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Higher Order Eigenvalues for Non-Local Schrödinger Operators*

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Abstract

Two-sided estimates for higher order eigenvalues are presented for a class of non-local Schrödinger operators by using the jump rate and the growth of the potential. For instance, let L be the generator of a Lévy process with Lévy measure $\nu(\mathrm{d}z) := \rho(z)\mathrm{d}z$ such that $\rho(z) = \rho(-z)$ and

$$c_1|z|^{-(d+\alpha_1)} \le \rho(z) \le c_2|z|^{-(d+\alpha_2)}, \quad |z| \le \kappa$$

for some constants $\kappa, c_1, c_2 > 0$ and $\alpha_1, \alpha_2 \in (0, 2)$, and let $c_3|x|^{\theta_1} \le V(x) \le c_4|x|^{\theta_2}$ for some constants $\theta_1, \theta_2, c_3, c_4 > 0$ and large $|x|$. Then the eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \lambda_n \le \cdots$ of $-L+V$ satisfies the following two-side estimate: for any $p > 1$, there exists a constant $C > 1$ such that

$$Cn^{\frac{\theta_2\alpha_2}{d(\theta_2+\alpha_2)}} \ge \lambda_n \ge C^{-1}n^{\frac{\theta_1\alpha_1}{d(\theta_1+\alpha_1)}}, \quad n \ge 1.$$

When α_1 is variable, a better lower bound estimate is derived.

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1 Introduction

Spectral asymptotics of Schrödinger operators and more general pseudo-differential operators is since the fundamental work of H. Weyl [39] a rather classical topic. The monographs of M.

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Reed and B. Simon [30] as well as [10] give a certain overview about techniques and results until the mid 1980s mainly based on functional analytical considerations.

Our understanding of spectral problems has undergone substantial changes with the emergence of micro-local analysis which in particular takes the symplectic structure of the co-tangent bundle and the Hamiltonian system associated with the (principal) symbol of the pseudo-differential operator under investigation into account. Of course this approach is most interesting for operators in mathematical physics, but it requires certain regularity properties of the symbols and the potentials, e.g. smoothness, the existence of a principal symbol, etc. In addition to the treatment in L. Hörmander [15] we refer to the monographs of M.A. Shubin [32] and V.Ya. Ivrii [16], as well as to the related work by C.L. Fefferman [12]. In [17] also a short historical account is provided. More recently, certain pseudo-differential operators which allow certain anisotropic behaviour are treated in [29].

A further shift of the topic was given when at the relation of heat semigroups to spectral geometry was looked in more detail, in particular in the context of the Atiyah-Singer index theorem, we refer just to the monographs of P.B. Gilkey [13] and [14]. The interplay of probability theory and positivity preserving one-parameter semi-groups with spectral theory lead to a clarification of the role played by functional inequality, we refer to the monographs [1] and [37].

In this paper, we investigate asymptotics of eigenvalues for the Schrödinger type operator $L - V$ by using the intrinsic super Poincaré inequality introduced in [36]. Here, L is the generator of a jump Markov process, and V is a measurable function. Comparing with Weyl's asymptotic formulae derived in the literature where regularities of L and V are required, we only use conditions on the finite-range jump rate and the growth of the potential V, due to the stability of functional inequalities under rough perturbations.

There is a lot of work done on non-local Schrödinger operators related to certain Lévy or Lévy-type processes, in particular the case of symmetric stable processes is well studied including the situation when it is restricted in some sense to a sub-domain, and this is followed by work on Schrödinger operators related to subordinate Brownian motion which includes the so-called relativistic Schrödinger operator. For a general discussion of subordinate Brownian motion we refer to [34, 5].

With respect to a study of such Schrödinger operators we refer to earlier papers of K. Bogdan and T. Byczkowski [3, 4], the important paper by Z.-Q. Chen and R. Song [9], to the string of papers by K.Kaleta and co-authors [21, 22, 23, 24], the work of T. Kulczycki [25], and that of M. Kwasnicki [26] as well as the papers by J. Lörinczi and co-authors[27, 33], just to mention a few contributions.

The background of our starting point differs a bit from other investigations. It is meanwhile apparent that Dirichlet form techniques and related stochastic analysis or methods from the theory of functional inequalities have lead to enormous progress in our understanding of jump-type processes, or more precisely Lévy-type processes which should be looked at as processes having as generator a pseudo-differential operator with a negative definite symbol, we refer to the recent survey initiated by R. Schilling [5]. However, it seems that certain problems are out of the reach of our current techniques, for example we lack a geometric interpretation of transition densities as we do have for local, sub-elliptic operators, or when discussing Feynman-Kac formulae and a possible semi-classical asymptotic we essentially do not have a "classical" counterpart.

In [19], see also [6], a suggestion was made to approach the first problem. In more recent work it was started to develop the Hamiltonian dynamics behind certain Lévy processes, i.e. to consider the symbol of a generators as Hamiltonian function. A good starting point is to look at $H(q,p) = \psi(p) + V(q)$ where ψ is a certain convex, coercive negative definite function of class C^1 and V is a suitable potential, see [20, 31]. As substitute for the harmonic oscillator it was proposed to consider $H(q,p) = \psi(p) + \psi^*(q)$ where ψ^* is the conjugate convex function (Legendre transform) of ψ. In this context now arises the question whether the study of the symbol $a(x,\xi) = \psi(\xi) + \psi^*(x)$ on the co-tangent bundle will allow us to derive for example spectral results as they are known for elliptic differential operators, we only refer to the monograph [15] of L. Hörmander.

Since functional inequalities also can lead to information on eigenvalues, see for instance [37], it was natural to raise the question posed in the beginning. Once the problem was laid out, it was possible to employ techniques and results from the theory of functional inequalities as developed in [35, 36, 38] and to come up with some eigenvalue asymptotic, see Theorem 1.3 below. Of special interest was to include an example constructed with the help of a non-smooth, i.e. not even C^2, convex and coercive negative definite function which is anisotropic, i.e. not a subordinate Brownian motion, see Examples 2.1 and 2.2 below.

In the next section we introduce two results for α-stable like Schrödinger operators, where the first allows the power α varies and the second compares α with a constant. Concrete examples are addressed to illustrate these results, which are proved in Sections 3 and 4 respectively.

2 Main results and example

By using the intrinsic super Poincaré inequality introduced in [36], the compactness of Schrödinger semigroups have been investigated in [38] under an abstract framework. Let E be a Polish space with a σ-finite measure μ. Let $(L_0, \mathscr{D}(L_0))$ be a symmetric Dirichlet operator on $L^2(\mu)$ such that the associated Markov semigroup P_t^0 is ultracontractive, i.e. $\|P_t^0\|_{L^1(\mu)\to L^\infty(\mu)} < \infty$ for $t > 0$. We consider the Schrödinger operator $L_V := L - V$, where V is a locally integrable nonnegative measurable function on E such that $\mathscr{D} := \{f \in \mathscr{D}(L_0) : \mu(Vf^2) < \infty\}$ is dense in $L^2(\mu)$. Then the Friedrichs extension $(L_V, \mathscr{D}(L_V))$ of $(L_V, \mathscr{D})$ is a negatively definite self-adjoint operator generating a sub-Markov semigroup P_t^V on $L^2(\mu)$. The following result follows from [38, Theorem 1.1] which indeed applies to a more general setting, see also [33] for an alternative proof when $L_0 = \Delta$ on $\mathbb{R}^d$.

Theorem 2.1 ([38]). *If $\mu(V \le r) < \infty$ for $r \ge 0$, then for any $t > 0$, P_t^V is compact in $L^2(\mu)$.*

It is well know that the compactness of P_t^V is equivalent to the absence of the essential spectrum of L_V. In this case $-L_V$ has purely discrete spectrum and all eigenvalues, including multiplicities, can be listed as

$$0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots$$

with $\lim_{n\to\infty}\lambda_n=\infty$. In this paper we aim to investigate upper and lower bound estimates of λ_n for L_0 being a non-local symmetric operator on $L^2(\mathbb{R}^d)$.

Let $J:\mathbb{R}^d\times\mathbb{R}^d\to[0,\infty]$ be measurable such that

(A) $J(x,y)=J(y,x)$ and

$$\sup_{x\in\mathbb{R}^d}\int_{\{|z|\le1\}}|z|\cdot|J(x,x+z)-J(x,x-z)|\mathrm{d}z<\infty, \tag{2.1}$$

$$\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}(|z|^2\wedge1)J(x,x+z)\mathrm{d}z<\infty. \tag{2.2}$$

Let $V\ge0$ be a locally integrable function on $\mathbb{R}^d$. Then the following non-local Schrödinger operator is well defined in $L^2(\mathbb{R}^d)$ for $f\in C_0^\infty(\mathbb{R}^d)$:

$$\begin{aligned}L_{J,V}f(x):=&\int_{\mathbb{R}^d}\big\{f(x+z)-f(x)-\langle\nabla f(x),z\rangle1_{\{|z|\le1\}}\big\}J(x,x+z)\mathrm{d}z\\&+\frac12\int_{\{|z|\le1\}}\langle\nabla f(x),z\rangle\big(J(x,x+z)-J(x,x-z)\big)\mathrm{d}z-(Vf)(x),\quad x\in\mathbb{R}^d.\end{aligned}$$

Moreover, we have the following integration by parts formula

$$\begin{aligned}&\int_{\mathbb{R}^d}(fL_{J,V}g)(x)\mathrm{d}x=-\mathscr{E}_{J,V}(f,g),\quad f,g\in C_0^\infty(\mathbb{R}^d),\\&\mathscr{E}_{J,V}(f,g):=\int_{\mathbb{R}^d\times\mathbb{R}^d}\big(f(x)-f(y)\big)\big(g(x)-g(y)\big)J(x,y)\mathrm{d}x\mathrm{d}y+\int_{\mathbb{R}^d}(Vfg)(x)\mathrm{d}x.\end{aligned}$$

Therefore, the form $(\mathscr{E}_{J,V},C_0^\infty(\mathbb{R}^d))$ in $L^2(\mathbb{R}^d)$ is closable and the closure $(\mathscr{E}_{J,V},\mathscr{D}(\mathscr{E}_{J,V}))$ is a symmetric Dirichlet form, the associated generator $(L_{J,V},\mathscr{D}(L_{J,V}))$ is the Friedrichs extension of $(L_{J,V},C_0^\infty(\mathbb{R}^d))$. Let $P_t^{J,V}$ be the associated sub-Markov semigroup.

According to [2, Theorem 1.1], under a suitable lower bound condition on J, the Markov semigroup $P_t^{J,0}$ generated by $L_{J,0}$ is ultracontractive with respect to the Lebesgue measure. By Theorem 2.2, if $V(x)\to\infty$ as $|x|\to\infty$, the essential spectrum of $L_{J,V}$ is empty. Let $\lambda_1\le\lambda_2\le\cdots$ be all eigenvalues of $-L_{J,V}$. We will estimate λ_n in terms of α_1,α_2 in $(A2)$ and the growth of $V(x)$ as $|x|\to\infty$. Obviously, $\mathscr{E}_{J,V}(f,f)=0$ if and only if $f=0$. This implies $\lambda_1>0$.

To estimate λ_n from below, we will use the following intrinsic super Poincaré inequality introduced in [36]:

$$\int_{\mathbb{R}^d}f^2(x)\mathrm{d}x\le s\mathscr{E}_{J,V}(f,f)+\beta(s)\bigg(\int_{\mathbb{R}^d}|f\phi|(x)\mathrm{d}x\bigg)^2,\quad s>0,f\in C_0^\infty(\mathbb{R}^d), \tag{2.3}$$

where $\beta:(0,\infty)\to(0,\infty)$ is a decreasing function, and $\phi\in L^2(\mathbb{R}^d)$ is a reference function. Let

$$\beta^{-1}(r)=\inf\{s\ge0:\beta(s)\le r\},\quad r>0,$$

where $\inf\emptyset=\infty$ by convention. The following result is essentially due to [36], see Section 2 for a complete proof.

Theorem 2.2 ([36])**.** *Let $\phi \in L^2(\mathbb{R}^d)$ be positive such that $P_t^{J,V}\phi \le \mathrm{e}^{\lambda t}\phi$ holds for some $\lambda \in \mathbb{R}$ and all $t \ge 0$. If (2.3) holds for some β with $\beta(\infty) := \lim_{s\to\infty}\beta(s) = 0$ and*

$$\Lambda(t) := \int_t^\infty \frac{\beta^{-1}(r)}{r}\mathrm{d}r < \infty, \quad t > 0,$$

then for any $\varepsilon \in (0,1)$, there exists a constant $c > 0$ such that

$$\lambda_n \ge \frac{c}{\Lambda(\varepsilon n)}, \quad n \ge 1. \tag{2.4}$$

According to this result, to derive sharp lower bound of λ_n, we need to prove the inequality (2.3) for as small as possible β. Intuitively, to establish (2.3) with smaller β, we should take larger $\phi \in L^2(\mathbb{R}^d)$. In this spirit, reasonable choices of ϕ will be $\phi(x) = (1+|x|^2)^{-p}$ for some constant $p > \frac{d}{4}$, or $\phi(x) = \varphi_k(|x|)$ for some $k \in \mathbb{Z}_+$ and $p > 1$, where

$$\varphi_k(s) := (1+s^2)^{-\frac{d}{4}} \cdot \left(\{\log^{k+1}(\mathrm{e}^{k+1}+s^2)\}^{\frac{p}{2}} \cdot \prod_{i=1}^{k}\sqrt{\log^i(\mathrm{e}^i+s^2)}\right)^{-1}, \quad s \ge 0, \tag{2.5}$$

$\log^{k+1} := \log^k \log$ for $k \ge 1$, and $\prod_{i=1}^k := 1$ if $k = 0$. In general, we consider $\phi(x) = \varphi(|x|)$ for $\varphi \in \mathscr{S}$, the class of decreasing functions $\varphi \in C^2([0,\infty);\mathbb{R}_+)$ such that

(i) $\int_0^\infty s^{d-1}\varphi(s)^2\mathrm{d}s < \infty$;

(ii) There exists a constant $c > 0$ such that

$$\sup_{r \le 1+s}\Big(|\varphi'(r)|(r+r^{-1}) + |\varphi''(r)|\Big) + \varphi(s/2) \le c\varphi(s), \quad s \ge 0.$$

By establishing the intrinsic Poincaré inequality (2.3) for $\phi(x) := \varphi(|x|)$, we will derive the following main result of the paper.

Theorem 2.3. *Let J satisfy (A). Assume that for some constant $c > 0$ and symmetric function $\alpha \in C(\mathbb{R}^d \times \mathbb{R}^d; (0,2))$ there holds*

$$J(x,y) \ge \frac{c_1}{|z|^{d+\alpha(x,y)}}1_{\{|x-y|\le\kappa\}}, \quad x,y \in \mathbb{R}^d. \tag{2.6}$$

Let $\mathbf{a}(r) = \inf_{|x|\vee|y|\le r}\alpha(x,y)$ and $\Phi(R) := \inf_{|x|\ge R}V(x) \uparrow \infty$ as $R \uparrow \infty$. For $\varphi \in \mathscr{S}$ and $\Phi^{-1}(r) = \inf\{s \ge 0 : \Phi(s) \ge r\}$, $r \ge 0$, let

$$\Gamma(r) = \inf\left\{s > 0 : s^{d/\mathbf{a}(\Phi^{-1}(2s^{-1})+1)}\varphi\big(\kappa + \Phi^{-1}(2s^{-1})\big)^2 \ge r^{-1}\right\}, \quad r > 0.$$

If

$$\lambda(t) := \int_t^\infty \frac{\Gamma(r)}{r}\mathrm{d}r < \infty, \quad t > 0,$$

then there exist constants $\delta_1, \delta_2 > 0$ such that

$$\lambda_n \ge \frac{\delta_1}{\lambda(\delta_2 n)}, \quad n \ge 1. \tag{2.7}$$

Remark 2.1. Condition (2.1) in (A) will be only used to verify the condition

$$P_t\phi(x) \le \mathrm{e}^{ct}\phi(x), \quad x \in \mathbb{R}^d, t \ge 0, \tag{2.8}$$

where P_t is the Markov for the jump process with jump kernel $J(x, y)$, $\phi \in \mathscr{S}$, and $c > 0$ is a constant depending on ϕ. If the heat kernel of P_t satisfies the upper bound estimate

$$p_t(x, y) \le ct(t^{\frac{1}{\alpha}} + |x - y|)^{-(d+\alpha)}, \quad x, y \in \mathbb{R}^d, t > 0 \tag{2.9}$$

for some constants $c > 0$ and $\alpha \in (0, 2)$, then $P_t\phi \le CP_t^\alpha\phi$ holds for some constant $C > 0$, where P_t^α is the semigroup of the jump process with jump kernel $J_\alpha(x, y) := |x - y|^{-(d+\alpha)}$ which trivially satisfies condition (2.1). According to the proof of Lemma 3.1 below for J_α replacing J, we have

$$P_t^\alpha\phi \le \mathrm{e}^{\lambda t}\phi$$

for some constant $\lambda > 0$, so that (2.8) follows. Therefore, under the heat kernel estimate (2.9), we can drop (2.1) from assumption (A) in Theorem 2.3.

Consider the following stable-like jump kernel J satisfying

$$J(x, y)1_{\{|x-y|\le\kappa\}} = \frac{n(x, y)}{|x - y|^{d+\alpha(x,y)}}1_{\{|x-y|\le\kappa\}}, \quad x, y \in \mathbb{R}^d, \tag{2.10}$$

where $\kappa > 0$ is a constant and

$$n, \alpha : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$$

are measurable and symmetric, such that

(a) There exists a constant $\varepsilon \in (0, 1)$ such that $\varepsilon \le n(x, y) \le \varepsilon^{-1}, x, y \in \mathbb{R}^d$;

(b) There exists constants $\alpha_2 \in (0, 2)$ such that $0 < \alpha(x, y) \le \alpha_2, x, y \in \mathbb{R}^d$;

(c) $\sup_{x\in\mathbb{R}^d} \int_{\{|z|\le1\}} \frac{|n(x,x+z)-n(x,x-z)|}{|z|^{d+\alpha_2-1}}\mathrm{d}z < \infty$.

Then assumptions (A) holds, so that Theorem 2.3 applies.

For instance, we have the following concrete example.

Example 2.1. Let J be in (2.10) with

$$\alpha(x, y) = \alpha_0 + \frac{\beta_1}{[\log(\beta_2 + |x| + |y|)]^{\frac{1}{2}}},$$

where $\alpha_0 \in (0, 2)$ and $\beta_1 > 0, \beta_2 > 1$ such that $\frac{\beta_1}{(\log\beta_2)^{\frac{1}{2}}} \in (0, 2 - \alpha_0)$. If $V(x) \ge c|x|^\theta$ holds for some constants $c, \theta > 0$, then for any $\delta \in \big(0, \frac{d\beta_1\sqrt{\theta}}{\alpha_0^2}\big(\frac{d(\alpha_0+\theta)}{\alpha_0\theta}\big)^{\frac{3}{2}}\big)$ there exists a constant $c(\delta) > 0$ such that

$$\lambda_n \ge c(\delta)n^{\frac{\theta\alpha_0}{d(\theta+\alpha_0)}} \exp\Big[\delta\sqrt{\log n}\Big], \quad n \ge 1. \tag{2.11}$$

Proof. We may take $\Phi(r) = cr^\theta$ for large $r > 0$ and $\mathbf{a}(r) = \alpha_0 + \frac{\beta_1}{\{\log(\beta_2+2r)\}^{\frac{1}{2}}}$ for $r > 0$. Taking $\varphi(r) = (1+r^2)^d$, for $s \in (0,1)$ we have

$$\Phi^{-1}(2s^{-1}) \le c_1 s^{-\frac{1}{\theta}}, \qquad \mathbf{a}(\Phi^{-1}(2s^{-1})+1) \ge \alpha_0 + \frac{\beta_1\sqrt{\theta}}{\sqrt{\log s^{-1}}} + \mathrm{o}\big((\log s^{-1})^{-\frac{1}{2}}\big).$$

So, there exist constants $c_1, c_2 > 0$ such that

$$\begin{aligned} s^{d/\mathbf{a}(\Phi(2s^{-1})+1)} &\le s^{\frac{d}{\alpha}0 - \frac{d\beta_1\sqrt{\theta}}{\alpha_0^2\sqrt{\log s^{-1}}+c_1}} \\ &\le c_2 s^{\frac{d}{\alpha}0} \exp\Big[-\frac{d\beta_1\sqrt{\theta}}{\alpha_0^2}\sqrt{\log s^{-1}}\Big], \quad s \in (0, s_0 \wedge \mathrm{e}^{-1}]. \end{aligned}$$

Let $\varphi(r) = (1+r^2)^{-\frac{d}{4}}\{\log(\mathrm{e}+r^2)\}^{-2}$. Then $\varphi \in \mathscr{S}$ and for any constant $\delta' \in (0, d\beta_1\sqrt{\theta}\alpha_0^{-2})$, there exists constant $c_1(\delta') > 0$ such that

$$s^{d/\mathbf{a}(\Phi(2s^{-1})+1)}\varphi\big(\kappa + \Phi^{-1}(2s^{-1})\big)^2 \le c(\delta') s^{\frac{d(\theta+\alpha_0)}{\theta\alpha_0}} \exp\Big[-\delta'(\log s^{-1})^{\frac{1}{2}}\Big], \quad s \in (0, s_0 \wedge \mathrm{e}^{-1}].$$

Then there exists a constant $c_2(\delta') > 0$ such that

$$\Gamma(r) \le c_2(\delta') r^{-\frac{\theta\alpha_0}{d(\theta+\alpha_0)}} \exp\Big[-\delta'\Big(\frac{d(\theta+\alpha_0)}{\theta\alpha_0}\Big)^{\frac{3}{2}}\sqrt{\log r}\Big], \quad r \ge 1,$$

so that for some constant $c_3(\delta') > 0$,

$$\lambda(t) := \int_t^\infty \frac{\Gamma(r)}{r}\mathrm{d}r \le c_3(\delta') t^{-\frac{\theta\alpha_0}{d(\theta+\alpha_0)}} \exp\Big[-\delta'\Big(\frac{d(\theta+\alpha_0)}{\theta\alpha_0}\Big)^{\frac{3}{2}}\sqrt{\log t}\Big], \quad t \ge 1.$$

Since $\delta' \in (0, d\beta_1\sqrt{\theta}\alpha_0^{-2})$ is arbitrary, this implies (2.11) for $\delta \in \big(0, \frac{d\beta_1\sqrt{\theta}}{\alpha_0^2}\big(\frac{d(\alpha_0+\theta)}{\alpha_0\theta}\big)^{\frac{3}{2}}\big)$. $\square$

When J is comparable with the α-stable kernel of finite range, we have the following sharp result. But in general the lower bound estimate (2.13) may be less sharp than that given in Theorem 2.3. For instance, applying Theorem 2.4(1) to Example 2.2 one only derives

$$\lambda_n \ge cn^{\frac{\theta\alpha_0}{d(\theta+\alpha_0)}}, \quad n \ge 1$$

which is less sharp than (2.11).

Theorem 2.4. *Let J satisfy (A) and let $\theta, \kappa > 0, \alpha \in (0,2)$ be constants.*

(1) *If there exists a constant $c > 0$ such that $V(x) \ge c|x|^\theta$ for large $|x|$, and*

$$J(x,y) \ge \frac{c}{|z|^{d+\alpha}} 1_{\{|z|\le\kappa\}}, \quad x, y \in \mathbb{R}^d, \tag{2.12}$$

then there exists a constant $\delta > 0$ such that

$$\lambda_n \ge \delta n^{\frac{\theta\alpha}{d(\theta+\alpha)}}, \quad n \ge 1. \tag{2.13}$$

(2) *If there exists a constant* $c > 0$ *such that* $V(x) \le c|x|^\theta$ *for large* $|x|$, *and*

$$J(x,y)1_{\{|x-y|\le\kappa\}} \le \frac{c}{|x-y|^{d+\alpha}}, \quad x,y \in \mathbb{R}^d, \tag{2.14}$$

then there exists a constant $\delta > 0$ *such that*

$$\lambda_n \le \delta n^{\frac{\theta\alpha}{d(\theta+\alpha)}}, \quad n \ge 1. \tag{2.15}$$

We now apply Theorem 2.4 to specific models induced by symbols of pseudo differential operators. Let

$$\psi(\xi) = \int_{\mathbb{R}^d} (1 - \cos x \cdot \xi)\nu(\mathrm{d}x), \quad \xi \in \mathbb{R}^d$$

for a Lévy measure ν with $\nu(1 \wedge |\cdot|^2) < \infty$. Typical examples of ψ include $\psi(\xi) = |\xi|^\alpha$ for $\alpha \in (0,2)$, or in general

$$\psi(\xi) = \sum_{i=1}^m c_i \Big(\sum_{j=1}^d |\xi_j|^{\alpha_{ij}} \Big)^{\beta_j}$$

for some constants $m \in \mathbb{N}$ and $c_i, \alpha_{ij}, \beta_j > 0$ with $\beta_j \max_i \alpha_{ij} < 2$. Let $\psi(D)$ be the pseudo differential operator induced by ψ, see for instance [18]. We consider the Schrödinger operator

$$L_{\psi,V} := -\psi(D) + V$$

for a nonnegative potential V.

Example 2.2. Assume that

$$c_1|\xi|^\alpha \le \psi(\xi) \le c_2(|\xi|^\alpha + |\xi|^{\alpha'}), \quad \xi \in \mathbb{R}^d \tag{2.16}$$

holds for some constants $2 > \alpha \ge \alpha' > 0$ and $c_1, c_2 > 0$. If

$$c_3|x|^\theta \le V(x) \le c_4|x|^\theta \ \text{ for large } |x|, \tag{2.17}$$

holds for some constants $\theta, c_3, c_4 > 0$, then

$$\delta_1 \delta n^{\frac{\theta\alpha}{d(\theta+\alpha)}} \le \lambda_n \le \delta_2 n^{\frac{\theta\alpha}{d(\theta+\alpha)}}, \quad n \ge 1 \tag{2.18}$$

holds for some constants $\delta_1, \delta_2 > 0$.

Proof. It is well known that $L_{\psi,V} = L_{J,V}$ for some jump rate J with $J(x,y) = J(0, x-y) = J(0, y-x)$ so that $(A1)$ holds, and by (2.16), there exists a constant $C > 1$ such that

$$\frac{1_{\{|x-y|\le1\}}}{C|x-y|^{d+\alpha}} \le J(x,y)1_{\{|x-y|\le1\}} \le \frac{C1_{\{|x-y|\le1\}}}{|x-y|^{d+\alpha}}, \quad x,y \in \mathbb{R}^d.$$

Then the proof is finished by Theorem 2.4. □

Example 2.3. Consider, for instance, $\psi(\xi) = (|\xi_1|^{r_1} + |\xi_2|^{\beta_1})^{\gamma_1} + (|\xi_1|^{r_2} + |\xi_2|^{\beta_2})^{\gamma_2}$ on $\mathbb{R}^2$ for some constants $r_i, \beta_i, \gamma_i > 0$ such that $r_1\gamma_1 = \beta_2\gamma_2, r_2\gamma_2 = \beta_1\gamma_1 \in (0, 2)$. Then condition (2.16) holds for

$$\alpha_1 := \max\{r_1\gamma_1, \beta_1\gamma_1\}, \quad \alpha_2 := \min\{r_1\gamma_1, \beta_1\gamma_1\}.$$

By Theorem 2.4, if (2.17) holds then eigenvalues $\{\lambda_n\}_{n\geq 1}$ of $-L_{\psi,V}$ satisfies (2.18) for some constants $\delta_1, \delta_2 > 0$.

Moreover, for $r_1\gamma_1 > r_2\gamma_2 > 1$ the function ψ is coercive, convex, negative definite and satisfies

$$K_0|\xi|^{r_1\gamma_1} \leq \psi(\xi) \leq K_1\big(|\xi|^{r_1\gamma_1} + |\xi|^{r_2\gamma_2}\big)$$

for some constants $K_0, K_1 > 0$. The convex conjugate function ψ^* is again coercive, i.e. $\lim_{|\xi|\to\infty} \frac{\psi^*(\xi)}{|\xi|} = \infty$, and satisfies $\psi^*(\xi) \leq K_2|\xi|^{(r_1\gamma_1)^*}$ for $(r_1\gamma_1)^* := \frac{r_1\gamma_1}{r_1\gamma_1 - 1}$. With the help of [11, Theorem 3 on page 87], we deduce that $\psi^* \geq 0$. Hence we may apply Theorem 2.4 to $\psi(D) + \psi^*(x)$.

3 Proof of Theorem 2.3

We first prove Theorem 2.2 using results in [36].

Proof of Theorem 2.2. Without loss of generality, we may and do assume that $\mu_\phi(\mathrm{d}x) := \phi(x)^2\mathrm{d}x$ is a probability measure. Simply denote $P_t = P_t^{J,V} = \mathrm{e}^{tL_{J,V}}$, the (sub)-Markov semigroup generated by $L_{J,V}$. We consider the following symmetric semigroup P_t^ϕ on $L^2(\mu_\phi)$:

$$P_t^\phi f(x) := \phi^{-1}P_t(f\phi), \quad f \in L^2(\mu_\phi), \ \ t \geq 0.$$

According to [36, Theorem 3.3(1)] for $\inf\beta = 0$, P_t^ϕ has a symmetric heat kernel $p_t^\phi(x, y)$ with respect to μ_ϕ, i.e.

$$P_t^\phi f(x) = \int_{\mathbb{R}^d} p_t^\phi(x, y)f(y)\phi(y)^2\mathrm{d}y, \quad f \in L^2(\mu_\phi), t > 0,$$

such that

$$\sup p_t^\phi = \|P_t^\phi\|_{L^1(\mu_\phi)\to L^\infty(\mu_\phi)} \leq \mathrm{e}^{\lambda t}\Lambda^{-1}(\varepsilon t)^2, \ \ t > 0, \varepsilon \in (0, 1)$$

where $\Lambda^{-1}(t) := \inf\{r > 0 : \Lambda(r) \leq t\} < \infty, \ \ t > 0$. Since Λ^{-1} is continuous, by letting $\varepsilon \to 0$ we obtain $p_t^\phi \leq \mathrm{e}^{\lambda t}\Lambda^{-1}(t)^2$. Consequently, the heat kernel p_t of P_t with respect to the Lebesgue measure has the upper bound estimate

$$p_t(x, y) \leq \phi(x)\phi(y)\mathrm{e}^{\lambda t}\Lambda^{-1}(t), \ \ t > 0, \ x, y \in \mathbb{R}^d.$$

In particular, $\int_{\mathbb{R}^d} p_t(x, x)\mathrm{d}x \leq \mathrm{e}^{\lambda t}\Lambda^{-1}(t), \ t > 0$. Combining this with [36, Theorem 2.4], we obtain

$$\lambda_n \geq \sup_{t>0} \frac{1}{t} \log \frac{n\mathrm{e}^{-\lambda t}}{\Lambda^{-1}(t)}, \ \ n \geq 1.$$

With $t = \Lambda(\varepsilon n)$ this implies

$$\lambda_n \geq \frac{\log(\varepsilon^{-1} \mathrm{e}^{-\lambda\Lambda(\varepsilon n)})}{\Lambda(\varepsilon n)}, \quad n \geq 1. \tag{3.1}$$

Since $\Lambda(r) \to 0$ as $r \to \infty$, there exists $n_0 \geq 1$ such that $c_0 := \varepsilon^{-1}\mathrm{e}^{-\lambda\Lambda(\varepsilon n_0)} > 1$. By the decreasing monotonicity of Λ, we have $\varepsilon^{-1}\mathrm{e}^{-\lambda(\varepsilon n)} \geq c_0 > 1$ for $n \geq n_0$. So, (3.1) implies (2.4) for $c := \log c_0 > 0$ and $n \geq n_0$. Combining this with $\lambda_1 > 0$, we conclude that (2.4) holds for $c := \min\{\log c_0, \lambda_1 \Lambda(\varepsilon n_0)\} > 0$ and all $n \geq 1$.

□

To prove Theorem 2.3 using Theorem 2.2, we first verify $P_t^{J,V}\phi \leq \mathrm{e}^{\lambda t}\phi$ for $\phi := \varphi(|\cdot|)$ and $\lambda > 0$.

Lemma 3.1. *For any positive $\varphi \in C^2([0,\infty)$ satisfying condition* (ii)*, there exists a constant $\lambda \geq 0$ such that*

$$P_t^{J,V}\varphi(|\cdot|)(x) \leq \mathrm{e}^{\lambda t}\varphi(|\cdot|)(x), \quad t \geq 0, x \in \mathbb{R}^d.$$

Proof. Let $\phi(x) = \varphi(|x|)$. Then condition (ii) implies

$$\begin{aligned} &\sup_{|z|\leq 1} |z|^{-2}\big|\phi(x+z) - \phi(x) - \langle \nabla\phi(x), z\rangle\big| \leq \sup_{|z|\leq 1} \|\mathrm{Hess}_\phi(x+z)\| \\ &\leq \sup_{r\in(0,1+|x|]} \Big(\frac{|\varphi'(r)|}{r} + |\varphi''(r)|\Big) \leq c_1\phi(x), \quad x \in \mathbb{R}^d \end{aligned} \tag{3.2}$$

for some constant $c_1 > 0$. Next, if $|x+z| \geq \frac{1}{2}|x|$, then (ii) implies

$$\phi(x+z) - \phi(x) \leq \phi(|x|/2) \leq c\phi(x)$$

for some constant $c > 0$; while if $|x+z| < \frac{1}{2}|x|$ then $\frac{1}{2}|x| \leq |z| \leq \frac{3}{2}|x|$, so that (ii) gives

$$\phi(x+z) - \phi(x) \leq |z| \int_0^1 |\varphi'(|x+sz|)|\mathrm{d}s \leq c \sup_{r\in(0,\frac{5}{2}|x|]} r|\varphi'(r)| \leq c'\phi(x)$$

for some constants $c, c' > 0$. Combining these with (3.2) we conclude that

$$\big|\phi(x+z) - \phi(x) - \langle\nabla\phi(x), z\rangle 1_{\{|z|\leq 1\}}\big| \leq c_2(|z|^2 \wedge 1)\phi(x)$$

holds for some constant $c_2 > 0$. Therefore, it follows from condition (2.2) in (A) that

$$\int_{\mathbb{R}^d} \big(\phi(x+z) - \phi(x) - \langle\nabla\phi(x), z\rangle 1_{\{|z|\leq 1\}}\big) J(x, x+z)\mathrm{d}z \leq c_3\phi(x), \quad x \in \mathbb{R}^d \tag{3.3}$$

holds for some constant $c_3 > 0$. Moreover, by (i) and condition (2.1) in (A), we have

$$\Big|\int_{\{|z|\leq 1\}} \langle\nabla\phi(x), z\rangle\big(J(x, x+z) - J(x, x-z)\big)\mathrm{d}z\Big|$$

$$\leq c_4\phi(x)\int_{\{|z|\leq 1\}}|z|\cdot\big|J(x,x+z)-J(x,x-z)\big|\mathrm{d}z\leq c_5\phi(x),\quad x\in\mathbb{R}^d$$

for some constants $c_4, c_5 > 0$. This together with (3.3) yields

$$\begin{aligned}L^{J,0}\phi(x):=&\int_{\mathbb{R}^d}\big(\phi(x+z)-\phi(x)-\langle\nabla\phi(x),z\rangle 1_{\{|z|\leq 1\}}\big)J(x,x+z)\mathrm{d}z\\&+\int_{\{|z|\leq 1\}}\langle\nabla\phi(x),z\rangle\big(J(x.x+z)-J(x,x-z)\big)\mathrm{d}z\leq\lambda\phi(x)\end{aligned}$$

for some constant $\lambda\geq 0$ and all $x\in\mathbb{R}^d$. Therefore, letting X_t^x be the jump process with jump rate J starting at x, we obtain

$$P_t^{J,0}\phi(x):=\mathbb{E}[\phi(X_t^x)]\leq \mathrm{e}^{\lambda t}\phi(x),\quad t\geq 0, x\in\mathbb{R}^d.$$

By Feynman-Kac formula and $V\geq 0$, we conclude that

$$P_t^{J,V}\phi(x)=\mathbb{E}\big[\phi(X_t^x)\mathrm{e}^{-\int_0^t V(X_s^x)\mathrm{d}s}\big]\leq P_t^{J,0}\phi(x)\leq \mathrm{e}^{\lambda t}\phi(x),\quad t\geq 0, x\in\mathbb{R}^d.$$

□

Proof of Theorem 2.3. According to Theorem 2.2 and Lemma 3.1, it suffices to estimate the rate function β in (2.3).

Let $\phi(x)=\varphi(|x|)$. We first prove the following local super Poincaré inequality

$$\int_{B(0,r)}f^2(x)dx\leq s\mathscr{E}_{J,V}(f,f)+c\alpha(r,s)\bigg(\int_{B(0,r+1)}(|f|\phi)(x)\mathrm{d}x\bigg)^2,\quad s>0, r\geq 1, \tag{3.4}$$

for some constant $c>0$, where

$$\alpha(r,s):=\frac{c(1+s^{-\frac{d}{\mathbf{a}(r+1)}})}{\varphi(r+1)^2}. \tag{3.5}$$

Indeed, for any $f\in L^2(\mathbb{R}^d)$ and $t\in(0,\kappa\wedge 1]$, let

$$f_t(x)=\frac{1}{|B(0,t)|}\int_{B(x,t)}f(y)\mathrm{d}y,\quad x\in\mathbb{R}^d.$$

According to step (ii) in the proof of [8, Proposition 3.5], we have

$$\begin{aligned}&\int_{B(0,r)}f^2(x)\mathrm{d}x\leq 2\int_{B(0,r)}|f(x)-f_t(x)|^2\mathrm{d}x+2\int_{B(0,r)}f_t^2(x)\mathrm{d}x\\&\leq\frac{2}{|B(0,t)|}\bigg\{\int_{B(0,r)\times B(x,t)}|f(x)-f(y)|^2\mathrm{d}x\mathrm{d}y+\bigg(\int_{B(0,r+t)}|f(y)|\mathrm{d}y\bigg)^2\bigg\}.\end{aligned} \tag{3.6}$$

Since $t\leq\kappa\wedge 1$, by (2.12) for $(x,y)\in B(0,r)\times B(x,t)$ we have

$$J(x,y)\geq\frac{c_1}{|x-y|^{d+\alpha(x,y)}}\geq\frac{c_2}{t^{d+\mathbf{a}(r+1)}},$$

so that

$$\int_{B(0,r)} |f(x)-f_t(x)|^2 \mathrm{d}x \le c_2^{-1} t^{d+\mathbf{a}(r+1)} \mathscr{E}_{J,V}(f,f). \tag{3.7}$$

Moreover, since $\phi(x)=\varphi(|x|)$ is decreasing in $|x|$, it holds that

$$\int_{B(0,r+t)} |f(y)|\mathrm{d}y \le \frac{1}{\varphi(r+1)} \int_{B(0,r+t)} |f\phi(y)|\mathrm{d}y, \quad t\in(0,1\wedge\kappa].$$

Combining this with (3.6) and (3.7), we may find a constant $c_3>0$ such that

$$\int_{B(0,r)} f^2(x)\mathrm{d}x \le c_3 t^{\mathbf{a}(r+1)} \mathscr{E}_{J,V}(f,f) + \frac{c_3}{t^d \varphi(r+1)^2}\bigg(\int_{\mathbb{R}^d} |f\phi|(x)\mathrm{d}x\bigg)^2, \quad t\in(0,1\wedge\kappa], r\ge 1.$$

Taking $s=c_3 t^{\mathbf{a}(r+1)}$ and noting that $\kappa^{\mathbf{a}(r+1)} \ge 1\wedge\kappa^2$, we may find out constants $c_4, s_0>0$ such that

$$\int_{B(0,r)} f^2(x)\mathrm{d}x \le s\mathscr{E}_{J,V}(f,f) + \frac{c_4 s^{\frac{d}{\mathbf{a}(r+1)}}}{\varphi(r+1)^2}\bigg(\int_{\mathbb{R}^d} |f\phi|(x)\mathrm{d}x\bigg)^2, \quad s\in(0,s_0], r\ge 1.$$

This implies (3.4).

We now estimate β in (2.3) using (3.4) and the growth function Φ of V. We first assume $\inf V>0$ then extend to the general case.

By [8, Proposition 3.5], condition (2.12) implies the following local super Poincaré inequality holds for

(a) Let $K:=\frac{1}{2}\inf V>0$. Then

$$\Phi(R) := \inf_{|x|\ge R, V(x)>K} V(x) = \inf_{|x|\ge R} V(x), \quad \Theta(R) := \mathrm{vol}(\{x: |x|\ge R, V(x)\le K\}) = 0. \quad R>0.$$

Combining this with (3.5), we may apply [8, Theorem 2.1] for $r_0=0$ to obtain (2.3) with

$$\begin{aligned}\beta(s) &= \delta_3 \alpha\big(\Phi^{-1}(2s^{-1}), s/2\big)\\ &\le \delta_4\big(1+s^{-d/\mathbf{a}(\Phi^{-1}(2s^{-1})+1)}\big)\varphi(1+\Phi^{-1}(2s^{-1}))^{-2}, \quad s>0\end{aligned}$$

for some constants $\delta_3, \delta_4>0$. Since $\lambda_1>0$, when $s\ge\lambda_1$, (2.3) holds for $\beta(s)=0$. Therefore, there exists a constant $\delta_5>0$ such that (2.3) holds for

$$\beta(s) := \delta_5 s^{-d/\mathbf{a}(\Phi^{-1}(2s^{-1})+1)}\varphi(1+\Phi^{-1}(2s^{-1}))^{-2} =: \delta_5\tilde\beta(s), \quad s>0.$$

Then, for Γ in Theorem 2.3, we have

$$\beta^{-1}(r) := \inf\{s>0: \beta(s)\le r\} = \inf\{s>0: \tilde\beta(s)\le \delta_5^{-1} r\} = \Gamma(\delta_5 r), \quad r>0.$$

Thus,

$$\Lambda(t) := \int_t^\infty \frac{\beta^{-1}(r)}{r}\mathrm{d}r = \int_t^\infty \frac{\Gamma(\delta_5 r)}{r}\mathrm{d}r = \lambda(\delta_5 t), \quad t>0.$$

According to Theorem 2.2 and Lemma 3.1, this implies (2.7) for some constants $\delta_1, \delta_2 > 0$.

(b) In general, let $\bar V = V + \lambda_1$. Since $\mathscr{E}_{J,V}(f,f) \le \mathscr{E}_{J,\bar V}(f,f)$, (2.3) also holds for $\mathscr{E}_{J,\bar V}$ replacing $\mathscr{E}_{J,V}$. Since $\inf \bar V \ge \lambda_1 > 0$, by (a) we see that (2.7) holds for the eigenvalues $\bar\lambda_n$ of $-L_{J,\bar V}$, i.e.

$$\bar\lambda_n \ge \frac{\delta_1}{\lambda(\delta_2 n)}, \quad n \ge 1$$

holds for some constants $\delta_1, \delta_2 > 0$. Noting that $\bar\lambda_n = \lambda_n + \lambda_1 \le 2\lambda_n$, we prove (2.7) for $2\delta_1$ replacing δ_1.

□

4 Proof of Theorem 2.4

We first present a general result on the lower bound estimate by using [36, Theorem 2.4]. Let $E, \mu, (L_0, \mathscr{D}(L_0)), P_t^0, (L_V, \mathscr{D}(L_V))$ and P_t^V be in the beginning of Section 2. Assume that P_t^0 and P_t^V are realized by a Markov process X_t; i.e. for any $x \in E$, X_t^x is a Markov process on E starting at x such that

$$P_t^0 f(x) = \mathbb{E} f(X_t^x), \quad P_t^v f(x) = \mathbb{E}\Big[f(X_t^x)\mathrm{e}^{-\int_0^t V(X_s^x)\mathrm{d}s}\Big] \tag{4.1}$$

holds for $x \in E, t \ge 0$ and $f \in \mathscr{B}_b(E) \cap L^2(\mu)$. We have the following result.

Proposition 4.1. *Assume that*

$$\rho_1(t) := \|P_t^0\|_{L^1(\mu)\to L^\infty(\mu)} < \infty, \quad \rho_2(t) := \int_E \mathrm{e}^{-2tV}\mathrm{d}\mu < \infty, \quad t > 0.$$

Then $-L_V$ has purely discrete specturm and the eigenvalues $(\lambda_n^V)_{n\ge1}$ satisfies

$$\lambda_n^V \ge \inf_{t>0} \frac{1}{2t} \log \frac{n+1}{\rho_1(t)\rho_2(t)}, \quad n \ge 1.$$

Proof. Let $f \in \mathscr{B}_b(E) \cap L^2(\mu)$. By (4.1) and the Schwarz/Jensen inequalities, we obtain

$$\begin{aligned} P_t^V f(x) &= \mathbb{E}\Big[f(X_t^x)\mathrm{e}^{-\int_0^t V(X_s^x)\mathrm{d}s}\Big] \le \sqrt{P_t^0 f^2(x)\mathbb{E}\mathrm{e}^{-2\int_0^t V(X_s^x)\mathrm{d}s}} \\ &\le \sqrt{\rho_1(t)}\|f\|_{L^2(\mu)}\bigg(\frac{1}{t}\int_0^t P_s^0 \mathrm{e}^{-2tV}(x)\mathrm{d}s\bigg)^{\frac{1}{2}}. \end{aligned}$$

This implies that P_t^V has a density $p_t^V(x,y)$ with respect to μ and

$$p_{2t}^V(x,x) = \int_E p_t^V(x,y)^2\mu(\mathrm{d}y) \le \frac{\rho_1(t)}{t}\int_0^t P_s^0\mathrm{e}^{-2tV}(x)\mathrm{d}s.$$

Since μ is P_s^0-invariant, this implies

$$\mu(p_{2t}^V) := \int_E \rho_{2t}^V(x,x)\mu(\mathrm{d}x) \le \rho_1(t)\rho_2(t), \quad t > 0.$$

Therefore, [36, Theorem 2.4] gives

$$\lambda_n^V \ge \inf_{t>0} \frac{1}{2t} \log \frac{n+1}{\mu(p_{2t}^V)} \ge \inf_{t>0} \frac{1}{2t} \log \frac{n+1}{\rho_1(t)\rho_2(t)}, \quad n \ge 1.$$

□

We will also use the following variational formula of λ_n:

$$\lambda_n = \inf_{(u_1,\cdots,u_n)\in\mathscr{S}_n} \sup_{u\in B(u_1,\cdots,u_n)} \mathscr{E}_{J,V}(u,u), \quad n \ge 1, \tag{4.2}$$

where $B(u_1,\cdots,u_n) := \big\{u \in \mathrm{span}\{u_1,\cdots,u_n\} : \int_{\mathbb{R}^d} u(x)^2 \mathrm{d}x = 1\big\}$, and $(u_1,\cdots,u_n) \in \mathscr{S}_n$ means that

$$u_i \in \mathscr{D}(\mathscr{E}_{J,V}), \quad \int_{\mathbb{R}^d} (u_i u_j)(x)\mathrm{d}x = 1_{\{i=j\}}, \quad 1 \le i,j \le n. \tag{4.3}$$

Proof of Theorem 2.4(1). Let $p_t^{\alpha,\theta}(x,y)$ be the heat kernel of the Schrödinger semigroup $P_t^{\alpha,\theta}$ generated by

$$L_{\alpha,\theta} := -(-\Delta)^{\frac{\alpha}{2}} - |\cdot|^\theta.$$

It is well known that

$$\rho_1(t) := \sup_{x,y\in\mathbb{R}^d} p_t^\alpha(x,y) \le c_1 t^{-\frac{d}{\alpha}}, \quad t > 0$$

holds for some constant $c_1 > 0$. Next, there exists a constant $c_2 > 0$ such that

$$\rho_2(t) := \int_{\mathbb{R}^d} \mathrm{e}^{-2|x|^\theta} \mathrm{d}x \le c_2 t^{-\frac{d}{\theta}}, \quad t > 0.$$

So, by Proposition 4.1, the eigenvalues $\{\lambda_n^{\alpha,\theta}\}_{n\ge1}$ of $L_{\alpha,\theta}$ satisfy

$$\lambda_n^{\alpha,\theta} \ge \inf_{t>0} \frac{1}{2t} \log \left[\frac{n+1}{c_1 c_2} t^{\frac{d(\alpha+\theta)}{\alpha\theta}}\right] \ge c_3 n^{\frac{\alpha\theta}{d(\alpha+\theta)}}, \quad n \ge 1 \tag{4.4}$$

for some constant $c_3 > 0$.

Now, let $\mathscr{E}_{\alpha,\theta}$ be the Dirichlet form associated to $L_{\alpha,\theta}$. By (2.12) and $V(x) \ge c|x|^\theta$ for large $|x|$,

$$\mathscr{E}_{J,V}(f,f) \ge c_4 \mathscr{E}_{\alpha,\theta}(f,f) - c_5 \int_{\mathbb{R}^d} f^2(x)\mathrm{d}x \tag{4.5}$$

holds for some constants $c_4, c_5 > 0$. Combining this with (4.2) and (4.4), we prove (2.13) for some constant $\delta > 0$ and large enough n. Since $\lambda_n \ge \lambda_1 > 0$, (2.13) holds for some constant $\delta > 0$ and all $n \ge 1$. □

Proof of Theorem 2.4(2). Similarly to (4.5), condition (2.14) and $V(x) \le c|x|^\theta$ for large $|x|$ imply

$$\mathscr{E}_{J,V}(f,f) \le c' \mathscr{E}_{\alpha,\theta}(f,f) + c' \int_{\mathbb{R}^d} f^2(x)\mathrm{d}x$$

for some constant $c' > 0$. Combining this with (4.2), we may and do assume that

$$V(x) = |x|^\theta, \quad J(x,y) = \frac{1}{|x-y|^{d+\alpha}}, \quad x, y \in \mathbb{R}^d. \tag{4.6}$$

(1) To construct suitable functions $(u_1, \cdots, u_n) \in \mathscr{S}_n$, let

$$\begin{aligned} &\xi(k) = k^{\frac{\alpha}{\theta+\alpha}}, \quad k \geq 1, \\ &h_k(s) = \min\big\{(s-\xi(k))^+, (\xi(k+1)-s)^+\big\}, \quad s \in \mathbb{R}, k \geq 1. \end{aligned}$$

Obviously, there exists a constant $c_0 > 1$ such that

$$c_0^{-1} k^{-\frac{\theta}{\theta+\alpha}} \leq \xi(k+1) - \xi(k) \leq c_0 k^{-\frac{\theta}{\theta+\alpha}}, \quad k \geq 1. \tag{4.7}$$

Next, for any $n \geq 1$, let

$$G_n = \{1, 2, \cdots, n\}^d = \big\{(k_1, \cdots, k_d) : \ 1 \leq k_i \leq n, 1 \leq i \leq d\big\}.$$

Define

$$u_{\mathbf{k}}(x) := \prod_{i=1}^d h_{k_i}(x_i), \quad \mathbf{k} := (k_1, \cdots, k_d) \in G_n.$$

Then

$$\int_{\mathbb{R}^d} (u_{\mathbf{k}} u_{\mathbf{k}'})(x) \mathrm{d}x = 0, \quad \mathbf{k} \neq \mathbf{k}',$$

and

$$\begin{aligned} \int_{\mathbb{R}^d} u_{\mathbf{k}}(x)^2 \mathrm{d}x &= \prod_{i=1}^d \bigg(2\int_{\xi(k_i)}^{\frac{1}{2}(\xi(k_i)+\xi(k_i+1))} (s-\xi(k_i))^2 \mathrm{d}s\bigg) \\ &= \prod_{i=1}^d \bigg(2\int_0^{\frac{1}{2}(\xi(k_i+1)-\xi(k_i))} s^2 \mathrm{d}s\bigg) = \frac{1}{6^d} \prod_{i=1}^d \big(\xi(k_i+1) - \xi(k_i)\big)^3. \end{aligned} \tag{4.8}$$

So, there exists a constant $c_1, c_2 > 0$ such that

$$c_1 \prod_{i=1}^d k_i^{-\frac{3\theta}{\theta+\alpha}} \leq I_{\mathbf{k}} := \int_{\mathbb{R}^d} u_{\mathbf{k}}(x)^2 \mathrm{d}x \leq c_2 \prod_{i=1}^d k_i^{-\frac{3\theta}{\theta+\alpha}}, \quad \mathbf{k} \in G_n, n \geq 1. \tag{4.9}$$

Since every $u_{\mathbf{k}}$ is Lipschitz continuous with compact support, we have $u_{\mathbf{k}} \in \mathscr{D}(\mathscr{E}_{J,V})$. Since G_n contains n^d many numbers, we have $\big(I_{\mathbf{k}}^{-\frac{1}{2}} u_{\mathbf{k}} : \mathbf{k} \in G_n\big) \in \mathscr{S}_{n^d}$ so that by (4.2),

$$\lambda_{n^d} \leq \sup_{u \in B(u_{\mathbf{k}}:\ \mathbf{k} \in G_n)} \mathscr{E}_{J,V}(u,u), \quad n \geq 1. \tag{4.10}$$

(2) To estimate the upper bound in (4.10), we first bound $\mathscr{E}_{J,V}(u_{\mathbf{k}}, u_{\mathbf{k}})$ by I_k. We Observe that

$$\begin{aligned} &u_{\mathbf{k}}(x+z) - u_{\mathbf{k}}(x) \neq 0 \text{ implies} \\ &\xi(k_i) - |z_i| \leq x_i \leq \xi(k_i+1) + |z_i| \text{ for all } 1 \leq i \leq d. \end{aligned} \tag{4.11}$$

Indeed, $u_{\mathbf{k}}(x+z)-u_{\mathbf{k}}(x)\neq 0$ only if at leat one of $u_{\mathbf{k}}(x+z)$ and $u_{\mathbf{k}}(x)$ is non-zero, so that either $\xi(k_i)\le x_i\le \xi(k_i+1)$ or $\xi(k_i)\le x_i+z_i\le \xi(k_i+1)$ holds, which ensures the assertion. For any $z=(z_1,\cdots,z_d)\in\mathbb{R}^d$, we take

$$z^0=0,\quad z^i=(z_1,z_2,\cdots,z_i,0,\cdots,0)\in\mathbb{R}^d,\quad 1\le i\le d.$$

By (4.6), we obtain

$$\begin{aligned}&\int_{\mathbb{R}^d\times\mathbb{R}^d}\big(u_{\mathbf{k}}(x+z)-u_{\mathbf{k}}(x)\big)^2 J(x,x+z)\mathrm{d}z\\ &\le d\sum_{i=1}^d\int_{\mathbb{R}^d\times\mathbb{R}^d}\big(u_{\mathbf{k}}(x+z^i)-u_{\mathbf{k}}(x+z^{i-1})\big)^2|z|^{-(d+\alpha)}\mathrm{d}z,\quad x\in\mathbb{R}^d.\end{aligned}\tag{4.12}$$

According to (4.11), $u_{\mathbf{k}}(x+z^i)-u_{\mathbf{k}}(x+z^{i-1})\neq 0$ implies

$$\xi(k_i)-|z_i|\le x_i\le \xi(k_i+1)+|z_i|.$$

This together with the Lipschitz continuity of $u_{\mathbf{k}}$ implies

$$\begin{aligned}&1_{\{|z_i|\le\xi(k_i+1)-\xi(k_i)\}}\big(u_{\mathbf{k}}(x+z^i)-u_{\mathbf{k}}(x+z^{i-1})\big)^2\\ &\le 1_{\{|z_i|\le\xi(k_i+1)-\xi(k_i)\}}1_{\{\xi(k_i)-|z_i|\le x_i\le\xi(k_i+1)+|z_i|\}}z_i^2\prod_{j\neq i}(h_{k_j}(x_j+(z^{i-1})_j))^2\\ &\le 1_{\{|z_i|\le\xi(k_i+1)-\xi(k_i)\}}1_{\{2\xi(k_i)-\xi(k_i+1)\le x_i\le 2\xi(k_i+1)-\xi(k_i)\}}z_i^2\prod_{j\neq i}(h_{k_j}(x_j+(z^{i-1})_j)))^2,\quad x\in\mathbb{R}^d.\end{aligned}$$

As in (4.8) we have

$$\int_{\mathbb{R}^{d-1}}\prod_{j\neq i}(h_{k_j}(x_j+(z^{i-1})_j)))^2\prod_{j\neq i}\mathrm{d}x_j=6^{1-d}\prod_{j\neq i}(\xi(k_j+1)-\xi(k_j))^3.$$

Combining these with (4.7) and (4.9), we arrive at

$$\begin{aligned}&\int_{\mathbb{R}^d}\mathrm{d}x\int_{\{z:|z_i|\le\xi(k_i+1)-\xi(k_i)\}}\big(u_{\mathbf{k}}(x+z^i)-u_{\mathbf{k}}(x+z^{i-1})\big)^2|z|^{-(d+\alpha)}\mathrm{d}z\\ &\le c_3\Big\{\prod_{j\neq i}(\xi(k_j+1)-\xi(k_j))^3\Big\}\cdot\big\{\xi(k_i+1)-\xi(k_i)\big\}\\ &\quad\times\int_{\{z:|z_i|\le\xi(k_i+1)-\xi(k_i)\}}z_i^2|z|^{-(d+\alpha)}\mathrm{d}z\\ &\le c_4 I_{\mathbf{k}}k_i^{\frac{2\theta}{\theta+\alpha}}\int_0^{\xi(k_i+1)-\xi(k_i)}s^2\mathrm{d}s\int_{\mathbb{R}^{d-1}}(|s+|\tilde z|)^{-(d+\alpha)}\mathrm{d}\tilde z\\ &\le c_5 I_{\mathbf{k}}k_i^{\frac{2\theta}{\theta+\alpha}}\int_0^{\xi(k_i+1)-\xi(k_i)}\frac{s^2}{s^{1+\alpha}}\mathrm{d}s\\ &=c_5 I_{\mathbf{k}}k_i^{\frac{2\theta}{\theta+\alpha}}\frac{(\xi(k_i+1)-\xi(k_i))^{2-\alpha}}{2-\alpha}\\ &\le c_6 I_{\mathbf{k}}k_i^{\frac{2\theta}{\theta+\alpha}}k_i^{-\frac{\theta(2-\alpha)}{\theta+\alpha}}=c_6 I_{\mathbf{k}}k_i^{\frac{\theta\alpha}{\theta+\alpha}},\quad k\in G_n,1\le i\le d,n\ge 1\end{aligned}\tag{4.13}$$

for some constants $c_3, c_4, c_5, c_6 > 0$. On the other hand, by (4.7),

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathrm{d}x \int_{\{z:|z_i|>\xi(k_i+1)-\xi(k_i)\}} \big(u_{\mathbf{k}}(x+z^i)-u_{\mathbf{k}}(x+z^{i-1})\big)^2 |z|^{-(d+\alpha)}\mathrm{d}z \\
&\le 2\int_{\{z:|z_i|>\xi(k_i+1)-\xi(k_i)\}} |z|^{-(d+\alpha)}\mathrm{d}z \int_{\mathbb{R}^d} \big(u_{\mathbf{k}}(x+z^i)^2+u_{\mathbf{k}}(x+z^{i-1})^2\big)\mathrm{d}x \\
&= 4I_{\mathbf{k}} \int_{\{z:|z_i|>\xi(k_i+1)-\xi(k_i)\}} |z|^{-(d+\alpha)}\mathrm{d}z \\
&\le 4I_{\mathbf{k}} \int_{\{|z|>\xi(k_i+1)-\xi(k_i)\}} |z|^{-(d+\alpha)}\mathrm{d}z \\
&\le c_7 I_{\mathbf{k}} \big(\xi(k_i+1)-\xi(k_i)\big)^{-\alpha} \\
&\le c_8 I_{\mathbf{k}} k_i^{\frac{\theta\alpha}{\theta+\alpha}}, \quad \mathbf{k}\in G_n, 1\le i\le d, n\ge 1
\end{aligned}$$

holds for some constants $c_7, c_8 > 0$. Combining this with (4.12), (4.13), and that $k_i \le n$ for $\mathbf{k} \in G_n$, we arrive at

$$\mathscr{E}_J(u_{\mathbf{k}}, u_{\mathbf{k}}) := \int_{\mathbb{R}^d\times\mathbb{R}^d} \big(u_{\mathbf{k}}(x+z)-u_{\mathbf{k}}(x)\big)^2 J(x,x+z)\mathrm{d}x\mathrm{d}z \le c_9 I_{\mathbf{k}} n^{\frac{\theta\alpha}{\theta+\alpha}}, \ \ \mathbf{k}\in G_n, n\ge 1 \tag{4.14}$$

for some constants $c, c_9 > 0$. Finally, by (4.6) and noting that $\operatorname{supp} u_{\mathbf{k}} \subset \{|\cdot| \le d\xi(n+1)\}$ for $\mathbf{k} \in G_n$, we have

$$\int_{\mathbb{R}^d} (u_{\mathbf{k}}^2 V)(x)\mathrm{d}x \le c_{10} I_{\mathbf{k}} \xi(n+1)^\theta \le c_{11} I_{\mathbf{k}} n^{\frac{\theta\alpha}{\theta+\alpha}}, \quad \mathbf{k}\in G_n, n\ge 1 \tag{4.15}$$

for some constants $c_{10}, c_{11} > 0$.

(3) Now, for any $u \in B(u_{\mathbf{k}}: \ \mathbf{k} \in G_n)$, we have

$$u = \sum_{\mathbf{k}\in G_n} a_k \frac{u_{\mathbf{k}}}{\sqrt{I_k}}, \quad \sum_{\mathbf{k}\in G_n} a_k^2 = 1.$$

Since $u_{\mathbf{k}} u_{\mathbf{k}'} = 0$ for $\mathbf{k} \ne \mathbf{k}'$, (4.15) implies

$$\int_{\mathbb{R}^d} (u^2 V)(x)\mathrm{d}x = \sum_{\mathbf{k}\in G_n} a_k^2 \int_{\mathbb{R}^d} (u_{\mathbf{k}}^2 V)(x)\mathrm{d}x \le c_{11} n^{\frac{\theta\alpha}{\theta+\alpha}}. \tag{4.16}$$

On the other hand, by (4.11), $u_{\mathbf{k}}(x+z) - u_{\mathbf{k}}(x) \ne 0$ implies

$$\xi(k_i) - |z_i| \le x_i \le \xi(k_i+1) + |z_i|, \quad 1 \le i \le d.$$

So, for any $\mathbf{k}, \mathbf{k}' \in G_n$,

$$|u_{\mathbf{k}}(x+z) - u_{\mathbf{k}}(x)| \cdot |u_{\mathbf{k}'}(x+z) - u_{\mathbf{k}'}(x)| \ne 0 \tag{4.17}$$

only if

$$|z_i| \ge \frac{1}{2}\big\{\xi(k_i \vee k_i') - \xi(k_i \wedge k_i' + 1)\big\}, \quad 1 \le i \le d.$$

Noting that $|k_i - k_i'| \geq 2$ implies

$$\begin{aligned}&\frac{1}{2}\big\{\xi(k_i \vee k_i') - \xi(k_i \wedge k_i' + 1)\big\} \geq \frac{1}{2}\big\{\xi(k_i \vee k_i') - \xi(k_i \vee k_i' - 1)\big\}\\ &\geq c_{12}(k_i \vee k_i')^{-\frac{\theta}{\theta+\alpha}} \geq c_{12}n^{-\frac{\theta}{\theta+\alpha}}\end{aligned}$$

for some constants $c_{12} > 0$. Then, when $\|\mathbf{k} - \mathbf{k}'\|_\infty := \max_{1\leq i\leq d}|k_i - k_i'| \geq 2$, (4.17) implies $|z| \geq c_{12}n^{-\frac{\theta}{\theta+\alpha}}$. Combining this with (4.6) and (4.14), we arrive at

$$\begin{aligned}&\mathscr{E}_J(u,u) \leq c_2 \int_{\mathbb{R}^d\times\mathbb{R}^d} 1_{\{|z|\geq c_{13}n^{-\frac{\theta}{\theta+\alpha}}\}} \frac{|u(x+z)-u(x)|^2}{|z|^{d+\alpha}}\mathrm{d}x\mathrm{d}z\\ &\quad + \int_{\mathbb{R}^d\times\mathbb{R}^d} 1_{\{|z|<c_{12}n^{-\frac{\theta}{\theta+\alpha}}\}}|u(x+z)-u(x)|^2 J(x,x+z)\mathrm{d}x\mathrm{d}z\\ &\leq 2c_2 \int_{\mathbb{R}^d\times\mathbb{R}^d} 1_{\{|z|\geq c_{12}n^{-\frac{\theta}{\theta+\alpha}}\}} \frac{u(x+z)^2+u(x)^2}{|z|^{d+\alpha}}\mathrm{d}x\mathrm{d}z\\ &\quad + \sum_{\mathbf{k},\mathbf{k}'\in G_n, \|\mathbf{k}-\mathbf{k}'\|_\infty\leq 1} |a_{\mathbf{k}}a_{\mathbf{k}'}| \int_{\mathbb{R}^d\times\mathbb{R}^d} |u_{\mathbf{k}}(x+z)-u_{\mathbf{k}}(x)|\cdot|u_{\mathbf{k}'}(x+z)-u_{\mathbf{k}'}(x)|J(x,x+z)\mathrm{d}x\mathrm{d}z\\ &\leq 4c_2 \int_{\{|z|\geq c_{12}n^{-\frac{\theta}{\theta+\alpha}}\}} \frac{\mathrm{d}z}{|z|^{d+\alpha}} + \frac{1}{2}\sum_{\mathbf{k},\mathbf{k}'\in G_n, \|\mathbf{k}-\mathbf{k}'\|_\infty\leq 1} \left(\frac{a_{\mathbf{k}}^2\mathscr{E}_J(u_{\mathbf{k}},u_{\mathbf{k}})}{I_{\mathbf{k}}} + \frac{a_{\mathbf{k}'}^2\mathscr{E}_J(u_{\mathbf{k}'},u_{\mathbf{k}'})}{I_{\mathbf{k}'}}\right)\\ &\leq c_{13}n^{\frac{\theta\alpha}{\theta+\alpha}}\end{aligned}$$

for some constant $c_{13} > 0$, where we have used the fact that

$$\int_{\mathbb{R}^d} u(x)^2\mathrm{d}x = \int_{\mathbb{R}^d} u(x+z)^2\mathrm{d}x = 1.$$

This together with (4.16) yields

$$\mathscr{E}_{J,V}(u,u) \leq cn^{\frac{\theta\alpha}{\theta+\alpha}}, \quad n \geq 1, u \in B(u_{\mathbf{k}} :\ \mathbf{k} \in G_n)$$

for some constant $c > 0$. Therefore, by (4.10) we obtain

$$\lambda_{n^d} \leq cn^{\frac{\theta\alpha}{\theta+\alpha}}, \quad n \geq 1.$$

For any $n \geq 1$, letting $r_n = \inf\{i \in \mathbb{Z}_+ : i \geq n^{\frac{1}{d}}\}$, we arrive at

$$\lambda_n \leq \lambda_{(1+r_n)^d} \leq c(1+r_n)^{\frac{\theta\alpha}{\theta+\alpha}} \leq Cn^{\frac{\theta\alpha}{d(\theta+\alpha)}}$$

for some constant $C > 0$. □

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