has a unique solution* $u\in C([0,T];C_b^1(\mathbb{H};\mathbb{H}_a))$ *satisfying*

$$\lim_{\lambda\to\infty}\|u\|_{T,\infty,a} + \|\nabla u\|_{T,\infty,a} = 0. \tag{3.8}$$

(2) If moreover (2.2) *holds, then we have*

$$\lim_{\lambda\to\infty}\left\|\nabla^2 u\right\|_{T,\infty} = 0. \tag{3.9}$$

Proof. (1) Let $\mathscr{H} = C([0,T];C_b^1(\mathbb{H};\mathbb{H}_a))$, which is a Banach space under the norm

$$\begin{aligned}\|u\|_{\mathscr{H}} :&= \|u\|_{T,\infty,a} + \|\nabla u\|_{T,\infty,a}\\ &= \sup_{t\in[0,T],x\in\mathbb{H}}|a(-A)u(t,x)| + \sup_{t\in[0,T],x\in\mathbb{H}}\|a(-A)\nabla u(t,x)\|, \quad u\in\mathscr{H}.\end{aligned}$$

For any $u \in \mathscr{H}$, define

$$(\Gamma u)(s,x) = \int_s^T \mathrm{e}^{-\lambda(t-s)} P^0_{s,t}(\nabla_{b(t,\cdot)} u(t,\cdot) + b(t,\cdot))(x)\mathrm{d}t, \quad s \in [0,T].$$

Then we have $\Gamma\mathscr{H} \subset \mathscr{H}$. In fact, for any $u \in \mathscr{H}$, by $(\mathbf{a2}')$, (2.3), (3.4), it holds that

$$\begin{aligned}
\|\Gamma u\|_{T,\infty,a} &= \sup_{s\in[0,T],x\in\mathbb{H}} \left| \int_s^T \mathrm{e}^{-\lambda(t-s)} P^0_{s,t}(a(-A)\nabla_{b(t,\cdot)} u(t,\cdot) + a(-A)b(t,\cdot))(x)\mathrm{d}t \right| \\
&\le \sup_{s\in[0,T]} \int_s^T \mathrm{e}^{-\lambda(t-s)}(\|b\|_{T,\infty}\|\nabla u\|_{T,\infty,a} + \|b\|_{T,\infty,a})\mathrm{d}t \\
&\le (\|b\|_{T,\infty}\|\nabla u\|_{T,\infty,a} + \|b\|_{T,\infty,a}) \int_0^T \mathrm{e}^{-\lambda t}\mathrm{d}t \\
&\le \frac{\|b\|_{T,\infty}\|\nabla u\|_{T,\infty,a} + \|b\|_{T,\infty,a}}{\lambda} < \infty.
\end{aligned}$$

Again by $(\mathbf{a2}')$, (2.3) and (3.4), we have

$$\begin{aligned}
\|\nabla\Gamma u\|_{T,\infty,a} &= \sup_{s\in[0,T],x\in\mathbb{H},|\eta|\le 1} \left\| \int_s^T \mathrm{e}^{-\lambda(t-s)} \nabla_\eta P^0_{s,t}(a(-A)\nabla_{b(t,\cdot)} u(t,\cdot) + a(-A)b(t,\cdot))(x)\mathrm{d}t \right\| \\
&\le C \sup_{s\in[0,T]} \int_s^T \frac{\mathrm{e}^{-\lambda(t-s)}}{\sqrt{t-s}}(\|b\|_{T,\infty}\|\nabla u\|_{T,\infty,a} + \|b\|_{T,\infty,a})\mathrm{d}t \\
&\le C(\|b\|_{T,\infty}\|\nabla u\|_{T,\infty,a} + \|b\|_{T,\infty,a}) \int_0^T \frac{\mathrm{e}^{-\lambda t}}{\sqrt{t}}\mathrm{d}t \\
&\le C\frac{\|b\|_{T,\infty}\|\nabla u\|_{T,\infty,a} + \|b\|_{T,\infty,a}}{\sqrt{\lambda}} < \infty.
\end{aligned}$$

So, $\Gamma\mathscr{H} \subset \mathscr{H}$. Next, by the fixed-point theorem, it suffices to show that for large enough $\lambda > 0$, Γ is contractive on $\mathscr{H}$. To do this, for any u, $\tilde{u} \in \mathscr{H}$, similarly to the estimates of $\|\Gamma u\|$ and $\|\nabla\Gamma u\|$, we obtain that

$$\begin{aligned}
\|\Gamma u - \Gamma\tilde{u}\|_{T,\infty,a} &\le \frac{\|b\|_{T,\infty}}{\lambda}\|\nabla u - \nabla\tilde{u}\|_{T,\infty,a}, \\
\|\nabla(\Gamma u - \Gamma\tilde{u})\|_{T,\infty,a} &\le C\frac{\|b\|_{T,\infty}}{\sqrt{\lambda}}\|\nabla u - \nabla\tilde{u}\|_{T,\infty,a}.
\end{aligned} \tag{3.10}$$

So we can find $\lambda(T) > 0$ such that Γ is contractive on $\mathscr{H}$ with $\lambda > \lambda(T)$, by fixed-point theorem, (3.6) has a unique solution $u \in C([0,T]; C^1_b(\mathbb{H};\mathbb{H}_a))$. Finally, substituting $\Gamma u = u$ into (3.10) and letting $\tilde{u}$=0, we obtain (3.8).

(2) This is a known result in [11, Lemma 2.3 (2)]. □

Remark 3.1. *Under the assumptions of Lemma 3.1 and a strengthened version of* (2.2)*:*

$$|a(-A)(b(t,x) - b(t,y))| \le \phi(|x-y|), \quad t \in [0,T], x,y \in \mathbb{H},$$

we can obtain that

$$\lim_{\lambda\to\infty} \left\|\nabla^2(a(-A)u)\right\|_{T,\infty} = 0,$$

see the proof of Lemma 5.2 with $a(x) = x^{\frac{1-\varepsilon}{2}}, x > 0$. *However,* (3.9) *is enough for the pathwise uniqueness of* (1.1).

Lemma 3.2. *Assume* (**a1**), (**a2**$'$), (**a3**$'$) *and* $\|b\|_{T,\infty} < \infty$ *for any* $T \geq 0$. *Then for any* $T > 0$, *there exists a constant* $\lambda(T) > 0$ *such that for any stopping time* τ, *any adapted continuous* $\mathscr{C}$*-valued process* $(X_t)_{t\in[0,T\wedge\tau]}$ *with* $\mathbb{P}$*-a.s.*

$$\begin{aligned}X(t) = \mathrm{e}^{At}X(0) &+ \int_0^t \mathrm{e}^{A(t-s)}(b(s,X(s)) + B(s,X_s))\mathrm{d}s\\ &+ \int_0^t \mathrm{e}^{A(t-s)}Q(s,X(s))\mathrm{d}W(s), \quad t\in[0,\tau\wedge T],\end{aligned}$$

and any $\lambda \geq \lambda(T)$, *there holds*

$$\begin{aligned}X(t) &= \mathrm{e}^{At}[X(0) + u(0,X(0))] - u(t,X(t))\\ &\quad + \int_0^t (\lambda - A)\mathrm{e}^{A(t-s)}u(s,X(s))\mathrm{d}s\\ &\quad + \int_0^t \mathrm{e}^{A(t-s)}[I + \nabla u(s,X(s))]B(s,X_s)\mathrm{d}s\\ &\quad + \int_0^t \mathrm{e}^{A(t-s)}[I + \nabla u(s,X(s))]Q(s,X(s))\mathrm{d}W(s), \quad t\in[0,\tau\wedge T],\end{aligned} \tag{3.11}$$

where u *solves* (3.6), *and* $\nabla u(s,z)v := [\nabla_v u(s,\cdot)](z)$ *for* $v,\ z \in \mathbb{H}$.

Proof. Since $\|B\|_{t,\infty} < \infty$ for any $t \geq 0$, the claim of this lemma can be obtained just repeating the proof of [11, Proposition 2.5]. To save space, we omit the detail here. $\square$

Now, we present a complete proof of the pathwise uniqueness to (1.1).

Proposition 3.3. *Assume* (**a1**) *and* (**a2**$'$)-(**a4**$'$). *Let* $\{X_t\}_{t\geq0}, \{Y_t\}_{t\geq0}$ *be two adapted continuous* $\mathscr{C}$*-valued processes with* $X_0 = Y_0 = \xi \in \mathscr{C}$. *For any* $n \geq 1$, *let*

$$\tau_n^X = n \wedge \inf\{t \geq 0 : |X(t)| \geq n\}, \quad \tau_n^Y = n \wedge \inf\{t\geq 0 : |Y(t)| \geq n\}.$$

If $\mathbb{P}$ *-a.s. for all* $t \in [0, \tau_n^X \wedge \tau_n^Y]$, *there holds :*

$$\begin{aligned}X(t) &= \mathrm{e}^{At}\xi(0) + \int_0^t \mathrm{e}^{A(t-s)}(b(s,X(s)) + B(s,X_s))\mathrm{d}s + \int_0^t \mathrm{e}^{A(t-s)}Q(s,X(s))\mathrm{d}W(s),\\ Y(t) &= \mathrm{e}^{At}\xi(0) + \int_0^t \mathrm{e}^{A(t-s)}(b(s,Y(s)) + B(s,Y_s))\mathrm{d}s + \int_0^t \mathrm{e}^{A(t-s)}Q(s,Y(s))\mathrm{d}W(s),\end{aligned}$$

then $\mathbb{P}$*-a.s.* $X(t) = Y(t)$, *for all* $t \in \left[0, \tau_n^X \wedge \tau_n^Y\right]$. *In particular,* $\mathbb{P}$*-a.s.* $\tau_n^X = \tau_n^Y$.

Proof. For any $n \geq 1$, let $\tau_n = \tau_n^X \wedge \tau_n^Y$. Then it suffices to prove that for any $T > 0$,

$$\mathbb{E} \sup_{s \in [0,T]} |X(s \wedge \tau_n) - Y(s \wedge \tau_n)|^{2p} = 0 \tag{3.12}$$

holds for some $p \in (1, \frac{1}{\varepsilon})$. In what follows, we fix $T > 0$ and $p \in (1, \frac{1}{\varepsilon})$. Take λ large enough such that assertions in Lemma 3.1 and Lemma 3.2 hold, and

$$\frac{5^{4p-1}}{2^{2p+1}} \left(\|\nabla u\|_{T,\infty,a} \int_0^T \|(-A)[a(-A)]^{-1} \mathrm{e}^{As}\| \mathrm{d}s \right)^{2p} + \|\nabla u\|_{T,\infty} \leq \frac{1}{5}. \tag{3.13}$$

By (3.11) for $\tau = \tau_n$, we have $\mathbb{P}$-a.s. for any $t \in [0, \tau_n \wedge T]$,

$$\begin{aligned} &[X(t) + u(t, X(t))] - [Y(t) + u(t, Y(t))] \\ &= \int_0^t (\lambda - A) \mathrm{e}^{A(t-s)} [u(s, X(s)) - u(s, Y(s))] \mathrm{d}s \\ &+ \int_0^t \mathrm{e}^{A(t-s)} \{[I + \nabla u(s, X(s))] B(s, X_s) - [I + \nabla u(s, Y(s))] B(s, Y_s)\} \mathrm{d}s \\ &+ \int_0^t \mathrm{e}^{A(t-s)} \{[I + \nabla u(s, X(s))] Q(s, X(s)) - [I + \nabla u(s, Y(s))] Q(s, Y(s))\} \mathrm{d}W(s). \end{aligned} \tag{3.14}$$

Then (3.13) yields that

$$\begin{aligned} &\mathbb{E} \sup_{t \in [0,q]} |X(t \wedge \tau_n) - Y(t \wedge \tau_n)|^{2p} \\ &\leq \frac{5^{4p-1}}{4^{2p}} \mathbb{E} \sup_{t \in [0,q]} \left| \int_0^{t \wedge \tau_n} (\lambda - A) \mathrm{e}^{A(t-s)} [u(s, X(s)) - u(s, Y(s))] \mathrm{d}s \right|^{2p} \\ &+ \frac{5^{4p-1}}{4^{2p}} \mathbb{E} \sup_{t \in [0,q]} \left| \int_0^{t \wedge \tau_n} \mathrm{e}^{A(t-s)} [I + \nabla u(s, X(s))][B(s, X_s) - B(s, Y_s)] \mathrm{d}s \right|^{2p} \\ &+ \frac{5^{4p-1}}{4^{2p}} \mathbb{E} \sup_{t \in [0,q]} \left| \int_0^{t \wedge \tau_n} e^{A(t-s)} [\nabla u(s, X(s)) - \nabla u(s, Y(s))] B(s, Y_s) \mathrm{d}s \right|^{2p} \\ &+ \frac{5^{4p-1}}{4^{2p}} \mathbb{E} \sup_{t \in [0,q]} \left| \int_0^{t \wedge \tau_n} \mathrm{e}^{A(t-s)} [\nabla u(s, X(s)) - \nabla u(s, Y(s))] Q(s, X(s)) \mathrm{d}W(s) \right|^{2p} \\ &+ \frac{5^{4p-1}}{4^{2p}} \mathbb{E} \sup_{t \in [0,q]} \left| \int_0^{t \wedge \tau_n} \mathrm{e}^{A(t-s)} (I + \nabla u(s, Y(s))) [Q(s, X(s)) - Q(s, Y(s))] \mathrm{d}W(s) \right|^{2p} \\ &=: I_1 + I_2 + I_3 + I_4 + I_5, \quad q \in [0, T]. \end{aligned} \tag{3.15}$$

First of all, let

$$\eta_q = \mathbb{E} \sup_{t \in [0,q]} |X(t \wedge \tau_n) - Y(t \wedge \tau_n)|^{2p}.$$

Then, by (3.13), there exists a constant $C(p, \lambda, T) > 0$ such that

$$I_1 - C(p, \lambda, T) \int_0^q \eta_s$$

$$\leq \frac{5^{4p-1}}{2^{2p+1}}\mathbb{E}\sup_{t\in[0,q]}\left[\int_0^{t\wedge\tau_n}\int_0^1 |Ae^{(t-s)A}\left[\nabla_{\Delta_s}u(s,\cdot)\right](X(s)+v\Delta_s)|\mathrm{d}v\mathrm{d}s\right]^{2p}$$

$$\leq \left[\frac{5^{4p-1}}{2^{2p+1}}\left(\|a(-A)\nabla u\|_{T,\infty}\int_0^T \|(-A)[a(-A)]^{-1}\mathrm{e}^{As}\|\mathrm{d}s\right)^{2p}\right]\eta_q \tag{3.16}$$

$$\leq \frac{1}{5}\eta_q,$$

where $\Delta_s = X(s) - Y(s)$.

Secondly, since A is negative definite, by (3.9), ($\mathbf{a2}'$), $X_0 = Y_0$ and Hölder inequality, it holds that

$$\begin{aligned} I_2 &\leq C\mathbb{E}\int_0^{q\wedge\tau_n}|B(s,X_s)-B(s,Y_s)|^{2p}\mathrm{d}s \\ &\leq C_1\mathbb{E}\int_0^q\sup_{t\in[0,s]}|X(t\wedge\tau_n)-Y(t\wedge\tau_n)|^{2p}\mathrm{d}s \\ &\leq C_1\int_0^q\eta_s\mathrm{d}s \end{aligned} \tag{3.17}$$

for a constant $C_1 > 0$. Similarly, combing (3.9) and $\|B\|_{T,\infty} < \infty$, we obtain

$$I_3 \leq C_2\int_0^q \eta_s\mathrm{d}s \tag{3.18}$$

for a constant $C_2 > 0$.

Finally, in view of ($\mathbf{a2}'$), Remark 1.4 and $p\in(1,\frac{1}{\varepsilon})$, Lemma 1.1 implies that

$$I_4 + I_5 \leq C_3\mathbb{E}\int_0^{q\wedge\tau_n}|X(s)-Y(s)|^{2p}\mathrm{d}s = C_3\int_0^q\eta_s\mathrm{d}s, \tag{3.19}$$

for a constant $C_3 > 0$. Combining (3.15)-(3.19), there exits a constant C_0 such that

$$\eta_l \leq \frac{1}{5}\eta_l + C_0\int_0^l \eta_q\mathrm{d}q, \quad l\in[0,T].$$

By Gronwall's inequality, we obtain $\eta_T = 0$, i.e. (3.12) holds. □

Remark 3.2. *The above result for the case without delay has been proved in [11, Proposition 3.1] under* ($\mathbf{a1}$), ($\mathbf{a2}'$), ($\mathbf{a3}'$) *and the condition* $\|b\|_{T,\infty} < \infty$ *for any* $T \geq 0$. *Now we explain the reason why* ($\mathbf{a4}'$) *is needed in the present case. Noticing that the Itô's formula is unavailable in the infinite dimension case, and the last term in* (3.11) *called stochastic convolution is not a local martingale, we can not apply the stochastic Gronwall Lemma [9, Lemma 5.2] which is an important tool in proving the pathwise uniqueness of functional SDEs in the finite dimension case. Moreover, since the drift term* B *is Lipschitz in* $\mathscr{C}$, *in the proof of Proposition 3.3, we need to take supremum instead of integration as in [11, Proposition*

3.1] and then obtain (3.15)*. Though we can also integrate both sides of* (3.15) *on* $[0,T]$ *as [11, Proposition 3.1] does, in general,*

$$
\begin{aligned}
&\mathbb{E}\int_0^T \sup_{t\in[0,q]}\left|\int_0^{t\wedge\tau_n}(\lambda-A)\mathrm{e}^{A(t-s)}[u(s,X(s))-u(s,Y(s))]\mathrm{d}s\right|^{2p}\mathrm{d}t\\
&\geq \mathbb{E}\sup_{q\in[0,T]}\int_0^q\left|\int_0^{t\wedge\tau_n}(\lambda-A)\mathrm{e}^{A(t-s)}[u(s,X(s))-u(s,Y(s))]\mathrm{d}s\right|^{2p}\mathrm{d}t.
\end{aligned}
$$

So, it is not available to treat I_1 *in* (3.15) *as in [11, Proposition 3.1] by Fubini Theorem. Noticing that* $\int_0^t\|\mathrm{e}^{A(t-s)}(-A)\|\mathrm{d}s=\infty$*, if* $u(s,\cdot)$ *is Lipschitz continuous uniformlly in* $s\in[0,T]$ *from* $\mathbb{H}$ *to some smaller space* $\mathbb{H}_a$ *with* $a\in\mathscr{A}_A$*, then by the definition of* $\mathscr{A}_A$*,* I_1 *can be treated as in* (3.16)*. Furthermore, by Lemma 3.1 (1),* (**a1**)*,* (**a2**$'$) *and* (**a4**$'$) *can also ensure* $\|a(-A)\nabla u\|_{T,\infty}<\infty$*. For more details, see the proof of the above Proposition 3.3. In fact, the above trick is used to prove the pathwise uniqueness of the neutral functional SPDE, see [3], where the condition [3, (H3)] is something like* $\|a(-A)\nabla u\|_{T,\infty}<\infty$ *with* $a(x)=x^\delta$ *for some* $\delta>0$*.*

4 Proof of Theorem 2.1

Proof of Theorem 2.1. (a) We first assume that (**a1**) and (**a2**$'$)-(**a4**$'$) hold. Consider the following SPDE on $\mathbb{H}$:

$$\mathrm{d}Z^\xi(t)=AZ^\xi(t)\mathrm{d}t+Q(t,Z^\xi(t))\mathrm{d}W(t),\quad Z^\xi(0)=\xi(0).$$

By (**a1**) and (**a2**$'$), the above equation has a uniqueness non-explosive mild solution:

$$Z^\xi(t)=\mathrm{e}^{At}\xi(0)+\int_0^t\mathrm{e}^{A(t-s)}Q(s,Z^\xi(s))\mathrm{d}W(s),\quad t\geq 0.$$

Letting $Z_0^\xi=\xi$ (i.e. $Z^\xi(\theta)=\xi(\theta)$ for $\theta\in[-r,0]\cap(-\infty,0)$), and taking

$$
\begin{aligned}
&W^\xi(t)=W(t)-\int_0^t\psi(s)\mathrm{d}s,\\
&\psi(s)=\big\{Q^*(QQ^*)^{-1}\big\}(s,Z^\xi(s))\big\{b(s,Z^\xi(s))+B(s,Z_s^\xi)\big\},\quad s,t\in[0,T],
\end{aligned}
$$

we have

$$
\begin{aligned}
Z^\xi(t)&=\mathrm{e}^{At}\xi(0)+\int_0^t\mathrm{e}^{A(t-s)}B(s,Z_s^\xi)\mathrm{d}s\\
&\quad+\int_0^t\mathrm{e}^{A(t-s)}b(s,Z^\xi(s))\mathrm{d}s+\int_0^t\mathrm{e}^{A(t-s)}Q(s,Z^\xi(s))\mathrm{d}W^\xi(s),\quad t\in[0,T].
\end{aligned}
$$

Since $\|B\|_{T,\infty}+\|b\|_{T,\infty}<\infty$, Girsanov theorem implies that $\{W^\xi(t)\}_{t\in[0,T]}$ is a cylindrical Brownian motion on $\bar{\mathbb{H}}$ under probability $\mathrm{d}\mathbb{Q}^\xi=R^\xi\mathrm{d}\mathbb{P}$, where

$$R^\xi:=\exp\left[\int_0^T\left\langle\psi(s),\mathrm{d}W(s)\right\rangle_{\bar{\mathbb{H}}}-\frac{1}{2}\int_0^T\left|\psi(s)\right|_{\bar{\mathbb{H}}}^2\mathrm{d}s\right].$$

Then, under the probability $\mathbb{Q}^\xi$, $(Z^\xi(t), W^\xi(t))_{t\in[0,T]}$ is a weak mild solution to (1.1). On the other hand, by Proposition 3.3, the pathwise uniqueness holds for the mild solution to (1.1). So, by the Yamada-Watanabe principle, the equation (1.1) has a unique mild solution. Moreover, in this case the solution is non-explosive.

(b) In general, take $\psi \in C_b^\infty([0,\infty))$ such that $0 \le \psi \le 1$, $\psi(v) = 1$ for $v \in [0,1]$ and $\psi(v) = 0$ for $v \in [2,\infty)$. For any $m \ge 1$, let

$$\begin{aligned} b^{[m]}(t,z) &= b(t\wedge m, z)\psi(|z|/m), \quad (t,z) \in [0,\infty)\times\mathbb{H}, \\ B^{[m]}(t,\xi) &= B(t\wedge m,\xi)\psi(\|\xi\|_\infty/m), \quad (t,\xi)\in[0,\infty)\times\mathscr{C}, \\ Q^{[m]}(t,z) &= Q(t\wedge m, z)\psi(|z|/m)), \quad (t,z)\in[0,\infty)\times\mathbb{H}. \end{aligned}$$

By (**a3**)-(**a4**) and the local boundedness of B, we know $B^{[m]}$, $Q^{[m]}$ and $b^{[m]}$ satisfy (**a2**$'$) – (**a4**$'$). Then by (a), (1.1) for $B^{[m]}$, $Q^{[m]}$ and $b^{[m]}$ in place of B, Q, b has a unique mild solution $X^{[m]}(t)$ starting at X_0 which is non-explosive. Let

$$\zeta_0 = 0, \quad \zeta_m = m \wedge \inf\{t\ge 0 : |X^{[m]}(t)| \ge m\}, \quad m\ge 1.$$

Then, since $B^{[m]}(s,\xi) = B(s,\xi)$, $Q^{[m]}(s,\xi(0)) = Q(s,\xi(0))$ and $b^{[m]}(s,\xi(0)) = b(s,\xi(0))$ hold for $s \le m$ and $\|\xi\|_\infty \le m$ with any $m\ge 1$, we can obtain that for any n, $m \ge 1$, $X^{[m]}(t) = X^{[n]}(t)$ for $t\in[0,\zeta_m\wedge\zeta_n]$ by Proposition 3.3. In particular, ζ_m is increasing in m. Let $\zeta = \lim_{m\to\infty}\zeta_m$ and

$$X(t) = \sum_{m=1}^\infty 1_{[\zeta_{m-1},\zeta_m)} X^{[m]}(t), \quad t\in[0,\zeta).$$

Then $X(t)_{t\in[0,\zeta)}$ is a mild solution to (1.1) with life time ζ and, due to Proposition 3.3, the mild solution is unique. So we prove Theorem 2.1 (1).

(c) Next, we prove the non-explosion.

Let Φ, h satisfy (2.1), and let $\{X(t)\}_{t\in[0,\zeta)}$ be the mild solution to (1.1) with lifetime ζ. Set $M(t) = \int_0^t \mathrm{e}^{A(t-s)}Q(s,X(s))\mathrm{d}W(s)$, $t\in[0,\zeta)$ and $M(t) = 0$, $t\in[-r,0]\cap(-\infty,0)$. Then, taking into account that $\|Q\|_{t,\infty}$ is locally bounded in t, $Y(t) := X(t) - M(t)$ is the mild solution to the following equation up to ζ,

$$\mathrm{d}Y(t) = (AY(t) + b(t, Y(t)+M(t)) + B(t, Y_t + M_t))\mathrm{d}t, \quad Y_0 = X_0.$$

Hence, (2.1) implies that for any $T>0$,

$$\begin{aligned} \mathrm{d}|Y(t)|^2 &\le 2\langle Y(t), b(t,Y(t)+M(t)) + B(t,Y_t+M_t)\rangle \mathrm{d}t \\ &\le 2\left(\Phi_{\zeta\wedge T}(\|Y_t\|_\infty^2) + h_{\zeta\wedge T}(\|M_t\|_\infty)\right)\mathrm{d}t. \end{aligned} \tag{4.1}$$

Let

$$\Psi_T(s) = \int_1^s \frac{\mathrm{d}v}{2\Phi_{\zeta\wedge T}(v)}, \quad \alpha_T = 2\|X_0\|_\infty^2 + 2\int_0^{\zeta\wedge T} h_{\zeta\wedge T}(\|M_s\|_\infty))\mathrm{d}s. \tag{4.2}$$

Since

$$||Y_q||_\infty^2 \le \sup_{t\in[0,q]} |Y(t)|^2 + ||Y_0||_\infty^2,$$

it follows from (4.1) that

$$\begin{aligned}||Y_q||_\infty^2 \le ||X_0||_\infty^2 + 2\int_0^{\zeta\wedge T} h_{\zeta\wedge T}(\|M_s\|_\infty))\mathrm{d}s \\ + 2\int_0^q \Phi_{\zeta\wedge T}\left(||Y_s||_\infty^2\right)\mathrm{d}s, \quad q\in[0,\zeta\wedge T).\end{aligned} \tag{4.3}$$

By Bihari-LaSalle inequality, (4.3) implies

$$||Y_t||_\infty^2 \le \Psi_T^{-1}(\Psi_T(\alpha_T)+t), \quad t\in[0,\zeta\wedge T). \tag{4.4}$$

Moreover, (**a1**), $\|Q\|_{T,\infty}<\infty$ and Lemma 1.1 yield

$$\mathbb{E}\sup_{t\in[0,\zeta\wedge T)} |M(t)|^2 < \infty. \tag{4.5}$$

So by the definition of ζ and Y, on the set $\{\zeta<\infty\}$, we have $\mathbb{P}$-a.s.

$$\limsup_{t\uparrow\zeta} |Y(t)| = \limsup_{t\uparrow\zeta} |X(t)| = \infty. \tag{4.6}$$

Moreover on the set $\{\zeta\le T\}$, $\mathbb{P}$-a.s. $\alpha_T<\infty$. Combining the property of Φ and (4.6), it holds that on the set $\{\zeta\le T\}$, $\mathbb{P}$-a.s.

$$\infty = \limsup_{t\uparrow\zeta} |Y(t)|^2 \le \Psi_T^{-1}(\Psi_T(\alpha_T)+T) < \infty.$$

So for any $T>0$, $\mathbb{P}\{\zeta\le T\}=0$. Therefore,

$$\mathbb{P}\{\zeta<\infty\} = \mathbb{P}\left(\bigcup_{m=1}^\infty \{\zeta\le m\}\right) \le \sum_{m=1}^\infty \mathbb{P}\{\zeta\le m\} = 0,$$

which implies the solution of (1.1) is non-explosive. □

5 Proof of Theorem 2.2

In Section 3, we have transform (1.1) into an equation with regular coefficients, so we study the Harnack inequalities for the new equation instead. To do this, we decompose the proof into two steps:

(1) In the finite dimension, using coupling by change of measure, Harnack inequalities were obtained by Lemma 6.1. Thus, we only need to check the condition **(A)** in Lemma 6.1.

(2) We can prove the desired result from step (1) by finite dimension approximation, since Harnack inequalities in Lemma 6.1 are dimension-free.

In this section, we fix $T > r$. Under (**a1**), (**a2**$'$), (**a3**$'$) and (**a4**$'$) with $a(x) = x^{\frac{1}{2}}$, by Lemma 3.1, we can take $\lambda(T) > 0$ large enough such that for any $\lambda \geq \lambda(T)$, the unique solution u to (3.6) satisfies

$$\|\nabla^2 u\|_{T,\infty} \leq \frac{1}{8}, \quad \|(-A)^{\frac{1}{2}} \nabla u\|_{T,\infty} \leq \frac{\sqrt{\lambda_1}}{8}. \tag{5.1}$$

To treat the delay part, set $u(s, \cdot) = u(0, \cdot)$ for $s \in [-r, 0]$ and still set $\theta(t, x) = x + u(t, x)$, $(t, x) \in [-r, T] \times \mathbb{H}$. By (5.1), $\{\theta(t, \cdot)\}_{t \in [-r,T]}$ is a family of diffeomorphisms on $\mathbb{H}$. For simplicity, we write $\theta^{-1}(t, x) = [\theta^{-1}(t, \cdot)](x)$, $\nabla\theta(t, x) = [\nabla\theta(t, \cdot)](x)$ and $\nabla\theta^{-1}(t, x) = [\nabla\theta^{-1}(t, \cdot)](x)$, $(t, x) \in [-r, T] \times \mathbb{H}$. By (5.1), we have

$$\frac{7}{8} \leq \|\nabla\theta(t, x)\| \leq \frac{9}{8}, \quad \frac{8}{9} \leq \|\nabla\theta^{-1}(t, x)\| \leq \frac{8}{7}, \quad (t, x) \in [-r, T] \times \mathbb{H}. \tag{5.2}$$

Since $u \in C([0, T], C_b^1(\mathbb{H}, \mathbb{H}_a))$, $\theta(t + \cdot, \xi(\cdot))$ is continuous for any $t \in [0, T]$, $\xi \in \mathscr{C}$. Then we can define $\theta_t : \mathscr{C} \to \mathscr{C}$ as

$$(\theta_t(\xi))(s) = \theta(t + s, \xi(s)), \quad \xi \in \mathscr{C}, s \in [-r, 0]. \tag{5.3}$$

On the other hand, (5.2) and (5.3) yield that

$$\begin{aligned}
&|\theta^{-1}(t + s + \Delta s, \xi(s + \Delta s)) - \theta^{-1}(t + s, \xi(s))| \\
&\quad \leq |\theta^{-1}(t + s + \Delta s, \xi(s)) - \theta^{-1}(t + s, \xi(s))| \\
&\qquad + |\theta^{-1}(t + s + \Delta s, \xi(s + \Delta s)) - \theta^{-1}(t + s + \Delta s, \xi(s))| \\
&\quad \leq |\theta^{-1}(t + s + \Delta s, \xi(s)) - \theta^{-1}(t + s + \Delta s, \theta(t + s + \Delta s, \theta^{-1}(t + s, \xi(s)))| \\
&\qquad + ||\nabla\theta^{-1}(t + s + \Delta s, \cdot)||_\infty \cdot |\xi(s + \Delta s) - \xi(s)| \\
&\quad \leq ||\nabla\theta^{-1}(t + s + \Delta s, \cdot)||_\infty \cdot |\xi(s) - \theta(t + s + \Delta s, \theta^{-1}(t + s, \xi(s)))| \\
&\qquad + ||\nabla\theta^{-1}(t + s + \Delta s, \cdot)||_\infty \cdot |\xi(s + \Delta s) - \xi(s)| \\
&\quad \leq \frac{8}{7}\Big\{|\xi(s + \Delta s) - \xi(s)| + |\xi(s) - \theta(t + s + \Delta s, \theta^{-1}(t + s, \xi(s)))|\Big\}.
\end{aligned}$$

Then $\theta^{-1}(t + \cdot, \xi(\cdot))$ is also continuous, and $\{\theta_t\}_{t \in [0,T]}$ is a family of homeomorphisms on $\mathscr{C}$ with

$$(\theta_t^{-1}(\xi))(s) = \theta^{-1}(t + s, \xi(s)), \ \xi \in \mathscr{C}, s \in [-r, 0], \ t \in [0, T]. \tag{5.4}$$

Furthermore, it follows from (5.2) and (5.3) that

$$\|\theta_t(\xi) - \theta_t(\eta)\|_\infty \leq \frac{9}{8}\|\xi - \eta\|_\infty, \quad t \in [0, T], \xi, \eta \in \mathscr{C}. \tag{5.5}$$

Similarly, we have

$$\|\theta_t^{-1}(\xi) - \theta_t^{-1}(\eta)\|_\infty \leq \frac{8}{7}\|\xi - \eta\|_\infty, \quad t \in [0, T], \xi, \eta \in \mathscr{C}. \tag{5.6}$$

Fix $\lambda \geq \lambda(T)$. Let $\{X^\xi(t)\}_{t\in[-r,T]}$ solve (1.1) with $X_0^\xi = \xi \in \mathscr{C}$. Then, by (3.11), $\{Y^\xi(t) = \theta(t, X^\xi(t))\}_{t\in[-r,T]}$ with $Y_t^\xi = \theta_t(X_t^\xi)$ satisfies

$$
\begin{aligned}
Y^\xi(t) = \mathrm{e}^{At}Y^\xi(0) &+ \int_0^t \mathrm{e}^{A(t-s)}(\lambda - A)u\left(s, \theta^{-1}\left(s, Y^\xi(s)\right)\right)\mathrm{d}s \\
&+ \int_0^t \mathrm{e}^{A(t-s)}\nabla\theta\left(s, \theta^{-1}\left(s, Y^\xi(s)\right)\right)B\left(s, \theta_s^{-1}\left(Y_s^\xi\right)\right)\mathrm{d}s \\
&+ \int_0^t \mathrm{e}^{A(t-s)}\nabla\theta\left(s, \theta^{-1}\left(s, Y^\xi(s)\right)\right)Q\left(s, \theta^{-1}\left(s, Y^\xi(s)\right)\right)\mathrm{d}W(s), \quad t \in [0, T].
\end{aligned} \tag{5.7}
$$

Set

$$\bar{b}(t,x) = (\lambda - A)u\left(t, \theta^{-1}(t,x)\right), \quad t \in [0,T], x \in \mathbb{H}. \tag{5.8}$$

$$\bar{B}(t,\xi) = \nabla\theta\left(t, \theta^{-1}(t, \xi(0))\right)B\left(t, \theta_t^{-1}(\xi)\right), \quad t \in [0,T], \xi \in \mathscr{C}. \tag{5.9}$$

$$\bar{Q}(t,x) = \nabla\theta\left(t, \theta^{-1}(t,x)\right)Q\left(t, \theta^{-1}(t,x)\right), \quad t \in [0,T], x \in \mathbb{H}. \tag{5.10}$$

Then, $\{\bar{X}^\xi(t) := Y^{\theta_0^{-1}(\xi)}(t)\}_{t\in[-r,T]}$ is a mild solution to the equation

$$\mathrm{d}\bar{X}^\xi(t) = \left[A\bar{X}^\xi(t) + \bar{b}\left(t, \bar{X}^\xi(t)\right) + \bar{B}\left(t, \bar{X}_t^\xi\right)\right]\mathrm{d}t + \bar{Q}\left(t, \bar{X}^\xi(t)\right)\mathrm{d}W(t), \quad \bar{X}_0^\xi = \xi. \tag{5.11}$$

Define

$$\bar{P}_t f(\xi) = \mathbb{E}f(\bar{X}_t^\xi), \quad t \in [0,T], f \in \mathscr{B}_b(\mathscr{C}).$$

Then

$$
\begin{aligned}
P_t f(\xi) :&= \mathbb{E}f(X_t^\xi) = \mathbb{E}(f\circ\theta_t^{-1})(Y_t^\xi) = \mathbb{E}(f\circ\theta_t^{-1})(\bar{X}_t^{\theta_0(\xi)}) \\
&= \bar{P}_t(f\circ\theta_t^{-1})(\theta_0(\xi)), \quad \xi \in \mathscr{C}, t \in [0,T], f \in \mathscr{B}_b(\mathscr{C}),
\end{aligned} \tag{5.12}
$$

and we shall turn to investigate the Harnack inequalities for $\bar{P}_t$.

To apply the method of coupling by change of measure, we will use the finite dimension approximation argument. More precisely, let $\{\bar{X}^{n,\xi_n}(t)\}_{t\in[-r,T]}$ solves the finite-dimensional equation on $\mathbb{H}_n := \mathrm{span}\{e_1, \cdots, e_n\}$ $(n \geq 1)$:

$$
\begin{aligned}
\mathrm{d}\bar{X}^{(n,\xi_n)}(t) &= \left[A\bar{X}^{(n,\xi_n)}(t) + \bar{b}^n\left(t, \bar{X}^{(n,\xi_n)}(t)\right) + \bar{B}^n\left(t, \bar{X}_t^{(n,\xi_n)}\right)\right]\mathrm{d}t \\
&\quad + \bar{Q}^n\left(t, \bar{X}^{(n,\xi_n)}(t)\right)\mathrm{d}W(t), \quad \bar{X}_0^{(n,\xi_n)} = \xi_n \in \mathscr{C}(\mathbb{H}_n) := C([-r,0];\mathbb{H}_n),
\end{aligned} \tag{5.13}
$$

where $\bar{b}^n = \pi_n\bar{b}$, $\bar{B}^n = \pi_n\bar{B}$, $\bar{Q}^n = \pi_n\bar{Q}$. Then we shall prove the convergence of the approximation

$$\lim_{n\to\infty}\mathbb{E}\left\|\bar{X}_t^\xi - \bar{X}_t^{(n,\pi_n\xi)}\right\|_\infty^\gamma = 0, \quad t \in [0,T]. \tag{5.14}$$

Lemma 5.1. *Assume* (**a1**), (**a2**$'$), (**a3**$'$) *and* (**a4**$''$), *then there exists a constant* $\tilde{\lambda}(T) \geq \lambda(T)$ *such that for any* $\lambda \geq \tilde{\lambda}(T)$ *and* $\gamma \in (2, \frac{2}{\varepsilon})$, (5.14) *holds.*

Proof. For simplicity, we omit ξ and $\pi_n\xi$ from the subscripts, i.e. we write $(\bar{X}_t, \bar{X}_t^{(n)})$ instead of $(\bar{X}_t^{\xi}, \bar{X}_t^{(n,\pi_n\xi)})$. Fix $\gamma \in (2, \frac{2}{\varepsilon})$. For any $t \in [0,T]$, let

$$\beta_n(t) = \mathbb{E} \sup_{s\in[-r,t]} |\bar{X}(s) - \bar{X}^{(n)}(s)|^\gamma, \quad \beta(t) = \limsup_{n\to\infty} \beta_n(t).$$

By (**a1**), (**a2**$'$) and (**a4**$''$), $\bar{b}^n$, $\bar{B}^n$ and $\bar{Q}^n$ are bounded uniformly in n. Thus Lemma 1.1 and $\gamma \in (2, \frac{2}{\varepsilon})$ imply

$$\mathbb{E} \sup_{t\in[0,T],n\geq 1} (\|\bar{X}_t\|_\infty^\gamma + \|\bar{X}_t^{(n)}\|_\infty^\gamma) < \infty,$$

so that $\beta(t) < \infty$ for any $t \in [0,T]$.

Combining (5.11) with (5.13), it holds that

$$\begin{aligned}
\beta_n(t) &\leq c(\gamma)\|\xi - \pi_n\xi\|_\infty^\gamma \\
&\quad + c(\gamma)\mathbb{E} \sup_{q\in[0,t]} \left| \int_0^q \mathrm{e}^{A(q-s)}[\bar{b}^n(s, \bar{X}(s)) - \bar{b}^n(s, \bar{X}^{(n)}(s))]\mathrm{d}s \right|^\gamma \\
&\quad + c(\gamma)\mathbb{E} \sup_{q\in[0,t]} \left| \int_0^q \mathrm{e}^{A(q-s)}[\bar{b}(s, \bar{X}(s)) - \bar{b}^n(s, \bar{X}(s))]\mathrm{d}s \right|^\gamma \\
&\quad + c(\gamma)\mathbb{E} \sup_{q\in[0,t]} \left| \int_0^q \mathrm{e}^{A(q-s)}[\bar{B}^n(s, \bar{X}_s) - \bar{B}^n(s, \bar{X}_s^{(n)})]\mathrm{d}s \right|^\gamma \\
&\quad + c(\gamma)\mathbb{E} \sup_{q\in[0,t]} \left| \int_0^q \mathrm{e}^{A(q-s)}[\bar{B}(s, \bar{X}_s) - \bar{B}^n(s, \bar{X}_s)]\mathrm{d}s \right|^\gamma \\
&\quad + c(\gamma)\mathbb{E} \sup_{q\in[0,t]} \left| \int_0^q \mathrm{e}^{A(q-s)}[\bar{Q}(s, \bar{X}(s)) - \bar{Q}^n(s, \bar{X}(s))]\mathrm{d}W(s) \right|^\gamma \\
&\quad + c(\gamma)\mathbb{E} \sup_{q\in[0,t]} \left| \int_0^q \mathrm{e}^{A(q-s)}[\bar{Q}^n(s, \bar{X}(s)) - \bar{Q}^n(s, \bar{X}^{(n)}(s))]\mathrm{d}W(s) \right|^\gamma \\
&=: \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 + \Gamma_5 + \Gamma_6 + \Gamma_7, \quad t \in [0,T].
\end{aligned} \tag{5.15}$$

First of all, for any $\xi \in \mathscr{C}$, $n \geq 1$, by the definition of π_n, we have $\|\xi - \pi_n\xi\|_\infty < \|\xi\|_\infty$ and $|(\xi - \pi_n\xi)(s_1) - (\xi - \pi_n\xi)(s_2)| \leq |\xi(s_1) - \xi(s_2)|$ for any s_1, $s_2 \in [-r, 0]$. Since for any $s \in [-r, 0]$, $|(\xi - \pi_n\xi)(s)| \to 0$ as $n \to \infty$, it follows from Arzela-Ascoli theorem that

$$\lim_{n\to\infty} \Gamma_1 = 0. \tag{5.16}$$

Similarly to the estimate of I_1 in Proposition 3.3, there exists a constant $\tilde{\lambda}(T) \geq \lambda(T)$ such that for any $\lambda \geq \tilde{\lambda}(T)$,

$$\Gamma_2 \leq C(\lambda, T) \int_0^t \beta_n(s)\mathrm{d}s + \frac{1}{5}\mathbb{E}\beta_n(t). \tag{5.17}$$

Next, by Hölder inequality and (**a4**$''$), for any $\delta \in (0, 2)$, it holds that

$$\Gamma_3 \leq C\mathrm{e}^{\lambda T} \left\{ \int_0^t \left\| (\lambda - A)^{\frac{1}{2}} e^{-(\lambda - A)s} \right\|^\delta \mathrm{d}s \right\}^{\frac{\gamma}{\delta}}$$

$$\times \mathbb{E}\left\{\int_0^t \left|(\lambda - A)^{\frac{1}{2}}(u-\pi_n u)\left(s, \theta^{-1}\left(s, \bar{X}(s)\right)\right)\right|^{\frac{\delta}{\delta-1}} \mathrm{d}s\right\}^{\frac{\gamma(\delta-1)}{\delta}}$$

$$\leq C(\lambda, T, \delta)\mathbb{E}\left\{\int_0^t \left|(\lambda - A)^{\frac{1}{2}}(u-\pi_n u)\left(s, \theta^{-1}\left(s, \bar{X}(s)\right)\right)\right|^{\frac{\delta}{\delta-1}} \mathrm{d}s\right\}^{\frac{\gamma(\delta-1)}{\delta}}.$$

Combing the definition of π_n and Lemma 3.1 (1) for $a(x) = x^{\frac{1}{2}}$, it follows from dominated convergence theorem that

$$\lim_{n\to\infty} \Gamma_3 = 0. \tag{5.18}$$

Moreover, $(\mathbf{a2}')$, Hölder inequality and $\|B\|_{T,\infty} < \infty$ yield that

$$\Gamma_4 \leq C(\lambda, T)\int_0^t \beta_n(s)\mathrm{d}s. \tag{5.19}$$

Again using $\|B\|_{T,\infty} < \infty$, Hölder inequality and dominated convergence theorem, we obtain

$$\lim_{n\to\infty} \Gamma_5 = 0. \tag{5.20}$$

Furthermore, combining $(\mathbf{a2}')$ with Lemma 1.1 and $\gamma \in (2, \frac{2}{\varepsilon})$, applying dominated convergence theorem, we have

$$\lim_{n\to\infty} \Gamma_6 = 0. \tag{5.21}$$

Finally, combining $(\mathbf{a2}')$ with Lemma 1.1 and $\gamma \in (2, \frac{2}{\varepsilon})$, we have

$$\Gamma_7 \leq C(\lambda, T)\int_0^t \beta_n(s)\mathrm{d}s. \tag{5.22}$$

Combining (5.15)-(5.22), applying dominated convergence theorem, it holds that

$$\beta(t) \leq C\int_0^t \beta(s)\mathrm{d}s, \quad t \in [0, T].$$

Since $\beta(t) < \infty$, Gronwall inequality yields $\beta(t) = 0$, $t \in [0, T]$, which implies (5.14). □

Lemma 5.2. *Assume* $(\mathbf{a1})$, $(\mathbf{a2}')$ *with* $B = 0$, $(\mathbf{a3}'')$ *and* $(\mathbf{a4}'')$. *Then for any* $\lambda \geq \lambda(T)$, *there exists a constant* $C(T) > 0$ *such that*

$$\|\nabla u(t,x) - \nabla u(t,y)\|_{\mathrm{HS}} \leq C(T)|x-y|, \quad x, y \in \mathbb{H}, t \in [0, T]. \tag{5.23}$$

Proof. In order to prove (5.23), by $(\mathbf{a1})$, it suffices to prove

$$\left\|(-A)^{\frac{1-\varepsilon}{2}}[\nabla u(t,x) - \nabla u(t,y)]\right\| \leq C(T)|x-y|, \quad x, y \in \mathbb{H}, t \in [0, T]. \tag{5.24}$$

In fact, if (5.24) holds, then

$$\begin{aligned}\|\nabla u(t,x)-\nabla u(t,y)\|_{\mathrm{HS}}^2 &= \left\|(-A)^{\frac{\varepsilon-1}{2}}(-A)^{\frac{1-\varepsilon}{2}}[\nabla u(t,x)-\nabla u(t,y)]\right\|_{\mathrm{HS}}^2 \\ &\le C(T)\left\|(-A)^{\frac{\varepsilon-1}{2}}\right\|_{\mathrm{HS}}^2|x-y|^2, \quad x,y\in\mathbb{H}, t\in[0,T].\end{aligned}$$

Define

$$(R_{s,t}^{\lambda}f)(x)=\int_s^t \mathrm{e}^{-(q-s)\lambda}[P_{s,q}^0 f(q,\cdot)](x)\mathrm{d}q, \quad x\in\mathbb{H}, \lambda\ge 0, t\ge s\ge 0, f\in\mathscr{B}_b([0,\infty)\times\mathbb{H};\mathbb{H}).$$

Firstly, by (2.5) and (5.1), we have

$$\|(-A)^{\frac{1}{2}}(\nabla_b u+b)\|_{T,\infty}<\infty \tag{5.25}$$

for any $\lambda\ge\lambda(T)$.

Secondly, due to (3.4), (3.6) and (5.25), [11, Lemma 2.2 (1)] implies that for any $\lambda\ge\lambda(T)$,

$$\begin{aligned}&\left\|(-A)^{\frac{1}{2}}[\nabla u(t,x)-\nabla u(t,y)]\right\| \\ =&\left\|\nabla\left(R_{t,T}^{\lambda}\left((-A)^{\frac{1}{2}}(\nabla_b u+b)\right)\right)(x)-\nabla\left(R_{t,T}^{\lambda}\left((-A)^{\frac{1}{2}}(\nabla_b u+b)\right)\right)(y)\right\| \\ \le& C|x-y|\log\left(\mathrm{e}+\frac{1}{|x-y|}\right) \quad x,y\in\mathbb{H}, t\in[0,T]\end{aligned} \tag{5.26}$$

holds for some constant $C>0$. Combining this with (5.26) and (2.4), for any $\lambda\ge\lambda(T)$, we obtain that

$$\left|(-A)^{\frac{1-\varepsilon}{2}}(\nabla_b u+b)(t,x)-(-A)^{\frac{1-\varepsilon}{2}}(\nabla_b u+b)(t,y)\right|\le\tilde\phi(|x-y|), \quad t\in[0,T], x,y\in\mathbb{H}, \tag{5.27}$$

where $\tilde\phi(s)=c\sqrt{\phi^2(s)+s}$ with a constant $c>0$.

Finally, by (3.4), (3.6), (5.27), and [11, Lemma 2.2 (3)], we conclude that, for any $\lambda\ge\lambda(T)$,

$$\begin{aligned}&\left\|(-A)^{\frac{1-\varepsilon}{2}}[\nabla u(t,x)-\nabla u(t,y)]\right\| \\ =&\left\|\nabla\left(R_{t,T}^{\lambda}\left((-A)^{\frac{1-\varepsilon}{2}}(\nabla_b u+b)\right)\right)(x)-\nabla\left(R_{t,T}^{\lambda}\left((-A)^{\frac{1-\varepsilon}{2}}(\nabla_b u+b)\right)\right)(y)\right\| \\ \le& C(T)|x-y|, \quad x,y\in\mathbb{H}, t\in[0,T].\end{aligned} \tag{5.28}$$

for a constant $C(T)>0$. Thus (5.24) holds, and we complete the proof. □

Lemma 5.3. *Assume* (**a1**), (**a2**$'$), (**a3**$''$) *and* (**a4**$''$). *If in addition*

$$\|Q(t,x)-Q(t,y)\|_{\mathrm{HS}}^2\le C(T)|x-y|^2, \quad t\in[0,T], x,y\in\mathbb{H}, \tag{5.29}$$

where $C(T)$ is a positive constant. Then for any $\lambda \geq \lambda(T)$, there exists $K_1 \geq 0$, $K_2 \geq 0$, $K_3 > 0$ and $K_4 \geq 0$ (K_1, K_2, K_3, K_4 only depend on T) such that

$$\left|(\bar{Q}^*(\bar{Q}\bar{Q}^*)^{-1})(t,\eta(0))\{\bar{B}(t,\xi) - \bar{B}(t,\eta)\}\right|_{\bar{\mathbb{H}}} \leq K_1\|\xi - \eta\|_\infty; \tag{5.30}$$

$$\left\|\bar{Q}(t,x) - \bar{Q}(t,y)\right\| \leq K_2(1 \wedge |(x-y|); \tag{5.31}$$

$$\left\|(\bar{Q}^*(\bar{Q}\bar{Q}^*)^{-1})(t,x)\right\| \leq K_3 \tag{5.32}$$

hold for $t \in [0,T]$, $\xi, \eta \in \mathscr{C}$, and $x, y \in \mathbb{H}$. Moreover, for any $t \in [0,T]$, $x, y \in \mathbb{H}_\infty := \cup_{n\geq 1}\mathbb{H}_n$, it holds that

$$\left\|\bar{Q}(t,x) - \bar{Q}(t,y)\right\|_{HS}^2 + 2\left\langle x-y, Ax - Ay + \bar{b}(t,x) - \bar{b}(t,y)\right\rangle \leq K_4|x-y|^2. \tag{5.33}$$

Proof. Fix $\lambda \geq \lambda(T)$.

(a) Since $\nabla\theta(t,\cdot) = I + \nabla u(t,\cdot)$, $t \in [0,T]$, (5.1) yields that for any $\lambda \geq \lambda(T)$ and $(t,x) \in [0,T] \times \mathbb{H}$, $\nabla\theta(t,x)$, $(\nabla\theta(t,x))^* \in \mathscr{L}(\mathbb{H},\mathbb{H})$ are invertible. Then from (5.10), we obtain

$$\left(\bar{Q}^*(\bar{Q}\bar{Q}^*)^{-1}\right)(t,x) = \left(Q^*(QQ^*)^{-1}\right)\left(t,\theta^{-1}(t,x)\right)\left[\nabla\theta\left(t,\theta^{-1}(t,x)\right)\right]^{-1}. \tag{5.34}$$

Combining this with $(\mathbf{a2}')$ (i'), (5.32) holds with $K_3 = \frac{8}{7}(C_Q(T))^2$.

(b) Due to (a), in order to prove (5.30), we only need to estimate $\left|\bar{B}(t,\xi) - \bar{B}(t,\eta)\right|$. From (5.9), $(\mathbf{a2}')$ (ii'), (5.1), (5.2) and (5.6), we have

$$\begin{aligned}
&|\bar{B}(t,\xi) - \bar{B}(t,\eta)| \\
=&\left|\nabla\theta(t,\theta^{-1}(t,\xi(0)))B(t,\theta_t^{-1}(\xi)) - \nabla\theta(t,\theta^{-1}(t,\eta(0)))B(t,\theta_t^{-1}(\eta))\right| \\
\leq&\left|\nabla\theta(t,\theta^{-1}(t,\xi(0)))B(t,\theta_t^{-1}(\xi)) - \nabla\theta(t,\theta^{-1}(t,\xi(0)))B(t,\theta_t^{-1}(\eta))\right| \\
+&\left|\nabla\theta(t,\theta^{-1}(t,\xi(0)))B(t,\theta_t^{-1}(\eta)) - \nabla\theta(t,\theta^{-1}(t,\eta(0)))B(t,\theta_t^{-1}(\eta))\right| \\
\leq&\frac{8}{7}\sup_{(t,z)\in[0,T]\times\mathbb{H}}\|\nabla\theta(t,z)\|C_B'(T)\|\xi-\eta\|_\infty \\
+&\sup_{(t,z)\in[0,T]\times\mathbb{H}}\|\nabla^2\theta(t,z)\|\sup_{(t,z)\in[0,T]\times\mathbb{H}}\|\nabla\theta^{-1}(t,z)\|\|B\|_{T,\infty}|\xi(0)-\eta(0)| \\
\leq&K\|\xi-\eta\|_\infty, \quad K > 0.
\end{aligned} \tag{5.35}$$

Combining (5.35) with (5.32), we prove (5.30).

(c) Similarly, from(5.10), again using $(\mathbf{a2}')$ (i'), (5.1), (5.2), we arrive at

$$\begin{aligned}
&\left\|\bar{Q}(t,x) - \bar{Q}(t,y)\right\| \\
=&\left\|\nabla\theta(t,\theta^{-1}(t,x))Q(t,\theta^{-1}(t,x)) - \nabla\theta(t,\theta^{-1}(t,y))Q(t,\theta^{-1}(t,y))\right\| \\
\leq&\left\|\nabla\theta(t,\theta^{-1}(t,x))Q(t,\theta^{-1}(t,x)) - \nabla\theta(t,\theta^{-1}(t,x))Q(t,\theta^{-1}(t,y))\right\| \\
+&\left\|\nabla\theta(t,\theta^{-1}(t,x))Q(t,\theta^{-1}(t,y)) - \nabla\theta(t,\theta^{-1}(t,y))Q(t,\theta^{-1}(t,y))\right\| \\
\leq&\sup_{(t,z)\in[0,T]\times\mathbb{H}}\|\nabla\theta(t,z)\|\|\nabla Q\|_{T,\infty}\sup_{(t,z)\in[0,T]\times\mathbb{H}}\|\nabla\theta^{-1}(t,z)\||x-y| \\
+&\sup_{(t,z)\in[0,T]\times\mathbb{H}}\|\nabla^2\theta(t,z)\|\|Q\|_{T,\infty}\sup_{(t,z)\in[0,T]\times\mathbb{H}}\|\nabla\theta^{-1}(t,z)\||x-y| \\
\leq&K'|x-y|, \quad K' > 0,
\end{aligned} \tag{5.36}$$

and

$$\|\bar{Q}(t,x)-\bar{Q}(t,y)\| \leq 2 \sup_{(t,z)\in[0,T]\times\mathbb{H}} \|\nabla\theta(t,z)\| \|Q\|_{T,\infty} \leq K'', \quad K''>0. \tag{5.37}$$

Then (5.36) and (5.37) yield (5.31).

(d) Applying ($\mathbf{a2}'$) (i'), (5.1), (5.2), (5.29) and Lemma 5.2, we obtain

$$\begin{aligned}
&\left\|\bar{Q}(t,x)-\bar{Q}(t,y)\right\|_{\mathrm{HS}} \\
=&\left\|\nabla\theta(t,\theta^{-1}(t,x))Q(t,\theta^{-1}(t,x))-\nabla\theta(t,\theta^{-1}(t,y))Q(t,\theta^{-1}(t,y))\right\|_{\mathrm{HS}} \\
\leq&\left\|\nabla\theta(t,\theta^{-1}(t,x))Q(t,\theta^{-1}(t,x))-\nabla\theta(t,\theta^{-1}(t,x))Q(t,\theta^{-1}(t,y))\right\|_{\mathrm{HS}} \\
&+\left\|\nabla\theta(t,\theta^{-1}(t,x))Q(t,\theta^{-1}(t,y))-\nabla\theta(t,\theta^{-1}(t,y))Q(t,\theta^{-1}(t,y))\right\|_{\mathrm{HS}} \\
\leq&\sup_{(t,z)\in[0,T]\times\mathbb{H}}\|\nabla\theta(t,z)\|\left\|Q(t,\theta^{-1}(t,x))-Q(t,\theta^{-1}(t,y)\right\|_{\mathrm{HS}} \\
&+\|Q\|_{T,\infty}\left\|\nabla u(t,\theta^{-1}(t,x))-\nabla u(t,\theta^{-1}(t,y))\right\|_{\mathrm{HS}} \\
\leq&C(T)\left[\sup_{(t,z)\in[0,T]\times\mathbb{H}}\|\nabla\theta(t,z)\|+\|Q\|_{T,\infty}\right]\sup_{(t,z)\in[0,T]\times\mathbb{H}}\left\|\nabla\theta^{-1}(t,z)\right\||x-y| \\
\leq&K_0|x-y|, \quad K_0>0.
\end{aligned} \tag{5.38}$$

Moreover, for any $x,y\in\mathbb{H}_\infty$, by (5.8), (5.1) and (5.2), we obtain

$$\begin{aligned}
&\langle A(x-y),x-y\rangle+\left\langle(-A)[u(t,\theta^{-1}(t,x))-u(t,\theta^{-1}(t,y))],x-y\right\rangle \\
&+\left\langle\lambda[u(t,\theta^{-1}(t,x))-u(t,\theta^{-1}(t,y))],x-y\right\rangle \\
=&-\left|(-A)^{\frac12}(x-y)\right|^2+\left\langle(-A)^{\frac12}\left[u\left(t,\theta^{-1}(t,x)\right)-u\left(t,\theta^{-1}(t,y)\right)\right],(-A)^{\frac12}(x-y)\right\rangle \\
&+\left\langle\lambda[u(t,\theta^{-1}(t,x))-u(t,\theta^{-1}(t,y))],x-y\right\rangle \\
\leq&-|(-A)^{\frac12}(x-y)|^2+c(\lambda)|x-y|^2+\frac12\left|(-A)^{\frac12}(x-y)\right|^2 \\
\leq&c(\lambda,\lambda_1)|x-y|^2
\end{aligned} \tag{5.39}$$

for a constant $c(\lambda)\geq 0$ depending on λ,λ_1. Combining (5.38) with (5.39), we obtain (5.33). $\square$

Proof of Theorem 2.2. In what follows, we only prove Theorem 2.2 (2), for (1) is completely the same with (2).

For any $n\geq 1$, let $\bar{X}_t^{(n,\xi_n)}$ be the solution to (5.13) with $\bar{X}_0^{(n,\xi_n)}=\xi_n\in\mathscr{C}(\mathbb{H}_n)$, and set $\bar{P}_T^n f(\xi_n)=\mathbb{E}f(\bar{X}_T^{(n,\xi_n)})$ for any $f\in\mathscr{B}_b(\mathscr{C}(\mathbb{H}_n))$. Combining Lemma 5.3 and Lemma 6.1 (2), for any $\lambda\geq\tilde{\lambda}(T)$ ($\tilde{\lambda}(T)$ is introduced in Lemma 5.1), we obtain Harnack inequalities with power for $\bar{P}_T^n$, i.e. for every $p>(1+K_2K_3)^2$,

$$\bar{P}_T^n f(\pi_n\eta)\leq\left(\bar{P}_T^n f^p(\pi_n\xi)\right)^{\frac1p}\exp\tilde{\Phi}_p(T;\pi_n\xi,\pi_n\eta),\ \ \xi,\eta\in\mathscr{C},\ \ f\in C_b^1(\mathscr{C}(\mathbb{H}_n)), \tag{5.40}$$

where

$$\tilde{\Phi}_p(T;\xi,\eta) = \tilde{C}(p)\left\{1 + \frac{|\xi(0)-\eta(0)|^2}{T-r} + \|\xi-\eta\|_\infty^2\right\}, \quad \xi,\eta \in \mathscr{C}(\mathbb{H}_n)$$

and $\tilde{C} : ((1+K_2K_3)^2,\infty) \to (0,\infty)$ is a decreasing function independent of n and f.

Letting $n\to\infty$ in (5.40), by Lemma 5.1 and $C_b^1(\mathscr{C}) \subset \bigcap_{n\geq 1} C_b^1(\mathscr{C}(\mathbb{H}_n))$, we have

$$\bar{P}_T f(\eta) \leq \left(\bar{P}_T f^p(\xi)\right)^{\frac{1}{p}} \exp \tilde{\Phi}_p(T;\xi,\eta), \quad \xi,\eta\in\mathscr{C},\ f\in C_b^1(\mathscr{C}). \tag{5.41}$$

By an approximation method or monotone class theorem, (5.41) holds for any non-negative function $f \in \mathscr{B}_b(\mathscr{C})$. Thus from (5.12), we obtain that for every $p > (1+K_2K_3)^2$ and non-negative function $f\in\mathscr{B}_b(\mathscr{C})$,

$$\begin{aligned} P_T f(\eta) &= \bar{P}_T(f\circ\theta_T^{-1})(\theta_0(\eta)) \\ &\leq \{\bar{P}_T(f^p\circ\theta_T^{-1})(\theta_0(\xi))\}^{\frac{1}{p}} \exp\tilde{\Phi}_p(T;\theta_0(\xi),\theta_0(\eta)) \\ &= \{P_T f^p(\xi)\}^{\frac{1}{p}} \exp\tilde{\Phi}_p(T;\theta_0(\xi),\theta_0(\eta)) \quad \xi,\eta\in\mathscr{C} \end{aligned} \tag{5.42}$$

Taking into account of (5.5), we have

$$\tilde{\Phi}_p(T;\theta_0(\xi),\theta_0(\eta)) \leq \frac{81}{64}\tilde{C}(p)\left\{1+\frac{|\xi(0)-\eta(0)|^2}{T-r} + \|\xi-\eta\|_\infty^2\right\} =: \Psi_p(T;\xi,\eta).$$

Taking $K = K_2K_3$, $C = \frac{81}{64}\tilde{C}$, (2.8) follows from (5.42) and the definition of Ψ_p. Similarly, we can obtain Theorem 2.2 (1). Thus, we finish the proof. □

6 Appendix

In this section, we give [13, Theorem 4.3.1 and Theorem 4.3.2] in detail as follows.

Fix a constant $r_0 > 0$. Let $\mathscr{C}(\mathbb{R}^d) := C([-r_0,0],\mathbb{R}^d)$. For simplicity, we denote $\mathscr{C}^d = \mathscr{C}(\mathbb{R}^d)$. Consider the functional SDE on $\mathbb{R}^d$:

$$\mathrm{d}Z(t) = \{a(t,Z(t)) + c(t,Z_t)\}\mathrm{d}t + \sigma(t,Z(t))\mathrm{d}w(t), \tag{6.1}$$

where w is a standard m-dimensional Brownian motion, $a:[0,\infty)\times\mathbb{R}^d\to\mathbb{R}^d$, $c:[0,\infty)\times\mathscr{C}^d\to\mathbb{R}^d$, and $\sigma:[0,\infty)\times\mathbb{R}^d\to\mathbb{R}^d\otimes\mathbb{R}^m$ are measurable and locally bounded (i.e. bounded on bounded sets).

To establish the Harnack inequality, we shall need the following assumption:

(A) For any $T > r_0$, there exist constants $K_1, K_2 \geq 0$, $K_3 > 0$ and $K_4\in\mathbb{R}$ (K_1, K_2, K_3, K_4 only depend on T) such that

$$\left|(\sigma^*(\sigma\sigma^*)^{-1})(t,\eta(0))\{c(t,\xi)-c(t,\eta)\}\right| \leq K_1\|\xi-\eta\|_\infty; \tag{6.2}$$

(6.3) $$\|\sigma(t,x)-\sigma(t,y)\| \le K_2(1\wedge|(x-y)|;$$

(6.4) $$\left\|(\sigma^*(\sigma\sigma^*)^{-1})(t,x)\right\| \le K_3;$$

(6.5) $$\|\sigma(t,x)-\sigma(t,y)\|_{HS}^2 + 2\langle x-y, a(t,x)-a(t,y)\rangle \le K_4|x-y|^2$$

hold for $t\in[0,T]$, $\xi,\eta\in\mathscr{C}^d$, and $x,y\in\mathbb{R}^d$.

(A) implies [13, **(A4.1)**], so by [13, Corollary 4.1.2], for any $\xi\in\mathscr{C}^d$, (6.1) has a unique strong solution Z_t^ξ with $Z_0=\xi$. Let P_T be the associated Markov semigroup defined as

$$P_Tf(\xi)=\mathbb{E}f(Z_T^\xi),\quad f\in\mathscr{B}_b(\mathscr{C}^d),\xi\in\mathscr{C}^d.$$

Lemma 6.1. *Assume* **(A)**. *Then for any $T>r_0$, every positive function $f\in\mathscr{B}_b(\mathscr{C})$,*

(1) the log-*Harnack inequality holds, i.e.*

(6.6) $$P_T\log f(\eta)\le \log P_Tf(\xi)+H(T,\xi,\eta),\ \ \xi,\eta\in\mathscr{C}^d$$

with

$$H(T,\xi,\eta)=C\left(\frac{|\xi(0)-\eta(0)|^2}{T-r}+\|\xi-\eta\|_\infty^2\right)$$

for some dimension-free constant $C>0$.

(2) For any $p>(1+K_2K_3)^2$, the Harnack inequality with power

(6.7) $$P_Tf(\eta)\le (P_Tf^p(\xi))^{\frac{1}{p}}\exp\Psi_p(T;\xi,\eta),\quad \xi,\eta\in\mathscr{C}^d$$

holds, where

$$\Psi_p(T;\xi,\eta)=C(p)\left\{1+\frac{|\xi(0)-\eta(0)|^2}{T-r}+\|\xi-\eta\|_\infty^2\right\}$$

for a dimension-free decreasing function $C:((1+K_2K_3)^2,\infty)\to(0,\infty)$.

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