Closability of Quadratic Forms Associated to Invariant Probability Measures of SPDEs *

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July 12, 2016

Abstract

By using the integration by parts formula of a Markov operator, the closability of quadratic forms associated to the corresponding invariant probability measure is proved. The general result is applied to the study of semilinear SPDEs, infinite-dimensional stochastic Hamiltonian systems, and semilinear SPDEs with delay.

AMS subject Classification: 60H10, 60H15, 60J75.
Keywords: Closability, Invariant probability measure, semi-linear SPDEs, integration by parts formula.

1 Introduction

Let $\mathbb{B}$ be a separable Banach space and μ a reference probability measure on $\mathbb{B}$. For any $k \in \mathbb{B}$, let ∂_k denote the directional derivative along k. According to [8], the form

$$\mathscr{E}_k(f,g) := \mu((\partial_k f)(\partial_k g)) := \int_{\mathbb{B}} (\partial_k f)(\partial_k g) \mathrm{d}\mu, \quad f, g \in C_b^2(\mathbb{B}),$$

is closable on $L^2(\mu)$ if $\rho_s := \frac{\mathrm{d}\mu(sk+\cdot)}{\mathrm{d}\mu}$ exists for any s such that $s \mapsto \rho_s$ is lower semi-continuous μ-a.e.; i.e. for some fixed μ-versions of $\rho_s, s \in \mathbb{R}$,

$$\liminf_{s \to t} \rho_s(x) \geq \rho_t(x), \quad \mu\text{−a.e. } x, \ t \in \mathbb{R}.$$

*Supported in part by NNSFC(11131003, 11431014).

In this paper, we aim to investigate the closability of $\mathscr{E}_k$ for μ being the invariant probability measure of a (degenerate/delay) semilinear SPDE. Since in this case the above lower semi-continuity condition is hard to check, in this paper we make use of the integration by parts formula for the associated Markov semigroup in the line of [10] using coupling arguments.

The main motivation to study the closability of $\mathscr{E}_k$ (respectively of ∂_k) on $L^2(\mu)$ is that it leads to a concept of weak differentiablity on $\mathbb{B}$ with respect to μ and one can define the corresponding Sobolev space on $\mathbb{B}$ in $L^p(\mu)$, $p \in [1, \infty)$. In particular, one can analyze the generator of a Markov process (e.g. arising from a solution of an SPDE) on these Sobolev spaces when μ is its (infinitesimally) invariant measure, see e.g. [7] for details.

Before considering specific models of SPDEs, we first introduce a general result on the closability of $\mathscr{E}_k$ using the integration by parts formula. To this end, we consider a family of $\mathbb{B}$-valued random variables $\{X^x\}_{x\in\mathbb{B}}$ measurable in x, and let $P(x, \mathrm{d}y)$ be the distribution of X^x for $x \in \mathscr{B}$. Then we have the following Markov operator on $\mathscr{B}_b(\mathbb{B})$:

$$Pf(x) := \int_{\mathbb{B}} f(y)P(x, \mathrm{d}y) = \mathbb{E}f(X^x), \quad x \in \mathbb{B}, f \in \mathscr{B}_b(\mathbb{B}).$$

A probability measure μ on $\mathbb{B}$ is called an invariant measure of P if $\mu(Pf) = \mu(f)$ for all $f \in \mathscr{B}_b(\mathbb{B})$.

Proposition 1.1. *Assume that the Markov operator P has an invariant probability measure μ. Let $k \in \mathbb{B}$. If there exists a family of real random variables $\{M_x\}_{x\in\mathbb{B}}$ measurable in x such that $M_\cdot \in L^2(\mathbb{P} \times \mu)$, i.e.*

$$(\mathbb{P} \times \mu)(|M_\cdot|^2) := \int_{\mathbb{B}} \mathbb{E}|M_x|^2 \mu(\mathrm{d}x) < \infty; \tag{1.1}$$

and the integration by parts formula

$$P(\partial_k f)(x) = \mathbb{E}\big\{f(X^x)M_x\big\}, \quad f \in C_b^2(\mathbb{B}), \mu\text{-a.e. } x \in \mathbb{B} \tag{1.2}$$

holds, then $(\mathscr{E}_k, C_b^2(\mathbb{B}))$ is closable in $L^2(\mu)$.

Proof. Since μ is P-invariant, by (1.1) and (1.2) we have

$$\mu(\partial_k f) = \int_{\mathbb{B}} P(\partial_k f)(x)\mu(\mathrm{d}x) = (\mathbb{P} \times \mu)\big(f(X^\cdot)M_\cdot\big), \quad f \in C_b^2(\mathbb{B}).$$

So,

$$\begin{aligned}\mathscr{E}_k(f, g) &:= \mu\big((\partial_k f)(\partial_k g)\big) = \mu(\partial_k\{f\partial_k g\}) - \mu(f\partial_k^2 g)\\ &= (\mathbb{P} \times \mu)\big(\{f\partial_k g\}(X^\cdot)M_\cdot\big) - \mu(f\partial_k^2 g), \quad f, g \in C_b^2(\mathbb{B}).\end{aligned}$$

It is standard that this implies the closability of the form $(\mathscr{E}_k, C_b^2(\mathbb{B}))$ in $L^2(\mu)$. Indeed, for $\{f_n\}_{n\geq1} \subset C_b^2(\mathbb{B})$ with $f_n \to 0$ and $\partial_k f_n \to Z$ in $L^2(\mu)$, it suffices to prove that $Z = 0$.

Since $\mu(f_n^2) \to 0$ and $(\mathbb{P}\times\mu)\big(|f_n\partial_k g|^2(X^\cdot)|\big) = \mu(|f_n\partial_k g|^2)$ as μ is P-invariant, the above formula yields

$$\begin{aligned}
|\mu(Zg)| &= \lim_{n\to\infty} |\mu(g\partial_k f_n)| \\
&= \lim_{n\to\infty} \big|(\mathbb{P}\times\mu)\big(\{f_n\partial_k g\}(X^\cdot)M_\cdot\big) - \mu(f_n\partial_k^2 g)\big| \\
&\le \liminf_{n\to\infty} \Big\{\sqrt{(\mathbb{P}\times\mu)\big(|f_n\partial_k g|^2(X^\cdot)\big)\cdot(\mathbb{P}\times\mu)(|M_\cdot|^2)} + \sqrt{\mu(f_n^2)\mu(|\partial_k^2 g|^2)}\Big\} \\
&\le \liminf_{n\to\infty} \Big\{\|\partial_k g\|_\infty\sqrt{\mu(f_n^2)\cdot(\mathbb{P}\times\mu)(|M_\cdot|^2)} + \|\partial_k^2 g\|_\infty\sqrt{\mu(f_n^2)}\Big\} = 0, \quad g\in C_b^2(\mathbb{B}).
\end{aligned}$$

Therefore, $Z = 0$. □

Remark 1.1. The integration by parts formula (1.2) implies the estimate

$$|\mu(\partial_k f)|^2 \le (\mathbb{P}\times\mu)(|M_\cdot|^2)\mu(f^2). \tag{1.3}$$

As the main result in [3] (Theorem 10), this type of estimate, called Fomin derivative estimate of the invariant measure, was derived as the main result for the following semi-linear SPDE on $\mathbb{H} := L^2(\mathscr{O})$ for any bounded open domain $\mathscr{O}\subset\mathbb{R}^n$ for $1\le n\le 3$:

$$\mathrm{d}X(t) = [\Delta X(t) + p(X(t))]\mathrm{d}t + (-\Delta)^{-\gamma/2}\mathrm{d}W(t),$$

where Δ is the Dirichlet Laplacian on $\mathscr{O}$, p is a decreasing polynomial with odd degree, $\gamma\in(\frac{n}{2}-1,1)$, and $W_(t)$ is the cylindrical Brownian motion on $\mathbb{H}$. The main point of the study is to apply the Bismut-Elworthy-Li derivative formula and the following formula for the semigroup P_t^α for the Yoshida approximation of this SPDE (see [3, Proposition 7]):

$$P_t^\alpha\partial_k f = \partial_k P_t^\alpha - \int_0^t P_{t-s}(\partial_{Ak+\partial_k p}P_s^\alpha f)\mathrm{d}s.$$

In this paper we will establish the integration by parts formula of type (1.2) for the associated semigroup which implies the estimate (1.3). Our results apply to a general framework where the operator $(-\Delta)^{-\gamma/2}$ is replaced by a suitable linear operator σ (see Section 2) which can be degenerate (see Section 3), and the drift $p(x)$ is replaced by a general map b which may include a time delay (see Section 4). However, the price we have to pay for the generalization is that the drift b should be regular enough.

2 Semilinear SPDEs

Let $(\mathbb{H},\langle\cdot,\cdot\rangle,|\cdot|)$ be a real separable Hilbert space, and $(W(t))_{t\ge0}$ a cylindrical Wiener process on $\mathbb{H}$ with respect to a complete probability space $(\Omega,\mathscr{F},\mathbb{P})$ with the natural filtration $\{\mathscr{F}_t\}_{t\ge0}$. Let $\mathscr{L}(\mathbb{H})$ and $\mathscr{L}_{HS}(\mathbb{H})$ be the spaces of all linear bounded operators and Hilbert-Schmidt operators on H respectively. Let $\|\cdot\|$ and $\|\cdot\|_{HS}$ denote the operator norm and the Hilbert-Schmidt norm respectively.

Consider the following semilinear SPDE

$$\mathrm{d}X(t) = \{AX(t) + b(X(t))\}\mathrm{d}t + \sigma\mathrm{d}W(t), \tag{2.1}$$

where

(A1) $(A, \mathscr{D}(A))$ is a negatively definite self-adjoint linear operator on $\mathbb{H}$ with compact resolvent.

(A2) Let $\mathbb{H}^{-2}$ be the completion of $\mathbb{H}$ under the inner product

$$\langle x, y\rangle_{\mathbb{H}^{-2}} := \langle A^{-1}x, A^{-1}y\rangle.$$

Let $b : \mathbb{H} \to \mathbb{H}^{-2}$ be such that

$$\int_0^1 |\mathrm{e}^{tA}b(0)|\mathrm{d}t < \infty, \quad |\mathrm{e}^{tA}(b(x) - b(y))| \le \gamma(t)|x - y|, \quad x, y \in \mathbb{H}, t > 0$$

holds for some positive $\gamma \in C((0,\infty))$ with $\int_0^1 \gamma(t)\mathrm{d}t < \infty$.

(A3) $\sigma \in \mathscr{L}(\mathbb{H})$ with $\mathrm{Ker}(\sigma\sigma^*) = \{0\}$ and $\int_0^1 \|\mathrm{e}^{tA}\sigma\|_{HS}^2\mathrm{d}t < \infty$.

According to **(A1)**, the spectrum of A is discrete with negative eigenvalues. Let $0 < \lambda_0 \le \cdots \le \lambda_n \cdots$ be all eigenvalues of $-A$ counting the multiplicities, and let $\{e_i\}_{i\ge1}$ be the corresponding unit eigen-basis. Denote $\mathbb{H}_{A,n} = \mathrm{span}\{e_i : 1 \le i \le n\}, n \ge 1$. Then $\mathbb{H}_A := \cup_{n=1}^{\infty}\mathbb{H}_{A,n}$ is a dense subspace of $\mathbb{H}$. In assumption **(A2)** we have used the fact that for any $t > 0$, the operator e^{tA} extends uniquely to a bounded linear operator from $\mathbb{H}^{-2}$ to $\mathbb{H}$, which is again denoted by e^{tA}.

Due to assumptions **(A1)**, **(A2)** and **(A3)**, by a standard iteration argument we conclude that for any $x \in \mathbb{H}$ the equation (2.1) has a unique mild solution $X^x(t)$ such that $X^x(0) = x$ (see [4]). Let

$$P_tf(x) = \mathbb{E}f(X^x(t)), \quad f \in \mathscr{B}_b(\mathbb{H}), x \in \mathbb{H}$$

be the associated Markov semigroup.

Let

$$\|x\|_\sigma = \inf\left\{|y| : \ y \in \mathbb{H}, \sqrt{\sigma\sigma^*}y = x\right\}, \quad x \in \mathbb{H},$$

where $\inf\emptyset := \infty$ by convention. Then $\|x\|_\sigma < \infty$ if and only if $x \in \mathrm{Im}(\sigma)$.

Theorem 2.1. *Assume that P_t has an invariant probability measure μ and $\mathbb{H}_A \subset \mathrm{Im}(\sqrt{\sigma\sigma^*})$.*

(1) *For any $k \in \mathbb{H}_A$ such that*

$$\sup_{x\in\mathbb{H}} \|\partial_k b(x)\|_\sigma := \sup_{x\in\mathbb{H}} \limsup_{\varepsilon\downarrow0} \frac{\|b(x + \varepsilon k) - b(x)\|_\sigma}{\varepsilon} < \infty, \tag{2.2}$$

the form $(\mathscr{E}_k, C_b^2(\mathbb{H}))$ is closable in $L^2(\mu)$.

(2) *If $\sigma\sigma^*$ is invertible and $b : \mathbb{H} \to \mathbb{H}$ is Lipschitz continuous, then $(\mathscr{E}_k, C_b^2(\mathbb{H}))$ is closable in $L^2(\mu)$ for any $k \in \mathscr{D}(A)$.*

Proof. Since $\mathrm{d}\tilde{W}_t := (\sigma\sigma^*)^{-1/2}\sigma\mathrm{d}W_t$ is also a cylindrical Brownian motion and $\sigma\mathrm{d}W_t = \sqrt{\sigma\sigma^*}\mathrm{d}\tilde{W}_t$, we may and do assume that σ is non-negatively definite.

(1) Without loss of generality, we may and do assume that k is an eigenvector of A, i.e. $Ak = \lambda k$ for some $\lambda \in \mathbb{R}$. We first prove the case where b is Fréchet differentiable along the direction k. By $Ak = \lambda k$ we have

$$k(t) := \int_0^t \mathrm{e}^{sA} k \mathrm{d}s = \frac{\mathrm{e}^{\lambda t} - 1}{\lambda} k, \quad t \geq 0,$$

where for $\lambda = 0$ we set $\frac{\mathrm{e}^{\lambda t}-1}{\lambda} = t$. Due to $\|k\|_\sigma < \infty$ and (2.2), the proof of [10, Theorem 5.1(1)] leads to the integration by parts formula

$$P_T(\partial_k f)(x) = \mathbb{E}\big\{f(X^x(T))M_{x,T}\big\}, \quad f \in C_b^1(\mathbb{H}), x \in \mathbb{H}, T > 0, \tag{2.3}$$

where

$$M_{x,T} := \frac{\lambda}{\mathrm{e}^{\lambda T} - 1} \int_0^T \Big\langle \sigma^{-1}\Big(k - \frac{\mathrm{e}^{\lambda t} - 1}{\lambda}(\partial_k b)(X^x(t))\Big), \mathrm{d}W(t)\Big\rangle.$$

Since (2.2) implies

$$\int_{\mathbb{B}} \mathbb{E}|M_{x,T}|^2 \, \mu(\mathrm{d}x) \leq \frac{\lambda^2}{(\mathrm{e}^{\lambda T} - 1)^2} \int_0^T \Big\|\sigma^{-1}\Big(k - \frac{\mathrm{e}^{\lambda t} - 1}{\lambda}\partial_k b\Big)\Big\|_\infty^2 \mathrm{d}t < \infty, \tag{2.4}$$

$(\mathscr{E}_k, C_b^2(\mathbb{H}))$ is closable in $L^2(\mu)$ according to Proposition 1.1.

In general, for any $\varepsilon > 0$ let

$$b_\varepsilon(x) = \frac{1}{\sqrt{2\pi\varepsilon}} \int_{\mathbb{R}} b(x + rk) \exp\Big[-\frac{r^2}{2\varepsilon}\Big] \mathrm{d}r, \quad x \in \mathbb{H}.$$

Then for any $\varepsilon > 0$, b_ε is Fréchet differentiable along k and (2.2) holds uniformly in ε with b_ε replacing b. Let P_t^ε be the semigroup for the solution $X_\varepsilon(t)$ associated to equation (2.1) with b_ε replacing b. By simple calculations we have:

(i) $\lim_{\varepsilon \downarrow 0} \mathbb{E}|X_\varepsilon^x(t) - X^x(t)|^2 = 0, \; t \geq 0, x \in \mathbb{H}$.

(ii) For any $T > 0$, the family

$$M_{\cdot,T}^\varepsilon := \frac{\lambda}{\mathrm{e}^{\lambda T} - 1} \int_0^T \Big\langle \sigma^{-1}\Big(k - \frac{\mathrm{e}^{\lambda t} - 1}{\lambda}(\partial_k b_\varepsilon)(X_\varepsilon^\cdot(t))\Big), \mathrm{d}W(t)\Big\rangle, \quad \varepsilon > 0$$

is bounded in $L^2(\mathbb{P} \times \mu)$; i.e. $\sup_{\varepsilon > 0} \int_{\mathbb{B}} \mathbb{E}|M_{x,T}|^2 \, \mu(\mathrm{d}x) < \infty$.

(iii) $P_T^\varepsilon(\partial_k f)(x) = \mathbb{E}\big(f(X_\varepsilon^x(T)M_{x,T}^\varepsilon\big), \; f \in C_b^1(\mathbb{H}), \varepsilon > 0.$

So, there exist $M_{\cdot,T} \in L^2(\mathbb{P}\times\mu)$ and a sequence $\varepsilon_n \downarrow 0$ such that $M^{\varepsilon_n}_{\cdot,T} \to M_{\cdot,T}$ weakly in $L^2(\mathbb{P}\times\mu)$. Thus, by taking $n\to\infty$ in (iii) and using (i), we prove (2.3) for μ-a.e. $x\in\mathbb{B}$. Then the proof of the first assertion is completed as in the first case.

(2) Since σ is invertible, **(A3)** implies $\alpha := \sum_{i=1}^\infty \frac{1}{\lambda_i} < \infty$. Next, since the Lipschitz constant $\|\partial b\|_\infty$ of b is finite, the integration by parts formula (2.3) also implies explicit Fomin derivative estimates on the invariant probability measure, which were investigated recently in [3]. Indeed, it follows from (2.3) and (2.4) that

$$\begin{aligned}|\mu(\partial_k f)| &= \inf_{T>0} |\mu(P_T(\partial_k f))| \le \inf_{T>0} \sqrt{\mu(P_T f^2)}\bigg(\int_{\mathbb{B}} \mathbb{E}|M_{x,T}|^2\,\mu(\mathrm{d}x)\bigg)^{\frac12}\\ &\le |k|\cdot\|f\|_{L^2(\mu)} \inf_{T>0}\frac{\lambda}{\mathrm{e}^{\lambda T}-1}\bigg(\int_0^T \Big\|\sigma^{-1}\Big(I-\frac{\mathrm{e}^{\lambda t}-1}{\lambda}\partial b\Big)\Big\|_\infty^2\mathrm{d}t\bigg)^{\frac12},\quad Ak=\lambda k.\end{aligned}$$

By taking $k=e_i, T=\lambda_i^{-1}$ and $\lambda=-\lambda_i$ in the above estimate, for any $k\in\mathscr{D}(A)$ we have

$$\begin{aligned}|\mu(\partial_k f)| &\le \sum_{i=1}^\infty |\langle k,e_i\rangle \mu(\partial_{e_i} f)| \le \bigg(\sum_{i=1}^\infty \lambda_i^2\langle k,e_i\rangle^2\bigg)^{\frac12}\bigg(\sum_{i=1}^\infty \frac{1}{\lambda_i^2}\mu(\partial_{e_i}f)^2\bigg)^{\frac12}\\ &\le |Ak|\bigg(\sum_{i=1}^\infty \frac{\|\sigma^{-1}\|^2}{\lambda_i(\mathrm{e}-1)^2}\Big(1+\frac{\mathrm{e}-1}{\lambda_i}\|\partial b\|_\infty\Big)^2\bigg)^{\frac12}\|f\|_{L^2(\mu)}\\ &\le C|Ak|\cdot\|f\|_{L^2(\mu)},\end{aligned} \tag{2.5}$$

where $C := \frac{\|\sigma^{-1}\|\sqrt{\alpha}}{\mathrm{e}-1}\Big(1+\frac{\mathrm{e}-1}{\lambda_1}\|\partial b\|_\infty\Big)$. This implies the closablity of $(\mathscr{E}_k, C_b^2(\mathbb{H}))$ as explained in the proof of Proposition 1.1. Indeed, if $\{f_n\}_{n\ge1}\subset C_b^2(\mathbb{B})$ satisfies $f_n\to 0$ and $\partial_k f_n\to Z$ in $L^2(\mu)$, then (2.5) implies

$$\begin{aligned}|\mu(gZ)| &= \lim_{n\to\infty}|\mu(g\partial_k f_n)| = \lim_{n\to\infty}|\mu(\partial_k(f_n g) - \mu(f_n\partial_k g)|\\ &\le C|Ak|\lim_{n\to\infty}\sqrt{\mu((f_n g)^2)} = 0,\quad g\in C_b^2(\mathbb{B}),\end{aligned}$$

so that $Z=0$. □

To conclude this section, let us recall a result concerning existence and stability of the invariant probability measure. Let $W_a(t)=\int_0^t \mathrm{e}^{A(t-s)}\sigma\mathrm{d}W(s), t\ge0$. Assume that b is Lipschitz continuous and $\int_0^\infty \|\mathrm{e}^{tA}\sigma\|_{HS}^2\mathrm{d}t<\infty$. We have

$$\sup_{t\ge0}\mathbb{E}\big(\|W_A(t)\|^2+|b(W_A(t))|^2\big)<\infty.$$

Therefore, by [5, Theorem 2.3], if there exist $c_1>0, c_2\in\mathbb{R}$ with $c_1+c_2>0$ such that

$$\langle A(x-y),x-y\rangle\le -c_1|x-y|^2,\ \langle b(x)-b(y),x-y\rangle\le -c_2|x-y|^2,\quad x,y\in\mathbb{H},$$

then P_t has a unique invariant probability measure such that $\lim_{t\to\infty}P_t f=\mu(f)$ holds for $f\in C_b(\mathbb{H})$.

3 Stochastic Hamiltonian systems on Hilbert spaces

Let $\tilde{\mathbb{H}}$ and $\mathbb{H}$ be two separable Hilbert spaces. Consider the following stochastic differential equation for $Z(t) := (X(t), Y(t))$ on $\tilde{\mathbb{H}} \times \mathbb{H}$:

$$\begin{cases} \mathrm{d}X(t) = BY(t)\mathrm{d}t, \\ \mathrm{d}Y(t) = \{AY(t) + b(t, X(t), Y(t))\}\mathrm{d}t + \sigma \mathrm{d}W(t), \end{cases} \tag{3.1}$$

where $B \in \mathscr{L}(\mathbb{H} \to \tilde{\mathbb{H}})$, $(A, \mathscr{D}(A))$ satisfies **(A1)**, σ satisfies **(A3)**, $W(t)$ is the cylindrical Brownian motion on $\mathbb{H}$, and $b : [0, \infty) \times \tilde{\mathbb{H}} \times \mathbb{H} \to \mathbb{H}^{-2}$ satisfies: for any $T > 0$ there exists $\gamma \in C((0, T])$ with $\int_0^T \gamma(t)\mathrm{d}t < \infty$ such that

$$\begin{aligned} &\sup_{s\in[0,T]} \int_0^T |\mathrm{e}^{tA} b(s, 0)|\mathrm{d}t < 1, \\ &\sup_{s\in[0,T]} |\mathrm{e}^{tA}(b(s, z) - b(s, z'))| \le \gamma(t)|z - z'|, \quad t \in [0, T], z, z' \in \tilde{\mathbb{H}} \times \mathbb{H}. \end{aligned} \tag{3.2}$$

Obviously, for any initial data $z := (x, y) \in \mathbb{H}$, the equation has a unique mild solution $Z^z(t)$. Let P_t be the associated Markov semigroup.

When $\tilde{\mathbb{H}}$ and $\mathbb{H}$ are finite-dimensional, the integration by parts formula of P_t has been established in [10, Theorem 3.1]. Here, we extend this result to the present infinite-dimensional setting.

Proposition 3.1. *Assume that $BB^* \in \mathscr{L}(\tilde{\mathbb{H}})$ with $\mathrm{Ker}(BB^*) = \{0\}$. Let $T > 0$ and $k := (k_1, k_2) \in \mathrm{Im}(BB^*) \times \mathbb{H}$ be such that*

$$Ak_2 = \theta_2 k_2, \quad AB^*(BB^*)^{-1}k_1 = \theta_1 B^*(BB^*)^{-1}k_1 \tag{3.3}$$

for some constants $\theta_1, \theta_2 \in \mathbb{R}$. For any $\phi, \psi \in C^1([0, T])$ such that

$$\phi(0) = \phi(T) = \psi(0) = \psi(T) - 1 = \int_0^T \mathrm{e}^{\theta_2 t}\psi(t)\mathrm{d}t = 0, \quad \int_0^T \phi(t)\mathrm{e}^{\theta_1 t}\mathrm{d}t = \mathrm{e}^{\theta_1 T}, \tag{3.4}$$

let

$$\begin{aligned} h(t) &= B^*(BB^*)^{-1}k_1 \int_0^t \phi'(s)\mathrm{e}^{\theta_1(s-T)}\mathrm{d}s + k_2 \int_0^t \psi'(s)\mathrm{e}^{\theta_2(s-T)}\mathrm{d}s, \\ \tilde{h}(t) &= \phi(t)\mathrm{e}^{\theta_1(t-T)}B^*(BB^*)^{-1}k_1 + \psi(t)\mathrm{e}^{\theta_2(t-T)}k_2, \\ \Theta(t) &= \bigg(\int_0^t B\tilde{h}(s)\mathrm{d}s, \ \tilde{h}(t)\bigg), \quad t \in [0, T]. \end{aligned}$$

If for any $t \in [0, T]$, $b(s, \cdot)$ is Fréchet differentiable along $\Theta(t)$ such that

$$\int_0^T \sup_{z\in\tilde{\mathbb{H}}\times\mathbb{H}} \big\|h'(t) - (\partial_{\Theta(t)}b(t, \cdot))(z)\big\|_\sigma^2 \mathrm{d}t < \infty, \tag{3.5}$$

then for any $f \in C_b^1(\tilde{\mathbb{H}} \times \mathbb{H})$,

$$P_T(\partial_k f) = \mathbb{E}\bigg\{f(Z(T)) \int_0^T \Big\langle (\sigma\sigma^*)^{-1/2}\big\{h'(t) - (\partial_{\Theta(t)}b(t, \cdot))(Z(t))\big\}, \ \mathrm{d}W(t)\Big\rangle\bigg\}.$$

Proof. As explained in the proof of Theorem 2.1, we simply assume that $\sigma = \sqrt{\sigma\sigma^*}$. Let $(X^0(t), Y^0(t)) = (X(t), Y(t))$ solve (3.1) with initial data (x, y), and for $\varepsilon \in (0, 1]$ let $(X^\varepsilon(t), Y^\varepsilon(t))$ solve the equation

$$\begin{cases} \mathrm{d}X^\varepsilon(t) = BY^\varepsilon(t)\mathrm{d}t, \quad X^\varepsilon(0) = x, \\ \mathrm{d}Y^\varepsilon(t) = \sigma \mathrm{d}W(t) + \big\{b(t, X(t), Y(t)) + AY^\varepsilon(t) + \varepsilon h'(t)\big\}\mathrm{d}t, \quad Y^\varepsilon(0) = y. \end{cases} \tag{3.6}$$

Then it is easy to see from (3.3) and (3.4) that

$$\begin{aligned} Y^\varepsilon(t) - Y(t) &= \varepsilon \int_0^t \mathrm{e}^{(t-s)A} h'(s)\mathrm{d}s \\ &= \varepsilon B^*(BB^*)^{-1}k_1 \int_0^t \phi'(s)\mathrm{e}^{\theta_1(s-T)}\mathrm{e}^{\theta_1(t-s)}\mathrm{d}s + \varepsilon k_2 \int_0^t \psi'(s)\mathrm{e}^{\theta_2(s-T)}\mathrm{e}^{\theta_2(t-s)}\mathrm{d}s \\ &= \varepsilon\big(\phi(t)\mathrm{e}^{\theta_1(t-T)}B^*(BB^*)^{-1}k_1 + \psi(t)\mathrm{e}^{\theta_2(t-T)}k_2\big) = \varepsilon\tilde{h}(t), \end{aligned}$$

and hence,

$$\begin{aligned} X^\varepsilon(t) - X(t) &= \varepsilon \int_0^t B\tilde{h}(s)\mathrm{d}s \\ &= \varepsilon\bigg(k_1 \int_0^t \phi(r)\mathrm{e}^{\theta_1(r-T)}\mathrm{d}r + (Bk_2)\int_0^t \psi(r)\mathrm{e}^{\theta_2(r-T)}\mathrm{d}r\bigg). \end{aligned}$$

So,

$$X^\varepsilon(t) - X(t) = \varepsilon\Theta(t), \quad t \in [0, T], \tag{3.7}$$

and in particular

$$(X^\varepsilon(T), Y^\varepsilon(T)) = (X(T), Y(T)) + \varepsilon k \tag{3.8}$$

due to (3.4). Next,

$$\xi_\varepsilon(s) = \varepsilon h'(s) + b(s, X(s), Y(s)) - b(s, X^\varepsilon(s), Y^\varepsilon(s)) \tag{3.9}$$

and

$$R_\varepsilon = \exp\bigg[-\int_0^T \big\langle \sigma^{-1}\xi_\varepsilon(s), \mathrm{d}W(s)\big\rangle - \frac{1}{2}\int_0^T |\sigma^{-1}\xi_\varepsilon(s)|^2\mathrm{d}s\bigg].$$

We reformulate (3.6) as

$$\begin{cases} \mathrm{d}X^\varepsilon(t) = BY^\varepsilon(t)\mathrm{d}t, \quad X^\varepsilon(0) = x, \\ \mathrm{d}Y^\varepsilon(t) = \sigma \mathrm{d}W^\varepsilon(t) + \{b(t, X^\varepsilon(t), Y^\varepsilon(t)) + AY^\varepsilon(t)\}\mathrm{d}t, \quad Y^\varepsilon(0) = y, \end{cases} \tag{3.10}$$

where by (3.5) and (3.7),

$$W^\varepsilon(t) := W(t) + \int_0^t \sigma^{-1}\xi_\varepsilon(s)\mathrm{d}s, \quad t \in [0, T]$$

is a cylindrical Brownian motion under the weighted probability measure $\mathbb{Q}_\varepsilon := R_\varepsilon \mathbb{P}$. Since $|\xi_\varepsilon|$ is uniformly bounded on $[0,T]$, by the dominated convergence theorem and (3.7), for any $f \in C_b^1(\tilde{\mathbb{H}} \times \mathbb{H})$ we obtain

$$\begin{aligned} P_T(\partial_k f) &= \lim_{\varepsilon \to 0} \mathbb{E} \frac{f((X(T),Y(T))+\varepsilon k) - f((X(t),Y(t)))}{\varepsilon} \\ &= \lim_{\varepsilon \to 0} \mathbb{E} \frac{f((X^\varepsilon(T),Y^\varepsilon(T))) - R_\varepsilon f((X^\varepsilon(T),Y^\varepsilon(T)))}{\varepsilon} \\ &= \mathbb{E}\bigg(f(Z(T)) \lim_{\varepsilon\to 0} \frac{1-R_\varepsilon}{\varepsilon}\bigg) \\ &= \mathbb{E}\bigg(f(Z(T)) \int_0^T \Big\langle \sigma^{-1}\big\{h'(t) - (\partial_{\Theta(t)} b)(Z(t))\big\},\ \mathrm{d}W(t)\Big\rangle\bigg). \end{aligned}$$

□

To apply this result, we present here a specific choice of (ϕ,ψ) such that (3.4) holds:

$$\phi(t) = \frac{\mathrm{e}^{\theta_1 T} t(T-t)}{\int_0^T s(T-s)\mathrm{e}^{\theta_1 s}\mathrm{d}s}, \quad \psi(t) = \frac{\mathrm{e}^{\theta_2(T-t)}}{T}\Big(\frac{3t^2}{T} - 2t\Big), \qquad t\in[0,T].$$

Theorem 3.2. *Let $\tilde{\mathbb{H}} = \mathbb{H} = \mathbb{H}$ and $\mathrm{Ker}(B) = \{0\}$. Let $b(t,\cdot) = b$ do not dependent on t such that P_t has an invariant probability measure μ. If*

$$\text{(3.11)} \qquad \sup_{(x,y)\in\mathbb{H}\times\mathbb{H}} \lim_{r\downarrow 0} \frac{\|b(x+rB^{-1}\tilde{k}, y+rk) - b(x,y)\|_\sigma}{r} < \infty, \quad (\tilde{k},k) \in (B\mathbb{H}_A)\times\mathbb{H}_A,$$

Then for any $(k_1,k_2)\in(B\mathbb{H}_A)\times\mathbb{H}_A$, the form $(\mathscr{E}_k, C_b^2(\mathbb{H}\times\mathbb{H}))$ is closable in $L^2(\mu)$.

Proof. It suffices to prove for $k=(k_1,k_2)$ such that $B^{-1}k_1$ and k_2 are eigenvectors of A, i.e. $AB^{-1}k_1 = \theta_1 B^{-1}k_1$ and $Ak_2 = \theta_2 k_2$ hold for some $\theta_1,\theta_2\in\mathbb{R}$. As explained above there exists $T>0$ such that (3.4) holds for some $\phi,\psi\in C^\infty([0,T])$. Moreover, as explained in the proof of Theorem 2.1, by taking

$$b_\varepsilon(s,x,y) = \frac{1}{\sqrt{2\pi\varepsilon}} \int_{\mathbb{R}} b\big((x,y)+r\Theta(s)\big)\exp\Big[-\frac{r^2}{2\varepsilon}\Big]\mathrm{d}r, \quad s\in[0,T], (x,y)\in\mathbb{H}\times\mathbb{H}$$

for $\varepsilon>0$, such that (3.11) holds uniformly in $\varepsilon>0$ and $s\in[0,T]$ with $b_\varepsilon(s,\cdot)$ replacing b, we may and do assume that $b(s,\cdot)$ is Fréchet differentiable along $\Theta(s)$. Then the integration by parts formula in Proposition 3.1 holds, and due to (3.11) we have

$$M_{\cdot,T} := \int_0^T \Big\langle (\sigma\sigma^*)^{-1/2}\big\{h'(t) - (\partial_{\Theta(t)} b(t,\cdot))(Z(t))\big\},\ \mathrm{d}W(t)\Big\rangle \in L^2(\mathbb{P}\times\mu).$$

Therefore, by Proposition 1.1, the form $(\mathscr{E}_k, C_b^2(\mathbb{H}\times\mathbb{H}))$ is closable on $L^2(\mu)$. □

Below are typical examples of the stochastic Hamiltonian system with invariant probability measure such that Theorem 3.2 applies.

Example 3.1. Let $\tilde{\mathbb{H}} = \mathbb{H} = \mathbb{H}$.

(1) Let $\mathbb{H} = \mathbb{R}^d$ for some $d \geq 1$. When $\sigma = B = I$, $A \leq -\lambda I$ for some $\lambda > 0$ is a negatively definite $d \times d$-matrix, and $b(x, y) = A^{-1}\nabla V(x)$ for some $V \in C^2(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \mathrm{e}^{-V(x)}\mathrm{d}x < \infty$. Then the unique invariant probability measure of P_t is

$$\mu(\mathrm{d}x, \mathrm{d}y) = C\mathrm{e}^{-V(x) + \frac{\lambda}{2}\langle Ay, y\rangle}\mathrm{d}x\mathrm{d}y,$$

where $C > 0$ is the normalization. See [2, 6, 9] for the study of hypercoercivity of the associated semigroup P_t with respect to μ, as well as [12] for the stronger property of hypercontractivity.

(2) In the infinite-dimensional setting, let $\sigma = B = I$ and A be negatively definite such that A^{-1} is of trace class. Take $b(x, y) = A^{-1}Qx$ for some positively definite self-adjoint operator $\mathbb{Q}$ on $\mathbb{H}$ such that Q^{-1} is of trace class and

$$\int_0^1 \|\mathrm{e}^{tA}A^{-1}Q\|\mathrm{d}t < 1.$$

Then it is easy to see that

$$\mu(\mathrm{d}x, \mathrm{d}y) = N_{Q^{-1}}(\mathrm{d}x)N_{-A^{-1}}(\mathrm{d}y)$$

is an invariant probability measure.

(3) More generally, let $\sigma = B = I$ and

$$b(x, y) = \tilde{b}(x) := A^{-1}\nabla V(x), \quad (x, y) \in \mathbb{H} \times \mathbb{H}_A$$

for some Fréchet differentiable $V : \mathbb{H}_A \to \mathbb{R}$ such that (3.11) holds. For any $n \geq 1$, let

$$V_n(r) = V \circ \varphi_n(r), \ \ \varphi_n(r) = \sum_{i=1}^n r_i e_i, \quad r = (r_1, \cdots, r_n) \in \mathbb{R}^n.$$

If $\int_{\mathbb{R}^n} \mathrm{e}^{-V_n(r)}\mathrm{d}r < \infty$ and when $n \to \infty$ the probability measure

$$\nu_n(D) := \frac{1}{\int_{\mathbb{R}^n} \mathrm{e}^{-V_n(r)}\mathrm{d}r} \int_{\varphi_n^{-1}(D)} \mathrm{e}^{-V_n(r)}\mathrm{d}r, \quad D \in \mathscr{B}(\mathbb{H})$$

converges weakly to some probability measure ν, then $\mu := \nu \times N_{-A^{-1}}$ is an invariant probability measure of P_t. This can be confirmed by (1) and a finite-dimensional approximation argument. Indeed, let $\pi_n : \mathbb{H} \to \mathbb{H}_{A,n}$ be the orthogonal projection, and let $A_n = \pi_n A, W_n = \pi_n W$ and $b_n(x, y) = \pi_n \nabla V(x)$. Let $X_n(t)$ solve the finite-dimensional equation

$$\begin{cases} \mathrm{d}X_n(t) = Y_n(t)\mathrm{d}t, \\ \mathrm{d}Y_n(t) = \{A_nY_n(t) + b_n(X_n(t))\}\mathrm{d}t + \mathrm{d}W_n(t) \end{cases}$$

with $(X_n(0), Y_n(0)) = (\pi_n X(0), \pi_n Y(0))$. Then the proof of [11, Theorem 2.1] yields that for every $t \geq 0$,

$$\lim_{n\to\infty} \mathbb{E}\big(|X_n(t) - X(t)|^2 + |Y_n(t) - Y(t)|^2\big) = 0$$

uniformly in the initial data $(X(0), Y(0)) \in \mathbb{H} \times \mathbb{H}$. Thus, letting $P_t^{(n)}$ be the semigroup for $(X_n(t), Y_n(t))$, we have

$$\lim_{n\to\infty} \sup_{(x,y)\in\mathbb{H}\times\mathbb{H}} |P_t^{(n)} f(\pi_n x, \pi_n y) - P_t f(x,y)| = 0, \quad f \in C_b^1(\mathbb{H}\times\mathbb{H}).$$

Combining this with the assertion in (1) and noting that $\nu_n \times (N_{-A^{-1}} \circ \pi_n^{-1}) \to \mu$ weakly as $n \to \infty$, we conclude that μ is an invariant probability measure of P_t.

4 Semilinear SPDEs with delay

For fixed $\tau > 0$, let $\mathscr{C}_\tau = C([-\tau, 0]; \mathbb{H})$ be equipped with the uniform norm $\|\eta\|_\infty := \sup_{\theta\in[-\tau,0]} |\eta(\theta)|$. For any $\xi \in C([-\tau,\infty); \mathbb{H})$, we define $\xi_\cdot \in C([0,\infty); \mathscr{C}_\tau)$ by letting

$$\xi_t(\theta) = \xi(t+\theta), \quad \theta \in [-\tau, 0], t \geq 0.$$

Consider the following stochastic differential equation with delay:

$$\text{(4.1)} \qquad \mathrm{d}X(t) = \big\{AX(t) + b(X_t)\big\}\mathrm{d}t + \sigma \mathrm{d}W(t), \quad X_0 \in \mathscr{C}_\tau,$$

where $(A, \mathscr{D}(A))$ satisfies **(A1)**, σ satisfies **(A3)**, and $b : \mathscr{C}_\tau \to \mathbb{H}$ satisfies: for any $T > 0$ there exists $\gamma \in C((0,T])$ with $\int_0^T \gamma(t)\mathrm{d}t < \infty$ such that

$$\text{(4.2)} \quad \int_0^T \sup_{s\in[0,T]} |\mathrm{e}^{tA} b(s,0)|^2 \mathrm{d}t < \infty, \ \ |\mathrm{e}^{tA}(b(s,\xi) - b(s,\eta))|^2 \leq \gamma(t)\|\xi-\eta\|_\infty^2, \ \ t,s \in [0,T].$$

Then for any initial datum $\xi \in \mathscr{C}_\tau$, the equation has a unique mild solution $X^\xi(t)$ with $X_0 = \xi$. Let P_t be the Markov semigroup for the segment solution X_t.

Let

$$\mathscr{C}_\tau^1 = \left\{\eta \in \mathscr{C}_\tau : \eta(\theta) \in \mathscr{D}(A) \text{ for } \theta \in [-\tau, 0], \int_{-\tau}^0 \big(|A\eta(\theta)|^2 + |\eta'(\theta)|^2\big)\mathrm{d}\theta < \infty\right\}.$$

The following result is an extension of [10, Theorem 4.1(1)] to the infinite-dimensional setting.

Proposition 4.1. *For any $\eta \in \mathscr{C}_\tau^1$ and $T > \tau$, let*

$$\Gamma(t) := \begin{cases} \frac{1}{T-\tau} \mathrm{e}^{(s+\tau-T)A}\eta(-\tau), & \text{if } s \in [0, T-\tau], \\ \eta'(s-T) - A\eta(s-T), & \text{if } s \in (T-\tau, T], \end{cases}$$

and

$$\Theta(t) := \int_0^{t\vee 0} \Gamma(s)\mathrm{d}s, \quad t \in [-\tau, T].$$

If $b(t,\cdot)$ is Fréchet differentiable along Θ_t for $t \in [0,T]$ such that

$$\text{(4.3)} \qquad \sup_{\xi\in\mathscr{C}_\tau} \int_0^T \big\|\Gamma(t) - (\nabla_{\Theta_t} b(T,\cdot))(\xi)\big\|_\sigma^2 \mathrm{d}t < \infty,$$

then

(4.4)

$$P_T(\partial_\eta f) = \mathbb{E}\bigg(f(X_T)\int_0^T \Big\langle (\sigma\sigma^*)^{-1/2}\big(\Gamma(t) - (\nabla_{\Theta_t} b(t,\cdot))(X_t)\big), \mathrm{d}W(t)\Big\rangle\bigg), \quad f \in C_b^1(\mathscr{C}_\tau).$$

Proof. Simply let $\sigma = \sqrt{\sigma\sigma^*}$ as in the proof of Theorem 2.1. For any $\varepsilon \in (0,1)$, let $X^\varepsilon(t)$ solve the equation

$$\mathrm{d}X^\varepsilon(t) = \{AX^\varepsilon(t) + b(t, X_t) + \varepsilon\Gamma(t)\}\mathrm{d}t + \sigma \mathrm{d}W(t), \quad X_0^\varepsilon = X_0. \tag{4.5}$$

We have

$$\begin{aligned} X^\varepsilon(t) - X(t) &= \varepsilon \int_0^{t^+} \mathrm{e}^{(t-s)A}\Gamma(s)\mathrm{d}s \\ &= \frac{\varepsilon t^+}{T-\tau}\mathrm{e}^{(\tau-T)A}\eta(-\tau)1_{[-\tau,T-\tau)}(t) + \varepsilon\eta(t-T)1_{[T-\tau,T]}(t), \quad t \in [-\tau, T]. \end{aligned} \tag{4.6}$$

In particular, we have $X_T^\varepsilon - X_T = \varepsilon\eta$. To formulate P_T using X_T^ε, rewrite (4.5) by

$$\mathrm{d}X^\varepsilon(t) = \{AX^\varepsilon(t) + b(t, X_t^\varepsilon)\}\mathrm{d}t + \sigma\mathrm{d}W_\varepsilon(t), \quad X_0^\varepsilon = X_0,$$

where

$$W_\varepsilon(t) := W(t) + \int_0^t \xi_\varepsilon(s)\mathrm{d}s, \quad \xi_\varepsilon(s) := b(s, X_s) - b(s, X_s^\varepsilon) + \varepsilon\Gamma(s).$$

By (4.3) and the Girsanov theorem, we see that $\{W_\varepsilon(t)\}_{t\in[0,T]}$ is a cylindrical Brownian motion on $\mathbb{H}$ under the probability measure $\mathrm{d}\mathbb{Q}_\varepsilon := R_\varepsilon \mathrm{d}\mathbb{P}$, where

$$R_\varepsilon := \exp\bigg[\int_0^T \Big\langle \sigma^{-1}\big(b(t, X_t^\varepsilon) - b(t, X_t) - \varepsilon\Gamma(t)\big),\ \mathrm{d}W(t)\Big\rangle\bigg].$$

Then

$$\mathbb{E}(f(X_T)) = P_T f = \mathbb{E}(R_\varepsilon f(X_T^\varepsilon)).$$

Combining this with $X_T^\varepsilon = X_T + \varepsilon\eta$ and using (4.6), we arrive at

$$\begin{aligned} P_T(\partial_\eta f) &= \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathbb{E}\{f(X_T + \varepsilon\eta) - f(X_T)\} = \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathbb{E}\{f(X_T^\varepsilon) - R_\varepsilon f(X_T^\varepsilon)\} \\ &= \mathbb{E}\Big(f(X_T)\lim_{\varepsilon\downarrow 0}\frac{1-R_\varepsilon}{\varepsilon}\Big) = \mathbb{E}\bigg\{f(X_T)\int_0^T \Big\langle \sigma^{-1}\big(\Gamma(t) - (\nabla_{\Theta_t} b(t,\cdot))(X_t)\big), \mathrm{d}W(t)\Big\rangle\bigg\}. \end{aligned}$$

□

Theorem 4.2. *Let $b(t,\cdot) = b$ be independent of t such that P_t has an invariant probability measure μ. If $\mathrm{Im}(\sigma) \supset \mathbb{H}_A$ and*

$$\sup_{\xi\in\mathscr{C}_\tau}\limsup_{\varepsilon\downarrow 0}\frac{\|b(\xi+\varepsilon\eta) - b(\xi)\|_\sigma}{\varepsilon} < \infty, \quad \eta \in \mathscr{C}_\tau^1 \cap \Big(\cup_{n\ge 1} C([-\tau,0];\mathbb{H}_{A,n})\Big), \tag{4.7}$$

then for any $\eta \in \mathscr{C}_\tau^1 \cap \big(\cup_{n\ge1} C([-\tau,0];\mathbb{H}_{A,n})\big)$*, which is dense in* $\mathscr{C}_\tau$*, the form*

$$\mathscr{E}_\eta(f,g) := \int_{\mathscr{C}_\tau} (\partial_\eta f)(\partial_\eta g)\mathrm{d}\mu, \quad f,g \in C_b^2(\mathscr{C}_\tau)$$

is closable in $L^2(\mu)$*.*

Proof. For any $\varepsilon \in (0,1)$ let

$$b_\varepsilon(t,\xi) = \frac{1}{\sqrt{2\pi\varepsilon}} \int_{\mathbb{R}} b(\xi + r\Theta_t) \exp\Big[-\frac{r^2}{2\varepsilon}\Big]\mathrm{d}r, \quad \xi \in \mathscr{C}_\tau.$$

Then $b_\varepsilon(t,\cdot)$ is Féchet differentiable along Θ_t and (4.7) holds uniformly in ε with $b_\varepsilon(t,\cdot)$ replacing b. Moreover, $\eta \in \mathscr{C}_\tau^1 \cap \big(\cup_{n\ge1} C([-\tau,0];\mathbb{H}_n)\big)$ implies that $\Theta_t \in \mathscr{C}_\tau^1 \cap \big(\cup_{n\ge1} C([-\tau,0];\mathbb{H}_n)\big)$ and (4.7) holds uniformly in $t \in [0,T]$ and $\varepsilon \in (0,1)$ with Θ_t and $b_\varepsilon(t,\cdot)$ replacing η and b respectively. Combining this with $\mathrm{Im}(\sigma) \supset \mathbb{H}_A$, we conclude that (4.3) holds uniformly in ε with b_ε replacing b. Therefore, as explained in the proof of Theorem 2.1, we may assume that b is Fréchet differentiable along $\Theta_t, t \in [0,T]$, and by Proposition 4.1 the integration by parts formula (4.4) holds. Moreover, (4.7) implies

$$M_{\cdot,T} := \int_0^T \Big\langle (\sigma\sigma^*)^{-1/2}\big(\Gamma(t) - (\nabla_{\Theta_t} b(t,\cdot))(X_t)\big), \mathrm{d}W(t)\Big\rangle \in L^2(\mathbb{P}\times\mu).$$

Then the proof is finished by Proposition 1.1. □

Finally, we introduce the following example to illustrate Theorem 4.2.

Example 4.1. Let $b(\xi) = F(\xi(-\tau)), \xi \in \mathscr{C}_\tau$, for some $F \in C_b^1(\mathbb{H})$. If σ is Hilbert-Schmidt and

$$\langle x, Ax + F(y) - F(y')\rangle \le -\lambda_1|x|^2 + \lambda_2|y-y'|^2, \quad x,y \in \mathbb{H},$$

for some constants $\lambda_1 > \lambda_2 \ge 0$, then according to [1, Theorem 4.9] P_t has a unique invariant probability measure μ. If moreover $\mathrm{Im}(\sigma) \supset \mathbb{H}_A$ and for any $y \in \mathbb{H}_A$ there exists a constant

$$\limsup_{\varepsilon\downarrow0} \sup_{x\in\mathbb{H}} \frac{\|F(x+\varepsilon y) - F(x)\|_\sigma}{\varepsilon} < \infty,$$

then by Theorem 4.2, for any $\eta \in \mathscr{C}_\tau^1 \cap \big(\cup_{n\ge1} C([-\tau,0];\mathbb{H}_{A,n})\big)$ the form $(\mathscr{E}_\eta, C_b^2(\mathscr{C}_\tau))$ is closable on $L^2(\mu)$.

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