Integrability Conditions for SDEs and Semi-Linear SPDEs *

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Abstract

By using the local dimension-free Harnack inequality established on incomplete Riemannian manifolds, integrability conditions on the coefficients are presented for SDEs to imply the non-explosion of solutions as well as the existence, uniqueness and regularity estimates of invariant probability measures. These conditions include a class of drifts unbounded on compact domains such that the usual Lyapunov conditions can not be verified. The main results are extended to second order differential operators on Hilbert spaces and semi-linear SPDEs.

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1 Introduction

In recent years, the existence and uniqueness of strong solutions up to life time have been proved under local integrability conditions for non-degenerate SDEs, see [6, 25, 22, 44, 45] and references within. See also [14, 15, 16, 39, 41, 42] for extensions to degenerate SDEs and semi-linear SPDEs.

As a further development in this direction, the present paper provides reasonable integrability conditions for the non-explosion of solutions, as well as the existence, uniqueness and regularity estimates of invariant probability measures. An essentially new point in the study

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is to make use of a local Harnack inequality in the spirit of [32]. With this inequality we are able to prove the non-explosion of a weak solution constructed from the Girsanov transform, see the proof of Lemma 3.1 below for details. Moreover, we use the hypercontractivity of the reference Markov semigroup to prove the boundedness of a Feyman-Kac semigroup induced by the singular SDE under study, which enables us to prove the existence of the invariant probability measure as well as a formula for the derivative of the density, see (4.3) and the proof of Lemma 4.2 below for details. To explain the motivation of the study more clearly, below we first recall some existing results in the literature, then present a simple example to show how far can we go beyond.

Let W_t be the d-dimensional Brownian motion on a complete filtration probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$. Consider the following SDE (stochastic differential equation) on $\mathbb{R}^d$:

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sqrt{2}\,\sigma(X_t)\mathrm{d}W_t, \tag{1.1}$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ is measurable and $\sigma \in W^{1,1}_{loc}(\mathbb{R}^d \to \mathbb{R}^d\otimes\mathbb{R}^d; \mathrm{d}x)$ such that $\sigma(x)$ is invertible for every $x \in \mathbb{R}^d$. According to [45, Theorem 1.1] (see also [6, 25, 22, 44] for earlier results), if $|b| + \|\nabla\sigma\| \in L^p_{loc}(\mathrm{d}x)$ for some $p > d$, then for any initial point x the SDE (1.1) has a unique solution $(X^x_t)_{t\in[0,\zeta^x)}$ up to life time ζ^x. We note that in [45] the global integrability and the uniform ellipticity conditions are assumed, but these conditions can be localized since for the existence and uniqueness up to life time one only needs to consider solutions before exiting bounded domains. On the other hand, the ODE

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t$$

does not have pathwise uniqueness if b is merely Hölder continuous (for instance, $d = 1$ and $b(x) := |x|^\alpha$ for some $\alpha \in (0, 1)$). So, the above result on SDE indicates that the Brownian noise may "regularize" the drift to make an ill-posed equation well-posed.

Next, sufficient integrability conditions for the non-explosion have also been presented in [44]. For instance, if σ is bounded and

$$|b| \leq C + F \quad \text{for some constant } C > 0 \text{ and } F \in L^p(\mathrm{d}x) \text{ for some } p > d, \tag{1.2}$$

then the solution to (1.1) is non-explosive. As the Lebesgue measure is infinite, this condition is very restrictive. So, one of our aims is to replace it by integrability conditions with respect to a probability measure, see Theorem 2.1 and Corollary 2.2 below.

We would like to indicate that when the invariant measure μ is given, there exist criteria on the conservativeness of non-symmetric Dirichlet forms, which imply the non-explosion of solutions for μ-a.e. initial points, see [31] and references within. However, in our study the invariant probability measure is unknown, which is indeed the main object to characterize. In general, to prove the existence of invariant probability measures one uses Lyapunov (or drift) conditions. For instance, if there exists a positive function $W_1 \in C^2(\mathbb{R}^d)$ and a positive compact function W_2 such that

$$LW_1 := \sum_{i,j=1}^{\infty} (\sigma\sigma^*)_{ij}\partial_i\partial_j W_1 + \sum_{i=1}^{d} b_i\partial_i W_1 \leq C - W_2 \tag{1.3}$$

holds for some constant $C > 0$, then the associated diffusion semigroup has an invariant probability measure μ with $\mu(W_2) \le C$, see for instances [23, 8, 10, 12]. Obviously, this condition is not available when b is unbounded on compact sets. Our second purpose is to present a reasonable integrability condition for the existence and uniqueness of invariant probability measures, which applies to a class of SDEs with locally unbounded coefficients.

Moreover, we also intend to investigate the regularity properties of the invariant probability measure. Recall that a probability measure μ on $\mathbb{R}^d$ is called an invariant probability measure of the generator L (denoted by $L^*\mu = 0$), if

$$\mu(Lf) := \int_{\mathbb{R}^d} Lf \mathrm{d}\mu = 0, \quad f \in C_0^\infty(\mathbb{R}^d). \tag{1.4}$$

Obviously, an invariant probability measure μ of the Markov semigroup P_t associated to (1.1) satisfies $L^*\mu = 0$. In the past two decades, the existence, uniqueness and regularity estimates for invariant probability measures of L have been intensively investigated in both finite and infinite dimensional spaces, see the survey paper [8] for concrete results and historical remarks. Here, we would like to recall a fundamental result on the regularity of the invariant probability measures. Let $W_{loc}^{1,1}(\mathrm{d}x)$ be the class of functions $f \in L_{loc}^1(\mathrm{d}x)$ such that

$$\int_{\mathbb{R}^d} f(x)(\mathrm{div} G)(x)\mathrm{d}x = -\int_{\mathbb{R}^d} \langle G, F\rangle(x)\mathrm{d}x, \quad G \in C_0^\infty(\mathbb{R}^d \to \mathbb{R}^d)$$

holds for some $F \in L_{loc}^1(\mathbb{R}^d \to \mathbb{R}^d; \mathrm{d}x)$, which is called the weak gradient of f and is denoted by $F = \nabla f$ as in the classical case. For any $p \ge 1$, let

$$W^{p,1}(\mathrm{d}x) = \big\{ f \in W_{loc}^{1,1}(\mathrm{d}x) : \ f, |\nabla f| \in L^p(\mathrm{d}x) \big\}.$$

Consider the elliptic differential operator $L := \Delta + b \cdot \nabla$ on $\mathbb{R}^d$ for some locally integrable $b : \mathbb{R}^d \to \mathbb{R}^d$. It has been shown in [9] that any invariant probability measure μ of L with $\mu(|b|^2) := \int_{\mathbb{R}^d} |b|^2 \mathrm{d}\mu < \infty$ has a density $\rho := \frac{\mathrm{d}\mu}{\mathrm{d}x}$ such that $\sqrt{\rho} \in W^{2,1}(\mathrm{d}x)$. In addition,

$$\int_{\mathbb{R}^d} \big|\nabla\sqrt{\rho}\big|^2 \mathrm{d}x \le \frac{1}{4}\int_{\mathbb{R}^d} |b|^2 \mathrm{d}\mu. \tag{1.5}$$

Since the invariant probability measure μ of L is in general unknown, the integrability condition $\mu(|b|^2) < \infty$ is not explicit. As mentioned above that to ensure the existence of μ one uses the Lyapunov condition (1.3) for some positive function $W_1 \in C^2(\mathbb{R}^d)$ and a compact function W_2, and to verify $\mu(|b|^2) < \infty$ one would further need $|b|^2 \le c + cW_2$ for some constant $c > 0$. As we noticed above that these conditions do not apply if the coefficients merely satisfy an integrability condition with respect to a reference probability measure.

In conclusion, we aim to search for explicit integrability conditions on b and σ with respect to a nice reference measure (for instance, the Gaussian measure) to imply the non-explosion of solutions to the SDE (1.1); the strong Feller property of the associated Markov semigroup; the existence, uniqueness and regularity estimates of the invariant probability measure. We also aim to extend the resulting assertions to the infinite-dimensional case.

The main results of this paper will be stated in Section 2. Their proofs are then presented in Sections 3-6 respectively. Finally, in Section 7 we present a local Harnack inequality which plays a crucial role in the study.

To conclude this section, we present below a simple example to compare our results with existing ones introduced above.

Example 1.1. Consider, for instance, the following SDE on $\mathbb{R}^d$:

$$\mathrm{d}X_t = \{Z(X_t) - \lambda_0 X_t\}\mathrm{d}t + \sqrt{2}\,\mathrm{d}W_t,$$

where $\lambda_0 \in \mathbb{R}$ is a constant and $Z : \mathbb{R}^d \to \mathbb{R}^d$ is measurable.

(1) By Theorem 2.1 below for $\psi(x) = |x|$, if

$$\int_{\mathbb{R}^d} \mathrm{e}^{\varepsilon|Z(x)|^2 - \varepsilon^{-1}|x|^2}\mathrm{d}x < \infty \text{ for some } \varepsilon \in (0,1), \tag{1.6}$$

then for any initial value the SDE has a unique strong solution which is non-explosive, and the associated Markov semigroup P_t is strong Feller with a strictly positive density. Obviously, there are a lot of maps Z satisfying (1.6) but (1.2) and the Lyapunov condition does not hold. For instance, it is the case when

$$Z(x) := x_0\bigg\{\sum_{n=1}^{\infty}\log(1 + |x - nx_0|^{-1})\bigg\}^{\theta} \tag{1.7}$$

for some $x_0 \in \mathbb{R}^d$ with $|x_0| = 1$ and $\theta \in (0, \frac{1}{2}]$.

(2) When $\lambda_0 > 0$, we let $\mu_0(\mathrm{d}x) = C\mathrm{e}^{-\frac{\lambda_0}{2}|x|^2}\mathrm{d}x$ be a probability measure with normalization constant $C > 0$. It is well known by Gross [21] that the log-Sobolev inequality in Assumption **(H1)** holds for $\kappa = \frac{2}{\lambda_0}$ and $\beta = 0$. By Theorem 2.3, if

$$\int_{\mathbb{R}^d} \mathrm{e}^{\lambda|Z(x)|^2 - \frac{\lambda_0}{2}|x|^2}\mathrm{d}x < \infty \text{ for some } \lambda > \frac{1}{2\lambda_0}, \tag{1.8}$$

then P_t has a unique invariant probability measure $\mu(\mathrm{d}x) = \rho(x)\mathrm{d}x$ such that

$$\mu_0\big(|\nabla\sqrt{\rho}|^2\big) \le \frac{\lambda_0}{4\lambda\lambda_0 - 2}\log\mu_0(\mathrm{e}^{\lambda|Z|^2}) < \infty,$$
$$\mu_0\big(|\nabla\log\rho|^2\big) \le \mu_0(|Z|^2) < \infty.$$

Obviously, for any $\theta \in (0, \frac{1}{2})$, condition (1.8) holds for Z defined by (1.7), but the Lyapunov condition (1.3) is not available.

2 Main results

In the following four subsections, we introduce the main results in finite-dimensions and their infinite-dimensional extensions respectively. To apply integrability conditions with respect to a reference measure μ_0, we regard the original SDE as a perturbation to the corresponding reference SDE whose semigroup is symmetric in $L^2(\mu_0)$.

2.1 Non-explosion and strong Feller for SDEs

Let $\sigma \in C^2(\mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d)$ with $\sigma(x)$ invertible for $x \in \mathbb{R}^d$ and denote $a = \sigma\sigma^* = (a_{ij})_{1\le i,j\le d}$. For $V \in C^2(\mathbb{R}^d)$, define

$$
(2.1)\qquad \begin{aligned}
&Z_0 = \sum_{i,j=1}^d \{\partial_j a_{ij} - a_{ij}\partial_j V\}e_i,\\
&L_0 = \mathrm{tr}(a\nabla^2) + Z_0\cdot\nabla = \sum_{i,j=1}^d a_{ij}\partial_i\partial_j + \sum_{i=1}^d \langle Z_0, e_i\rangle\partial_i,
\end{aligned}
$$

where $\{e_i\}_{i=1}^d$ is the canonical orthonormal basis of $\mathbb{R}^d$, and ∂_i is the directional derivative along e_i.

By the integration by parts formula, L_0 is symmetric in $L^2(\mu_0)$ for $\mu_0(\mathrm{d}x) := \mathrm{e}^{-V(x)}\mathrm{d}x$:

$$\mu_0(fL_0g) = -\mu_0(\langle a\nabla f, \nabla g\rangle),\quad f,g \in C_0^\infty(\mathbb{R}^d).$$

Then

$$\mathscr{E}_0(f,g) := \mu_0(\langle a\nabla f, \nabla g\rangle),\quad f,g \in H_\sigma^{2,1}(\mu_0)$$

is a symmetric Dirichlet form generated by L_0, where $H_\sigma^{2,1}(\mu_0)$ is the closure of $C_0^\infty(\mathbb{R}^d)$ under the norm

$$\|f\|_{H_\sigma^{2,1}(\mu_0)} := \big\{\mu_0(|f|^2 + |\sigma^*\nabla f|^2)\big\}^{\frac{1}{2}}.$$

When $\sigma \equiv I$ (the identity matrix), we simply denote $H_\sigma^{2,1}(\mu_0) = H^{2,1}(\mu_0)$.

Let W_t be the d-dimensional Brownian motion as in Introduction. Consider the reference SDE

$$(2.2)\qquad \mathrm{d}X_t = Z_0(X_t)\mathrm{d}t + \sqrt{2}\,\sigma(X_t)\mathrm{d}W_t.$$

Since σ and Z_0 are locally Lipschitz continuous, for any initial point $x \in \mathbb{R}^d$ the SDE (2.2) has a unique solution X_t^x up to the explosion time ζ^x. Let P_t^0 be the associated (sub-)Markov semigroup:

$$P_t^0 f(x) := \mathbb{E}\big\{1_{\{\zeta^x > t\}} f(X_t^x)\big\},\quad f \in \mathscr{B}_b(\mathbb{R}^d), t \ge 0, x \in \mathbb{R}^d.$$

When $\mu_0(\mathrm{d}x) := \mathrm{e}^{-V(x)}\mathrm{d}x$ is finite and $1 \in H_\sigma^{2,1}(\mu_0)$ with $\mathscr{E}_0(1,1) = 0$, we have $P_t^0 1 = 1$ μ_0-a.e. Since $P_t^0 1$ is continuous (indeed, differentiable) for $t > 0$, we have $P_t^0 1(x) = 1$ for all $t \ge 0$ and $x \in \mathbb{R}^d$. Therefore, in this case the solution to (2.2) is non-explosive for any initial points. By the symmetry of P_t^0 in $L^2(\mu_0)$, μ_0 is P_t^0-invariant.

Now, for a measurable drift $Z : \mathbb{R}^d \to \mathbb{R}^d$, we consider the perturbed SDE

$$(2.3)\qquad \mathrm{d}X_t = \big\{Z + Z_0\big\}(X_t)\mathrm{d}t + \sqrt{2}\,\sigma(X_t)\mathrm{d}W_t.$$

By Itô's formula, the generator of the solution is $L := L_0 + Z\cdot\nabla$. According to [45, Theorem 1.1], if $|Z| \in L_{loc}^p(\mathrm{d}x)$ for some $p > d$, then for any initial point $x \in \mathbb{R}^d$, the SDE (2.3) has a unique solution X_t^x up to the life time ζ^x. We let P_t be the associated (Dirichlet) semigroup:

$$P_t f(x) = \mathbb{E}\big[1_{\{t<\zeta^x\}} f(X_t^x)\big],\quad x \in \mathbb{R}^d, t \ge 0, f \in \mathscr{B}_b(\mathbb{R}^d).$$

If $\mathbb{P}(\zeta^x = \infty) = 1$ for all $x \in \mathbb{R}^d$, the solution is called non-explosive. In this case P_t is a Markov semigroup. More generally, for any non-empty open set $\mathscr{O} \subset \mathbb{R}^d$, let

$$T^x_{\mathscr{O}} = \zeta^x \wedge \inf\{t \in [0, \zeta^x) : X^x_t \notin \mathscr{O}\}, \quad \inf \emptyset := \infty.$$

Then the associated Dirichlet semigroup on $\mathscr{O}$ is given by

$$P^{\mathscr{O}}_t f(x) = \mathbb{E}\big[1_{\{t<T^x_{\mathscr{O}}\}} f(X^x_t)\big], \quad x \in \mathscr{O}, t \geq 0, f \in \mathscr{B}_b(\mathscr{O}).$$

Let ρ_σ be the intrinsic metric induced by σ as follows:

$$\rho_\sigma(x, y) := \sup\big\{|f(x) - f(y)| : \ f \in C^\infty(\mathbb{R}^d), |\sigma^*\nabla f| \leq 1\big\}, \quad x, y \in \mathbb{R}^d.$$

We have the following result.

Theorem 2.1. *Let $\sigma \in C^2(\mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d)$ with $\sigma(x)$ invertible for $x \in \mathbb{R}^d$, and let $V \in C^2(\mathbb{R}^d)$ such that*

$$\int_{\mathbb{R}^d} \Big(|\sigma^*\nabla\psi(x)|^2 + \mathrm{e}^{\varepsilon|(\sigma^{-1}Z)(x)|^2}\Big) \mathrm{e}^{-V(x)-\varepsilon^{-1}\rho_\sigma(0,x)^2} \mathrm{d}x < \infty \tag{2.4}$$

holds for some constant $\varepsilon \in (0, 1)$ and a local Lipschitz continuous compact function ψ on $\mathbb{R}^d$. Then (2.3) has a unique non-explosive solution for any initial points, and the associated Markov semigroup P_t is strong Feller with at most one invariant probability measure. Moreover, for any non-empty open set $\mathscr{O} \subset \mathbb{R}^d$ and $t > 0$, $P^{\mathscr{O}}_t$ is strong Feller and has a strictly positive density $p^{\mathscr{O}}_t$ with respect to the Lebesgue measure on $\mathscr{O}$.

Remark 2.1. (1) Typical choices of ψ include $|x|, \log(1+|x|), \log\log(\mathrm{e}+|x|)$... For instance, with $\psi(x) := \log\log(\mathrm{e}+|x|)$ one may replace the term $|\sigma^*\nabla\psi(x)|^2$ in (2.4) by $\frac{\|\sigma(x)\|^2}{(\mathrm{e}+|x|)^2\{\log(\mathrm{e}+|x|)\}^2}$. So, if $V = 0$ and $\int_{\mathbb{R}^d} \mathrm{e}^{-\lambda\rho_\sigma(0,\cdot)^2}\mathrm{d}x < \infty$ for some $\lambda > 0$, the condition (2.4) holds provided

$$\log \frac{\|\sigma\|^2}{(\mathrm{e} + |\cdot|)^2\{\log(\mathrm{e} + |\cdot|)\}^2} + |\sigma^{-1}Z|^2 \leq C(1 + \rho_\sigma(0, \cdot))^2 + f$$

for some constant $C > 0$ and some function f with $\mathrm{e}^{\varepsilon f-\varepsilon^{-1}\rho_\sigma(0,\cdot)^2} \in L^1(\mathrm{d}x)$ for some $\varepsilon \in (0, 1)$.

(2) Let $\bar\sigma(r) = \sup_{|x|=r} \|\sigma(x)\|$ for $r \geq 0$. Then

$$\rho_\sigma(0, x) \geq U(x) := \int_0^{|x|} \frac{\mathrm{d}r}{\bar\sigma(r)}, \quad x \in \mathbb{R}^d.$$

So, in (2.4) we may replace $\rho_\sigma(0, \cdot)$ by the more explicit function U.

(3) The condition $\sigma \in C^2(\mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d)$ is stronger than $\sigma \in W^{p,1}_{loc}(\mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d; \mathrm{d}x)$ for some $p > 1$ as required for the existence and uniqueness of solutions according to [45, Theorem 1.1]. This stronger condition is introduced because it together with the invertibility of σ implies the local Harnack inequality (see Theorem 7.1 below), which is a crucial tool in our study. If the local Harnack inequality could be established under weaker conditions,

this condition would be weakened automatically. Indeed, under an additional assumption, this condition will be replaced by $\sigma \in W^{p,1}_{loc}(\mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d; \mathrm{d}x)$ for some $p > 1$, see Theorem 2.4 below for details.

Intuitively, the non-explosion is a long distance property of the solution. So, it is natural for us to weaken the integrability condition (2.4) by taking the integral outside a compact set. But under this weaker condition we are not able to prove other properties included in Theorem 2.1.

Corollary 2.2. *Let $\sigma \in C^2(\mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d)$ with $\sigma(x)$ invertible for $x \in \mathbb{R}^d$, and let $V \in C^2(\mathbb{R}^d)$ such that*

$$\int_{D^c} \Big(|\sigma^* \nabla \psi(x)|^2 + \mathrm{e}^{\varepsilon |(\sigma^{-1} Z)(x)|^2} \Big) \mathrm{e}^{-V(x) - \varepsilon^{-1} \rho_\sigma(0,x)^2} \mathrm{d}x < \infty \tag{2.5}$$

holds for some compact set $D \subset \mathbb{R}^d$, some constant $\varepsilon \in (0,1)$, and some local Lipschitz continuous compact function ψ on $\mathbb{R}^d$. If $Z \in L^p_{loc}(\mathrm{d}x)$ for some constant $p > d$, then the SDE (2.3) has a unique non-explosive solution for any initial points.

2.2 Invariant probability measure for SDEs

To investigate the invariant probability measures for the SDE (2.3), we need the non-explosion of solutions such that the standard tightness argument for the existence of invariant probability measure applies. To this end, we will apply Theorem 2.1 above, for which we first assume that σ is C^2-smooth (see **(H1)** below) then extend to less regular σ by approximations (see **(H1′)** below).

Assumption (H1)

(1) $\sigma \in C^2(\mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d)$ with $\sigma(x)$ invertible for $x \in \mathbb{R}^d$, $V \in C^2(\mathbb{R}^d)$ such that $\mu_0(\mathrm{d}x) := \mathrm{e}^{-V(x)}\mathrm{d}x$ is a probability measure satisfying

$$H^{2,1}_\sigma(\mu_0) = W^{2,1}_\sigma(\mu_0) := \big\{ f \in W^{1,1}_{loc}(\mathrm{d}x) : \ f, |\sigma^* \nabla f| \in L^2(\mu_0) \big\}. \tag{2.6}$$

(2) The (defective) log-Sobolev inequality

$$\mu_0(f^2 \log f^2) \le \kappa \mu_0(|\sigma^* \nabla f|^2) + \beta, \quad f \in C_0^\infty(\mathbb{R}^d), \mu_0(f^2) = 1 \tag{2.7}$$

holds for some constants $\kappa > 0, \beta \ge 0$.

Since $\mu_0(\mathrm{d}x) := \mathrm{e}^{-V(x)}\mathrm{d}x$ is finite, (2.6) implies $1 \in H^{2,1}_\sigma(\mu_0)$ with $\mathscr{E}_0(1,1) = 0$, so that the solution to (2.2) is non-explosive as explained above. We note that (2.6) holds if the metric ρ_σ is complete. Indeed, in this case the function $\rho_\sigma(0,\cdot)$ is compact with $|\sigma^* \nabla \rho_\sigma(0,\cdot)| = 1$, so that for any $f \in W^{2,1}_\sigma(\mu_0)$ we have $f_n := f\{1 \wedge (n + 1 - \rho_\sigma(0,\cdot))^+\} \in H^{2,1}_\sigma(\mu_0)$ for $n \ge 1$, and it is easy to see that $f_n \to f$ in the norm $\| \cdot \|_{H^{2,1}_\sigma(\mu_0)}$.

There are plentiful sufficient conditions for the log-Sobolev inequality (2.7) to hold. For instance, if $\sigma\sigma^* \geq \alpha I$ and $\mathrm{Hess}_V \geq KI$ for some constants $\alpha, K > 0$, then the Bakry-Emery criterion [5] implies (2.7) for $\kappa = \frac{2}{K\alpha}$. In the case that K is not positive, the log-Sobolev inequality holds for some constant $\kappa > 0$ if $\mu_0(e^{\lambda|\cdot|^2}) < \infty$ for some $\varepsilon > -\frac{K}{2}$, see [36, Theorem 1.1]. See also [13] for a Lyapunov type sufficient condition of the log-Sobolev inequality.

Theorem 2.3. *Assume* **(H1)** *and that*

$$\mu_0(e^{\lambda|\sigma^{-1}Z|^2}) := \int_{\mathbb{R}^d} e^{\lambda|\sigma^{-1}Z|^2} d\mu_0 < \infty \tag{2.8}$$

holds for some constant $\lambda > \frac{\kappa}{4}$. *Let* P_t *be the semigroup associated to* (2.3)*, and let* $L = L_0 + Z \cdot \nabla$ *for* L_0 *in* (2.1)*. Then:*

(1) *L has an invariant probability measure μ, which is absolutely continuous with respect to μ_0 such that the density function $\rho := \frac{d\mu}{d\mu_0}$ is strictly positive with $\sqrt{\rho}, \log\rho \in H^{2,1}_\sigma(\mu_0)$ and*

$$\mu_0\big(|\sigma^*\nabla\sqrt{\rho}|^2\big) \leq \frac{1}{4\lambda - \kappa}\big\{\log\mu_0(e^{\lambda|\sigma^{-1}Z|^2}) + \beta\big\} < \infty; \tag{2.9}$$

$$\mu_0(|\sigma^*\nabla\log\rho|^2) := \lim_{\delta\downarrow 0}\int_{\mathbb{R}^d} \frac{|\sigma^*\nabla\rho|^2}{(\rho+\delta)^2} d\mu_0 \leq \mu_0(|\sigma^{-1}Z|^2) < \infty. \tag{2.10}$$

(2) *The measure μ is the unique invariant probability measure of L and P_t provided*

$$\mu_0(e^{\varepsilon\|\sigma\|^2}) < \infty \text{ for some constant } \varepsilon > 0. \tag{2.11}$$

Remark 2.2. (1) Simply consider the case that $\sigma = \sigma_0 = I$. If $\mathrm{Hess}_V \geq K$ for some $K > 0$, then **(H1)** holds for $\kappa = \frac{2}{K}$ and $\beta = 0$. So, when $\mu_0(e^{\lambda|Z|^2}) < \infty$ holds for some $\lambda > \frac{1}{2K}$, Theorem 2.3 implies that L and P_t have a unique invariant probability measure μ, which is absolutely continuous with respect to μ_0, and the density function satisfies $\rho := \frac{d\mu}{d\mu_0}$ satisfies $\sqrt{\rho}$, $\log\rho \in H^{2,1}(\mu_0)$ with

$$\mu_0(|\nabla\sqrt{\rho}|^2) \leq \frac{K}{4K\lambda - 2}\log\mu_0(e^{\lambda|Z|^2}) < \infty; \quad \mu_0(|\nabla\log\rho|^2) \leq \mu_0(|Z|^2) < \infty.$$

(2) Under **(H1)**, if the super log-Sobolev inequality

$$\mu_0(f^2\log f^2) \leq r\mu_0(|\sigma^*\nabla f|^2) + \beta(r), \quad r > 0, f \in C_0^\infty(\mathbb{R}^d), \mu_0(f^2) = 1$$

holds for some $\beta : (0,\infty) \to (0,\infty)$, then Theorem 2.3 applies when (2.8) holds for some $\lambda > 0$, and in this case (2.9) reduces to

$$\mu_0\big(|\sigma^*\nabla\sqrt{\rho}|^2\big) \leq \inf_{\lambda>0, r\in(0,4\lambda)} \frac{1}{4\lambda - r}\big\{\log\mu_0(e^{\lambda|\sigma^{-1}Z|^2}) + \beta(r)\big\} < \infty.$$

According to e.g. [28, Theorems 2.1(1) and 2.3(2)] for $M = \mathbb{R}^d$, the super log-Sobolev inequality holds provided $a \geq \alpha I$ for some constant $\alpha > 0$ and Hess_V is bounded below with $\mu(\mathrm{e}^{\lambda|\cdot|^2}) < \infty$ for any $\lambda > 0$. In particular, it is the case when $a = I, V(x) = c_1 + c_2|x|^p$ for some constants $c_1 \in \mathbb{R}, c_2 > 0$ and $p > 2$. See [18, 21, 35] and references within for more discussions on the super log-Sobolev inequality and the corresponding semigroup property.

(3) To illustrate the sharpness of condition (2.8) for some $\lambda > \frac{\kappa}{4}$, let us consider $\sigma = \sigma_0 = I$ and $V(x) = c + \frac{1}{2}|x|^2$ for some constant $c \in \mathbb{R}$, so that **(H1)** holds for $\kappa = 2$ and $\beta = 0$. Let $Z(x) = rx = \frac{r}{2}\nabla|\cdot|^2(x)$ for some constant $r \geq 0$. It is trivial that L has an invariant probability measure if and only if $r < 1$, which is equivalent to $\mu_0(\mathrm{e}^{\lambda|Z|^2}) < \infty$ for some $\lambda > \frac{\kappa}{4} = \frac{1}{2}$.

Now, we extend Theorem 2.3 to less regular σ by using the following assumption to replace **(H1)**.

Assumption (H1′)

(1) $\sigma \in W^{p,1}_{loc}(\mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d; \mathrm{d}x)$ for some $p > d$, $\sigma(x)$ is invertible for every $x \in \mathbb{R}^d$, and $a := \sigma\sigma^* \geq \alpha I$ for some constants $\alpha > 0$.

(2) $V \in C^2(\mathbb{R}^d)$ such that $\mu_0(\mathrm{d}x) := \mathrm{e}^{-V(x)}\mathrm{d}x$ is a probability measure satisfying (2.6) and

$$\mu_0(f^2 \log f^2) \leq \kappa'\mu_0(|\nabla f|^2) + \beta, \quad f \in C_0^\infty(\mathbb{R}^d), \mu_0(f^2) = 1 \tag{2.12}$$

for some constants $\kappa' > 0, \beta \geq 0$.

(3) There exists a constant $p > 1$ such that $a_{ij} \in H^{2,1}(\mu_0) \cap L^{2p}(\mu_0)$ for any $1 \leq i, j \leq d$ and $|\nabla V| \in L^{\frac{2p}{p-1}}(\mu_0)$.

Let L and P_t be in Theorem 2.3 associated to the SDE (2.3).

Theorem 2.4. *Assume* **(H1′)** *and let $\mu_0\big(\exp[\lambda|Z|^2]\big) < \infty$ hold for some $\lambda > \frac{\kappa'}{4\alpha^2}$. Then L and P_t have a unique invariant probability measure $\mu(\mathrm{d}x) := \rho(x)\mu_0(\mathrm{d}x)$ for some strictly positive function ρ such that $\sqrt{\rho}$, $\log\rho \in H^{2,1}(\mu_0)$ with*

$$\mu_0\big(|\nabla\sqrt{\rho}|^2\big) \leq \frac{1}{4\alpha^2\lambda - \kappa'}\big\{\log\mu_0\big(\mathrm{e}^{\lambda|Z|^2}\big) + \beta\big\} < \infty; \tag{2.13}$$

$$\mu_0(|\nabla\log\rho|^2) := \lim_{\delta\downarrow 0}\int_{\mathbb{R}^d}\frac{|\nabla\rho|^2}{(\rho+\delta)^2}\,\mathrm{d}\mu_0 \leq \frac{1}{\alpha^2}\mu_0(|Z|^2) < \infty. \tag{2.14}$$

2.3 Elliptic differential operators on Hilbert spaces

We first consider the invariant probability measure of second order differential operators on a separable Hilbert space, then apply to semi-linear SPDEs. We will take a Gaussian measure as the reference measure.

Let $(\mathbb{H}, \langle\cdot,\cdot\rangle, |\cdot|)$ be a separable Hilbert space, let $(A, \mathscr{D}(A))$ be a positive definite self-adjoint operator on $\mathbb{H}$ having discrete spectrum with all eigenvalues $(0 <)\lambda_1 \le \lambda_2 \le \cdots$ counting multiplicities such that

$$\sum_{i=1}^{\infty} \lambda_i^{-1} < \infty. \tag{2.15}$$

Let $\{e_i\}_{i\ge1}$ be the corresponding eigenbasis of A. Let μ_0 be the Gaussian measure on $\mathbb{H}$ with covariance operator A^{-1}. In coordinates with respect to the basis $\{e_i\}_{i\ge1}$, we have

$$\mu_0(\mathrm{d}x) = \prod_{i=1}^{\infty} \Big(\frac{\sqrt{\lambda_i}}{\sqrt{2\pi}} \mathrm{e}^{-\frac{\lambda_i x_i^2}{2}} \,\mathrm{d}x_i \Big), \quad x_i := \langle x, e_i\rangle, i \ge 1. \tag{2.16}$$

For any $n \ge 1$, let $\mathbb{H}_n = \mathrm{span}\{e_i : 1 \le i \le n\}$ and define the probability measure

$$\mu_0^{(n)}(\mathrm{d}x) = \prod_{i=1}^{n} \Big(\frac{\sqrt{\lambda_i}}{\sqrt{2\pi}} \mathrm{e}^{-\frac{\lambda_i x_i^2}{2}} \,\mathrm{d}x_i \Big) \text{ on } \mathbb{H}_n.$$

We have $\mu_0^{(n)} = \mu_0 \circ \pi_n^{-1}$ for the orthogonal projection $\pi_n : \mathbb{H} \to \mathbb{H}_n$.

Let $(\mathscr{L}(\mathbb{H}), \|\cdot\|)$ be the space of bounded linear operators on $\mathbb{H}$ with operator norm $\|\cdot\|$, and let $\mathscr{L}_s(\mathbb{H})$ be the class of all symmetric elements in $\mathscr{L}(\mathbb{H})$. For any $a \in \mathscr{L}_s(\mathbb{H})$ let $a_{ij} = \langle ae_i, e_j\rangle$ for $i, j \ge 1$. We make the following assumption.

Assumption (H2)

(1) $a_{ij} \in C^2(\mathbb{H})$ for $i, j \ge 1$, and $a \ge \alpha I$ for some constant $\alpha > 0$.

(2) For $n \ge 1$ and $\sigma_n := \sqrt{(a_{ij})_{1\le i,j\le n}}$, $H^{2,1}_{\sigma_n}(\mu_0^{(n)}) = W^{2,1}_{\sigma_n}(\mu_0^{(n)})$ holds.

(3) For any $i, j \ge 1$, there exists $\varepsilon_{ij} \in (0, 1)$ such that

$$\sup_{n\ge1} \int_{\mathbb{R}^n} \exp\big[\varepsilon_{ij}|a_{ij}|^{1+\varepsilon_{ij}}\big] \mathrm{d}\mu_0^{(n)} < \infty. \tag{2.17}$$

We note that

$$\int_{\mathbb{R}^n} \exp\big[\varepsilon_{ij}|a_{ij}|^{1+\varepsilon_{ij}}\big] \mathrm{d}\mu_0^{(n)} = \int_{\mathbb{H}} \exp\big[\varepsilon_{ij}|a_{ij} \circ \pi_n|^{1+\varepsilon_{ij}}\big] \mathrm{d}\mu_0.$$

As mentioned above that $H^{2,1}_{\sigma_n}(\mu_0^{(n)}) = W^{2,1}_{\sigma_n}(\mu_0^{(n)})$ is implied by the completeness of the metric on $\mathbb{R}^n$ induced by σ_n, and the later holds if for any $i, j \ge 1$ there exists $\varepsilon_{ij} > 0$ such that

$$|a_{ij}(x)| \le \frac{1}{\varepsilon_{ij}}(1 + |x|)^2, \quad x \in \mathbb{H}. \tag{2.18}$$

The condition (2.17) will be used for finite-dimensional approximations in the end of the proof of Theorem 2.5(1) below. According to (2.15) and the definition of $\mu_0^{(n)}$, the conditions (2.17) and (2.18) hold provided for any $i, j \geq 1$ there exists a constant $\varepsilon'_{ij} \in (0,1)$ such that $|a_{ij}(x)| \leq \frac{1}{\varepsilon'_{ij}}(1+|x|)^{\frac{2}{1+\varepsilon'_{ij}}}$.

Let ∂_i be the directional derivative along $e_i, i \geq 1$. For a measurable drift $Z : \mathbb{H} \to \mathbb{H}$, consider the operators

$$L := L_0 + Z \cdot \nabla, \quad L_0 := \sum_{i,j=1}^{\infty} \Big(a_{ij}\partial_i\partial_j + \big\{\partial_j a_{ij} - a_{ij}\lambda_j\big\}\partial_i \Big), \tag{2.19}$$

which are well defined on the class of smooth cylindrical functions with compact support:

$$\mathscr{F}C_0^\infty := \big\{\mathbb{H} \ni x \mapsto f(\langle x, e_1\rangle, \cdots, \langle x, e_n\rangle) : n \geq 1, f \in C_0^\infty(\mathbb{R}^n)\big\}.$$

It is easy to see that L_0 is symmetric in $L^2(\mu_0)$:

$$\mu_0(fL_0 g) = -\mu_0(\langle a\nabla f, \nabla g\rangle), \quad f, g \in \mathscr{F}C_0^\infty.$$

Let $H^{2,1}(\mu_0)$ be the completion of $\mathscr{F}C_0^\infty$ with respect to the inner product

$$\langle f, g\rangle_{H^{2,1}(\mu_0)} := \mu_0(fg) + \mu_0(\langle \nabla f, \nabla g\rangle).$$

A probability measure μ on $\mathbb{H}$ is called an invariant probability measure of L (denoted by $L^*\mu = 0$), if for any $f \in \mathscr{F}C_0^\infty$ we have $Lf \in L^1(\mu)$ and $\mu(Lf) = 0$.

Theorem 2.5. *Assume* (2.15) *and* **(H2)**.

(1) *If $\mu_0(\mathrm{e}^{\lambda|Z|^2}) < \infty$ for some $\lambda > \frac{1}{2\lambda_1\alpha^2}$, then L has an invariant probability measure μ, which is absolutely continuous with respect to μ_0, and the density function $\rho := \frac{\mathrm{d}\mu}{\mathrm{d}\mu_0}$ satisfies $\sqrt{\rho}$, $\log\rho \in H^{2,1}(\mu_0)$ with*

$$\mu_0\big(\big|\nabla\sqrt{\rho}\big|^2\big) \leq \frac{\lambda_1}{4\alpha^2\lambda_1\lambda - 2}\log\mu_0(\mathrm{e}^{\lambda|Z|^2}) < \infty \tag{2.20}$$

and

$$\mu_0\big(\big|\nabla\log\rho\big|^2\big) := \lim_{\delta\downarrow 0}\mu_0\big(\big|\nabla\log\rho + \delta\big|^2\big) \leq \frac{\mu_0(|Z|^2)}{\alpha^2} < \infty. \tag{2.21}$$

(2) *If moreover $\|a\|_\infty < \infty$, then $(L, \mathscr{F}C_0^\infty)$ is closable in $L^1(\mu)$ and the closure generates a Markov C_0-semigroup T_t on $L^1(\mu)$ with μ as an invariant probability. Moreover, there exists a standard Markov process $\{\bar{\mathbb{P}}_x\}_{x\in\mathbb{H}\cup\{\partial\}}$ on $\mathbb{H}\cup\{\partial\}$ which is continuous and non-explosive for $\mathscr{E}^\mu$-q.e. x, such that the associated Markov semigroup $\bar{P}_t$ is a μ-version of T_t; that is, $\bar{P}_t f = T_t f$ μ-a.e. for all $t \geq 0$ and $f \in \mathscr{B}_b(\mathbb{H})$.*

For readers' convenience, we would like to recall here the notion of standard Markov process involved in Theorem 2.5(2). Let ∂ be an extra point and extend the topology of $\mathbb{H}$ to $\mathbb{H}\cup\{\partial\}$ by letting the set $\{\partial\}$ open. A family of probability measures $\{\mathbb{P}_x\}_{x\in\mathbb{H}\cup\{\partial\}}$ on

$$\Omega := \big\{\omega : [0,\infty) \to \mathbb{H}\cup\{\partial\} : \text{if } \omega_t = \partial \text{ then } \omega_s = \partial \text{ for } s \geq t\big\}$$

equipped with the σ-field $\mathscr{F} := \sigma(\omega_t : t \geq 0)$ is called a standard Markov process, if $\mathbb{P}_x(\omega_0 = x) = 1$ and the distribution $P_t(x, \mathrm{d}y)$ of $\Omega \ni \omega \mapsto \omega_t$ under $\mathbb{P}_x$ gives rise to a Markov transition kernel on $\mathbb{H}\cup\{\partial\}$. When the process is non-explosive, i.e.

$$\mathbb{P}_x\big(\inf\{t \geq 0 : \omega_t = \partial\} = \infty\big) = 1, \quad x \in \mathbb{H},$$

the sub-family $\{\mathbb{P}_x\}_{x\in\mathbb{H}}$ is a standard Markov process on $\mathbb{H}$. In this case, the process (or the associated Markov semigroup P_t) is called Feller if $P_tC_b(\mathbb{H}) \subset C_b(\mathbb{H})$ for all $t \geq 0$, and is called strong Feller if $P_t\mathscr{B}_b(\mathbb{H}) \subset C_b(\mathbb{H})$ for all $t > 0$. If moreover $\mathbb{P}_x(C([0,\infty) \to \mathbb{H})) = 1$ holds for all $x \in \mathbb{H}$, then the process is continuous.

Next, we extend Theorem 2.4 to the infinite-dimensional case, for which we need the following assumption.

Assumption (H2′)

(1) $a \geq \alpha I$ for some constant $\alpha > 0$, and for every $n \geq 1$ there exists a constant $p > n$ such that $a_n \in W^{p,1}_{loc}(\mathbb{R}^n \to \mathbb{R}^n \otimes \mathbb{R}^n; \mathrm{d}x)$.

(2) For any $i, j \in \mathbb{N}$ there exists $\varepsilon_{ij} \in (0,1)$ such that (2.17) and $\mu_0^{(n)}(|\nabla a_{ij} \circ \pi_n|^{2+\varepsilon_{ij}}) < \infty$ hold for any $n \geq 1$.

Theorem 2.6. *Under* (2.15) *and* **(H2′)**, *assertions* (1) *and* (2) *in Theorem* 2.5 *hold.*

2.4 Semi-linear SPDEs

We intend to investigate the existence, uniqueness and non-explosion of the SPDE corresponding to L in (2.19), and to show that the probability measure in Theorem 2.5 is the unique invariant probability measure of the associated Markov semigroup. For technical reasons, we only consider the case that $a = I$, for which the corresponding SPDE reduces to the standard semi-linear SPDE

$$\mathrm{d}X_t = \big\{Z(X_t) - AX_t\big\}\mathrm{d}t + \sqrt{2}\,\mathrm{d}W_t, \tag{2.22}$$

where $Z : \mathbb{H} \to \mathbb{H}$ is measurable, W_t is the cylindrical Brownian motion, i.e.

$$W_t = \sum_{i=1}^{\infty} \beta_t^i e_i, \quad t \geq 0$$

for a sequence of independent one-dimensional Brownian motions $\{\beta_t^i\}_{i\geq 1}$. An adapted continuous process X_t on $\mathbb{H}$ is called a mild solution to (2.22), if

$$X_t = \mathrm{e}^{-At}X_0 + \int_0^t \mathrm{e}^{-A(t-s)}Z(X_s)\mathrm{d}s + \int_0^t \mathrm{e}^{-(t-s)A}\mathrm{d}W_s, \quad t \geq 0.$$

We assume

(H3) $\sum_{i=1}^\infty \frac{1}{\lambda_i^\theta} < \infty$ for some $\theta \in (0,1)$, and $\mu_0(\mathrm{e}^{\lambda|Z|^2}) < \infty$ for some constant $\lambda > 0$.

According to the recent paper [15], **(H3)** implies the existence and pathwise uniqueness of mild solutions to (2.22) for μ_0-a.e. starting points. Below we intend to prove the weak uniqueness of (2.22) for any initial points. A standard continuous Markov process on $\mathbb{H}$ is called a weak solution to (2.22), if it solves the martingale problem for $(L, \mathscr{F}C_0^\infty)$. In this case one may construct a cylindrical Brownian motion W_t on the probability space $(C([0,\infty) \to \mathbb{H}); \mathscr{F}, \mathbb{P}_x)$, where $\mathscr{F} := \sigma(\{\omega \mapsto \omega_t : t \geq 0\})$, such that the coordinate process $X_t(\omega) := \omega_t$ is a mild solution to (2.22) with $X_0 = x$. See e.g. [24, Proposition IV.2.1] for the explanation in the finite-dimensional case, which works also in the present case as the cylindrical Brownian motion is determined by its finite-dimensional projections.

Theorem 2.7. *Assume that* **(H3)** *holds.*

(1) *There exists a standard continuous Markov process* $\{\mathbb{P}_x\}_{x\in\mathbb{H}}$ *solving* (2.22) *weakly for every initial point, and the associated Markov semigroup* P_t *is strong Feller having a strictly positive density with respect to* μ_0.

(2) *If* Z *is bounded on bounded sets, then there exists a unique standard Markov process solving* (2.22) *weakly for every initial point such that the associated Markov semigroup is Feller.*

(3) *If* Z *is bounded on bounded sets and* $\mu_0(\mathrm{e}^{\lambda|Z|^2}) < \infty$ *holds for some* $\lambda > \frac{1}{2\lambda_1}$, *then* P_t *has a unique invariant probability measure* μ, *which is absolutely continuous with respect to* μ_0 *and the density function* $\rho := \frac{\mathrm{d}\mu}{\mathrm{d}\mu_0}$ *is strictly positive with* $\sqrt{\rho}, \log\rho \in H^{2,1}(\mu_0)$ *such that estimates* (2.20) *and* (2.21) *hold for* $\alpha = 1$.

Remark 2.3. Unlike in the finite-dimensional case where $Z \in L^p_{loc}(\mathrm{d}x)$ for some $p > d$ implies the pathwise uniqueness of the solution for any initial points, in the infinite-dimensional case this is unknown without any continuity conditions on Z. It is shown in [39] (also for the multiplicative noise case) that if Z is Dini continuous then the pathwise uniqueness holds for any initial points.

3 Proof of Theorem 2.1

The main idea is to show that the solution to the reference SDE (2.2) is a weak solution to (2.3) under a weighted probability, so that the non-explosion of (2.2) implies that of (2.3).

To this end, we will apply the local Harnack inequality (3.2) below to verify the Novikov condition for the Girsanov transform. To realize the idea, we first consider the case that

$$\int_{\mathbb{R}^d} \mathrm{e}^{\varepsilon|(\sigma^{-1}Z)(x)|^2 - V(x)} \mathrm{d}x < \infty \tag{3.1}$$

holds for some $\varepsilon > 0$, then reduce back to the original condition (2.4).

Lemma 3.1. *Assume that* (3.1) *holds for some constant* $\varepsilon > 0$ *and* $1 \in H_\sigma^{2,1}(\mu_0)$ *with* $\mathscr{E}_0(1,1) = 0$*, then all assertions in Theorem* 2.1 *hold.*

Proof. Obviously, (3.1) implies that $\mu_0(\mathrm{d}x) := \mathrm{e}^{-V(x)}\mathrm{d}x$ is a finite measure. Since the coefficients in (2.2) is locally Lipschitz continuous, it is classical that the SDE has a unique solution up to the explosion time. Since $1 \in H_\sigma^{2,1}(\mu_0)$ with $\mathscr{E}_0(1,1) = 0$, as explained after (2.2) that the solution to (2.2) is non-explosive and μ_0 is P_t^0-invariant. Moreover, since the drift in (2.3) is locally bounded, according to [44], this SDE has a unique solution for any initial points. So, it remains to show that the solution is non-explosive, and the associated Markov semigroup P_t is strong Feller with at most one invariant probability measure.

A crucial tool in the proof is the following local Harnack inequality. Consider $\mathbb{R}^d$ with the C^2-Riemannian metric

$$\langle u, v\rangle_\sigma := \langle \sigma\sigma^* u, v\rangle, \quad u, v \in \mathbb{R}^d,$$

and let $\Delta_\sigma, \nabla_\sigma$ be the corresponding Laplace-Beltrami operator and the gradient operator. Then L_0 can be rewritten as

$$L_0 = \Delta_\sigma + \nabla_\sigma \bar{V}$$

for some $\bar{V} \in C^2(\mathbb{R}^d)$. Since the intrinsic distance ρ_σ is locally equivalent to the Euclidean distance, according to Theorem 7.1 below, for any $p > 1$ there exists positive $\Phi_p \in C(\mathbb{R}^d)$ such that

$$(P_t^0 f)^p(x) \le (P_t^0 f^p(y)) \exp\left[\Phi_p(x)\Big(1 + \frac{|x-y|^2}{1 \wedge t}\Big)\right] \quad x, y \in \mathbb{R}^d,\ |x-y| \le \frac{1}{\Phi_p(x)} \tag{3.2}$$

holds for all $t > 0$ and $f \in \mathscr{B}_b^+(\mathbb{R}^d) := \{f \in \mathscr{B}_b(\mathbb{R}^d) :\ f \ge 0\}$.

(a) **Non-explosion**. It suffices to find out a constant $t_0 > 0$ such that for any initial points, the solution to (2.3) is non-explosive before time t_0. To this end, we construct a weak solution by using the reference SDE (2.2). We intend to find out $t_0 > 0$ such that for any initial point x, the solution to (2.2) for $X_0 = x$ is a weak solution to (2.3) for $t \in [0, t_0]$. So, by the weak uniqueness of (2.3), which follows from the strong uniqueness, we conclude that the strong solution to (2.3) is non-explosive before t_0. To this end, we verify the Novikov condition

$$\mathbb{E}\exp\left[\frac{1}{4}\int_0^{t_0} |(\sigma^{-1}Z)(X_s)|^2 \mathrm{d}s\right] < \infty, \quad X_0 = x \in \mathbb{R}^d, \tag{3.3}$$

so that $\mathbb{Q} := \exp[\frac{1}{\sqrt{2}}\int_0^{t_0}\langle(\sigma^{-1}Z)(X_s), \mathrm{d}W_s\rangle - \frac{1}{4}\int_0^{t_0}|(\sigma^{-1}Z)(X_s)|^2]\mathbb{P}$ is a probability measure. In this case, by the Girsanov theorem,

$$\tilde{W}_t := W_t - \frac{1}{\sqrt{2}}\int_0^t (\sigma^{-1}Z)(X_s)\,\mathrm{d}s, \quad t \in [0, t_0]$$

is a Brownian motion under $\mathbb{Q}$. Thus, rewriting (2.2) as

$$\mathrm{d}X_t = (Z + Z_0)(X_t) + \sqrt{2}\,\sigma(X_t)\mathrm{d}\tilde{W}_t, \quad t \in [0, t_0],$$

we see that $(X_t, \tilde{W}_t)_{t\in[0,t_0]}$ is a weak solution to (2.3) under the probability measure $\mathbb{Q}$.

To prove (3.3), we use the Harnack inequality (3.2) for $p = d + 1$ to derive

$$\begin{aligned}
&\big\{\mathbb{E}\mathrm{e}^{\lambda(|(\sigma^{-1}Z)(X_t)|^2\wedge N)}\big\}^{d+1} = \big(P_t^0\mathrm{e}^{\lambda(|\sigma^{-1}Z|^2\wedge N)}(x)\big)^{d+1}\\
&\le P_t^0\mathrm{e}^{(d+1)\lambda(|\sigma^{-1}Z|^2\wedge N)}(y)\mathrm{e}^{\Phi_{d+1}(x)(1+|x-y|^2/t)}, \quad t \in (0,1], N > 0, |y - x| \le \frac{1}{\Phi_{d+1}(x)}.
\end{aligned}$$

Since μ_0 is P_t^0-invariant, for $B_{x,t} := \big\{y : |y - x| \le \frac{1}{\Phi_{d+1}(x)} \wedge \sqrt{t}\big\}$ this implies

$$\begin{aligned}
&\big\{\mathbb{E}\exp\big[\lambda(|(\sigma^{-1}Z)(X_t)|^2 \wedge N)\big]\big\}^{d+1}\mu_0(B_{x,t})\mathrm{e}^{-2\Phi_{d+1}(x)}\\
&\le \int_{B_{x,t}} \big(P_t^0\mathrm{e}^{\lambda(|\sigma^{-1}Z|^2\wedge N)}\big)^{d+1}(x)\,\exp\Big[-\Phi_{d+1}(x)\Big(1 + \frac{|x-y|^2}{t}\Big)\Big]\mu_0(\mathrm{d}y)\\
&\le \int_{B_{x,t}} P_t^0\mathrm{e}^{(d+1)\lambda(|\sigma^{-1}Z|^2\wedge N)}(y)\mu_0(\mathrm{d}y) \le \mu_0(\mathrm{e}^{\varepsilon|\sigma^{-1}Z|^2}) < \infty, \quad t \in (0,1], \lambda \in \Big(0, \frac{\varepsilon}{d+1}\Big].
\end{aligned}$$

Since μ_0 has strictly positive and continuous density e^{-V} with respect to $\mathrm{d}x$, there exists $G \in C(\mathbb{R}^d \to (0,\infty))$ such that $\mu_0(B_{x,t}) \ge G(x)t^{\frac{d}{2}}$ for $t \in (0,1]$ and $x \in \mathbb{R}^d$. By taking $\lambda = \varepsilon/(d+1)$ and letting $N \to \infty$ in the above display, we arrive at

$$\mathbb{E}\mathrm{e}^{\frac{\varepsilon}{d+1}|(\sigma^{-1}Z)(X_t)|^2} \le \frac{H(x)}{\sqrt{t}} < \infty, \quad t \in (0,1], x \in \mathbb{R}^d$$

for some positive $H \in C(\mathbb{R}^d)$. Therefore, by Jensen's inequality, we have

$$\begin{aligned}
&\mathbb{E}\exp\Big[\gamma\int_0^r |(\sigma^{-1}Z)(X_s)|^2\mathrm{d}s\Big] \le \frac{1}{r}\int_0^r \mathbb{E}\mathrm{e}^{\gamma r|(\sigma^{-1}Z)(X_s)|^2}\mathrm{d}s\\
&\le \frac{1}{r}\int_0^r \frac{H(x)}{\sqrt{t}}\mathrm{d}t = \frac{2H(x)}{\sqrt{r}} < \infty, \quad x \in \mathbb{R}^d, r \in (0,1], \gamma \in \Big(0, \frac{\varepsilon}{(d+1)r}\Big].
\end{aligned} \tag{3.4}$$

This implies (3.3) by taking $\gamma = \frac{1}{4}$ and $t_0 = r = 1 \wedge \frac{4\varepsilon}{d+1}$.

(b) **Strong Feller of P_t and uniqueness of invariant probability measure**. According to [7, Theorem 4.1], the Markov semigroup P_t^0 is strong Feller. For any $x \in \mathbb{R}^d$, we let X_s^x solve (2.2) with initial point x and define

$$R_r^x = \exp\Big[\frac{1}{\sqrt{2}}\int_0^r \langle(\sigma^{-1}Z)(X_s^x), \mathrm{d}W_s\rangle - \frac{1}{4}\int_0^r |(\sigma^{-1}Z)(X_s^x)|^2\mathrm{d}s\Big], \quad r \in [0, t_0].$$

By (3.3) and the Girsanov theorem, we have

$$P_t f(x) = \mathbb{E}\big[f(X_t^x) R_t^x\big], \quad t \in [0, t_0], f \in B_b(\mathbb{R}^d), x \in \mathbb{R}^d.$$

Then for any $t > 0, x \in \mathbb{R}^d$ and $f \in \mathscr{B}_b(\mathbb{R}^d)$, the semigroup property of P_s and the strong Feller property of P_s^0 imply

$$\begin{aligned}
&\limsup_{y\to x} |P_t f(y) - P_t f(x)| = \limsup_{y\to x} |P_r(P_{t-r}f)(y) - P_r(P_{t-r}f)(x)| \\
&= \limsup_{y\to x} \big|\mathbb{E}[R_r^y P_{t-r} f)(X_r^y) - R_r^x (P_{t-r}f)(X_r^x)]\big| \\
&\le \limsup_{y\to x} \Big\{\big|P_r^0(P_{t-r}f)(y) - P_r^0(P_{t-r}f)(x)\big| + \mathbb{E}\big(|R_r^y - 1| + |R_r^x - 1|\big)\Big\} \\
&\le \sup_{y:|y-x|\le 1} \mathbb{E}\big(|R_r^y - 1| + |R_r^x - 1|\big), \quad r \in (0, t).
\end{aligned}$$

Noting that $\mathbb{E}|R_r^y - 1|^2 = \mathbb{E}(R_r^y)^2 - 1$ for small $r > 0$, then the strong Feller property follows provided

$$\limsup_{r\to 0} \sup_{y:|y-x|\le 1} \mathbb{E}(R_r^y)^2 \le 1. \tag{3.5}$$

To prove this, we let $M_r = \frac{1}{\sqrt{2}} \int_0^r \langle (\sigma^{-1}Z)(X_s^x), \mathrm{d}W_s\rangle$. Then for small $r > 0$

$$\mathbb{E}(R_r^y)^2 = \mathbb{E}\mathrm{e}^{2M_r - \langle M\rangle_r} \le \big(\mathbb{E}\mathrm{e}^{4M_r - 8\langle M\rangle_r}\big)^{\frac{1}{2}} \big(\mathbb{E}\mathrm{e}^{6\langle M\rangle_r}\big)^{\frac{1}{2}} = \big(\mathbb{E}\mathrm{e}^{3\int_0^r |(\sigma^{-1}Z)(X_s^x)|^2 \mathrm{d}s}\big)^{\frac{1}{2}}.$$

So, applying (3.4) with $\gamma = \frac{\varepsilon}{(d+1)r}$ for small $r > 0$, and using Jensen's inequality, we obtain

$$\begin{aligned}
&\limsup_{r\to 0} \sup_{y:|y-x|\le 1} \big\{\mathbb{E}(R_r^y)^2\big\}^2 \le \limsup_{r\to 0} \sup_{y:|y-x|\le 1} \mathbb{E}\mathrm{e}^{3\int_0^r |(\sigma^{-1}Z)(X_s^x)|^2 \mathrm{d}s} \\
&\le \limsup_{r\to 0} \sup_{y:|y-x|\le 1} \big(\mathbb{E}\mathrm{e}^{\gamma\int_0^r |(\sigma^{-1}Z)(X_s^x)|^2 \mathrm{d}s}\big)^{\frac{3}{\gamma}} \le \limsup_{r\to 0} \sup_{y:|y-x|\le 1} \Big(\frac{2H(y)}{\sqrt{r}}\Big)^{\frac{3(d+1)r}{\varepsilon}} = 1.
\end{aligned}$$

This implies (3.5).

Next, as already mentioned above, every invariant probability measure of P_t has strictly positive density with respect to the Lebesgue measure, so that any two invariant probability measures are equivalent each other. Therefore, the invariant probability measure has to be unique, since it is well known that any two different extremal invariant probability measures of a strong Feller Markov operator are singular each other.

(c) **The assertion for $P_t^{\mathscr{O}}$.** Due to the semigroup property ensured by the pathwise uniqueness, it suffices to prove for small enough $t > 0$. Let $T_{\mathscr{O}}^x$ be the hitting time of X_t^x to the boundary of $\mathscr{O}$. By the Girsanov theorem we have

$$P_t^{\mathscr{O}} f(x) = \mathbb{E}\big[1_{\{T_{\mathscr{O}}^x > t\}} f(X_t^x) R_t^x\big], \quad f \in \mathscr{B}_b(\mathscr{O}), x \in \mathscr{O}. \tag{3.6}$$

Let $P_t^{\mathscr{O},0} f(x) = \mathbb{E}\big[1_{\{T_{\mathscr{O}}^x > t\}} f(X_t^x)\big]$ be the Dirichlet semigroup associated to (2.3). Since σ is invertible, by the C^2-regularity of σ and V we see that $P_t^{\mathscr{O},0}$ is strong Feller having strictly positive density with respect to the Lebesgue measure (see [4] for gradient estimates and log-Harnack inequalities of $P_t^{\mathscr{O},0}$). Then the strong Feller property can be proved as in (b) using $P_t^{\mathscr{O},0}$ in place of P_t^0.

Next, by (3.4) we have $\mathbb{E}\{(R_t^x)^{-1}\} < \infty$ for small $t > 0$. Then for any measurable set A such that $P_t^{\mathscr{O}} 1_A(x) = 0$, (3.6) implies

$$\{P_t^{\mathscr{O},0} 1_A(x)\}^2 = \big\{\mathbb{E}\big[1_{\{T_{\mathscr{O}}^x > t\}} 1_A(X_t^x)\big]\big\}^2 \le \{P_t^{\mathscr{O}} 1_A(x)\}\mathbb{E}\{(R_t^x)^{-1}\} = 0.$$

Thus, the measure $P_t^{\mathscr{O},0} 1_{\mathrm{d}z}(x)$ is absolutely continuous to the measure $P_t^{\mathscr{O}} 1_{\mathrm{d}z}(x)$. Since $P_t^{\mathscr{O},0}$ has a strictly positive density, so does $P_t^{\mathscr{O}}$. □

Proof of Theorem 2.1. Since $|\sigma^*\nabla\rho_\sigma(0,\cdot)| = 1$, for any $\delta > 0$ the function $\rho_\sigma(0,\cdot)$ can be uniformly approximated by smooth ones f_n with $|\sigma^*\nabla f_n| \le 1+\delta$. In particular, we may take $\tilde\rho \in C^2(\mathbb{R}^d)$ such that $|\rho_\sigma(0,\cdot) - \tilde\rho| \le 1$ and $|\sigma^*\nabla\tilde\rho|^2 \le 2$, so that (2.4) holds for some $\varepsilon \in (0,1)$ if and only if

$$\int_{\mathbb{R}^d} \Big(|\sigma^*\nabla\psi(x)|^2 + \mathrm{e}^{\varepsilon|(\sigma^{-1}Z)(x)|^2}\Big)\mathrm{e}^{-V(x)-\varepsilon^{-1}\tilde\rho(x)^2}\mathrm{d}x < \infty \tag{3.7}$$

holds for some $\varepsilon \in (0,1)$.

To apply Lemma 3.1, we take

$$\bar\mu_0(\mathrm{d}x) := \frac{\mathrm{e}^{-V(x)-2\varepsilon^{-1}\tilde\rho(x)^2}\mathrm{d}x}{\int_{\mathbb{R}^d}\mathrm{e}^{-V(x)-2\varepsilon^{-1}\tilde\rho(x)^2}\mathrm{d}x},$$

which is a probability measure by (3.7). Let

$$\bar Z_0(x) = Z_0(x) - 2\varepsilon^{-1}a(x)\nabla\tilde\rho(x)^2,\quad \bar Z(x) = Z(x) + 2\varepsilon^{-1}a(x)\nabla\tilde\rho(x)^2.$$

By (3.7) we have $\bar\mu_0(|\sigma^*\nabla\psi|^2) < \infty$, so that $f_n := (n-\psi)^+ \wedge 1 \to 1$ in $L^2(\bar\mu_0)$ and

$$\lim_{n\to\infty}\bar\mu_0(|\sigma^*\nabla f_n|^2) = \lim_{n\to\infty}\int_{1+n\ge\psi\ge n}|\sigma^*\nabla\psi|^2\mathrm{d}\bar\mu_0 = 0.$$

Thus, $1 \in H_\sigma^{2,1}(\bar\mu_0)$ and $\bar{\mathscr{E}}_0(1,1) = 0$. Then by Lemma 3.1 for $(\bar Z_0, \bar Z, \bar\mu_0)$ in place of (Z_0, Z, μ_0), and due to (3.7), it remains to prove $\bar\mu_0(\mathrm{e}^{\varepsilon'|\sigma^{-1}\bar Z|^2}) < \infty$ for some $\varepsilon' > 0$. Since $|\sigma^*\nabla\tilde\rho|^2 \le 2$, we have

$$|\sigma^{-1}\bar Z|^2(x) \le 2|\sigma^{-1}Z|^2(x) + 8\varepsilon^{-2}|(\sigma^*\nabla\tilde\rho(x)^2|^2(x) \le 2|\sigma^{-1}Z|^2(x) + 64\varepsilon^{-2}\tilde\rho(x)^2.$$

By (2.4), for $\varepsilon' \in (0, \frac{\varepsilon}{64}]$ we have

$$\begin{aligned}\bar\mu_0(\mathrm{e}^{\varepsilon'|\sigma^{-1}\bar Z|^2}) &\le \frac{1}{\int_{\mathbb{R}^d}\mathrm{e}^{-V(x)-2\varepsilon^{-1}\tilde\rho(x)^2}\mathrm{d}x}\int_{\mathbb{R}^d}\mathrm{e}^{2\varepsilon'|\sigma^{-1}Z|^2(x)+64\varepsilon'\varepsilon^{-2}\tilde\rho(x)^2-V(x)-2\varepsilon^{-1}\tilde\rho(x)^2}\mathrm{d}x\\ &\le \frac{1}{\int_{\mathbb{R}^d}\mathrm{e}^{-V(x)-2\varepsilon^{-1}\tilde\rho(x)^2}\mathrm{d}x}\int_{\mathbb{R}^d}\mathrm{e}^{\varepsilon|\sigma^{-1}Z|^2(x)-V(x)-\varepsilon^{-1}\tilde\rho(x)^2}\mathrm{d}x < \infty.\end{aligned}$$

Therefore, the proof is finished. □

Proof of Corollary 2.2. Let $n \ge 1$ such that $B_n := \{|\cdot| \le n\} \supset D$. It suffices to show that for any $l \ge n+1$ and any $x \in S_l := \{|\cdot| = l\}$, the solution $\bar X_t^x$ to (2.3) is non-explosive. Let

$$\zeta^x = \lim_{m\to\infty} \inf\{t \ge 0 : |\bar X_t^x| \ge m\}, \quad \sigma_n^x = \inf\{t \ge 0 : |\bar X_t| \le n\}, \quad m > l \ge n+1, x \in S_l.$$

Let $\tilde X_t^x$ solve the SDE (2.3) for $Z1_{B_n^c}$ in place of Z. Due to (2.5), Theorem 2.1 applies to $\tilde X_t$. In particular, $\tilde X_t^x$ is non-explosive, i.e.

$$\tilde\zeta^x := \lim_{m\to\infty} \inf\{t \ge 0 : |\tilde X_t^x| \ge m\} = \infty, \tag{3.8}$$

where and in the following, $\inf\emptyset := \infty$. Moreover, since $|Z| \in L^p_{loc}(\mathrm{d}x)$ for some $p > d$, [45, Theorem 1.1] implies the pathwise uniqueness of the SDE (2.3). So,

$$\tilde X_t^x = \bar X_t^x, \quad t \le \sigma_n^x.$$

Then

$$\sigma_n^x = \tilde\sigma_n^x := \inf\{t \ge 0 : |\tilde X_t^x| \le n\} \tag{3.9}$$

and

$$\zeta^x = \tilde\zeta^x \ \text{ if } \ \zeta^x \le \sigma_n^x. \tag{3.10}$$

Obviously, for

$$\theta_n^x := \inf\{t \ge \sigma_n^x : \ |X_t^x| \ge l\}$$

we have

$$\{\sigma_n^x < \zeta^x\} = \{\theta_n^x < \zeta^x\}. \tag{3.11}$$

By (3.8), (3.11) and the strong Markov property ensured by the uniqueness (see [24, Theorem 5.1]), we have

$$\begin{aligned}
\mathbb{P}(\zeta^x \le T) &= \mathbb{P}(\zeta^x \le T, \sigma_n^x \ge \zeta^x) + \mathbb{P}(\zeta^x \le T, \sigma_n^x < \zeta^x)\\
&\le \mathbb{P}(\tilde\zeta^x \le T) + \mathbb{P}(\theta_n^x < \zeta^x \le T) = \mathbb{E}\big[1_{\{\theta_n^x \le T\}}\mathbb{P}(\theta_n^x < \zeta^x \le T|\mathscr{F}_{\theta_n^x})\big]\\
&= \mathbb{E}\big[1_{\{\theta_n^x \le T\}}\{\mathbb{P}(\zeta^z \le T-s)|_{s=\theta_n^x, z=X^x_{\theta_n^x}}\}\big]\\
&\le \mathbb{P}(\theta_n^x \le T)\sup_{z\in S_l}\mathbb{P}(\zeta^z \le T) \le \mathbb{P}(\sigma_n^x \le T)\sup_{z\in S_l}\mathbb{P}(\zeta^z \le T), \quad T > 0, x \in S_l.
\end{aligned}$$

Combining this with (3.9) we obtain

$$\sup_{x\in S_l}\mathbb{P}(\zeta^x \le T) \le \Big\{\sup_{x\in S_l}\mathbb{P}(\tilde\sigma_n^x \le T)\Big\}\sup_{x\in S_l}\mathbb{P}(\zeta^x \le T), \quad T > 0. \tag{3.12}$$

Let $\tilde P_t^{\mathscr{O}}$ be the Dirichlet semigroup of $\tilde X_t$ for $\mathscr{O} = B_n^c$. By applying Theorem 2.1 for $Z1_{B_n^c}$ in place of Z, we obtain

$$\mathbb{P}(\tilde\sigma_n^x \le T) = 1 - \mathbb{P}(\tilde\sigma_n^x > T) = 1 - \tilde P_T^{\mathscr{O}}1(x) < 1$$

and that $\mathbb{P}(\tilde\sigma_n^x \le T)$ is continuous in $x \in \mathscr{O}$. So,

$$\varepsilon_T := \sup_{x\in S_l}\mathbb{P}(\sigma_n^x \le T) < 1.$$

This together with (3.12) implies $\mathbb{P}(\zeta^x \le T) = 0$ for any $T > 0$ and $x \in S_l$. Since $l \ge n+1$ is arbitrary and the solution is continuous, we have $\mathbb{P}(\zeta^x = \infty) = 1$ for all $x \in \mathbb{R}^d$. □

4 Proofs of Theorem 2.3 and Theorem 2.4

Since the uniqueness of invariant probability measure is ensured by the irreducibility and the strong Feller property, we only prove the existence and regularity estimates on the density. The new technique in the proof of the existence is to reduce the usual tightness condition to the boundedness of a Feyman-Kac semigroup, which follows from the hypercontractivity of P_t^0 under the given integrability condition. Moreover, to estimate the derivative of the density, the formula (4.3) below will play a crucial role.

Lemma 4.1. *Let $V \in W_{loc}^{1,1}(\mathrm{d}x)$ and $\sigma \in W_{loc}^{1,1}(\mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d; \mathrm{d}x)$ such that $\mu_0(\mathrm{d}x) = \mathrm{e}^{-V(x)}\mathrm{d}x$ is a probability measure satisfying (2.6) and the Poincaré inequality*

$$\mu_0(f^2) \le C\mu_0(|\sigma^*\nabla f|^2) + \mu_0(f)^2, \quad f \in C_0^\infty(\mathbb{R}^d) \tag{4.1}$$

for some constant $C > 0$. Let L_0 be in (2.1) and let $L := L_0 + Z \cdot \nabla$ for some measurable $Z : \mathbb{R}^d \to \mathbb{R}^d$. If Z has compact support and $|Z| + |\nabla\sigma| \in L^p(\mathrm{d}x)$ for some $p \in [2,\infty) \cap (d,\infty)$, then any invariant probability measure μ of L is absolutely continuous with respect to μ_0 with density $\rho := \frac{\mathrm{d}\mu}{\mathrm{d}\mu_0} \in H_\sigma^{2,1}(\mu_0)$ satisfying

$$\mu_0(\rho^2 + |\sigma^*\nabla\rho|^2) \le (C+1)\,\mu_0(\rho^2|\sigma^* Z|^2) < \infty. \tag{4.2}$$

Moreover,

$$\int_{\mathbb{R}^d} \langle \sigma^*\nabla f, \sigma^*\nabla\rho\rangle \mathrm{d}\mu_0 = \int_{\mathbb{R}^d} \langle Z, \nabla f\rangle\,\mathrm{d}\mu, \quad f \in H_\sigma^{2,1}(\mu_0). \tag{4.3}$$

Proof. Let μ be an invariant probability measure of L. Since $|Z| + |\nabla\sigma|$ is in $L_{loc}^p(\mathrm{d}x)$ for some $p \in [2,\infty) \cap (d,\infty)$, by the local boundedness of Z_0 so is $|Z + Z_0|$. Then according to [8, Corollary 1.2.8], for any invariant probability measure μ of L, $\mu(\mathrm{d}x) = \hat\rho(x)\mathrm{d}x$ holds for some $\hat\rho \in W_{loc}^{p,1}(\mathrm{d}x)$. Since $\mu_0(\mathrm{d}x) = \mathrm{e}^{-V(x)}\mathrm{d}x$ and $V \in C^1(\mathbb{R}^d)$, this implies $\mu = \rho\mu_0$ for some $\rho \in W_{loc}^{2,1}(\mathrm{d}x)$. In particular, we may take a continuous version ρ which is thus locally bounded. By the integration by parts formula,

$$\begin{aligned} &\int_{\mathbb{R}^d} \langle \sigma^*\nabla\rho, \sigma^*\nabla f\rangle \mathrm{d}\mu_0 = -\int_{\mathbb{R}^d} \rho L_0 f \mathrm{d}\mu_0 \\ &= -\int_{\mathbb{R}^d} Lf \mathrm{d}\mu + \int_{\mathbb{R}^d} \langle Z, \nabla f\rangle\,\mathrm{d}\mu = \int_{\mathbb{R}^d} \langle \sigma^{-1}Z, \sigma^*\nabla f\rangle \rho\,\mathrm{d}\mu_0, \quad f \in C_0^\infty(\mathbb{R}^d). \end{aligned} \tag{4.4}$$

Since Z has compact support with $|Z| \in L^2(\mathrm{d}x)$, and $\rho + \|\sigma^{-1}\|$ is locally bounded, (4.4) implies

$$\left| \int_{\mathbb{R}^d} \langle \sigma^*\nabla\rho, \sigma^*\nabla f\rangle \mathrm{d}\mu_0 \right| \le \mu_0(\rho^2|\sigma^{-1}Z|^2)^{\frac12} \mu_0(|\sigma^*\nabla f|^2)^{\frac12} < \infty, \quad f \in C_0^\infty(\mathbb{R}^d).$$

Hence, $\mu_0(|\sigma^*\nabla\rho|^2) \le \mu_0(\rho^2|\sigma^{-1}Z|^2) < \infty$. This and (2.6) imply $\rho \wedge N \in H_\sigma^{2,1}(\mu_0)$ for any $N \in (0,\infty)$. By the Poincaré inequality (4.1) we obtain

$$\mu_0((\rho\wedge N)^2) \le C\mu_0(|\sigma^*\nabla(\rho\wedge N)|^2) + \mu_0(\rho)^2 \le C\mu_0(|\sigma^*\nabla\rho|^2) + 1 < \infty, \quad N \in (0,\infty).$$

By letting $N \to \infty$ we prove $\rho \in H_\sigma^{2,1}(\mu_0)$ and (4.2), so that (4.3) follows from (4.4). □

Below we will often use the following version of Young's inequality on a probability space $(E, \mathscr{B}, \nu)$ (see [3, Lemma 2.4]):

$$\nu(fg) \le \log \nu(\mathrm{e}^f) + \nu(g \log g), \quad f, g \ge 0, \nu(g) = 1. \tag{4.5}$$

The next lemma ensures the existence of invariant probability measure of P_t for bounded $\sigma^{-1}Z$.

Lemma 4.2. *Assume* **(H1)**. *If $\sigma^{-1}Z$ is bounded then the Markov semigroup P_t associated to the SDE* (2.3) *has a unique invariant probability measure.*

Proof. According to (b) in the proof of Lemma 3.1, P_t has at most one invariant probability measure. So, it suffices to prove the existence. Letting $\mu_0 P_t$ be the distribution at time t of the solution to (2.3) with initial distribution μ_0, we intend to show that the sequence $\{\frac{1}{n}\int_0^n \mu_0 P_t \mathrm{d}t\}_{n \ge 1}$ is tight, so that the weak limit of a weakly convergent subsequence provides an invariant probability measure of P_t. To this end, it suffices to find out a positive compact function F on $\mathbb{R}^d$ such that

$$\frac{1}{n}\int_0^n \mu_0(P_t F)\,\mathrm{d}t \le C, \quad n \ge 1 \tag{4.6}$$

holds for some constant $C > 0$.

According to Gross [21], **(H1)** implies the hyperboundedness of P_t^0. Precisely, by [21, Theorem 1] (see for instance also [35, Theorem 5.1.4]), we have

$$\|P_t^0\|_{L^q(\mu_0) \to L^{q(t)}(\mu_0)} \le \exp\Big[\beta\Big(\frac{1}{q} - \frac{1}{q(t)}\Big)\Big], \quad t > 0, q > 1, q(t) := 1 + (q-1)\mathrm{e}^{\frac{4t}{\kappa}}. \tag{4.7}$$

Since μ_0 is a probability measure, there exists a compact function $W \ge 1$ such that $\mu_0(W) < \infty$. Letting $F = \sqrt{\log W}$ which is again a compact function, we have $\mu_0(\mathrm{e}^{V^2}) < \infty$. We now prove (4.6) for this function F. To this end, we consider the Feyman-Kac semigroup

$$P_t^F f(x) := \mathbb{E}\Big[f(X_t^x)\mathrm{e}^{\int_0^t F(X_s^x)\mathrm{d}s}\Big], \quad t \ge 0, x \in \mathbb{R}^d.$$

Since $\mu_0(\mathrm{e}^{F^2}) < \infty$, P_t^F is a bounded linear operator from $L^p(\mu_0)$ to $L^1(\mu_0)$ for every $t \ge 0$ and $p > 1$. We first observe that P_t^F is bounded on $L^p(\mu_0)$ for any $t \ge 0$ and $p > 1$. Let $q = \sqrt{p}$. For any non-negative $f \in L^p(\mu_0)$, by Schwarz's and Jensen's inequalities, and that μ_0 is P_t^0-invariant, we have

$$\begin{aligned}
&\mu_0\big(|P_t^F f|^p\big) = \int_{\mathbb{R}^d} \Big(\mathbb{E}\Big[f(X_t^x)\mathrm{e}^{\int_0^t F(X_s^x)\mathrm{d}s}\Big]\Big)^p \mu_0(\mathrm{d}x)\\
&\le \int_{\mathbb{R}^d} \bigg(\big\{\mathbb{E}f^q(X_t^x)\big\}\Big\{\mathbb{E}\mathrm{e}^{\frac{q}{q-1}\int_0^t F(X_s^x)\mathrm{d}s}\Big\}^{q-1}\bigg)^q \mu_0(\mathrm{d}x)\\
&= \int_{\mathbb{R}^d} (P_t^0 f^q)^q \bigg\{\frac{1}{t}\int_0^t P_s^0 \mathrm{e}^{\frac{qt}{q-1}F}\mathrm{d}s\bigg\}^{q(q-1)} \mathrm{d}\mu_0\\
&\le \big\{\mu_0((P_t^0 f^q)^{q(t)})\big\}^{\frac{q}{q(t)}} \bigg\{\int_{\mathbb{R}^d}\bigg(\frac{1}{t}\int_0^t P_s^0\mathrm{e}^{\frac{qt}{q-1}F}\mathrm{d}s\bigg)^{\frac{q(t)q(q-1)}{q(t)-q}}\mathrm{d}\mu_0\bigg\}^{\frac{q(t)-q}{q(t)}}\\
&\le \|P_t^0\|^q_{L^q(\mu_0)\to L^{q(t)}(\mu_0)} \mu_0\big(\{f^q\}^q\big) \max\bigg\{\Big(\mu_0\Big(\mathrm{e}^{\frac{q^2 q(t)t}{q(t)-q}F}\Big)\Big)^{\frac{q(t)-q}{q(t)}},\ \Big(\mu_0\Big(\mathrm{e}^{\frac{qt}{q-1}F}\Big)\Big)^{q(q-1)}\bigg\}\\
&= \|P_t^0\|^q_{L^q(\mu_0)\to L^{q(t)}(\mu_0)} \max\bigg\{\Big(\mu_0\Big(\mathrm{e}^{\frac{q^2 q(t)t}{q(t)-q}F}\Big)\Big)^{\frac{q(t)-q}{q(t)}},\ \Big(\mu_0\Big(\mathrm{e}^{\frac{qt}{q-1}F}\Big)\Big)^{q(q-1)}\bigg\}\mu_0(f^p),\quad t\ge 0.
\end{aligned}$$

By $\mu_0(\mathrm{e}^{F^2}) < \infty$ and (4.7), this implies $\|P_t^F\|_{L^p(\mu_0)} < \infty$ for any $t > 0$, and moreover, $\limsup_{t\downarrow 0}\|P_t^F\|_{L^p(\mu_0)} \le 1$. Since $F \ge 0$ implies $P_t^F 1 \ge 1$, we have $\lim_{t\downarrow 0}\|P_t^F\|_{L^p(\mu_0)} = 1$. In particular, by taking $p = 2$ and using the semigroup property, we obtain

$$\mathbb{E}\mathrm{e}^{\int_0^n F(X_t^{\mu_0})} = \mu_0(P_n^F 1) \le \|P_n^F\|_{L^2(\mu_0)} \le \|P_1^F\|^n_{L^2(\mu_0)} =: c_0^n < \infty,\quad n\ge 1, \tag{4.8}$$

where $X_t^{\mu_0}$ is the solution to (2.2) with initial distribution μ_0. Now, define

$$R_n = \exp\bigg[\frac{1}{\sqrt{2}}\int_0^n \langle(\sigma^{-1}Z)(X_s^{\mu_0}),\mathrm{d}W_s\rangle - \frac{1}{4}\int_0^n |(\sigma^{-1}Z)(X_s^{\mu_0})|^2\mathrm{d}s\bigg],\quad n\ge 0.$$

Since $\sigma^{-1}Z$ is bounded, by Girsanov's theorem we have

$$\mu_0(P_t F) = \mathbb{E}\big\{F(X_t^{\mu_0})R_n\big\},\quad t\in[0,n].$$

Then (4.5) and (4.8) imply

$$\begin{aligned}
\frac{1}{n}\int_0^n \mu_0(P_tF)\mathrm{d}t &= \frac{1}{n}\int_0^n \mathbb{E}\big\{F(X_t^{\mu_0})R_n\big\}\mathrm{d}t\\
&\le \frac{1}{n}\log\mathbb{E}\mathrm{e}^{\int_0^n F(X_t^{\mu_0})\mathrm{d}t} + \frac{1}{n}\mathbb{E}\big\{R_n\log R_n\big\} \le c_0 + \frac{1}{n}\mathbb{E}\big\{R_n\log R_n\big\}.
\end{aligned}\tag{4.9}$$

Since by Girsanov's theorem

$$\tilde{W}_t := W_t - \frac{1}{\sqrt{2}}\int_0^t (\sigma^{-1}Z)(X_s^{\mu_0})\mathrm{d}s,\quad t\in[0,n]$$

is a d-dimensional Brownian motion under the probability $\mathbb{Q}_n := R_n\mathbb{P}$, we have

$$\begin{aligned}
&\mathbb{E}\big\{R_n\log R_n\big\} = \mathbb{E}_{\mathbb{Q}_n}\log R_n\\
&= \mathbb{E}_{\mathbb{Q}_n}\bigg(\frac{1}{\sqrt{2}}\int_0^n\langle(\sigma^{-1}Z)(X_s^{\mu_0}),\mathrm{d}\tilde{W}_s\rangle + \frac{1}{4}\int_0^n |(\sigma^{-1}Z)(X_s^{\mu_0})|^2\mathrm{d}s\bigg) \le \frac{n\|\sigma^{-1}Z\|_\infty^2}{4}.
\end{aligned}$$

Combining this with (4.9), we prove (4.6), and hence finish the proof. □

Proof of Theorem 2.3(1). By Lemma 3.1, **(H1)** implies that (2.3) has a unique non-explosive solution and the associated Markov semigroup P_t is strong Feller with at most one invariant probability measure. To apply Lemma 4.1, we first consider bounded Z with compact support, then pass to the general situation by using an approximation argument.

(a) Let Z be bounded with compact support. By Lemma 4.2, P_t has a unique invariant probability measure μ. In particular, $L^*\mu = 0$, so that by Lemma 4.1(1) we have $\mu = \rho\mu_0$ for some $\rho \in H_\sigma^{2,1}(\mu_0)$ such that (4.3) holds.

Since $\rho \in H_\sigma^{2,1}(\mu_0)$, $f := \log(\rho+\delta) \in H_\sigma^{2,1}(\mu_0)$ for all $\delta > 0$. Taking this f in (4.3) we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} \frac{|\sigma^*\nabla\rho|^2}{\rho+\delta}\mathrm{d}\mu_0 \le \int_{\mathbb{R}^d} \{|\sigma^{-1}Z|\cdot|\sigma^*\nabla\log(\rho+\delta)|\}\,\mathrm{d}\mu\\
&= \int_{\mathbb{R}^d} \{|\sigma^{-1}Z|\cdot|\sigma^*\nabla\log(\rho+\delta)|\}\rho\,\mathrm{d}\mu_0 \le \bigg(\int_{\mathbb{R}^d}\rho|\sigma^{-1}Z|^2\mathrm{d}\mu_0\bigg)^{\frac{1}{2}}\bigg(\int_{\mathbb{R}^d}\frac{\rho|\sigma^*\nabla\rho|^2}{(\rho+\delta)^2}\,\mathrm{d}\mu_0\bigg)^{\frac{1}{2}}\\
&\le \bigg(\int_{\mathbb{R}^d}\rho|\sigma^{-1}Z|^2\mathrm{d}\mu_0\bigg)^{\frac{1}{2}}\bigg(\int_{\mathbb{R}^d}\frac{|\sigma^*\nabla\rho|^2}{\rho+\delta}\,\mathrm{d}\mu_0\bigg)^{\frac{1}{2}},\quad \delta>0.
\end{aligned}$$

Since $\mu_0(\frac{|\sigma^*\nabla\rho|^2}{\rho+\delta}) < \infty$ due to $\rho \in H_\sigma^{2,1}(\mu_0)$, this implies

$$\int_{\mathbb{R}^d} \frac{|\sigma^*\nabla\rho|^2}{\rho+\delta}\mathrm{d}\mu_0 \le \int_{\mathbb{R}^d}\rho|\sigma^{-1}Z|^2\mathrm{d}\mu_0,\quad \delta>0.$$

By letting $\delta \to 0$ we obtain

$$\int_{\mathbb{R}^d}\big|\sigma^*\nabla\sqrt{\rho}\big|^2\mathrm{d}\mu_0 \le \frac{1}{4}\int_{\mathbb{R}^d}\rho|\sigma^{-1}Z|^2\mathrm{d}\mu_0 < \infty \tag{4.10}$$

since $\sigma^{-1}Z$ is bounded and $\mu_0(\rho) = 1$. So, $\sqrt{\rho} \in H_\sigma^{2,1}(\mu_0)$ by (2.6), and the log-Sobolev inequality (2.7) implies

$$\mu(\rho\log\rho) \le \kappa\int_{\mathbb{R}^d}\big|\sigma^*\nabla\sqrt{\rho}\big|^2\mathrm{d}\mu_0 + \beta. \tag{4.11}$$

Combining this with (4.10) and the Young inequality (4.5), we obtain

$$\begin{aligned}
\mu_0(|\sigma^*\nabla\sqrt{\rho}|^2) &\le \frac{1}{4\lambda}\log\mu_0(\mathrm{e}^{\lambda|\sigma^{-1}Z|^2}) + \frac{1}{4\lambda}\mu_0(\rho\log\rho)\\
&\le \frac{1}{4\lambda}\log\mu_0(\mathrm{e}^{\lambda|\sigma^{-1}Z|^2}) + \frac{\kappa}{4\lambda}\mu_0(|\sigma^*\nabla\sqrt{\rho}|^2) + \frac{\beta}{4\lambda}.
\end{aligned}$$

This and (4.10) imply (2.9).

Similarly, $\rho \in H_\sigma^{2,1}(\mu_0)$ implies $f = (\rho+\delta)^{-1} \in H_\sigma^{2,1}(\mu_0)$ for $\delta > 0$, so that by (4.3) we have

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{|\sigma^*\nabla\rho|^2}{(\rho+\delta)^2}\mathrm{d}\mu_0 &\le \int_{\mathbb{R}^d} \big\{\rho|\sigma^{-1}Z|\cdot|\sigma^*\nabla(\rho+\delta)^{-1}|\big\}\mathrm{d}\mu_0\\ &\le \bigg(\int_{\mathbb{R}^d}\frac{(\rho|\sigma^{-1}Z|)^2}{(\rho+\delta)^2}\mathrm{d}\mu_0\bigg)^{\frac{1}{2}}\bigg(\int_{\mathbb{R}^d}\frac{|\sigma^*\nabla\rho|^2}{(\rho+\delta)^2}\mathrm{d}\mu_0\bigg)^{\frac{1}{2}}\\ &\le \sqrt{\mu_0(|\sigma^{-1}Z|^2)}\bigg(\int_{\mathbb{R}^d}\frac{|\sigma^*\nabla\rho|^2}{(\rho+\delta)^2}\mathrm{d}\mu_0\bigg)^{\frac{1}{2}},\quad \delta>0.\end{aligned}$$

Therefore, (2.10) holds.

Finally, by [9] the density function ρ is strictly positive, so that by (2.10) and $H_\sigma^{2,1}(\mu_0) = W_\sigma^{2,1}(\mu_0)$ we have $\log\rho \in H_\sigma^{2,1}(\mu_0)$ if $\log\rho \in L^2(\mu_0)$. To prove $\mu_0(|\log\rho|^2) < \infty$, we use the Poincaré inequality. As explained above that the defective log-Sobolev inequality implies that the spectrum of L_0 is discrete, by the irreducibility of the Dirichlet form we see that L_0 has a spectral gap, equivalently, the Poincaré inequality

$$\mu_0(f^2) \le C\mu_0(|\sigma^*\nabla f|^2) + \mu(f)^2,\quad f \in H_\sigma^{2,1}(\mu_0)$$

holds for some constant $C > 0$. Since ρ is strictly positive, we take $\varepsilon \in (0,1)$ such that $\mu_0(\rho \le \varepsilon) \le \frac{1}{4}$. By (2.10) and $\mu_0(\rho) = 1$, for any $\delta > 0$ we have $\log(\rho+\delta) \in H_\sigma^{2,1}(\mu_0)$. Moreover, by the Poincaré inequality, (2.10) and $|\log(\rho+\delta)| \le \rho+\delta+\log\varepsilon^{-1}$ for $\rho \ge \varepsilon$, there exist constants $C_1, C_2 > 0$ such that

$$\begin{aligned}&\mu_0(|\log(\rho+\delta)|^2) \le C\mu_0(|\sigma^*\nabla\log(\rho+\delta)|^2) + \mu_0(\log(\rho+\delta))^2\\ &\le C_1 + 2\mu_0(\log(\rho+\delta)1_{\{\rho\le\varepsilon\}})^2 + 2\mu_0(\log(\rho+\delta)1_{\{\rho>\varepsilon\}})^2\\ &\le C_1 + 2\mu_0(|\log(\rho+\delta)|^2)\mu_0(\rho\le\varepsilon) + 2\mu_0(\rho+\delta+\log\varepsilon^{-1})^2\\ &\le \frac{1}{2}\mu_0(|\log(\rho+\delta)|^2) + C_2,\quad \delta\in(0,1).\end{aligned}$$

Since $\mu(|\log(\rho+\delta)|^2) < \infty$, this implies

$$\mu(|\log\rho|^2) = \lim_{\delta\downarrow 0}\mu(|\log(\rho+\delta)|^2) \le 2C_2 < \infty.$$

(b) In general, for any $n \ge 1$ let

$$Z_n(x) = 1_{\{|x|+|Z(x)|\le n\}}Z(x),\quad L_n = L_0 + Z_n\cdot\nabla.$$

By (a) and $|\sigma^{-1}Z_n| \le |\sigma^{-1}Z|$, L_n has an invariant probability measure $\mathrm{d}\mu_n = \rho_n\mathrm{d}\mu_0$ such that

$$\begin{aligned}&\mu_0\big(|\sigma^*\nabla\sqrt{\rho_n}|^2\big) \le \frac{1}{4\lambda-\kappa}\big\{\log\mu_0(\mathrm{e}^{\lambda|\sigma^{-1}Z|^2}) + \beta\big\} < \infty,\\ &\mu_0(|\sigma^*\nabla\log\rho_n|^2) \le \mu_0(|\sigma^{-1}Z|^2) < \infty.\end{aligned}$$

Then the family $\{\sqrt{\rho_n}\}_{n\geq 1}$ is bounded in $H^{2,1}_\sigma(\mu_0)$. Moreover, the defective log-Sobolev inequality (2.7) implies the existence of a super Poincaré inequality, and hence the essential spectrum of L_0 is empty, see [33, Theorem 2.1 and Corollary 3.3]. So, $H^{2,1}_\sigma(\mu_0)$ is compactly embedded into $L^2(\mu_0)$, i.e. a bounded set in $H^{2,1}_\sigma(\mu_0)$ is relatively compact in $L^2(\mu_0)$. Therefore, for some subsequence $n_k \to \infty$ we have $\sqrt{\rho_{n_k}} \to \sqrt{\rho}$ in $L^2(\mu_0)$ for some nonnegative ρ which satisfies (2.9) and (2.10). In particular, $\rho_{n_k} \to \rho$ in $L^1(\mu_0)$ so that $\mu := \rho\mu_0$ is a probability measure. Moreover, by using the Poincaré inequality as in (a), we prove $\log\rho \in L^2(\mu_0)$ so that $\log\rho \in H^{2,1}_\sigma(\mu_0)$. It remains to show that $L^*\mu = 0$.

Since $(L_{n_k})^*\mu_{n_k} = 0$, for any $f \in C_0^\infty(\mathbb{R}^d)$, there exists a constant $C > 0$ and a compact set D such that

$$\begin{aligned}\Big|\int_{\mathbb{R}^d} Lf\,\mathrm{d}\mu\Big| &= \Big|\int_{\mathbb{R}^d}\big(\rho Lf - \rho_{n_k}L_{n_k}f\big)\mathrm{d}\mu_0\Big| \\ &\leq C\int_D \Big\{|Z - Z_{n_k}|\rho + (1+|Z|)|\rho_{n_k} - \rho|\Big\}\mathrm{d}\mu_0.\end{aligned} \tag{4.12}$$

Since $\mu_0(\mathrm{e}^{\lambda|\sigma^{-1}Z|^2}) < \infty$, we have $|Z_n| \leq |Z| \in L^q_{loc}(\mathrm{d}x)$ for any $q > 1$. Then $\mu_0(1_D|Z - Z_n|^q) \to 0$ as $n \to \infty$ holds for any $q > 1$. Moreover, the local Harnack inequality (see [8, Corollary 1.2.11]) implies that $\{\rho_{n_k}, \rho\}_{k\geq 0}$ is uniformly bounded on the compact set D. Combining these with $\mu_0(|\rho_{n_k} - \rho|) \to 0$, we may use the dominated convergence theorem to prove $\mu(Lf) = 0$ by taking $k \to \infty$ in (4.12). Therefore, $L^*\mu = 0$. Then the proof is complete. □

Proof of Theorem 2.3(2). By Theorem 2.1, the SDE (2.3) has a unique solution and the associated semigroup P_t is strong Feller having at most one invariant probability measure. So, it suffices to prove that the above constructed probability measure μ is the unique invariant probability measure of L and P_t. This can be done according to [29] and [8] as follows.

Let $b_0 = Z_0 + a\nabla\log\rho$ and $b = Z + Z_0$. Then $L = \mathrm{tr}(a\nabla^2) + b\cdot\nabla$, and $\hat{L}_0 := \mathrm{tr}(a\nabla^2) + b_0\cdot\nabla$ is symmetric in $L^2(\mu)$. Obviously, **(H1)** and (2.8) imply that conditions (1.1′)-(1.3′) and (1.4) in [29] hold for $U = \mathbb{R}^d$; that is, $a_{ij} \in W^{2,1}_{loc}(\mathrm{d}x)$, a is locally uniformly positive definite, and $b \in L^2_{loc}(\mathrm{d}x)$. Moreover, by the Young inequality (4.5), (2.9), (2.8), (2.11) and (4.11), for small enough $r > 0$ we have

$$\begin{aligned}&\mu(\|a\| + |b - b_0|) \leq \mu_0(\rho|Z| + \|\sigma\|\cdot|\sigma^*\nabla\rho| + \rho\|\sigma\|^2)\\ &\leq \frac{1}{2}\mu_0(\rho(|\sigma^{-1}Z|^2 + 3\|\sigma\|^2)) + \mu_0(\|\sigma\|\cdot|\sigma^*\nabla\rho|)\\ &\leq \frac{1}{2r}\mu_0(\rho\log\rho) + \frac{1}{2r}\log\mu_0(\mathrm{e}^{r(|\sigma^{-1}Z|^2+3\|\sigma\|^2)}) + 2\sqrt{\mu_0(\rho\|\sigma\|^2)\mu_0(|\sigma^*\nabla\sqrt{\rho}|^2)}\\ &\leq \frac{1}{2r}\mu_0(\rho\log\rho) + \frac{1}{2r}\log\mu_0(\mathrm{e}^{r(|\sigma^{-1}Z|^2+3\|\sigma\|^2)})\\ &\quad + 2\sqrt{\{\varepsilon^{-1}\mu_0(\rho\log\rho) + \varepsilon^{-1}\log\mu_0(\mathrm{e}^{\varepsilon\|\sigma\|^2})\}\mu_0(|\sigma^*\nabla\sqrt{\rho}|^2)} < \infty.\end{aligned}$$

Therefore, by [29, Theorem 1.5, Proposition 1.9 and Proposition 1.10(a)], $(L, C_0^\infty(\mathbb{R}^d))$ has a unique closed extension in $L^1(\mu)$ which generates a Markov C_0-semigroup T_t^μ in $L^1(\mu)$ such that μ is an invariant probability measure. Then, according to [8, Corollary 1.7.3], μ is the unique invariant probability measure of L.

On the other hand, according to [29, Theorem 3.5], there is a standard Markov process $\{\bar{\mathbb{P}}_x\}_{x\in\mathbb{R}^d\cup\{\partial\}}$ which is continuous and non-explosive for μ-a.e. x, such that the associated semigroup $\bar{P}_t$ satisfies

$$\int_0^\infty \mathrm{e}^{-\lambda t}\bar{P}_t f\mathrm{d}t = \int_0^t \mathrm{e}^{-\lambda t}T_t^\mu f\mathrm{d}t, \quad \mu\text{-a.e.}$$

holds for any $f \in \mathscr{B}_b(\mathbb{R}^d)$ and $\lambda > 0$. So, for any $f \in \mathscr{B}_b(\mathbb{R}^d)$, $\bar{P}_t f = T_t^\mu f$ holds $\mathrm{d}t \times \mu$-a.e. By the continuity of the process and the strong continuity of T_t^μ in $L^1(\mu)$, $\bar{P}_t f = T_t^\mu f$ μ-a.e. for any $t \geq 0$ and $f \in C_b(\mathbb{R}^d)$, and hence also for $f \in L^1(\mu)$ since $C_b(\mathbb{R}^d)$ is dense in $L^1(\mu)$. That is, $\bar{P}_t$ is a μ-version of T_t^μ. In particular, μ is $\bar{P}_t$-invariant and the probability measure

$$\bar{\mathbb{P}}_\mu := \int_{\mathbb{R}^d} \mathbb{P}_x\mu(\mathrm{d}x) \text{ on } \bar{\Omega} := C([0,\infty) \to \mathbb{R}^d)$$

solves the martingale problem of $(L, C_0^\infty(\mathbb{R}^d))$, so that under this probability space the coordinate process $\bar{X}_t(\bar{\omega}) := \bar{\omega}_t$ for $t \geq 0$ and $\bar{\omega} \in \bar{\Omega}$ is a weak solution to (2.3) with initial distribution μ (c.f. [24, Proposition 2.1] or [30, §5.0]). By the uniqueness of solutions, this implies $\mu(P_t f) = \mu(\bar{P}_t f)$ for $t \geq 0$ and $f \in \mathscr{B}_b(\mathbb{R}^d)$. Therefore, μ is an invariant probability measure of P_t. □

Proof of Theorem 2.4. Obviously, the proof of Theorem 2.3(2) also works if we replace **(H1)** by **(H1′)**. So, we only need to prove assertion (1). Next, by repeating (b) in the proof of Theorem 2.3(1), we may and do assume that Z is bounded having compact support, and only prove that L has an invariant probability measure $\mathrm{d}\mu = \rho\mathrm{d}\mu_0$ with ρ satisfying the required estimates (2.13) and (2.14). Here, the only thing we need to clarify is that in the right hand side of (4.12) the term $(1 + |Z|)$ should be replaced by $(1 + |Z| + |\nabla\sigma|)$ since $\nabla\sigma$ is no longer locally bounded. This does not make any trouble since $|\nabla\sigma| \in L^2_{loc}(\mathrm{d}x)$ by **(H1′)**, and $(\rho_{n_k} - \rho)1_D$ is uniformly bounded according to [8, Corollarty 1.2.11].

Now, we assume that Z is bounded with compact support. Let $\tilde{V} \in C^\infty(\mathbb{R}^d)$ with $\|\tilde{V} - V\|_\infty \leq 1$, and let $\tilde{P}_t$ be the Markov semigroup generated by $\Delta - \nabla\tilde{V}$. Then $H^{2,1}(\mu_0) = H^{2,1}(\mathrm{e}^{-\tilde{V}(x)}\mathrm{d}x)$, so that **(H1′)** together with the smoothness and positivity-preserving of $\tilde{P}_t$ implies

$$\begin{aligned} &\tilde{a}_n := \tilde{P}_{\frac{1}{n}} a \in C^2(\mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d), \quad \tilde{a}_n \geq \alpha I, \\ &\text{and } (\tilde{a}_n)_{ij} \to a_{ij} \text{ in } H^{2,1}(\mu_0) \cap L^{2p}(\mu_0), \ 1 \leq i,j \leq d. \end{aligned} \tag{4.13}$$

Let $\tilde{L}_n$ be defined as L for $\tilde{a}_n$ in place of a; that is,

$$\tilde{L}_n = \mathrm{tr}(\tilde{a}_n\nabla^2) + \sum_{i,j=1}^d \big\{Z_i + \partial_j(\tilde{a}_n)_{ij} - (\tilde{a}_n)_{ij}\partial_j V\big\}e_i.$$

By Lemmas 4.1 and 4.2, $\tilde{L}_n$ has an invariant probability measure $\tilde{\mu}_n(\mathrm{d}x) := \tilde{\rho}_n(\mathrm{d}x)\mu_0(\mathrm{d}x)$ with $\tilde{\rho}_n \in H^{2,1}(\mu_0)$ such that

$$\mu_0(\tilde{\rho}_n^2 + |\nabla\tilde{\rho}_n|^2) \leq C\mu_0(\tilde{\rho}_n^2|Z|^2) < \infty.$$

According to [8, Corollarty 1.2.11], $\{\tilde\rho_n\}_{n\ge1}$ is uniformly bounded on the compact set $D := \operatorname{supp} Z$, so this implies that $\{\tilde\rho_n\}_{n\ge1}$ is bounded in $H^{2,1}(\mu_0)$, and hence $\tilde\rho_{n_k}\to\rho$ in $L^2(\mu_0)$ for some subsequence $n_k\to\infty$ and some $\rho\in H^{2,1}(\mu_0)$. In particular, $\mu(\mathrm{d}x) := \rho(x)\mathrm{d}x$ is a probability measure. We intend to prove $L^*\mu = 0$.

For any $f\in C_0^\infty(\mathbb{R}^d)$ there exists a constant $C(f)>0$ such that

$$|\tilde L_n f - Lf| \le C(f)\big(\|\nabla\tilde a_n - \nabla a\| + |\nabla V|\cdot\|\tilde a_n - a\|\big),$$
$$|\tilde L_n f| \le C(f)(\|\nabla \tilde a_n\| + \|a\|\cdot|\nabla V|).$$

By (4.13), $|\nabla V|\in L^{\frac{2p}{p-1}}(\mu_0)$ included in **(H1′)**, $\tilde\rho_{n_k}\to\rho$ in $L^2(\mu_0)$, and $\tilde L_n^*\tilde\mu_n = 0$, we are able to use the dominated convergence theorem to derive

$$|\mu(Lf)| = \lim_{k\to\infty}|\mu(Lf) - \tilde\mu_{n_k}(\tilde L_{n_k}f)| \le \limsup_{k\to\infty}\mu_0(|Lf - \tilde L_{n_k}f|\rho + |\tilde L_{n_k}f|\cdot|\tilde\rho_{n_k} - \rho|) = 0.$$

So, $L^*\mu = 0$.

Since (5.3) and $\tilde a_n\ge\alpha I$ imply (2.7) for $\big(\sqrt{\tilde a_n},\frac{\kappa'}{\alpha}\big)$ in place of (σ,κ), by Theorem 2.3 we have

$$\begin{aligned}\alpha\mu_0\Big(|\nabla\sqrt{\tilde\rho_{n_k}}|^2\Big) &\le \mu_0\Big(\Big|\sqrt{\tilde a_{n_k}}\nabla\sqrt{\tilde\rho_{n_k}}\Big|^2\Big)\\ &\le \frac{1}{4\alpha\lambda - \frac{\kappa'}{\alpha}}\Big\{\log\mu_0\big(\mathrm{e}^{\alpha\lambda|(\tilde a_{n_k})^{-1/2}Z|^2}\big) + \beta\Big\} \le \frac{\alpha}{4\alpha^2\lambda - \kappa'}\Big\{\log\mu_0\big(\mathrm{e}^{\lambda|Z|^2}\big) + \beta\Big\},\\ \alpha\mu_0(|\nabla\log\tilde\rho_{n_k}|^2) &\le \mu_0\big(\big|(\tilde a_{n_k})^{-1/2}\nabla\log\tilde\rho_{n_k}\big|^2\big) \le \frac{1}{\alpha}\mu_0(|Z|^2).\end{aligned}$$

By using $\rho_{n_k}+\delta$ to replace ρ_{n_k}, and letting first $k\to\infty$ then $\delta\downarrow 0$, we prove (2.13) and (2.14) from these two inequalities respectively. □

5 Proofs of Theorem 2.5 and Theorem 2.6

The following Sobolev embedding theorem is crucial in the proof. This result can be deduced from existing ones, for instance, [26, Corollary 1.4] in the framework of generalized Mehler semigroup. We include below a brief proof by using the dimension-free Harnack inequality for the O-U semigroup.

Lemma 5.1. *Let* (2.15) *hold. Then* $H^{2,1}(\mu_0)$ *is compactly embedded into* $L^2(\mu_0)$*; i.e. bounded sets in* $H^{2,1}(\mu_0)$ *are relatively compact in* $L^2(\mu_0)$.

Proof. Consider the linear SPDE

$$\mathrm{d}X_t = -AX_t\mathrm{d}t + \sqrt{2}\,\mathrm{d}W_t, \tag{5.1}$$

By (2.15), for any initial point x this equation has a unique mild solution

$$X_t^x = \mathrm{e}^{-At}x + \sqrt{2}\int_0^t \mathrm{e}^{-A(t-s)}\mathrm{d}W_s,\quad t\ge 0,$$

and the associated Markov semigroup

$$P_t^0 f(x) := \mathbb{E} f(X_t^x), \quad t \geq 0, f \in \mathscr{B}_b(\mathbb{H}), x \in \mathbb{H}$$

is symmetric in $L^2(\mu_0)$ with Dirichlet form

$$\mathscr{E}_0(f,g) := \mu_0(\langle \nabla f, \nabla g \rangle), \quad f, g \in H^{2,1}(\mu_0),$$

see for instance [17]. So, by the spectral theory, $H^{2,1}(\mu_0)$ is compactly embedded into $L^2(\mu_0)$ if and only if P_t^0 is compact for some (equivalently, all) $t > 0$, both are equivalent to the absence of the essential spectrum of the generator. By [38, Theorem 3.2.1] with $b = 0$ and $\sigma = \sqrt{2}$ so that $K = 0$ and $\lambda = \frac{1}{2}$, P_t^0 satisfies the Harnack inequality

$$(P_t^0 f(x))^2 \leq (P_t^0 f(y))^2 \mathrm{e}^{2|x-y|^2/t}, \quad t > 0, x, y \in \mathbb{H}, f \in \mathscr{B}_b(\mathbb{H}), \tag{5.2}$$

which implies that P_t^0 has a density with respect to the invariant probability measure μ_0 (see [38, Theorem 1.4.1]). Next, it is well known that the Gaussian measure μ_0 satisfies the log-Sobolev inequality (see for instance [21])

$$\mu_0(f^2 \log f^2) \leq \frac{2}{\lambda_1} \mu_0(|\nabla f|^2), \quad f \in H^{2,1}(\mu_0), \mu_0(f^2) = 1. \tag{5.3}$$

This, together with the existence of density of P_t^0 with respect to μ_0 for any $t > 0$, implies that P_t^0 is compact in $L^2(\mu_0)$ for all $t > 0$, see [20, Theorem 1.2], [34, Theorems 1.1 and 3.1] or [37, Theorem 1.6.1]. □

Proof of Theorem 2.5(1). For any $n \geq 1$, let $\mathbb{H}_{\langle n \rangle} = \{x \in \mathbb{H} : \langle x, e_i \rangle = 0, 1 \leq i \leq n\}$ be the orthogonal complement of $\mathbb{H}_n := \mathrm{span}\{e_1, \cdots, e_n\}$. Let $\pi_n : \mathbb{H} \to \mathbb{H}_n$ and $\pi_{\langle n \rangle} : \mathbb{H} \to \mathbb{H}_{\langle n \rangle}$ be orthogonal projections. For convenience, besides the orthogonal decomposition $\mathbb{H} = \mathbb{H}_n \oplus \mathbb{H}_{\langle n \rangle}$ we may regard $\mathbb{H}$ as the product space $\mathbb{H} = \mathbb{H}_n \times \mathbb{H}_{\langle n \rangle}$, so that $\mu_0 = \mu_0^{(n)} \times \mu_0^{\langle n \rangle}$ for $\mu_0^{(n)} = \mu_0 \circ \pi_n^{-1}$ and $\mu_0^{\langle n \rangle} = \mu_0 \circ \pi_{\langle n \rangle}^{-1}$ being the marginal distributions of μ_0 on $\mathbb{H}_n$ and $\mathbb{H}_{\langle n \rangle}$ respectively. Let

$$a_n(x) = \pi_n a(x), \quad Z_n(x) = \pi_n \int_{\mathbb{H}_{\langle n \rangle}} Z(x,y) \mu_0^{\langle n \rangle}(\mathrm{d}y), \quad x \in \mathbb{H}_n. \tag{5.4}$$

By **(H2)** we have

$$\langle a_n v, v \rangle \geq \alpha |v|^2, \quad v \in \mathbb{H}_n, \tag{5.5}$$

and due to Jensen's inequality,

$$\mu_0^{(n)}(\mathrm{e}^{\lambda |Z_n|^2}) \leq \int_{\mathbb{H}_n} \mathrm{e}^{\lambda \int_{\mathbb{H}_{\langle n \rangle}} |Z(x,y)|^2 \mu_0^{\langle n \rangle}(\mathrm{d}y)} \mu_0^{(n)}(\mathrm{d}x) \leq \int_{\mathbb{H}} \mathrm{e}^{\lambda |Z|^2} \mathrm{d}\mu_0 < \infty, \quad n \geq 1. \tag{5.6}$$

Let $V_n(x) = \frac{1}{2} \sum_{i=1}^n \lambda_i x_i^2$ and $L^{(n)} = L_0^{(n)} + Z_n \cdot \nabla$ on $\mathbb{H}_n$, where

$$L_0^{(n)} = \sum_{i,j=1}^n \Big(a_{ij} \partial_i \partial_j + \big\{ \partial_j a_{ij} - a_{ij} \partial_j V_n \big\} \partial_i \Big).$$

Noting that (5.3) and **(H2)** imply

$$\mu_0(f^2\log f^2)\le \frac{2}{\lambda_1\alpha}\mu_0(\langle a\nabla f,\nabla f\rangle),\quad f\in H^{2,1}(\mu_0),\mu_0(f^2)=1, \tag{5.7}$$

and that (5.5) implies $\alpha|a_n^{-1/2}Z_n|^2\le|Z_n|^2$, we may apply Theorem 2.3(1) to L_n on $\mathbb{R}^n\equiv\mathbb{H}_n$ for $\kappa=\frac{2}{\lambda_1\alpha},\beta=0$ and $\lambda\alpha$ in place of λ, to conclude that $L^{(n)}$ has an invariant probability measure μ_n with density function $\rho_n:=\frac{\mathrm{d}\mu^{(n)}}{\mathrm{d}\mu_0^{(n)}}$ satisfying $\sqrt{\rho}_n\in H^{2,1}(\mu_0^{(n)})$ and

$$\begin{aligned}\mu_0^{(n)}\big(\big|\nabla\sqrt{\rho}_n\big|^2\big)&\le\frac{1}{\alpha}\mu_0^{(n)}(|\sqrt{a_n}\nabla\sqrt{\rho_n}|^2)\le\frac{\lambda_1}{4\alpha^2\lambda_1\lambda-2}\log\mu_0^{(n)}(\mathrm{e}^{\lambda\alpha|a_n^{-1/2}Z_n|^2})\\&\le\frac{\lambda_1}{4\alpha^2\lambda_1\lambda-2}\log\mu_0^{(n)}(\mathrm{e}^{\lambda|Z_n|^2})\le\frac{\lambda_1}{4\alpha^2\lambda_1\lambda-2}\log\mu_0(\mathrm{e}^{\lambda|Z|^2})<\infty,\ n\ge1,\end{aligned}\tag{5.8}$$

where the last step is due to Jensen's inequality and the definitions of Z_n and $\mu_0^{(n)}$. Moreover,

$$\begin{aligned}\mu_0^{(n)}(|\nabla\log\rho_n|^2)&\le\frac{1}{\alpha}\mu_0^{(n)}(|\sqrt{a_n}\nabla\log\rho_n|^2)\\&\le\frac{\mu_0^{(n)}(|a_n^{-1/2}Z_n|^2)}{\alpha}\le\frac{\mu_0^{(n)}(|Z_n|^2)}{\alpha^2}\le\frac{\mu_0(|Z|^2)}{\alpha^2}<\infty,\quad n\ge1.\end{aligned}\tag{5.9}$$

Letting $\bar\rho_n=\rho_n\circ\pi_n$, (5.8) implies that $\{\sqrt{\bar\rho_n}\}_{n\ge1}$ is bounded in $H^{2,1}(\mu_0)$. By Lemma 5.1, there exists a subsequence $n_k\to\infty$ and some positive $\rho\in L^1(\mu_0)$ with $\sqrt{\rho}\in H^{2,1}(\mu_0)$ such that $\sqrt{\bar\rho_{n_k}}\to\sqrt{\rho}$ in $L^2(\mu_0)$, (2.20) and (2.21) hold. Then $\log\rho\in H^{2,1}(\mu_0)$ as shown in the end of the proof of Theorem 2.3(1) using the Poincaré inequality. In particular, $\mu:=\rho\mu_0$ is a probability measure on $\mathbb{H}$. It remains to show that $L^*\mu=0$.

By the definition of Z_n, we have $\bar Z_n:=Z_n\circ\pi_n=\pi_n\mu_0(Z|\pi_n)$, where $\mu_0(\cdot|\pi_n)$ is the conditional expectation of μ_0 given π_n. Since $\mu_0(|Z|^2)<\infty$, by the martingale converges theorem, $\mu_0(Z|\pi_n)\to Z$ in $L^2(\mu_0)$, and hence, $\bar Z_n\to Z$ in $L^2(\mu_0)$ as well. By the continuity of a, $\bar a_n:=a_n\circ\pi_n\to a$ pointwise. Noting that for any $f\in\mathscr{F}C_0^\infty$ there exist $l\in\mathbb{N}$ and a constant $C(f)>0$ such that

$$\begin{aligned}&|\mu(Lf)|=|\mu(Lf)-\mu_{n_k}(L_{n_k}f)|\\&\le C(f)\mu_0\Big(\rho\{|Z-\bar Z_{n_k}|+\sum_{i,j=1}^l|(a-\bar a_{n_k})_{ij}|\}\Big)+C(f)\mu_0\Big(\Big\{|\bar Z_{n_k}|+\sum_{i,j=1}^l|(\bar a_{n_k})_{ij}|\Big\}|\rho-\bar\rho_{n_k}|\Big)\end{aligned}$$

holds for $n_k\ge l$, to prove $\mu(Lf)=0$ by using the dominated convergence theorem, it suffices to verify the uniform integrability of $\{\bar\rho_n(|\bar Z_n|+|a_{ij}\circ\pi_n|)\}_{n\ge1}$ in $L^1(\mu_0)$ for every $i,j\ge1$. Obviously, for any $\varepsilon\in(0,1)$ there exists a constant $C(\varepsilon)>0$ such that

$$(|\bar Z_n|+|a_{ij}\circ\pi_n|)\bar\rho_n\le\mathrm{e}^{\varepsilon|\bar Z_n|^{1+\varepsilon}}+\mathrm{e}^{\varepsilon|a_{ij}\circ\pi_n|^{1+\varepsilon}}+C\bar\rho_n\{\log(\mathrm{e}+\bar\rho_n)\}^{\frac{1}{1+\varepsilon}},\quad n\ge1.$$

Since $\mu_0^{(n)}(f)=\mu_0(f\circ\pi_n)$ for $f\in L^1(\mu_0^{(n)})$, this implies the desired the uniform integrability by (2.17), (5.6), (5.8) and

$$\mu_0(\bar\rho_n\log\bar\rho_n)\le\frac{2}{\lambda_1}\mu_0(|\nabla\sqrt{\bar\rho_n}|^2)=\frac{2}{\lambda_1}\mu_0(|\nabla\sqrt{\rho_n}|^2)$$

due to the log-Sobolev inequality (5.3). □

Proof of Theorem 2.5(2). The desired assertion can be deduced from [29]. Since a is bounded and **(H2)** holds, we have $H^{2,1}(\mu_0) = H^{2,1}_\sigma(\mu_0)$. Let μ be a probability measure μ on $\mathbb{H}$ such that the form

$$\mathscr{E}^\mu(f,g) := \mu(\langle a\nabla f, \nabla g\rangle), \quad f,g \in \mathscr{F}C_0^\infty$$

is closable in $L^2(\mu)$, and let $(L^\mu, \mathscr{D}(L^\mu))$ be the generator of the closure $(\mathscr{E}^\mu, H^{2,1}(\mu))$. Moreover, let $\beta \in L^2(\mathbb{H} \to \mathbb{H}; \mu)$ such that

$$\mu(\langle \beta, \nabla f\rangle) = 0, \quad f \in H^{2,1}(\mu). \tag{5.10}$$

Then, according to Proposition 1.3, Theorem 1.9 and Proposition 1.10 in [29, Part II], we have the following assertions for $L := L^\mu + \beta \cdot \nabla$:

(i) $(L, \mathscr{F}C_b^\infty)$ is dissipative and hence closable in $L^1(\mu)$, whose closure $(\bar{L}, \mathscr{D}(\bar{L}))$ generates a Markovian C_0-semigroup of contraction operators $(T_t)_{t\ge 0}$ on $L^1(\mu)$, $\mathscr{D}(\bar{L}) \subset H^{2,1}(\mu)$, and

$$\mu(\langle \nabla f, \beta - a\nabla g\rangle) = \mu(g\bar{L}f), \quad f \in \mathscr{D}(\bar{L}) \cap \mathscr{B}_b(\mathbb{H}), g \in H^{2,1}(\mu) \cap \mathscr{B}_b(\mathbb{H}). \tag{5.11}$$

(ii) There exists a standard continuous Markov process $\{\bar{\mathbb{P}}_x\}_{x\in\mathbb{H}}$ whose semigroup $\bar{P}_t$ satisfies

$$\int_0^\infty \mathrm{e}^{-\lambda t}\bar{P}_t f \mathrm{d}t = \int_0^\infty T_t f \mathrm{d}t, \quad \mu\text{-a.e.}, \ \lambda > 0, f \in \mathscr{B}_b(\mathbb{H}). \tag{5.12}$$

As shown in the proof of Theorem 2.3(2), (5.12) implies that $\bar{P}_t$ is a μ-version of T_t.

Now, let $L = L_0 + Z \cdot \nabla$ and $\mu = \rho\mu_0$ be in Theorem 2.5. We intend to verify the above conditions such that assertions (i) and (ii) hold.

Firstly, $\sqrt{\rho} \in H^{2,1}(\mu_0)$ implies $\nabla \log\rho \in L^2(\mu)$ and

$$\mu_0(|\nabla\rho|) \le 2\sqrt{\mu_0(|\nabla\sqrt{\rho}|^2)\mu_0(\rho)} < \infty.$$

Consider the operator

$$L^\mu := L_0 + a\nabla\log\rho, \quad f \in \mathscr{F}C_0^\infty.$$

By the symmetry of L_0 in $L^2(\mu_0)$, the boundedness of a, $\nabla\log\rho \in L^2(\mu_0), \nabla\rho \in L^1(\mu_0)$ and noting that $H^{2,1}(\mu_0)$ is dense in $H^{1,1}(\mu_0)$, we obtain

$$\begin{aligned}\mu(fL^\mu g) &= \mu(f\langle\nabla\log\rho, a\nabla g\rangle) + \mu_0(f\rho L_0 g)\\ &= \mu(f\langle\nabla\log\rho, a\nabla g\rangle) - \mu_0(\nabla(f\rho), a\nabla g\rangle) = -\mu(\langle\nabla f, a\nabla g\rangle), \quad f,g \in \mathscr{F}C_0^\infty.\end{aligned}$$

Thus, the form $(\mathscr{E}^\mu, \mathscr{F}C_0^\infty)$ is closable in $L^2(\mu)$ with generator extending $(L^\mu, \mathscr{F}C_0^\infty)$.

Next, let $\beta = Z - a\nabla\log\rho$. We have $L = L^\mu + \beta\cdot\nabla$ on $\mathscr{F}C_0^\infty$. Since $L^*\mu = 0$ and $\mu_0(\langle\nabla\rho, \nabla f\rangle) = -\mu_0(\rho L_0 f)$ for $f \in \mathscr{F}C_0^\infty$, we have

$$\begin{aligned}\mu(\langle\beta, \nabla f\rangle) &= \mu_0(\langle\rho Z - a\nabla\rho, \nabla f\rangle)\\ &= \mu(\langle Z, \nabla f\rangle) + \mu_0(\rho L_0 f) = \mu(Lf) = 0, \quad f \in \mathscr{F}C_0^\infty.\end{aligned} \tag{5.13}$$

Noting that (2.20) and the boundedness of a imply

$$\mu(|a\nabla \log \rho|^2) \le 4\|a\|^2\mu_0(|\nabla\sqrt{\rho}|^2) < \infty,$$

while by the Young inequality (4.5) and the log-Sobolev inequality (5.3)

$$\begin{aligned}\mu(|Z|^2) = \mu_0(\rho|Z|^2) &\le \frac{1}{\lambda}\log\mu_0(\mathrm{e}^{\lambda|Z|^2}) + \frac{1}{\lambda}\mu_0(\rho\log\rho)\\ &\le \frac{1}{\lambda}\log\mu_0(\mathrm{e}^{\lambda|Z|^2}) + \frac{2}{\lambda_1\lambda}\mu_0(|\nabla\sqrt{\rho}|^2) + \frac{1}{\lambda} < \infty,\end{aligned}$$

we have $\mu(|\beta|^2) < \infty$ for $\beta := Z - a\nabla\log\rho$. So, (5.10) follows from (5.13).

In conclusion, the above assertions (i) and (ii) hold for the present situation. Combining (5.10) with (5.11) for $g = 1$ and $T_t f$ in place of f, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu(T_t f) = \mu(LT_t f) = \mu(\langle\nabla T_t f, \beta\rangle) = 0, \quad f \in \mathscr{F}C_0^\infty, t \ge 0.$$

Therefore, μ is an invariant probability measure of T_t, and the proof is finished since $\bar{P}_t$ is a μ-version of T_t. □

Proof of Theorem 2.6. Since $V_n(x) := \frac{1}{2}\sum_{i=1}^n \lambda_i x_i^2$ on $\mathbb{H}_n$ satisfies $|\nabla V_n| \in L^1(\mu_0^{(n)})$ for all $q > 1$, **(H2′)** and (5.7) imply that **(H1′)** holds for $(a_n, V_n, \mu_0^{(n)})$ in place of (a, V, μ_0) with $\kappa' = \frac{2}{\alpha\lambda_1}$ and $\beta = 0$. So, by repeating the proof of Theorem 2.5 using Theorem 2.4 in place of Theorem 2.3(1), we prove the desired assertions. □

6 Proof of Theorem 2.7

We first prove the non-explosion of the weak solution constructed from the Girsnaov transform of the linear SPDE (5.1), then prove the strong Feller property of the associated Markov semigroup. The Feller property, together with the pathwise uniqueness for μ_0-a.e. starting points due to [15], implies that the constructed Markov process is the unique Feller process solving (2.3) weakly. Noting that in the present case we have $d = \infty$, the estimate (3.4) derived in the finite-dimensional case does not make sense. To construct the desired weak solution we need to establish a reasonable infinite-dimensional version of (3.4). We will soon find out that this is non-trivial at all. If we start from the Harnack inequality (5.2), it is standard that

$$(P_t f(x))^p \le \frac{\mu_0(f^p)}{\mu_0(\mathrm{e}^{-|x-\cdot|^p/t})} \approx \mathrm{e}^{c(x)/t}$$

for some constant $c(x) > 0$ and small $t > 0$. The hard point is that $\int_0^t \mathrm{e}^{c(x)/(ps)}\mathrm{d}s = \infty$ for any $t > 0$ and $p > 1$, so that the argument we used in the finite-dimensional case is invalid. To kill this high singularity for small time t, we will use a refined version of the Harnack inequality and make a clever choice of reference measure ν_t on $[0, t]$ to replace the Lebesgue measure.

6.1 Construction of the weak solution

We first construct weak solutions to (2.3) using the Girsanov transform. For any $x \in \mathbb{H}$, let X_t^x solve (5.1) with $X_0 = x$. Let

$$R_{s,t}^x := \exp\left[\frac{1}{\sqrt{2}}\int_s^t \langle Z(X_r^x), \mathrm{d}W_r\rangle - \frac{1}{4}\int_s^t |Z(X_r^x)|^2 \mathrm{d}r\right], \quad t \geq s \geq 0. \tag{6.1}$$

By Girsanov's theorem, if $(R_t^x)_{t\geq 0} := (R_{0,t}^x)_{t\geq 0}$ is a martingale, then for any $T > 0$ the process

$$\widetilde{W}_t^x := W_t - \frac{1}{\sqrt{2}}\int_0^t Z(X_s^x)\mathrm{d}s, \quad t \in [0, T]$$

is a cylindrical Brownian motion under the weighted probability $Q_T^x := R_T^x \mathbb{P}$, so that $(X_t^x, \widetilde{W}_t^x)_{t\in[0,T]}$ is a weak solution to (2.22) starting at x. To prove that $(R_t^x)_{t\geq 0}$ is a martingale, it suffices to verify the Novikov condition

$$\mathbb{E}\mathrm{e}^{\frac{1}{4}\int_0^{t_0} |Z(X_s^x)|^2 \mathrm{d}s} < \infty, \quad x \in \mathbb{H} \tag{6.2}$$

for some $t_0 > 0$. Indeed, by the Markov property, this condition implies that $(R_{s,t}^x)_{t\in[s,s+t_0]}$ is a martingale for all $x \in \mathbb{H}$ and $s \geq 0$, and thus $(R_t^x)_{t\geq 0}$ is a martingale for all $x \in \mathbb{H}$ by induction: if $(R_t^x)_{t\in[0,nt_0]}$ is a martingale for some $n \geq 1$, then for any $nt_0 \leq s < t \leq (n+1)t_0$ we have

$$\mathbb{E}(R_t^x|\mathscr{F}_s) = R_s^x \mathbb{E}(R_{s,t}^x|\mathscr{F}_s) = R_s^x.$$

Therefore, the condition (6.2) implies that $(X_t^x, \widetilde{W}_t^x)_{t\in[0,T]}$ is a weak solution to (2.22) for any $T > 0$ and $x \in \mathbb{H}$. Let $P_t(x, \mathrm{d}y)$ be the distribution of X_t^x under $\mathbb{Q}_t^x$, and let

$$P_t f(x) = \mathbb{E}_{\mathbb{Q}_t^x} f(X_t^x) = \mathbb{E}\big\{f(X_t^x)R_t^x\big\}, \quad f \in \mathscr{B}_b(\mathbb{H}), t \geq 0, x \in \mathbb{H}. \tag{6.3}$$

By the Markov property of X_t under $\mathbb{P}$, it is easy to see that P_t is a Markov semigroup on $\mathscr{B}_b(\mathbb{H})$, i.e. $\{P_t(x, \mathrm{d}y) : t \geq 0, x \in \mathbb{H}\}$ is a Markov transition kernel.

To verify condition (6.2), we introduce a refined version of the Harnack inequality (5.2). For each $i \geq 1$ let $P_t^{0,i}$ be the diffusion semigroup generated by $L_{0,i}f := f'' - \lambda_i f'$ on $\mathbb{R}$. By [32, Lemma 2.1] for $K = -\lambda_i$ and $g(s) = \mathrm{e}^{-Ks}$, we have

$$(P_t^{0,i} f(x))^p \leq (P_t^{0,i} f^p(y)) \exp\left[\frac{p\lambda_i |x-y|^2}{2(p-1)(\mathrm{e}^{2\lambda_i t} - 1)}\right], \quad t > 0, p > 1, f \in \mathscr{B}^+(\mathbb{R}), x, y \in \mathbb{R}.$$

By regarding $P_t^{0,i}$ as a linear operator on $\mathscr{B}_b(\mathbb{H})$ acting on the i-th component $x_i := \langle x, e_i\rangle$, we have $P_t^0 = \prod_{i=1}^\infty P_t^{0,i}$, so that this Harnack inequality leads to

$$(P_t^0 f(x))^p \leq P_t^0 f^p(y) \exp\left[\frac{p}{2(p-1)}\sum_{i=1}^\infty \frac{\lambda_i |x_i - y_i|^2}{\mathrm{e}^{2\lambda_i t} - 1}\right], \quad t > 0, f \in \mathscr{B}_b^+(\mathbb{H}), x, y \in \mathbb{H}$$

for any $p > 1$. Noting that μ_0 is an invariant probability measure of P_t^0, by taking $p = 2$ we obtain

$$(6.4) \quad (P_t^0 f(x))^2 \int_{\mathbb{H}} \exp\Big[-\sum_{i=1}^\infty \frac{\lambda_i (x_i - y_i)^2}{e^{2\lambda_i t} - 1}\Big] \mu_0(dy) \le \mu_0(f^2), \quad x \in \mathbb{H}, t > 0, f \in L^2(\mu_0).$$

Observing that

$$\frac{\lambda_i (x_i - y_i)^2}{e^{2\lambda_i t} - 1} + \frac{\lambda_i y_i^2}{2} = \frac{\lambda_i (e^{2\lambda_i t} + 1)}{2(e^{2\lambda_i t} - 1)} \Big(y_i - \frac{2x_i}{e^{2\lambda_i t} + 1}\Big)^2 + \frac{\lambda_i x_i^2}{e^{2\lambda_i t} + 1},$$

by (2.16) we have

$$\begin{aligned} &\int_{\mathbb{H}} \exp\Big[-\sum_{i=1}^\infty \frac{\lambda_i (x_i - y_i)^2}{e^{2\lambda_i t} - 1}\Big] \mu_0(dy) \\ &= \prod_{i=1}^\infty \frac{\sqrt{\lambda_i}}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\Big[-\frac{\lambda_i (x_i - y_i)^2}{e^{2\lambda_i t} - 1} - \frac{\lambda_i y_i^2}{2}\Big] dy_i \\ &= \exp\Big[-\sum_{i=1}^\infty \frac{\lambda_i x_i^2}{e^{2\lambda_i t} + 1}\Big] \Big(\prod_{i=1}^\infty \frac{e^{2\lambda_i t} - 1}{e^{2\lambda_i t} + 1}\Big)^{\frac{1}{2}}, \quad t > 0, x \in \mathbb{H}. \end{aligned}$$

So, (6.4) reduces to

$$(6.5) \qquad P_t^0 f(x) \le \sqrt{\mu_0(f^2)}\, \Gamma_x(t), \quad x \in \mathbb{H}, t > 0, f \in L^2(\mu_0),$$

where due to (2.15),

$$(6.6) \qquad \begin{aligned} \Gamma_x(t) &:= \exp\Big[\frac{1}{2} \sum_{i=1}^\infty \frac{\lambda_i x_i^2}{e^{2\lambda_i t} + 1}\Big] \Big(\prod_{i=1}^\infty \frac{e^{2\lambda_i t} + 1}{e^{2\lambda_i t} - 1}\Big)^{\frac{1}{4}} \\ &\le \exp\Big[\frac{1}{2} \sum_{i=1}^\infty \frac{\lambda_i x_i^2}{e^{2\lambda_i t} + 1}\Big] \Big(\prod_{i=1}^\infty \Big(1 + \frac{1}{\lambda_i t}\Big)\Big)^{\frac{1}{4}} < \infty, t > 0, x \in \mathbb{H}. \end{aligned}$$

Moreover, using the stronger condition $\sum_{i=1}^\infty \lambda_i^{-\theta} < \infty$ for some $\theta \in (0,1)$ included in **(H3)**, and noting that $\log(1 + r) \le c r^\theta$ for some constant $c > 0$ and all $r \ge 0$, we obtain

$$(6.7) \qquad \begin{aligned} \Psi(t, x) &:= \int_0^t \log \Gamma_x(s) ds = \frac{1}{4} \sum_{i=1}^\infty \int_0^t \Big\{\frac{2\lambda_i x_i^2}{e^{2\lambda_i s} + 1} + \log\Big(1 + \frac{1}{\lambda_i s}\Big) ds\Big\} ds \\ &\le \frac{1}{4} \sum_{i=1}^\infty \Big\{2 \int_0^t \lambda_i x_i^2 e^{-2\lambda_i s} ds + \frac{c}{\lambda_i^\theta} \int_0^t r^{-\theta} dr\Big\} \\ &\le \frac{1}{4} \sum_{i=1}^\infty x_i^2 (1 - e^{-2\lambda_i t}) + C t^{1-\theta} < \infty, \quad t > 0, x \in \mathbb{H} \end{aligned}$$

for some constant $C>0$. For later use we deduce from this that

$$\limsup_{t\to 0}\sup_{y\to x}\Psi(t,y)\le \frac{1}{2}\limsup_{t\to 0}\sup_{y\to x}\bigg\{\sum_{i=1}^\infty x_i^2(1-\mathrm{e}^{-2\lambda_i t})+|x-y|^2+Ct^{1-\theta}\bigg\}=0. \tag{6.8}$$

Since (6.6) implies $\Gamma_x(s)\in(1,\infty)$, for every $t>0$ we have

$$\beta_x(t):=\int_0^t\frac{\mathrm{d}s}{\Gamma_x(s)}\in(0,t],$$

so that

$$\nu_{t,x}(\mathrm{d}s):=\frac{1_{[0,t]}(s)}{\beta_x(t)\Gamma_x(s)}\mathrm{d}s$$

is a probability measure on $[0,t]$. Noting that $\frac{\beta_x(t)}{t}\int_0^t\Gamma_x(s)\nu_{t,x}(\mathrm{d}s)=1$ and $\log\big(\frac{\beta_x(t)}{t}\Gamma_x(s)\big)\le\log\Gamma_x(s)$, the Young inequality (4.5) yields

$$\begin{aligned}
&\int_0^t|Z(X_s^x)|^2\mathrm{d}s=\frac{2t}{\lambda}\int_0^t\Big(\frac{\lambda}{2}|Z(X_s^x)|^2\Big)\Big(\frac{\beta_x(t)}{t}\Gamma_x(s)\Big)\nu_{t,x}(\mathrm{d}s)\\
&\le\frac{2t}{\lambda}\log\nu_{t,x}\big(\mathrm{e}^{\frac{\lambda}{2}|Z(X_\cdot^x)|^2}\big)+\frac{2t}{\lambda}\int_0^t\Big\{\frac{\beta_x(t)}{t}\Gamma_x(s)\log\Big(\frac{\beta_x(t)}{t}\Gamma_x(s)\Big)\Big\}\nu_{t,x}(\mathrm{d}s)\\
&\le\frac{2t}{\lambda}\log\nu_{t,x}\big(\mathrm{e}^{\frac{\lambda}{2}|Z(X_\cdot^x)|^2}\big)+\frac{2}{\lambda}\Psi(t,x),\quad t\ge0,x\in\mathbb{H}.
\end{aligned}$$

Combining this with (6.5) for $f=\mathrm{e}^{\frac{\lambda}{2}|Z|^2}$, (6.7) and $\mu_0(\mathrm{e}^{\lambda|Z|^2})<\infty$, we arrive at

$$\begin{aligned}
&\mathbb{E}\exp\bigg[\gamma\int_0^t|Z(X_s^x)|^2\mathrm{d}s\bigg]\le\mathrm{e}^{\frac{2\gamma}{\lambda}\Psi(t,x)}\mathbb{E}\bigg\{\int_0^t\mathrm{e}^{\frac{\lambda}{2}|Z(X_s^x)|^2}\nu_{t,x}(\mathrm{d}s)\bigg\}^{\frac{2\gamma t}{\lambda}}\\
&\le\mathrm{e}^{\frac{2\gamma}{\lambda}\Psi(t,x)}\bigg\{\int_0^t\big\{P_s^0\mathrm{e}^{\frac{\lambda}{2}|Z|^2}(x)\big\}\nu_{t,x}(\mathrm{d}s)\bigg\}^{\frac{2\gamma t}{\lambda}}\\
&\le\mathrm{e}^{\frac{2\gamma}{\lambda}\Psi(t,x)}\bigg\{\int_0^t\sqrt{\mu_0(\mathrm{e}^{\lambda|Z|^2})}\Gamma_x(s)\nu_{t,x}(\mathrm{d}s)\bigg\}^{\frac{2\gamma t}{\lambda}}\\
&=\mathrm{e}^{\frac{2\gamma}{\lambda}\Psi(t,x)}\Big(\frac{t}{\beta_x(t)}\sqrt{\mu_0(\mathrm{e}^{\lambda|Z|^2})}\Big)^{\frac{2\gamma t}{\lambda}}=:\Lambda(t,x,\gamma)<\infty,\quad x\in\mathbb{H},\gamma>0,t\in\Big(0,\frac{\lambda}{2\gamma}\Big].
\end{aligned}\tag{6.9}$$

By taking $\gamma=\frac{1}{4}$, we prove (6.2) for $t_0=2\lambda$.

6.2 Strong Feller and strictly positive density of P_t

By the Harnack inequality (5.2), P_t^0 is strong Feller having strictly positive density with respect to μ_0 (see [40, Proposition 3.1(1)]). Then as in (b) and (c) in the proof of Lemma 3.1, we may prove the same property for P_t using (6.3) and (6.9). To save space, we only prove here the strong Feller property.

For any $t > 0$, by the semigroup group property of P_t, (6.3), and the strong Feller property of P_t^0, we obtain

$$
\begin{aligned}
\limsup_{y\to x} |P_t f(x) - P_t f(y)| &= \limsup_{r\to 0} \limsup_{y\to x} |P_r(P_{t-r}f)(x) - P_r(P_{t-r}f)(y)| \\
&\le \limsup_{r\to 0} \limsup_{y\to x} \Big\{ |P_r^0(P_{t-r}f)(x) - P_r^0(P_{t-r}f)(x)| \\
&\qquad + \big|\mathbb{E}\big[(P_{t-r}f)(X_r^x)(R_r^x - 1) - (P_{t-r}f)(X_r^y)(R_r^r - 1)\big]\big| \Big\} \\
&\le \|f\|_\infty \limsup_{r\to 0} \limsup_{y\to x} \mathbb{E}\big(|R_r^x - 1| + |R_r^y - 1|\big).
\end{aligned} \tag{6.10}
$$

Recalling that $R_r^y = R_{0,r}^y$, by (6.1) we have

$$
\mathbb{E}|R_r^y - 1|^2 = \mathbb{E}(R_r^y)^2 - 1 \le \Big(\mathbb{E}\mathrm{e}^{3\int_0^r |Z(X_s^y)|^2 \mathrm{d}s}\Big)^{\frac{1}{2}} - 1, \quad y \in \mathbb{R}^d.
$$

So, according to (6.10), P_t is strong Feller provided

$$
\limsup_{r\to 0} \limsup_{y\to x} \mathbb{E}\exp\left[3\int_0^r |Z(X_s^y)|^2 \mathrm{d}s\right] = 1. \tag{6.11}
$$

Recall that $\beta_x(t) = \int_0^t \frac{\mathrm{d}s}{\Gamma_x(s)}\,\mathrm{d}s$. By Jensen's inequality and (6.7) we have

$$
\log\frac{\beta_x(t)}{t} = -\log\left(\frac{1}{t}\int_0^t \frac{\mathrm{d}s}{\Gamma_x(s)}\right) \le -\frac{1}{t}\int_0^t \Big\{\log\frac{1}{\Gamma_x(s)}\Big\}\mathrm{d}s = \frac{\Psi(t,x)}{t}.
$$

Combining this with (6.8) and (6.9), we obtain

$$
\begin{aligned}
\lim_{r\to 0}\limsup_{y\to x} \Lambda(r,y,3) &\le \lim_{r\to 0}\limsup_{y\to x} \mathrm{e}^{\frac{6}{\lambda}\Psi(r,y)}\Big(\mathrm{e}^{\frac{1}{r}\Psi(r,y)}\sqrt{\mu_0(\mathrm{e}^{\lambda|Z|^2})}\Big)^{\frac{6r}{\lambda}} \\
&= \lim_{r\to 0}\limsup_{y\to x} \mathrm{e}^{\frac{12}{\lambda}\Psi(r,y)} = 1.
\end{aligned}
$$

Combining this with (6.9), we prove (6.11).

6.3 Uniqueness of the Feller semigroup and invariant probability measure

To prove that P_t is the unique Feller Markov semigroup associated to (2.22), we recall the pathwise uniqueness for μ_0-a.e. initial points. By [15, Theorem 1], there exists an μ_0-null set $\mathbb{H}_0$ such that for any $x \notin \mathbb{H}_0$, the SPDE (2.22) has at most one mild solution starting at x up to life time. Combining this with the weak solution constructed in (a), we see that for any initial point $x \notin \mathbb{H}_0$, the SPDE (2.22) has a unique mild solution X_t^x which is non-explosive with distribution $P_t(x, \mathrm{d}y)$. So, if there exists another Feller transition probability kernel $\bar{P}_t(x, \mathrm{d}y)$ associated to (2.22), then $\bar{P}_t(x, \mathrm{d}y) = P_t(x, \mathrm{d}y)$ for $x \notin \mathbb{H}_0$. Since $\mathbb{H} \setminus \mathbb{H}_0$ is dense in $\mathbb{H}$, by the Feller property these transition probability kernels are weak continuous in x, so that $\bar{P}_t(x, \mathrm{d}y) = P_t(x, \mathrm{d}y)$ for all $x \in \mathbb{H}$.

Next, according to [40, Proposition 3.1(3)], to show that P_t has at most one invariant probability measure, it suffices to prove for instance the Harnack inequality

$$(P_t f)^6(x) \le (P_t f^6)(y) H_t(x,y), \quad x, y \in \mathbb{H}, f \in \mathscr{B}_b(\mathbb{H}) \tag{6.12}$$

for some $t > 0$ and measurable function $H_t : \mathbb{H}^2 \to (0, \infty)$. By (6.3) and (5.2), we have

$$\begin{aligned}(P_t f(x))^6 &= \big\{\mathbb{E}[f(X_t^x)R_t^x]\big\}^6 \le \big\{P_t^0 f^2(x)\mathbb{E}(R_t^x)^2\big\}^4 \\ &\le \big\{(P_t^0 f^4)(y)\big\}^2 \mathbb{E}(R_t^x)^6 = \big\{\mathbb{E} f^4(X_t^y)\big\}^2 [\mathbb{E}(R_t^x)^6] \\ &\le \big\{\mathbb{E}[f^6(X_t^y)R_t^y]\big\} \cdot \big\{\mathbb{E}(R_t^y)^{-1}\big\}\mathbb{E}(R_t^x)^6 = \{P_t f^6(y)\} \cdot \big\{\mathbb{E}(R_t^y)^{-1}\big\}[\mathbb{E}(R_t^x)^6].\end{aligned}$$

By (6.9) and the definition of $R_t^\cdot$, it is easy to see that when $t > 0$ is small enough, $\{\mathbb{E}(R_t^y)^{-1}\}[\mathbb{E}(R_t^x)^6] \le H_t(x,y)$ holds for some measurable function $H_t : \mathbb{H}^2 \to (0,\infty)$. Therefore, (6.12) holds.

6.4 P_t-invariance of μ and estimates on the density

Finally, we prove that μ in Theorem 2.5 is an invariant probability measure of P_t. Let μ and $\bar{\mathbb{P}}_x$ be in Theorem 2.5, according to the proof of Theorem 2.3(2) we conclude that $\bar{\mathbb{P}}_\mu := \int_{\mathbb{H}} \bar{\mathbb{P}}_x \mu(\mathrm{d}x)$ is the distribution of a weak solution to (2.22) with initial distribution μ. Since μ is absolutely continuous with respect to μ_0, the uniqueness for μ_0-a.e. initial points implies that the weak solution starting from μ is unique, so that $\mu(P_t f) = \mu(\bar{P}_t f)$ for $t \ge 0$ and $f \in \mathscr{B}_b(\mathbb{H})$. Since μ is $\bar{P}_t$-invariant, it is P_t-invariant as well. Since Theorem 2.5 implies $\sqrt{\rho} \in H^{2,1}(\mu_0)$, (2.20) and (2.21), it remains to prove $\log \rho \in H^{2,1}(\mu_0)$.

By $\mu(\rho) = 1$ and $\sqrt{\rho} \in H^{2,1}(\mu_0)$, we have $\log(\rho + \delta) \in H^{2,1}(\mu_0)$ for all $\delta > 0$. Combining this with (2.21) we conclude that $\log \rho \in H_\sigma^{2,1}(\mu_0)$ provided $\mu_0(\rho > 0) = 1$ with $\mu_0(|\log \rho|^2) < \infty$. It is well known that the Gaussian measure μ_0 satisfies the Poincaré inequality

$$\mu_0(f^2) \le \frac{1}{\lambda_1} \mu_0(|\nabla f|^2) + \mu_0(f)^2, \quad f \in H_\sigma^{2,1}(\mu_0).$$

Then, as shown in the last step in the proof of Theorem 2.3(1), $\mu_0(|\log \rho|^2) < \infty$ follows from (2.21) if $\mu_0(\rho > 0) = 1$. Thus, we only need to prove $\mu_0(\rho > 0) = 1$.

Recalling that $R_t^x = R_{0,t}^x$ for $R_{s,t}^x$ defined in (6.1), by (6.3) and (6.9) we may find out a constant $t_0 > 0$ and some function $H \in C(\mathbb{H} \to (0, \infty))$ such that for any $f \in \mathscr{B}_b^+(\mathbb{H})$,

$$\begin{aligned}\big(P_{t_0}^0 f(x)\big)^2 &= \big(\mathbb{E} f(X_{t_0}^x)\big)^2 \le \big(\mathbb{E}[f^2(X_{t_0}^x)R_t^x]\big)\mathbb{E}\big[(R_{t_0}^x)^{-2}\big] \\ &= (P_{t_0} f^2(x))\mathbb{E}\big[(R_{t_0}^x)^{-2}\big] \le H(x) P_{t_0} f^2(x), \quad x \in \mathbb{H}.\end{aligned}$$

Then for any measurable set $A \subset \mathbb{H}$ with $\mu_0(A) > 0$, we have

$$\mu(A) = \mu(P_{t_0} 1_A^2) \ge \mu\bigg(\frac{(P_{t_0}^0 1_A)^2}{H}\bigg). \tag{6.13}$$

On the other hand, by $\mu_0(P^0_{t_0}1_A)=\mu_0(A)>0$, there exists $y\in\mathbb{H}$ such that $P_{t_0}1_A(y)>0$ so that (5.2) implies

$$P^0_{t_0}1_A(x)\geq (P^0_{t_0}1_A(y))^2 e^{-\frac{C|x-y|^2}{t_0}}>0,\quad x\in\mathbb{H}.$$

Combining this with (6.13) and $\frac{1}{H}>0$. Therefore, μ_0 is absolutely continuous with respect to μ and hence, $\mu_0(\rho>0)=1$.

7 Local Harnack inequality on incomplete manifolds

Let M be a d-dimensional differential manifold without boundary which is equipped with a (not necessarily complete) C^2-metric such that the curvature is well defined and continuous. Let Δ and ∇ be the corresponding Laplace-Beltrami operator and the gradient operator respectively. Then for any $V\in C^2(M)$, the operator $L:=\Delta+\nabla V$ generates a unique diffusion process up to life time. Let $(X^x_t)_{t\in[0,\zeta(x)]}$ be the diffusion process starting at x with life time $\zeta(x)$. Then the associated Dirichlet semigroup is given by

$$P_tf(x):=\mathbb{E}\big\{1_{\{t<\zeta(x)\}}f(X^x_t)\big\},\quad x\in M, t\geq 0, f\in\mathscr{B}_b(M).$$

For any $f\in\mathscr{B}^+_b(M):=\{f\in\mathscr{B}_b(M): f\geq 0\}$, define

$$E_{P_t}(f)=P_t(f\log f)-(P_tf)\log P_tf,\quad t\geq 0.$$

Let ρ be the Riemannian distance. By the locally compact of the manifold we may take $R\in C(M\to(0,\infty))$ such that

$$B_\rho(x,R(x)):=\big\{y\in M:\ \rho(x,y)\leq R(x)\big\}$$

is compact for all $x\in M$. When the metric is complete this is true for all $R\in C(M\to(0,\infty)$. We will use this function R to establish the local Harnack inequality.

Theorem 7.1. *There exists a function $H\in C(M\to(0,\infty))$ such that*

$$|\nabla P_tf(x)|\leq \delta E_{P_t}(f)(x)+H(x)\Big(\delta+\frac{1}{\delta(t\wedge 1)}\Big),\quad t>0, \delta\geq\frac{160}{R(x)}, f\in\mathscr{B}^+_b(M). \tag{7.1}$$

Consequently, for any $p>1$ there exists a function $F\in C(M\to(0,\infty))$ such that for any $t>0$ and $f\in\mathscr{B}^+_b(M)$,

$$(P_tf(x))^p\leq (P_tf^p(y))\exp\left[\frac{F(x)\rho(x,y)^2}{t\wedge 1}+F(x)\right],\quad x,y\in M \text{ with } \rho(x,y)\leq\frac{1}{F(x)}. \tag{7.2}$$

Proof. According to [2], it is easy to prove (7.2) from (7.1). When the metric is complete, an estimate of type (7.1) for all $\delta>0$ has been proved in [3]. The only difference comes from the incompleteness of the metric for which we can not take $R(x)$ arbitrarily large as in [3]. Below we figure out the proof in the present case.

(1) To prove (7.1), we fix $f \in B_b^+(M)$. By using $\frac{f}{P_t f(x)}$ replace f, we may and do assume that $P_t f(x) = 1$ at a fixed point x so that $E_{P_t}(f)(x) = P_t(f \log f)(x)$.

Now, let us check the proof of Theorem 1.1 in [3] (pages 3666-3667), where the part before (4.5) has nothing to do with the completeness; that is, with the compact set $D := B_\rho(x, R(x))$, all estimates therein before (4.5) apply to the present setting. More precisely, letting

$$\tau(x) = \inf\{t \ge 0 : X_t^x \notin D\},$$

we have ((4.1) in [3])

$$|\nabla P_t f(x)| \le I_1 + I_2, \tag{7.3}$$

where ((4.2) in [3])

$$I_1 \le \delta \mathbb{E}\big\{1_{\{t<\tau(x)\}}(f \log f)(X_t^x)\big\} + \frac{\delta}{\mathrm{e}} + C(x)\Big(1 + \frac{1}{\delta t}\Big), \quad \delta > 0, t > 0 \tag{7.4}$$

holds for function $C \in C(M \to (0, \infty))$ depending only on d and curvature of the operator L; and moreover ((4.5) in [3]),

$$I_2 \le \delta \mathbb{E}\big\{1_{\{\tau(x)\le t<\zeta(x)\}}(f \log f)(X_t^x)\big\} + \frac{\delta}{\mathrm{e}} + \delta \log \mathbb{E}\mathrm{e}^{\frac{9R(x)}{\delta\tau(x)}} + A(x), \quad \delta > 0, t > 0 \tag{7.5}$$

holds for $A(x) := \sup_{r>0}\big\{C(x)\sqrt{r}\log(\mathrm{e}+r) - r\big\}$, which is finite and continuous in x. Now, due to the restriction of $R(x)$, we have to take large enough $\delta > 0$ and can not replace δ by $\delta \wedge 1$ as in (4.5) of [3]. This will lead to less harp estimate but it is enough for our study in the present paper. More precisely, using δ to replace $\alpha \wedge 1$ in the display after (4.5) of [3], we have

$$\mathbb{E}\mathrm{e}^{\frac{9R(x)}{\delta\tau(x)}} \le 1 + 9\int_0^\infty (9u+1)\mathrm{e}^{-u}\mathrm{d}u =: A' < \infty, \quad \delta \ge \frac{160}{R(x)}.$$

Combining the with (7.3)-(7.5), we prove (7.1) for some $H \in C(M \to (0, \infty))$.

(2) Since H, R are strictly positive and continuous, and $B_\rho(x, R(x))$ is compact for every x,

$$\bar H(x) := \sup_{B_\rho(x,R(x))} H \quad \text{and} \quad \hat R(x) := \inf_{B_\rho(x,R(x))} R$$

are strictly positive continuous functions in x. For any $p > 1$, let

$$G(x) = \frac{p-1}{p\bar H(x)} \wedge \hat R(x), \quad x \in M.$$

Then (7.1) implies

$$|\nabla P_t f(y)| \le \delta E_{P_t}(f)(y) + \bar H(x)\Big(\frac{1}{\delta(1\wedge t)} + \delta\Big), \quad y \in B_\rho(x, G(x)), \delta \ge \frac{160}{\hat R(x)}$$

for $f \in \mathscr{B}_b^+(M)$. So, letting $\gamma : [0,1] \to M$ be the minimal geodesic from x to y with $|\dot\gamma_s| = \rho(x,y)$ for $s \in [0,1]$, letting $\beta(s) = 1 + s(p-1)$, and applying the above inequality with $\delta := \frac{p-1}{p\rho(x,y)} \geq \frac{160}{\hat R(x)}$, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}s}\big\{\log P_t f^{\beta(s)}\big\}^{\frac{p}{\beta(s)}} = \frac{p(p-1)E_{P_t}(f^{\beta(s)})}{\beta(s)^2 P_t f^{\beta(s)}} + \frac{p\langle \nabla P_t f^{\beta(s)}, \dot\gamma_s\rangle}{\beta(s) P_t f^{\beta(s)}}(\gamma_s)\\ &\geq \frac{p\rho(x,y)}{\beta(s) P_t f^{\beta(s)}(\gamma_s)}\bigg\{\frac{p-1}{p\rho(x,y)} E_{P_t}(f^{\beta(s)}) - |\nabla P_t f^{\beta(s)}|\bigg\}(\gamma_s)\\ &\geq -\frac{p\rho(x,y)}{\beta(s) P_t f^{\beta(s)}(\gamma_s)}\bigg\{\bar H(x)\big(P_t f^{\beta(s)}(\gamma_s)\big)\Big(\frac{p\rho(x,y)}{(p-1)(t\wedge 1)} + \frac{p-1}{p\rho(x,y)}\Big)\bigg\}\\ &\geq -\bar H(x)\Big(\frac{p^2\rho(x,y)^2}{(p-1)(t\wedge 1)} + 1\Big), \quad s \in [0,1],\ \rho(x,y) \leq G(x).\end{aligned}$$

Integrating over $[0,1]$ with respect to ds, we prove (7.2) for $F := \frac{p^2 \bar H}{p-1} \vee \frac{1}{G}$.

□

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