Ranks of overpartitions modulo 6 and 10

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Abstract. Lovejoy and Osburn proved formulas for the generating functions for rank differences of overpartitions modulo 3 and 5. In this paper, we derive formulas for the generating functions for ranks of overpartitions modulo 6 and 10. With these generating functions, we obtain some equalities and inequalities on ranks of overpartitions modulo 6 and 10. We also relate these generating functions to the third order mock theta functions $\omega(q)$ and $\rho(q)$ and the tenth order mock theta functions $\phi(q)$ and $\psi(q)$.

Keywords: Overpartition, Dyson's rank, rank differences, generalized η-functions, modular functions, mock theta functions.

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1 Introduction

The rank of a partition introduced by Dyson [11] is an important statistic in the theory of partitions. It was defined to be the largest part minus the number of parts. Dyson [11] conjectured that this partition statistic provided combinatorial interpretations of Ramanujan's congruences $p(5n+4) \equiv 0 \pmod 5$ and $p(7n+5) \equiv 0 \pmod 7$, where $p(n)$ is the number of partitions of n. More precisely, let $N(m,n)$ denote the number of partitions of n with rank m and let $N(s,\ell,n)$ denote the number of partitions of n with rank congruent to s modulo ℓ. Dyson conjectured

$$N(k,5,5n+4) = \frac{p(5n+4)}{5}, \quad 0 \leq k \leq 4, \tag{1.1}$$

$$N(k,7,7n+5) = \frac{p(7n+5)}{7}, \quad 0 \leq k \leq 6. \tag{1.2}$$

These two assertions were confirmed by Atkin and Swinnerton-Dyer [5]. In fact, they established generating functions for every rank difference $N(s,\ell,\ell n+d) - N(t,\ell,\ell n+d)$ with $\ell = 5$ or 7 and for $0 \leq d,s,t < \ell$, many of which are in terms of infinite products and generalized Lambert series. Although Dyson's rank fails to explain Ramanujan's congruence $p(11n+6) \equiv 0 \pmod{11}$ combinatorially, the generating functions for the rank differences $N(s,\ell,\ell n+d) - N(t,\ell,\ell n+d)$ with $\ell = 11$ have also been determined by Atkin and Hussain [4]. Since then, the rank differences modulo other numbers have been extensively studied, see, for example, Lewis established the rank differences modulo 2 in [20] and the rank differences modulo 9 in [19]. Santa-Gadea [28] obtained generating functions of ranks of partitions modulo 9 and 12. Recently, Mao [25] established the generating function of ranks of partitions modulo 10.

Dyson's rank can be extended to overpartitions in the obvious way. Recall that an overpartition [10] is a partition in which the first occurrence of a part may be overlined. The rank of an overpartition is defined to be the largest part of an overpartition minus its number of parts. Similarly, let $\overline{N}(m,n)$ denote the number of overpartitions of n with rank m, and let $\overline{N}(s,\ell,n)$ denote the number of overpartitions of n with rank congruent to s modulo ℓ. Lovejoy [21] gave the following generating function for $\overline{N}(m,n)$,

$$\overline{R}(z;q) := \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\overline{N}(m,n)z^m q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^n q^{n^2+n}}{(1-zq^n)(1-z^{-1}q^n)}. \quad (1.3)$$

Analogous to the rank of a partition, Lovejoy and Osburn [22] studied the rank differences $\overline{N}(s,\ell,\ell n+d) - \overline{N}(t,\ell,\ell n+d)$ with $\ell = 3$ or 5 for $0 \le d,s,t < \ell$. The rank differences with $\ell = 7$ have been recently determined by Jennings-Shaffer [18]. It has been shown in [9] that there are no congruences of the form $\overline{p}(\ell n+d) \equiv 0 \pmod{\ell}$ for primes $\ell \ge 3$. The generating functions for these rank differences provide a measure of the extent to which the rank fails to produce a congruence $\overline{p}(\ell n+d) \equiv 0 \pmod{\ell}$. On the other hand, as remarked by Jennings-Shaffer in [18], determining these three difference formulas is equivalent to determining the 3-dissection of $\overline{R}(\exp(2i\pi/3);q)$, the 5-dissection of $\overline{R}(\exp(2i\pi/5);q)$ and the 7-dissection of $\overline{R}(\exp(2i\pi/7);q)$.

In this paper, we establish the generating functions for ranks of overpartitions modulo 6 and 10. To this end, we consider the 3-dissection of $\overline{R}(\exp(i\pi/3);q)$ and the 5-dissection of $\overline{R}(\exp(i\pi/5);q)$. The main results are summarized in Theorems 1.1, 1.2 and 1.3 below, which are stated in terms of rank differences for overpartitions. Here and throughout, we use the notation

$$\begin{aligned}
&(x_1,x_2,\ldots,x_k;q)_\infty := \prod_{n=0}^{\infty}(1-x_1q^n)(1-x_2q^n)\cdots(1-x_kq^n),\\
&j(z;q) := (z,q/z,q;q)_\infty,\\
&J_{a,m} := j(q^a;q^m),\ J_m := (q^m;q^m)_\infty,\ \overline{J}_{a,m} := j(-q^a;q^m).
\end{aligned}$$

and we require $|q| < 1$ for absolute convergence.

Theorem 1.1. *We have*

$$\sum_{n=0}^{\infty}(\overline{N}(0,6,n) + \overline{N}(1,6,n) - \overline{N}(2,6,n) - \overline{N}(3,6,n))q^n$$

$$=\frac{J_{18}^3 J_{9,18}}{J_6 J_{3,18}^2}+q\frac{2J_{18}^3}{J_6 J_{3,18}}+q^2\left\{\frac{4J_{18}^3}{J_6 J_{9,18}}-\frac{2}{J_{9,18}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{9n^2+9n}}{1+q^{9n+3}}\right\}. \tag{1.4}$$

Theorem 1.2. *We have*

$$\sum_{n=0}^{\infty}(\overline{N}(0,10,n)+\overline{N}(1,10,n)-\overline{N}(4,10,n)-\overline{N}(5,10,n))q^n$$
$$=2A_0+2q\left(A_1+\frac{q^5}{J_{25,50}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{25n^2+25n}}{1+q^{25n+10}}\right)$$
$$+2q^2A_2+2q^3A_3+2q^4A_4, \tag{1.5}$$

where

$$A_0:=\frac{J_{10,50}^2 J_{15,50}^2 J_{25,50}^4}{2J_{5,10}^3 J_{20,50} J_{50}^3}+4q^{10}\frac{J_{5,50}J_{50}^3}{J_{5,10}^2 J_{20,50}},$$

$$A_1:=\frac{J_{20,50}J_{25,50}^4 J_{50}^3}{J_{5,10}^4 J_{10,50}^2 J_{15,50}}-4q^5\frac{J_{5,50}^4 J_{20,50}^3 J_{25,50}^4}{J_{5,10}^5 J_{10,50}^2 J_{50}^3}-8q^{15}\frac{J_{10,50}^4 J_{15,50}^2 J_{25,50} J_{50}^3}{J_{5,10}^4 J_{20,50}^5},$$

$$A_2:=\frac{J_{5,50}^3 J_{20,50}^3 J_{25,50}^5}{J_{5,10}^5 J_{10,50}^2 J_{50}^3}+4q^{10}\frac{J_{10,50}^5 J_{25,50} J_{50}^7}{J_5^4 J_{5,10} J_{5,50}^3 J_{20,50}^4}-16q^{10}\frac{J_{5,50}J_{15,50}^2 J_{50}^3}{J_{5,10}^4 J_{20,50}},$$

$$A_3:=\frac{2J_{10,50}^4 J_{15,50}^5 J_{25,50}^4}{J_{5,10}^5 J_{5,50} J_{20,50}^3 J_{50}^3}+2q^5\frac{J_{10,50}J_{25,50}^3 J_{50}^3}{J_{5,10}^4 J_{20,50}^2}-16q^5\frac{J_{15,50}^2 J_{25,50} J_{50}^3}{J_{5,10}^4 J_{20,50}}$$
$$+8q^{10}\frac{J_{5,50}^3 J_{20,50} J_{25,50} J_{50}^3}{J_{5,10}^4 J_{10,50}^2 J_{15,50}},$$

$$A_4:=\frac{4J_{10,50}^4 J_{15,50}^6 J_{25,50}^3}{J_{5,10}^5 J_{5,50} J_{20,50}^3 J_{50}^3}-\frac{J_{10,50}J_{25,50}^4 J_{50}^3}{J_{5,10}^4 J_{5,50} J_{20,50}^2}-16q^5\frac{J_{15,50}^3 J_{50}^3}{J_{5,10}^4 J_{20,50}}$$
$$-8q^5\frac{J_{25}J_{50}^5}{J_5 J_{5,10}^3 J_{10,50}}.$$

Theorem 1.3. *We have*

$$\sum_{n=0}^{\infty}(\overline{N}(1,10,n)+\overline{N}(2,10,n)-\overline{N}(3,10,n)-\overline{N}(4,10,n))q^n$$
$$=2B_0+2q\left(B_1-\frac{q^5}{J_{25,50}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{25n^2+25n}}{1+q^{25n+10}}\right)+2q^2B_2$$
$$+2q^3B_3+2q^4\left(B_4-\frac{1}{J_{25,50}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{25n^2+25n}}{1+q^{25n+20}}\right), \tag{1.6}$$

where

$$B_0 := \frac{4q^5 J_5^5 J_{25}^5 J_{5,50}^4 J_{15,50}^2}{J_{5,10}^6 J_{50}^6 J_{10,50}^3} - \frac{q^5 J_{50}^3 J_{25,50}^2}{J_{5,10}^2 J_{15,50} J_{20,50}},$$

$$B_1 := \frac{4q^5 J_5^8 J_{50}^7 J_{25,50}}{J_{5,10}^6 J_{10,50}^4 J_{20,50}^5} - \frac{4q^{10} J_{50}^3 J_{5,50}^2 J_{25,50}^2}{J_{5,10}^4 J_{10,50} J_{15,50}} + \frac{8q^{15} J_{50}^6 J_{10,50}^6}{J_5^6 J_{5,50}^2 J_{20,50}^3},$$

$$B_2 := -\frac{J_{5,50}^7 J_{20,50}^7 J_{25,50}^6}{J_{5,10}^6 J_{10,50}^4 J_{50}^9} + \frac{2J_{50} J_{20,50}^3 J_{25,50}}{J_5^4} - \frac{4q^{10} J_{50}^6 J_{10,50}^6 J_{25,50}}{J_5^6 J_{5,50}^3 J_{20,50}^3} + \frac{16q^{10} J_{50}^8}{J_5^2 J_{5,10}^2 J_{10,50} J_{15,50}^2},$$

$$B_3 := -\frac{J_{50}^3 J_{25,50}^4}{J_{5,10}^4 J_{10,50} J_{15,50}} + \frac{2J_{50}^3 J_{15,50} J_{25,50}}{J_{5,10}^2 J_{5,50} J_{20,50}} + \frac{4q^5 J_{50}^6 J_{20,50}^3 J_{25,50}^2}{J_5^6 J_{15,50}^4} + \frac{16q^{15} J_{50}^3 J_{5,50}^3}{J_{5,10}^4 J_{20,50}},$$

$$B_4 := \frac{4J_{50}^3 J_{15,50}^2}{J_{5,10}^2 J_{5,50} J_{20,50}} - \frac{2J_{5,50}^3 J_{20,50}^3 J_{25,50}^5}{J_5^2 J_{5,10}^4 J_{50}^4} + \frac{8q^{10} J_{50}^8}{J_5^2 J_{5,10}^2 J_{15,50}^2 J_{20,50}} + \frac{16q^{10} J_{50}^6 J_{10,50}^3}{J_5^6 J_{5,50} J_{25,50}} - \frac{16q^{15} J_{50}^3 J_{5,50}^4}{J_{5,10}^4 J_{10,50} J_{15,50}}.$$

Besides the equalities on ranks of partitions, like (1.1) and (1.2), some inequalities have also been obtained by Andrews [3], Garvan [12], Mao [25], and so on. In particular, Bringmann and Kane [6] characterized the sign of the rank differences of partitions for all odd moduli. In this paper, we obtain the following equalities and inequalities between the ranks of overpartitions modulo 6 and 10 with the aid of the generating functions in Theorems 1.1, 1.2 and 1.3 together with identities (1.2), (1.3) and (1.4) of [22].

Theorem 1.4. *We have*

$$\overline{N}(1,6,3n) = \overline{N}(3,6,3n) \text{ for } n \geq 1, \tag{1.7}$$

$$\overline{N}(0,6,3n) > \overline{N}(2,6,3n) \text{ for } n \geq 1, \tag{1.8}$$

$$\overline{N}(1,6,3n+1) = \overline{N}(3,6,3n+1) \text{ for } n \geq 0, \tag{1.9}$$

$$\overline{N}(0,6,3n+1) > \overline{N}(2,6,3n+1) \text{ for } n \geq 0, \tag{1.10}$$

$$\overline{N}(0,6,3n+2) < \overline{N}(2,6,3n+2) \text{ for } n \geq 0, \tag{1.11}$$

$$\overline{N}(1,6,3n+2) > \overline{N}(3,6,3n+2) \text{ for } n \geq 0. \tag{1.12}$$

Theorem 1.5. *For $n \geq 0$, we have*

$$\overline{N}(0,10,5n) + \overline{N}(1,10,5n) > \overline{N}(4,10,5n) + \overline{N}(5,10,5n). \tag{1.13}$$

Computer evidence suggests that the following inequalities hold, but we fail to prove them and so we leave them in the following two conjectures.

Conjecture 1.6. *For $n \geq 0$ and $1 \leq i \leq 4$, we have*

$$\overline{N}(0,10,5n+i) + \overline{N}(1,10,5n+i) \geq \overline{N}(4,10,5n+i) + \overline{N}(5,10,5n+i). \tag{1.14}$$

Conjecture 1.7. *For $n \geq 0$ and $0 \leq i \leq 4$, we have*

$$\overline{N}(1,10,5n+i) + \overline{N}(2,10,5n+i) \geq \overline{N}(3,10,5n+i) + \overline{N}(4,10,5n+i). \tag{1.15}$$

Many of the classical mock theta functions can be written in terms of rank differences of partitions. For example, Andrews and Garvan [1] found that the fifth order mock theta functions $\chi_0(q)$ and $\chi_1(q)$ can be expressed in terms of rank differences of partitions modulo 5, which was later proved by Hickerson [15]. Subsequently, Hickerson [16] showed that the seventh order mock theta functions $\mathscr{F}_0(q)$, $\mathscr{F}_1(q)$ and $\mathscr{F}_2(q)$ are related to rank differences of partitions modulo 7. Recently, Lovejoy [23] has proved that the tenth order mock theta functions $\phi(q)$ and $\psi(q)$ can be expressed in terms of rank differences of overpartitions modulo 5. In this paper, we establish a relation between the third order mock theta functions $\omega(q)$ and $\rho(q)$ and the ranks of overpartitions modulo 6.

Theorem 1.8. *We have*

$$\begin{aligned}&\sum_{n=0}^{\infty}(\overline{N}(0,6,3n+2)+\overline{N}(1,6,3n+2)-\overline{N}(2,6,3n+2)-\overline{N}(3,6,3n+2))q^n\\&=\frac{4}{3}\omega(q)+\frac{2}{3}\rho(q),\end{aligned} \tag{1.16}$$

where the third order mock theta functions $\omega(q)$ and $\rho(q)$ are defined by [29]:

$$\omega(q)=\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2} \quad \textit{and} \quad \rho(q)=\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}(q;q^2)_{n+1}}{(q^3;q^6)_{n+1}}.$$

In light of Theorem 1.2 and Theorem 1.3, we obtain the following relations between the tenth order mock theta functions $\phi(q)$ and $\psi(q)$ and the ranks of overpartitions modulo 10.

Theorem 1.9. *We have*

$$\begin{aligned}&\sum_{n=0}^{\infty}(\overline{N}(0,10,5n+1)+\overline{N}(1,10,5n+1)-\overline{N}(4,10,5n+1)-\overline{N}(5,10,5n+1))q^n\\&=-\phi(q)+M_1(q),\end{aligned} \tag{1.17}$$

$$\sum_{n=0}^{\infty}(\overline{N}(1,10,5n+1)+\overline{N}(2,10,5n+1)-\overline{N}(3,10,5n+1)-\overline{N}(4,10,5n+1))q^n$$

$$= \phi(q) + M_2(q), \tag{1.18}$$

$$\sum_{n=0}^{\infty}(\overline{N}(1,10,5n+4)+\overline{N}(2,10,5n+4)-\overline{N}(3,10,5n+4)-\overline{N}(4,10,5n+4))q^n$$
$$= q^{-1}\psi(q) + M_3(q), \tag{1.19}$$

where the tenth order mock theta functions $\phi(q)$ *and* $\psi(q)$ *are defined as* [8]:

$$\phi(q) = \sum_{n=0}^{\infty} \frac{q^{\binom{n+1}{2}}}{(q;q^2)_{n+1}} \quad \text{and} \quad \psi(q) = \sum_{n=0}^{\infty} \frac{q^{\binom{n+2}{2}}}{(q;q^2)_{n+1}}. \tag{1.20}$$

and

$$M_1(q) = \frac{J_5 J_{10} J_{4,10}}{J_{2,5} J_{2,10}} + 2q \frac{J_{10}^3 \overline{J}_{0,10} \overline{J}_{5,10}}{J_{2,10} \overline{J}_{2,10} J_{3,10} \overline{J}_{3,10}} + \frac{2 J_{4,10} J_{5,10}^4 J_{10}^3}{J_{1,2}^4 J_{2,10}^2 J_{3,10}}$$
$$- 8q \frac{J_{1,10}^4 J_{4,10}^3 J_{5,10}^4}{J_{1,2}^5 J_{2,10}^2 J_{10}^3} - 16q^3 \frac{J_{2,10}^4 J_{3,10}^2 J_{5,10} J_{10}^3}{J_{1,2}^4 J_{4,10}^5},$$

$$M_2(q) = -\frac{J_5 J_{10} J_{4,10}}{J_{2,5} J_{2,10}} - 2q \frac{J_{10}^3 \overline{J}_{0,10} \overline{J}_{5,10}}{J_{2,10} \overline{J}_{2,10} J_{3,10} \overline{J}_{3,10}} + \frac{8q J_1^8 J_{10}^7 J_{5,10}}{J_{1,2}^6 J_{2,10}^4 J_{4,10}^5}$$
$$- \frac{8q^2 J_{10}^3 J_{1,10}^2 J_{5,10}^2}{J_{1,2}^4 J_{2,10} J_{3,10}} + \frac{16 q^3 J_{10}^6 J_{2,10}^6}{J_1^6 J_{1,10}^2 J_{4,10}^3},$$

$$M_3(q) = \frac{J_5 J_{10} J_{2,10}}{J_{1,5} J_{4,10}} - \frac{2 J_{10}^3 \overline{J}_{0,10} \overline{J}_{5,10}}{J_{1,10} \overline{J}_{1,10} J_{4,10} \overline{J}_{4,10}} + \frac{8 J_{10}^3 J_{3,10}^2}{J_{1,2}^2 J_{1,10} J_{4,10}} - \frac{4 J_{1,10}^3 J_{4,10}^3 J_{5,10}^5}{J_1^2 J_{1,2}^4 J_{10}^4}$$
$$+ \frac{16 q^2 J_{10}^8}{J_1^2 J_{1,2}^2 J_{3,10}^2 J_{4,10}} + \frac{32 q^2 J_{10}^6 J_{2,10}^3}{J_1^6 J_{1,10} J_{5,10}} - \frac{32 q^3 J_{10}^3 J_{1,10}^4}{J_{1,2}^4 J_{2,10} J_{3,10}},$$

are (explicit) weakly holomorphic modular forms.

This paper is organized as follows. In Section 2, we prove Theorem 1.1 by establishing the 3-dissection of $\overline{R}(\exp(i\pi/3); q)$. In Section 3, we give the proofs of Theorem 1.2 and Theorem 1.3 by investigating the 5-dissection of $\overline{R}(\exp(i\pi/5); q)$ and using standard computational techniques from the theory of modular forms. Section 4 is devoted to establishing some equalities and inequalities on ranks of overpartitions modulo 6 and 10 with the aid of Theorem 1.1, Theorem 1.2 and Theorem 1.3. In Section 5, we show the relations between the rank differences of overpartitions and mock theta functions as stated in Theorem 1.8 and Theorem 1.9.

2 Proof of Theorem 1.1

In this section, we will give a proof of Theorem 1.1. To this end, we need to determine the 3-dissection of $\overline{R}(\exp(i\pi/3); q)$. First, we show that $\overline{R}(\exp(i\pi/3); q)$ can be simplified as follows.

Lemma 2.1. *We have*

$$\overline{R}(\exp(i\pi/3);q) = \sum_{n=0}^{\infty}(\overline{N}(0,6,n)+\overline{N}(1,6,n)-\overline{N}(2,6,n)-\overline{N}(3,6,n))q^n$$
$$= \frac{2(-q;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+n}}{1+q^{3n}}. \tag{2.1}$$

Proof. Replacing z by $\xi_6 = \exp\left(\frac{\pi i}{3}\right)$ in (1.3), we have

$$\overline{R}(\exp(i\pi/3);q) = \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\overline{N}(m,n)\xi_6^m q^n$$
$$= \frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(1-\xi_6)(1-\xi_6^{-1})(-1)^n q^{n^2+n}}{(1-\xi_6 q^n)(1-\xi_6^{-1}q^n)}. \tag{2.2}$$

Using the fact that $\overline{N}(s,\ell,n)=\overline{N}(\ell-s,\ell,n)$ in [21], we find that the left-hand side of (2.2) reduces to

$$\overline{R}(\exp(i\pi/3);q) = \sum_{n=0}^{\infty}\sum_{s=0}^{5}\overline{N}(s,6,n)\xi_6^s q^n$$
$$= \sum_{n=0}^{\infty}\left\{\overline{N}(0,6,n)+(\xi_6+\xi_6^5)\overline{N}(1,6,n)+(\xi_6^2+\xi_6^4)\overline{N}(2,6,n)+\xi_6^3\overline{N}(3,6,n)\right\}q^n.$$

Observe that $1-\xi_6+\xi_6^2=0$ and $\xi_6^3=-1$, so we have

$$\overline{R}(\exp(i\pi/3);q) = \sum_{n=0}^{\infty}(\overline{N}(0,6,n)+\overline{N}(1,6,n)-\overline{N}(2,6,n)-\overline{N}(3,6,n))q^n.$$

We proceed to simplify the right-hand side of (2.2). In light of the fact that $1-\xi_6^{-1}-\xi_6=0$, we get

$$\overline{R}(\exp(i\pi/3);q) = \frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(2-\xi_6^{-1}-\xi_6)(-1)^n q^{n^2+n}}{(1-\xi_6^{-1}q^n-\xi_6 q^n+q^{2n})}$$
$$= \frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+n}}{1-q^n+q^{2n}}$$
$$= \frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+n}(1+q^n)}{1+q^{3n}}$$
$$= \frac{(-q;q)_\infty}{(q;q)_\infty}\left\{\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+n}}{1+q^{3n}}+\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2-2n}}{1+q^{-3n}}\right\}$$
$$= \frac{2(-q;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+n}}{1+q^{3n}},$$

as desired. Thus, we complete the proof of Lemma 2.1. ∎

We are now in position to prove Theorem 1.1.

Proof of Theorem 1.1. By Lemma 2.1, it suffices to show that

$$\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+n}}{1+q^{3n}}$$
$$=\frac{J_{18}^3J_{9,18}}{2J_6J_{3,18}^2}+q\frac{J_{18}^3}{J_6J_{3,18}}+q^2\left\{\frac{2J_{18}^3}{J_6J_{9,18}}-\frac{1}{J_{9,18}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{9n^2+9n}}{1+q^{9n+3}}\right\}. \tag{2.3}$$

First, we split the sum on the left-hand side of (2.3) into three sums according to the summation index n modulo 3,

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+n}}{1+q^{3n}}=\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{9n^2+3n}}{1+q^{9n}}-\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{9n^2+9n+2}}{1+q^{9n+3}}+\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{9n^2+15n+6}}{1+q^{9n+6}}$$
$$:=S_0-S_1+S_2. \tag{2.4}$$

We claim that

$$S_0+S_2=\frac{2qJ_{3,18}}{J_{9,18}}S_1+\frac{J_6^3J_{3,18}^6J_{9,18}^2}{2J_{18}^9}. \tag{2.5}$$

The identity (2.5) can be justified by using the following identity in [22, Lemma 4.1].

$$\sum_{n=-\infty}^{\infty}(-1)^nq^{n^2+n}\left(\frac{\zeta^{-2n}}{1-z\zeta^{-1}q^n}+\frac{\zeta^{2n+2}}{1-z\zeta q^n}\right)$$
$$=\frac{\zeta(\zeta^2,q\zeta^{-2},-1,-q;q)_\infty}{(\zeta,q\zeta^{-1},-\zeta,-q\zeta^{-1};q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+n}}{1-zq^n}$$
$$+\frac{(\zeta,q\zeta^{-1},\zeta^2,q\zeta^{-2},-z,-qz^{-1};q)_\infty(q;q)_\infty^2}{(z,qz^{-1},z\zeta,qz^{-1}\zeta^{-1},z\zeta^{-1},q\zeta z^{-1},-\zeta,-q\zeta^{-1};q)_\infty}. \tag{2.6}$$

Replacing q, z and ζ in (2.6) by q^9, $-q^3$ and q^3, we find that

$$\sum_{n=-\infty}^{\infty}(-1)^nq^{9n^2+9n}\left(\frac{q^{-6n}}{1+q^{9n}}+\frac{q^{6n+6}}{1+q^{9n+6}}\right)$$
$$=\frac{q^3(-1,-q^9;q^9)_\infty}{(-q^3,-q^6;q^9)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{9n^2+9n}}{1+q^{9n+3}}+\frac{(q^3,q^6;q^9)_\infty^3(q^9;q^9)_\infty^2}{(-q^3,-q^6;q^9)_\infty^3(-1,-q^9;q^9)_\infty}$$
$$=\frac{2qJ_{3,18}}{J_{9,18}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{9n^2+9n+2}}{1+q^{9n+3}}+\frac{J_6^3J_{3,18}^6J_{9,18}^2}{2J_{18}^9},$$

which gives (2.5), and hence the claim is verified.

Substituting (2.5) into (2.4), we have

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+n}}{1+q^{3n}}=-\left(1-\frac{2qJ_{3,18}}{J_{9,18}}\right)S_1+\frac{J_6^3J_{3,18}^6J_{9,18}^2}{2J_{18}^9}.$$

Using the identity in [22, Lemma 3.1]

$$\frac{(q;q)_\infty}{(-q;q)_\infty} = \frac{(q^9;q^9)_\infty}{(-q^9;q^9)_\infty} - 2q(q^3,q^{15},q^{18};q^{18})_\infty = J_{9,18} - 2qJ_{3,18}, \tag{2.7}$$

it follows that

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+n}}{1+q^{3n}} = -\frac{(q;q)_\infty}{(-q;q)_\infty} \cdot \frac{S_1}{J_{9,18}} + \frac{J_6^3 J_{3,18}^6 J_{9,18}^2}{2J_{18}^9}.$$

Hence it remains to show that

$$\frac{J_6^3 J_{3,18}^6 J_{9,18}^2}{J_{18}^9} = \frac{(q;q)_\infty}{(-q;q)_\infty} \left\{ \frac{J_{18}^3 J_{9,18}}{J_6 J_{3,18}^2} + q\frac{2J_{18}^3}{J_6 J_{3,18}} + q^2 \frac{4J_{18}^3}{J_6 J_{9,18}} \right\}. \tag{2.8}$$

Substituting (2.7) into (2.8), we find that the right-hand side of (2.8) can be simplified as

$$\begin{aligned}
&(J_{9,18} - 2qJ_{3,18}) \left\{ \frac{J_{18}^3 J_{9,18}}{J_6 J_{3,18}^2} + q\frac{2J_{18}^3}{J_6 J_{3,18}} + q^2 \frac{4J_{18}^3}{J_6 J_{9,18}} \right\} \\
&= \left(\frac{J_{18}^3 J_{9,18}^2}{J_{3,18}^2 J_6} - 8q^3 \frac{J_{18}^3 J_{3,18}}{J_6 J_{9,18}} \right) + q\left(\frac{2J_{18}^3 J_{9,18}}{J_{3,18} J_6} - \frac{2J_{18}^3 J_{9,18}}{J_{3,18} J_6} \right) + q^2 \left(\frac{4J_{18}^3}{J_6} - \frac{4J_{18}^3}{J_6} \right) \\
&= \frac{J_{18}^3 J_{9,18}^2}{J_{3,18}^2 J_6} - 8q^3 \frac{J_{18}^3 J_{3,18}}{J_6 J_{9,18}}.
\end{aligned}$$

Hence (2.8) is equivalent to

$$\frac{J_{18}^3 J_{9,18}^2}{J_{3,18}^2 J_6} - 8q^3 \frac{J_{18}^3 J_{3,18}}{J_6 J_{9,18}} = \frac{J_6^3 J_{3,18}^6 J_{9,18}^2}{J_{18}^9}, \tag{2.9}$$

which can be simplified as follows.

$$J_{9,18}^3 - 8q^3 J_{3,18}^3 = \frac{J_6^4 J_{3,18}^8 J_{9,18}^3}{J_{18}^{12}}. \tag{2.10}$$

This identity can be verified by using the following identity in [5]

$$j(x;q)^2 j(yz;q) j(yz^{-1};q) = j(y;q)^2 j(xz;q) j(xz^{-1};q) - yz^{-1} j(z;q)^2 j(xy;q) j(xy^{-1};q). \tag{2.11}$$

Letting $q \to q^9$, $x = -q^3$, $y = q^3$ and $z = -1$ in (2.11), we obtain

$$j(-q^3;q^9)^3 - q^3 j(-1;q^9)^3 = \frac{j(q^3;q^9)^4}{j(-q^3;q^9)},$$

which is equivalent to

$$\frac{J_{18}^3}{J_{3,18}^3} - 8q^3 \frac{J_{18}^3}{J_{9,18}^3} = \frac{J_{3,18}^5 J_6^4}{J_{18}^9}. \tag{2.12}$$

If we multiply both sides of (2.12) by $J_{3,18}^3 J_{9,18}^3 / J_{18}^3$, we obtain (2.10), and hence (2.8) is verified. Thus, we complete the proof of Theorem 1.1. ∎

3 Proofs of Theorem 1.2 and Theorem 1.3

To prove Theorem 1.2 and Theorem 1.3, we are required to consider the 5-dissection of $\overline{R}(\exp(i\pi/5);q)$.

Lemma 3.1. *We have*

$$\begin{aligned}\overline{R}(\exp(i\pi/5);q) &= \sum_{n=0}^{\infty}(\overline{N}(0,10,n)+\overline{N}(1,10,n)-\overline{N}(4,10,n)-\overline{N}(5,10,n))q^n \\ &\quad + (\xi_{10}^2-\xi_{10}^3)\sum_{n=0}^{\infty}(\overline{N}(1,10,n)+\overline{N}(2,10,n)-\overline{N}(3,10,n)-\overline{N}(4,10,n))q^n \\ &= F_1(q)+(\xi_{10}^2-\xi_{10}^3)F_2(q), \qquad (3.1)\end{aligned}$$

where

$$F_1(q) := \frac{2(-q;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+n}}{1+q^{5n}},$$
$$F_2(q) := \frac{2(-q;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+n}(q^n-1)}{1+q^{5n}}.$$

Proof. Plugging $z=\xi_{10}=\exp\left(\frac{\pi i}{5}\right)$ into (1.3), we have

$$\begin{aligned}\overline{R}(\exp(i\pi/5);q) &= \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\overline{N}(m,n)\xi_{10}^m q^n \\ &= \frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(1-\xi_{10})(1-\xi_{10}^{-1})(-1)^n q^{n^2+n}}{(1-\xi_{10}q^n)(1-\xi_{10}^{-1}q^n)}. \qquad (3.2)\end{aligned}$$

Note that $\overline{N}(s,\ell,n)=\overline{N}(\ell-s,\ell,n)$ and $\xi_{10}^5=-1$, we find that the left-hand side of (3.2) can be simplified as

$$\begin{aligned}\overline{R}(\exp(i\pi/5);q) &= \sum_{n=0}^{\infty}\sum_{t=0}^{9}\overline{N}(t,10,n)\xi_{10}^t q^n \\ &= \sum_{n=0}^{\infty}\left\{\overline{N}(0,10,n)+(\xi_{10}-\xi_{10}^4)\overline{N}(1,10,n)+(\xi_{10}^2-\xi_{10}^3)\overline{N}(2,10,n)\right. \\ &\quad \left.+(\xi_{10}^3-\xi_{10}^2)\overline{N}(3,10,n)+(\xi_{10}^4-\xi_{10})\overline{N}(4,10,n)-\overline{N}(5,10,n)\right\}q^n.\end{aligned}$$

Using the fact that $1-\xi_{10}+\xi_{10}^2-\xi_{10}^3+\xi_{10}^4=0$, we have

$$\overline{R}(\exp(i\pi/5);q) = \sum_{n=0}^{\infty}\left\{\overline{N}(0,10,n)+(1+\xi_{10}^2-\xi_{10}^3)\overline{N}(1,10,n)+(\xi_{10}^2-\xi_{10}^3)\overline{N}(2,10,n)\right.$$

$$+(\xi_{10}^3-\xi_{10}^2)\overline{N}(3,10,n)-(1+\xi_{10}^2-\xi_{10}^3)\overline{N}(4,10,n)-\overline{N}(5,10,n)\Big\}\, q^n$$
$$=\sum_{n=0}^{\infty}(\overline{N}(0,10,n)+\overline{N}(1,10,n)-\overline{N}(4,10,n)-\overline{N}(5,10,n))q^n$$
$$+(\xi_{10}^2-\xi_{10}^3)\sum_{n=0}^{\infty}(\overline{N}(1,10,n)+\overline{N}(2,10,n)-\overline{N}(3,10,n)-\overline{N}(4,10,n))q^n.$$

We now turn to simplify the right-hand side of (3.2). Using the fact that $1-\xi_{10}-\xi_{10}^{-1}-\xi_{10}^{-3}-\xi_{10}^3=0$, we deduce that

$$(1-\xi_{10}q^n)(1-\xi_{10}^{-1}q^n)(1-\xi_{10}^3q^n)(1-\xi_{10}^{-3}q^n)(1+q^n)=1+q^{5n}$$

and

$$\begin{aligned}&(1-\xi_{10})(1-\xi_{10}^{-1})(1-\xi_{10}^3q^n)(1-\xi_{10}^{-3}q^n)(1+q^n)\\&=(1-\xi_{10}^2+\xi_{10}^3)+(-1+\xi_{10}^{-1}+\xi_{10})q^n+(\xi_{10}^{-1}-1+\xi_{10})q^{2n}+(1-\xi_{10}^2+\xi_{10}^3)q^{3n}\\&=(1-\xi_{10}^2+\xi_{10}^3)+(\xi_{10}^2-\xi_{10}^3)q^n+(\xi_{10}^2-\xi_{10}^3)q^{2n}+(1-\xi_{10}^2+\xi_{10}^3)q^{3n}.\end{aligned}$$

Hence the right-hand side of (3.2) can be simplified as

$$\begin{aligned}&\overline{R}(\exp(i\pi/5);q)\\&=\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(1-\xi_{10})(1-\xi_{10}^{-1})(1-\xi_{10}^3q^n)(1-\xi_{10}^{-3}q^n)(1+q^n)(-1)^nq^{n^2+n}}{(1-\xi_{10}q^n)(1-\xi_{10}^{-1}q^n)(1-\xi_{10}^3q^n)(1-\xi_{10}^{-3}q^n)(1+q^n)}\\&=\frac{(-q;q)_\infty}{(q;q)_\infty}\left\{\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+n}(1+q^{3n})}{1+q^{5n}}+(\xi_{10}^2-\xi_{10}^3)\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+n}(q^n+q^{2n}-1-q^{3n})}{1+q^{5n}}\right\}\\&=\frac{(-q;q)_\infty}{(q;q)_\infty}\left\{\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+n}}{1+q^{5n}}+\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2-4n}}{1+q^{-5n}}\right\}+(\xi_{10}^2-\xi_{10}^3)\frac{(-q;q)_\infty}{(q;q)_\infty}\\&\quad\times\left\{\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+2n}}{1+q^{5n}}+\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2-3n}}{1+q^{-5n}}-\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+n}}{1+q^{5n}}-\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2-4n}}{1+q^{-5n}}\right\}\\&=\frac{2(-q;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+n}}{1+q^{5n}}-(\xi_{10}^2-\xi_{10}^3)\frac{2(-q;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n^2+n}(1-q^n)}{1+q^{5n}},\end{aligned}$$

as desired. Thus, we complete the proof of Lemma 3.1. ∎

Since the coefficients of $F_1(q)$ and $F_2(q)$ are all integers and $[\mathbb{Q}(\xi_{10}):\mathbb{Q}]=4$, we equate the coefficients of ξ_{10}^k on the both side of (3.1) and find the following corollaries are true.

Corollary 3.2. *We have*

$$\sum_{n=0}^{\infty}(\overline{N}(0,10,n)+\overline{N}(1,10,n)-\overline{N}(4,10,n)-\overline{N}(5,10,n))q^n$$

$$= \frac{2(-q;q)_\infty}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+n}}{1+q^{5n}}. \tag{3.3}$$

Corollary 3.3. *We have*

$$\sum_{n=0}^{\infty} (\overline{N}(1,10,n) + \overline{N}(2,10,n) - \overline{N}(3,10,n) - \overline{N}(4,10,n))q^n$$
$$= \frac{2(-q;q)_\infty}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+n}(q^n-1)}{1+q^{5n}}. \tag{3.4}$$

Hence it suffices to determine the 5-dissection of right-hand sides of (3.3) and (3.4) in order to prove Theorem 1.2 and Theorem 1.3.

Lemma 3.4. *Let*

$$U_1 := \frac{J_{5,50}^2 J_{10,50}^3 J_{15,50}^4 J_{25,50}^2}{J_{50}^9},$$
$$U_2 := \frac{J_{5,50}^3 J_{10,50}^2 J_{15,50}^4 J_{20,50} J_{25,50}}{J_{50}^9}.$$

We have

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+n}}{1+q^{5n}} = \frac{1}{2}U_1 - q^2 U_2 + \frac{(q;q)_\infty}{(-q;q)_\infty} \cdot \frac{q^6}{J_{25,50}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{25n^2+25n}}{1+q^{25n+10}}. \tag{3.5}$$

Proof. First, we split the sum on the left-hand side of (3.5) into five sums according to the summation index n modulo 5,

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+n}}{1+q^{5n}} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{25n^2+5n}}{1+q^{25n}} - \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{25n^2+15n+2}}{1+q^{25n+5}} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{25n^2+25n+6}}{1+q^{25n+10}}$$
$$- \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{25n^2+35n+12}}{1+q^{25n+15}} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{25n^2+45n+20}}{1+q^{25n+20}}$$
$$:= P_0 - P_1 + P_2 - P_3 + P_4. \tag{3.6}$$

We will establish the two relations,

$$P_0 + P_4 = \frac{2q^4 J_{5,50}}{J_{25,50}} P_2 + \frac{1}{2}U_1, \tag{3.7}$$

$$P_1 + P_3 = \frac{2q J_{15,50}}{J_{25,50}} P_2 + q^2 U_2. \tag{3.8}$$

Replacing q, z and ζ in (2.6) by q^{25}, $-q^{10}$ and q^{10} respectively, we find that

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty}(-1)^n q^{25n^2+25n}\left(\frac{q^{-20n}}{1+q^{25n}}+\frac{q^{20n+20}}{1+q^{25n+20}}\right)\\
&=\frac{(-1,q^5,q^{20},-q^{25};q^{25})_\infty}{(q^{10},-q^{10},q^{15},-q^{15};q^{25})_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{25n^2+25n+10}}{1+q^{25n+10}}\\
&\quad+\frac{(q^5,q^{20};q^{25})_\infty(q^{10},q^{15},q^{25};q^{25})_\infty^2}{(-1,-q^5,-q^{20},-q^{25};q^{25})_\infty(-q^{10},-q^{15};q^{25})_\infty^2}\\
&=\frac{2q^4J_{5,50}}{J_{25,50}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{25n^2+25n+6}}{1+q^{25n+10}}+\frac{J_{5,50}^2J_{10,50}^3J_{15,50}^4J_{25,50}^2}{2J_{50}^9},
\end{aligned}$$

which gives (3.7).

Replacing q, z and ζ in (2.6) by q^{25}, $-q^{10}$ and q^5 respectively, we have

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty}(-1)^n q^{25n^2+25n}\left(\frac{q^{-10n}}{1+q^{25n+5}}+\frac{q^{10n+10}}{1+q^{25n+15}}\right)\\
&=\frac{(-1,q^{10},q^{15},-q^{25};q^{25})_\infty}{(q^5,-q^5,q^{20},-q^{20};q^{25})_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{25n^2+25n+5}}{1+q^{25n+10}}\\
&\quad+\frac{(q^5,q^{20};q^{25})_\infty(q^{10},q^{15},q^{25};q^{25})_\infty^2}{(-q^5,-q^{10},-q^{15},-q^{20};q^{25})_\infty^2}\\
&=\frac{2J_{15,50}}{qJ_{25,50}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{25n^2+25n+6}}{1+q^{25n+10}}+\frac{J_{5,50}^3J_{10,50}^2J_{15,50}^4J_{20,50}J_{25,50}}{J_{50}^9}.
\end{aligned}$$

This yields (3.8). Substituting (3.7) and (3.8) into (3.6), we find that

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+n}}{1+q^{5n}}=\left(1-\frac{2qJ_{15,50}}{J_{25,50}}+\frac{2q^4J_{5,50}}{J_{25,50}}\right)P_2+\frac{1}{2}U_1-q^2U_2.$$

Next, by [22, Lemma 3.1]

$$\begin{aligned}
\frac{(q;q)_\infty}{(-q;q)_\infty}&=\frac{(q^{25};q^{25})_\infty}{(-q^{25};q^{25})_\infty}-2q(q^{15},q^{35},q^{50};q^{50})_\infty+2q^4(q^5,q^{45},q^{50};q^{50})_\infty\\
&=J_{25,50}-2qJ_{15,50}+2q^4J_{5,50},
\end{aligned}\tag{3.9}$$

we derive that

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+n}}{1+q^{5n}}=\frac{(q;q)_\infty}{(-q;q)_\infty}\cdot\frac{P_2}{J_{25,50}}+\frac{1}{2}U_1-q^2U_2.$$

This completes the proof of (3.5). ∎

Lemma 3.5. *Let*

$$V_1 := \frac{J_{5,50}^4 J_{15,50}^2 J_{20,50}^3 J_{25,50}^2}{J_{50}^9},$$
$$V_2 := \frac{J_{5,50}^4 J_{10,50} J_{15,50}^3 J_{20,50}^2 J_{25,50}}{J_{50}^9}.$$

We have

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+2n}}{1+q^{5n}} = \frac{1}{2}V_1 - q^3 V_2 - \frac{(q;q)_\infty}{(-q;q)_\infty} \cdot \frac{q^4}{J_{25,50}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{25n^2+25n}}{1+q^{25n+20}}. \tag{3.10}$$

Proof. First, split the sum on the left-hand side of (3.10) into five sums according to the summation index n modulo 5,

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+2n}}{1+q^{5n}} &= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{25n^2+10n}}{1+q^{25n}} - \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{25n^2+20n+3}}{1+q^{25n+5}} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{25n^2+30n+8}}{1+q^{25n+10}} \\ &\quad - \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{25n^2+40n+15}}{1+q^{25n+15}} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{25n^2+50n+24}}{1+q^{25n+20}} \\ &:= T_0 - T_1 + T_2 - T_3 + T_4. \end{aligned} \tag{3.11}$$

We will establish the two relations,

$$T_0 - T_3 = -\frac{2qJ_{15,50}}{J_{25,50}} T_4 + \frac{1}{2}V_1, \tag{3.12}$$

$$-T_1 + T_2 = \frac{2q^4 J_{5,50}}{J_{25,50}} T_4 - q^3 V_2. \tag{3.13}$$

These two relations can be verified by using the following identity

$$\begin{aligned}&\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \left(\frac{\zeta^{-2n} q^{2n}}{1+z\zeta^{-1}q^n} - \frac{q^{-1}\zeta^{2n+2}}{1+z\zeta q^{n-1}} \right) \\ &= -\frac{(-1,-q,\zeta^2 q^{-1}, q^2\zeta^{-2};q)_\infty}{(-\zeta,-q\zeta^{-1},\zeta,q\zeta^{-1};q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+2n+1}}{1+zq^n} \\ &+ \frac{(\zeta, q\zeta^{-1}, z, qz^{-1}, \zeta^2 q^{-1}, q^2\zeta^{-2};q)_\infty (q;q)_\infty^2}{(-z,-qz^{-1},-\zeta,-q\zeta^{-1},-z\zeta^{-1},-q\zeta z^{-1},-z\zeta q^{-1},-q^2\zeta^{-1}z^{-1};q)_\infty}, \end{aligned} \tag{3.14}$$

which follows by setting $r=1, s=3$ and replacing $a_1=-b_3=z,\ b_1=-z\zeta^{-1},\ b_2=-z\zeta q^{-1}$ respectively in [7, Theorem 2.1],

$$\frac{(a_1, q/a_1, \ldots, a_r, q/a_r;q)_\infty (q;q)_\infty^2}{(b_1, q/b_1, \ldots, b_s, q/b_s;q)_\infty}$$

$$= \frac{(a_1/b_1, qb_1/a_1, \ldots, a_r/b_1, qb_1/a_r; q)_\infty}{(b_2/b_1, qb_1/b_2, \ldots, b_s/b_1, qb_1/b_s; q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^{(s-r)n} q^{(s-r)n(n+1)/2}}{1-b_1q^n} \left(\frac{a_1 \cdots a_r b_1^{s-r-1}}{b_2 \cdots b_s} \right)^n$$

$$+ \, idem(b_1; b_2, \ldots, b_s).$$

Here we use the notation

$$\begin{aligned} &F(b_1, b_2, \cdots, b_m) + idem(b_1; b_2, \cdots, b_m) \\ &:= F(b_1, b_2, \cdots, b_m) + F(b_2, b_1, b_3, \cdots, b_m) + \cdots + F(b_m, b_2, \cdots, b_{m-1}, b_1). \end{aligned}$$

Replacing q, z and ζ in (3.14) by q^{25}, q^{20} and q^{20} respectively, we find that

$$\begin{aligned} &\sum_{n=-\infty}^{\infty} (-1)^n q^{25n^2} \left(\frac{q^{10n}}{1+q^{25n}} - \frac{q^{40n+15}}{1+q^{25n+15}} \right) \\ &= -\frac{(-1, q^{10}, q^{15}, -q^{25}; q^{25})_\infty}{(q^5, -q^5, q^{20}, -q^{20}; q^{25})_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{25n^2+50n+25}}{1+q^{25n+20}} \\ &\quad + \frac{(q^{10}, q^{15}; q^{25})_\infty (q^5, q^{20}, q^{25}; q^{25})_\infty^2}{(-1, -q^{10}, -q^{15}, -q^{25}; q^{25})_\infty (-q^5, -q^{20}; q^{25})_\infty^2} \\ &= -\frac{2qJ_{15,50}}{J_{25,50}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{25n^2+50n+24}}{1+q^{25n+20}} + \frac{J_{5,50}^4 J_{15,50}^2 J_{20,50}^3 J_{25,50}^2}{2J_{50}^9}, \end{aligned}$$

which gives (3.12).

Replacing q, z and ζ in (3.14) by q^{25}, q^{20} and q^{15} respectively, we have

$$\begin{aligned} &\sum_{n=-\infty}^{\infty} (-1)^n q^{25n^2} \left(\frac{q^{20n}}{1+q^{25n+5}} - \frac{q^{30n+5}}{1+q^{25n+10}} \right) \\ &= -\frac{(-1, q^5, q^{20}, -q^{25}; q^{25})_\infty}{(q^{10}, -q^{10}, q^{15}, -q^{15}; q^{25})_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{25n^2+50n+25}}{1+q^{25n+20}} \\ &\quad + \frac{(q^{10}, q^{15}; q^{25})_\infty (q^5, q^{20}, q^{25}; q^{25})_\infty^2}{(-q^5, -q^{10}, -q^{15}, -q^{20}; q^{25})_\infty^2} \\ &= -\frac{2qJ_{5,50}}{J_{25,50}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{25n^2+50n+24}}{1+q^{25n+20}} + \frac{J_{5,50}^4 J_{10,50} J_{15,50}^3 J_{20,50}^2 J_{25,50}}{J_{50}^9}. \end{aligned}$$

This yields (3.13). Substituting (3.12) and (3.13) into (3.11), we have

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+2n}}{1+q^{5n}} = \left(1 - \frac{2qJ_{15,50}}{J_{25,50}} + \frac{2q^4 J_{5,50}}{J_{25,50}} \right) T_4 + \frac{1}{2} V_1 - q^3 V_2. \qquad (3.15)$$

Substituting (3.9) into (3.15), and noting that

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{25n^2+50n+24}}{1+q^{25n+20}} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{25n^2+25n+4}(1+q^{25n+20}-1)}{1+q^{25n+20}} = -\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{25n^2+25n+4}}{1+q^{25n+20}}.$$

we obtain (3.10). Thus, we complete the proof of Lemma 3.5. ∎

The following two lemmas will be of key importance in proofs of Theorem 1.2 and Theorem 1.3.

Lemma 3.6. *Recall that U_1 and U_2 are defined in Lemma 3.4. The following identity holds.*

$$\frac{1}{2}U_1 - q^2U_2 = \frac{(q;q)_\infty}{(-q;q)_\infty}\left\{A_0 + A_1q + A_2q^2 + A_3q^3 + A_4q^4\right\}, \tag{3.16}$$

where A_0, A_1, A_2, A_3, A_4 are defined in Theorem 1.2.

Lemma 3.7. *Recall that U_1 and U_2 are defined in Lemma 3.4 and V_1 and V_2 are defined in Lemma 3.5. The following identity holds.*

$$\frac{1}{2}V_1 - \frac{1}{2}U_1 + q^2U_2 - q^3V_2 = \frac{(q;q)_\infty}{(-q;q)_\infty}\left\{B_0 + B_1q + B_2q^2 + B_3q^3 + B_4q^4\right\}, \tag{3.17}$$

where B_0, B_1, B_2, B_3, B_4 are defined in Theorem 1.3.

Before verifying Lemma 3.6 and Lemma 3.7, we will show Theorem 1.2 and Theorem 1.3 based on Lemma 3.6 and Lemma 3.7. We begin with Theorem 1.2.

Proof of Theorem 1.2. Substituting (3.5) into (3.3), we find that

$$\begin{aligned}&\sum_{n=0}^{\infty}(\overline{N}(0,10,n) + \overline{N}(1,10,n) - \overline{N}(4,10,n) - \overline{N}(5,10,n))q^n\\ &\quad= \frac{2(-q;q)_\infty}{(q;q)_\infty}\left(\frac{1}{2}U_1 - q^2U_2\right) + \frac{2q^6}{J_{25,50}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{25n^2+25n}}{1+q^{25n+10}}.\end{aligned} \tag{3.18}$$

Substituting (3.16) into (3.18), we obtain (1.5), and so Theorem 1.2 is verified. ∎

We now turn to prove Theorem 1.3 by using Lemma 3.7.

Proof of Theorem 1.3. Substituting (3.5) and (3.10) into (3.4), we have

$$\begin{aligned}&\sum_{n=0}^{\infty}(\overline{N}(1,10,n) + \overline{N}(2,10,n) - \overline{N}(3,10,n) - \overline{N}(4,10,n))q^n\\ &= \frac{2(-q;q)_\infty}{(q;q)_\infty}\left(\frac{1}{2}V_1 - q^3V_2\right) - \frac{2q^4}{J_{25,50}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{25n^2+25n}}{1+q^{25n+20}}\\ &\quad - \left\{\frac{2(-q;q)_\infty}{(q;q)_\infty}\left(\frac{1}{2}U_1 - q^2U_2\right) + \frac{2q^6}{J_{25,50}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{25n^2+25n}}{1+q^{25n+10}}\right\}\\ &= \frac{2(-q;q)_\infty}{(q;q)_\infty}\left(\frac{1}{2}V_1 - \frac{1}{2}U_1 + q^2U_2 - q^3V_2\right)\end{aligned}$$

$$-\frac{2q^4}{J_{25,50}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{25n^2+25n}}{1+q^{25n+20}}-\frac{2q^6}{J_{25,50}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{25n^2+25n}}{1+q^{25n+10}}. \tag{3.19}$$

Then we obtain (1.6) after substituting (3.17) into (3.19). Thus, we complete the proof of Theorem 1.3. ∎

We conclude this section by giving proofs of Lemma 3.6 and Lemma 3.7 with the aid of standard computational techniques from the theory of modular forms. Recall that the Dedekind η-function is defined by

$$\eta(\tau)=q^{\frac{1}{24}}(q;q)_\infty,$$

where $\tau\in\mathcal{H}:=\{\tau\in\mathcal{C}:Im\tau>0\}$ and $q=e^{2\pi i\tau}$, and the generalized Dedekind η-function is defined to be

$$\eta_{\delta,g}(\tau)=q^{P(g/\delta)\delta/2}\prod_{\substack{n>0\\ n\equiv g \pmod{\delta}}}(1-q^n)\prod_{\substack{n>0\\ n\equiv -g \pmod{\delta}}}(1-q^n), \tag{3.20}$$

where $g,\delta\in\mathbb{Z}^+$ and $0\le g<\delta$, $P(t)=\{t\}^2-\{t\}+\frac{1}{6}$ is the second Bernoulli function, and $\{t\}:=t-[t]$ is the fractional part of t, so

$$\eta_{\delta,0}(\tau)=q^{\frac{\delta}{12}}(q^\delta;q^\delta)_\infty^2$$

and

$$\eta_{\delta,\frac{\delta}{2}}(\tau)=q^{-\frac{\delta}{24}}(q^{\frac{\delta}{2}};q^\delta)_\infty^2.$$

Let N be a fixed positive integer. A generalized Dedekind η-product of level N has the form

$$f(\tau)=\prod_{\substack{\delta|N\\ 0<g<\delta}}\eta_{\delta,g}^{r_{\delta,g}}(\tau),$$

where

$$r_{\delta,g}\in\begin{cases}\frac{1}{2}\mathbb{Z} & \text{if } g=0 \text{ or } g=\delta/2,\\ \mathbb{Z} & \text{otherwise.}\end{cases}$$

Suppose f is a modular function with respect to the congruence subgroup Γ of $\Gamma_0(1)$. For $A\in\Gamma_0(1)$ we have a cusp given by $\zeta=A^{-1}\infty$. If

$$f(A^{-1}\tau)=\sum_{m=m_0}^{\infty}b_m q^{m/N}$$

and $b_{m_0}\neq 0$, then we say m_0 is the order of f at ζ with respect to Γ and we denote this value by $ORD(f,\zeta,\Gamma)$.

In [27], Robins determined the sufficient conditions on which a generalized η-product is a modular function on $\Gamma_1(N)$ and the order of a generalized η-product at the cusps.

By the valence formula for modular functions and note that any generalized Dedekind η-product has weight $k = 0$ and has no zeros and no poles on the upper-half plane $\mathcal{H}$, Garvan and Liang [13] established the following sufficient and necessary condition to prove generalized η-quotient identities. This theorem is based on the valence formula for modular functions, along with the fact that a generalized η-quotient has no zeros nor poles in the upper-half plane.

Theorem 3.8 (Garvan-Liang)**.** *Let $f_1(\tau), f_2(\tau), \ldots, f_n(\tau)$ be generalized η-products that are modular functions on $\Gamma_1(N)$. Let $\mathcal{S}_N$ be a set of inequivalent cusps for $\Gamma_1(N)$. Define the constant*

$$B = \sum_{\substack{s\in\mathcal{S}_N \\ s\neq i\infty}} min(\{ORD(f_j, s, \Gamma_1(N)) : 1 \leq j \leq n\} \cup \{0\}), \tag{3.21}$$

and consider

$$g(\tau) := \alpha_1 f_1(\tau) + \alpha_2 f_2(\tau) + \cdots + \alpha_n f_n(\tau) + 1, \tag{3.22}$$

where each $\alpha_j \in \mathbb{C}$. Then

$$g(\tau) \equiv 0$$

if and only if

$$ORD(g(\tau), i\infty, \Gamma_1(N)) > -B. \tag{3.23}$$

We are now in a position to prove Lemmas 3.6 and 3.7 with the aid of Theorem 3.8. We begin with Lemma 3.6.

Proof of Lemma 3.6. It is equivalent to show that

$$\begin{aligned}
&\frac{1}{2}\frac{J_{5,50}^2 J_{10,50}^3 J_{15,50}^4 J_{25,50}^2}{J_{50}^9} - q^2\frac{J_{5,50}^3 J_{10,50}^2 J_{15,50}^4 J_{20,50} J_{25,50}}{J_{50}^9} \\
&= \frac{(q;q)_\infty}{(-q;q)_\infty}\left\{\frac{J_{10,50}^2 J_{15,50}^2 J_{25,50}^4}{2J_{5,10}^3 J_{20,50} J_{50}^3} + 4q^{10}\frac{J_{5,50} J_{50}^3}{J_{5,10}^2 J_{20,50}}\right. \\
&+ q\left(\frac{J_{20,50} J_{25,50}^4 J_{50}^3}{J_{5,10}^4 J_{10,50}^2 J_{15,50}} - 4q^5\frac{J_{5,50}^4 J_{20,50}^3 J_{25,50}^4}{J_{5,10}^5 J_{10,50}^2 J_{50}^3} - 8q^{15}\frac{J_{10,50}^4 J_{15,50}^2 J_{25,50} J_{50}^3}{J_{5,10}^4 J_{20,50}^5}\right) \\
&+ q^2\left(\frac{J_{5,50}^3 J_{20,50}^3 J_{25,50}^5}{J_{5,10}^5 J_{10,50}^2 J_{50}^3} + 4q^{10}\frac{J_{10,50}^5 J_{25,50} J_{50}^7}{J_5^4 J_{5,10} J_{5,50}^3 J_{20,50}^4} - 16q^{10}\frac{J_{5,50} J_{15,50}^2 J_{50}^3}{J_{5,10}^4 J_{20,50}}\right) \\
&+ q^3\left(\frac{2J_{10,50}^4 J_{15,50}^5 J_{25,50}^4}{J_{5,10}^5 J_{5,50} J_{20,50}^3 J_{50}^3} + 2q^5\frac{J_{10,50} J_{25,50}^3 J_{50}^3}{J_{5,10}^4 J_{20,50}^2} - 16q^5\frac{J_{15,50}^2 J_{25,50} J_{50}^3}{J_{5,10}^4 J_{20,50}}\right. \\
&\qquad\left. +8q^{10}\frac{J_{5,50}^3 J_{20,50} J_{25,50} J_{50}^3}{J_{5,10}^4 J_{10,50}^2 J_{15,50}}\right)
\end{aligned}$$

$$+ q^4 \left(\frac{4J_{10,50}^4 J_{15,50}^6 J_{25,50}^3}{J_{5,10}^5 J_{5,50} J_{20,50}^3 J_{50}^3} - \frac{J_{10,50} J_{25,50}^4 J_{50}^3}{J_{5,10}^4 J_{5,50} J_{20,50}^2} - 16q^5 \frac{J_{15,50}^3 J_{50}^3}{J_{5,10}^4 J_{20,50}} \right.$$
$$\left. - 8q^5 \frac{J_{25} J_{50}^5}{J_5 J_{5,10}^3 J_{10,50}} \right) \Bigg\}.$$

If we multiply both sides of above identity by $2J_{50}^9 J_{5,50}^{-2} J_{10,50}^{-3} J_{15,50}^{-4} J_{25,50}^{-2}$, we find that the obtained identity can be expressed in terms of the generalized η-product as follows.

$$1 - \frac{2\eta_{50,5}(\tau)\eta_{50,20}(\tau)}{\eta_{50,10}(\tau)\eta_{50,25}(\tau)} = \frac{\eta_{1,0}(\tau)\eta_{50,0}(\tau)^{\frac{5}{2}}}{\eta_{2,0}(\tau)^{\frac{1}{2}}\eta_{5,0}(\tau)\eta_{10,0}(\tau)^2} \left\{ \frac{\eta_{10,0}(\tau)^{\frac{1}{2}}\eta_{50,10}(\tau)\eta_{50,20}(\tau)\eta_{50,25}(\tau)^3}{\eta_{50,0}(\tau)^{\frac{1}{2}}\eta_{10,5}(\tau)^3} \right.$$
$$+ \frac{8\eta_{10,0}(\tau)\eta_{50,5}(\tau)\eta_{50,20}(\tau)}{\eta_{50,0}(\tau)\eta_{10,5}(\tau)^2\eta_{50,10}(\tau)\eta_{50,15}(\tau)^2\eta_{50,25}(\tau)} + \frac{2\eta_{50,20}(\tau)^3\eta_{50,25}(\tau)^3}{\eta_{10,5}(\tau)^4\eta_{50,10}(\tau)^3\eta_{50,15}(\tau)^3}$$
$$- \frac{8\eta_{50,0}(\tau)^{\frac{1}{2}}\eta_{50,5}(\tau)^4\eta_{50,20}(\tau)^5\eta_{50,25}(\tau)^3}{\eta_{10,0}(\tau)^{\frac{1}{2}}\eta_{10,5}(\tau)^5\eta_{50,10}(\tau)^3\eta_{50,15}(\tau)^2} - \frac{16\eta_{50,10}(\tau)^3}{\eta_{10,5}(\tau)^4\eta_{50,20}(\tau)^3}$$
$$+ \frac{2\eta_{50,0}(\tau)^{\frac{1}{2}}\eta_{50,5}(\tau)^3\eta_{50,20}(\tau)^5\eta_{50,25}(\tau)^4}{\eta_{10,0}(\tau)^{\frac{1}{2}}\eta_{10,5}(\tau)^5\eta_{50,10}(\tau)^3\eta_{50,15}(\tau)^2} + \frac{8\eta_{10,0}(\tau)^{\frac{3}{2}}\eta_{50,0}(\tau)^{\frac{1}{2}}\eta_{50,10}(\tau)^4}{\eta_{5,0}(\tau)^2\eta_{10,5}(\tau)\eta_{50,5}(\tau)^3\eta_{50,15}(\tau)^2\eta_{50,20}(\tau)^2}$$
$$- \frac{32\eta_{50,5}(\tau)\eta_{50,20}(\tau)}{\eta_{10,5}(\tau)^4\eta_{50,10}(\tau)\eta_{50,25}(\tau)} + \frac{4\eta_{50,0}(\tau)^{\frac{1}{2}}\eta_{50,10}(\tau)^3\eta_{50,15}(\tau)^3\eta_{50,25}(\tau)^3}{\eta_{10,0}(\tau)^{\frac{1}{2}}\eta_{10,5}(\tau)^5\eta_{50,5}(\tau)\eta_{50,20}(\tau)}$$
$$+ \frac{4\eta_{50,25}(\tau)^2}{\eta_{10,5}(\tau)^4\eta_{50,15}(\tau)^2} - \frac{32\eta_{50,20}(\tau)}{\eta_{10,5}(\tau)^4\eta_{50,10}(\tau)} + \frac{16\eta_{50,5}(\tau)^3\eta_{50,20}(\tau)^3}{\eta_{10,5}(\tau)^4\eta_{50,10}(\tau)^3\eta_{50,15}(\tau)^3}$$
$$+ \frac{8\eta_{50,0}(\tau)^{\frac{1}{2}}\eta_{50,10}(\tau)^3\eta_{50,15}(\tau)^4\eta_{50,25}(\tau)^2}{\eta_{10,0}(\tau)^{\frac{1}{2}}\eta_{10,5}(\tau)^5\eta_{50,5}(\tau)\eta_{50,20}(\tau)} - \frac{2\eta_{50,25}(\tau)^3}{\eta_{10,5}(\tau)^4\eta_{50,5}(\tau)\eta_{50,15}(\tau)^2}$$
$$\left. - \frac{32\eta_{50,15}(\tau)\eta_{50,20}(\tau)}{\eta_{10,5}(\tau)^4\eta_{50,10}(\tau)\eta_{50,25}(\tau)} - \frac{16\eta_{10,0}(\tau)^{\frac{1}{2}}\eta_{25,0}(\tau)^{\frac{1}{2}}\eta_{50,20}(\tau)^2}{\eta_{5,0}(\tau)^{\frac{1}{2}}\eta_{50,0}(\tau)^{\frac{1}{2}}\eta_{10,5}(\tau)^3\eta_{50,10}(\tau)^2\eta_{50,15}(\tau)^2\eta_{50,25}(\tau)} \right\}.$$

It can be shown that each term of the above identity is a modular function with respect to $\Gamma_1(50)$. Using the algorithm in [13], we calculate the constant B in (3.21) which is equal to -145. Thus, by Theorem 3.8, we need only verify the identity in the q-expansion past q^{145}. Hence we complete the proof of Lemma 3.6. ∎

We turn to Lemma 3.7.

Proof of Lemma 3.7. It is equivalent to show that

$$\frac{1}{2}\frac{J_{5,50}^4 J_{15,50}^2 J_{20,50}^3 J_{25,50}^2}{J_{50}^9} - \frac{1}{2}\frac{J_{5,50}^2 J_{10,50}^3 J_{15,50}^4 J_{25,50}^2}{J_{50}^9}$$
$$+ q^2 \frac{J_{5,50}^3 J_{10,50}^2 J_{15,50}^4 J_{20,50} J_{25,50}}{J_{50}^9} - q^3 \frac{J_{5,50}^4 J_{10,50} J_{15,50}^3 J_{20,50}^2 J_{25,50}}{J_{50}^9}$$
$$= \frac{(q;q)_\infty}{(-q;q)_\infty} \left\{ \frac{4q^5 J_5^5 J_{25}^5 J_{5,50}^4 J_{15,50}^2}{J_{5,10}^6 J_{50}^6 J_{10,50}^3} - \frac{q^5 J_{50}^3 J_{25,50}^2}{J_{5,10}^2 J_{15,50} J_{20,50}} \right.$$

$$+ q\left(\frac{4q^5 J_5^8 J_{50}^7 J_{25,50}}{J_{5,10}^6 J_{10,50}^4 J_{20,50}^5} - \frac{4q^{10} J_{50}^3 J_{5,50}^2 J_{25,50}^2}{J_{5,10}^4 J_{10,50} J_{15,50}} + \frac{8q^{15} J_{50}^6 J_{10,50}^6}{J_5^6 J_{5,50}^2 J_{20,50}^3}\right)$$

$$+ q^2\left(-\frac{J_{5,50}^7 J_{20,50}^7 J_{25,50}^6}{J_{5,10}^6 J_{10,50}^4 J_{50}^9} + \frac{2J_{50} J_{20,50}^3 J_{25,50}}{J_5^4} - \frac{4q^{10} J_{50}^6 J_{10,50}^6 J_{25,50}}{J_5^6 J_{5,50}^3 J_{20,50}^3} + \frac{16q^{10} J_{50}^8}{J_5^2 J_{5,10}^2 J_{10,50} J_{15,50}^2}\right)$$

$$+ q^3\left(-\frac{J_{50}^3 J_{25,50}^4}{J_{5,10}^4 J_{10,50} J_{15,50}} + \frac{2J_{50}^3 J_{15,50} J_{25,50}}{J_{5,10}^2 J_{5,50} J_{20,50}} + \frac{4q^5 J_{50}^6 J_{20,50}^3 J_{25,50}^2}{J_5^6 J_{15,50}^4} + \frac{16q^{15} J_{50}^3 J_{5,50}^3}{J_{5,10}^4 J_{20,50}}\right)$$

$$+ q^4\left(\frac{4J_{50}^3 J_{15,50}^2}{J_{5,10}^2 J_{5,50} J_{20,50}} - \frac{2J_{5,50}^3 J_{20,50}^3 J_{25,50}^5}{J_5^2 J_{5,10}^4 J_{50}^4} + \frac{8q^{10} J_{50}^8}{J_5^2 J_{5,10}^2 J_{15,50}^2 J_{20,50}}\right.$$

$$\left.\left. + \frac{16q^{10} J_{50}^6 J_{10,50}^3}{J_5^6 J_{5,50} J_{25,50}} - \frac{16q^{15} J_{50}^3 J_{5,50}^4}{J_{5,10}^4 J_{10,50} J_{15,50}}\right)\right\}.$$

If we multiply both sides of above identity by $2J_{50}^9 J_{5,50}^{-2} J_{10,50}^{-3} J_{15,50}^{-4} J_{25,50}^{-2}$, we find that the obtained identity can be expressed in terms of the generalized η-product as follows.

$$-1 + \frac{\eta_{50,5}(\tau)^2\eta_{50,20}(\tau)^3}{\eta_{50,10}(\tau)^3\eta_{50,15}(\tau)^2} + \frac{2\eta_{50,5}(\tau)\eta_{50,20}(\tau)}{\eta_{50,10}(\tau)\eta_{50,25}(\tau)} - \frac{2\eta_{50,5}(\tau)^2\eta_{50,20}(\tau)^2}{\eta_{50,10}(\tau)^2\eta_{50,15}(\tau)\eta_{50,25}(\tau)}$$

$$= \frac{\eta_{1,0}(\tau)\eta_{50,0}(\tau)^{\frac{5}{2}}}{\eta_{2,0}(\tau)^{\frac{1}{2}}\eta_{5,0}(\tau)^2\eta_{10,0}(\tau)^2}\left\{\frac{8\eta_{5,0}(\tau)^{\frac{7}{2}}\eta_{25,0}(\tau)^{\frac{5}{2}}\eta_{50,5}(\tau)^4\eta_{50,20}(\tau)^2}{\eta_{10,0}(\tau)\eta_{50,0}(\tau)^4\eta_{10,5}(\tau)^6\eta_{50,10}(\tau)^4\eta_{50,25}(\tau)}\right.$$

$$- \frac{2\eta_{5,0}(\tau)\eta_{10,0}(\tau)\eta_{50,20}(\tau)\eta_{50,25}(\tau)}{\eta_{50,0}(\tau)\eta_{10,5}(\tau)^2\eta_{50,10}(\tau)\eta_{50,15}(\tau)^3} + \frac{8\eta_{5,0}(\tau)^4\eta_{50,5}(\tau)^2\eta_{50,25}(\tau)}{\eta_{10,0}(\tau)\eta_{10,5}(\tau)^6\eta_{50,0}(\tau)^2\eta_{50,10}(\tau)^3\eta_{50,20}(\tau)}$$

$$- \frac{8\eta_{5,0}(\tau)\eta_{50,5}(\tau)^2\eta_{50,20}(\tau)^2\eta_{50,25}(\tau)}{\eta_{10,5}(\tau)^4\eta_{50,10}(\tau)^2\eta_{50,15}(\tau)^3} + \frac{16\eta_{10,0}(\tau)^2\eta_{50,0}(\tau)\eta_{50,10}(\tau)^5}{\eta_{5,0}(\tau)^2\eta_{50,5}(\tau)^2\eta_{50,15}(\tau)^2\eta_{50,20}(\tau)\eta_{50,25}(\tau)}$$

$$- \frac{2\eta_{5,0}(\tau)\eta_{50,0}(\tau)\eta_{50,5}(\tau)^7\eta_{50,20}(\tau)^9\eta_{50,25}(\tau)^5}{\eta_{10,0}(\tau)\eta_{10,5}(\tau)^6\eta_{50,10}(\tau)^5\eta_{50,15}(\tau)^2} + \frac{4\eta_{10,0}(\tau)^2\eta_{50,20}(\tau)^5}{\eta_{5,0}(\tau)\eta_{50,10}(\tau)\eta_{50,15}(\tau)^2}$$

$$- \frac{8\eta_{10,0}(\tau)^2\eta_{50,0}(\tau)\eta_{50,10}(\tau)^5}{\eta_{5,0}(\tau)^2\eta_{50,5}(\tau)^3\eta_{50,15}(\tau)^2\eta_{50,20}(\tau)} + \frac{32\eta_{10,0}(\tau)\eta_{50,20}(\tau)^2}{\eta_{10,5}(\tau)^2\eta_{50,10}(\tau)^2\eta_{50,15}(\tau)^4\eta_{50,25}(\tau)}$$

$$- \frac{2\eta_{5,0}(\tau)\eta_{50,20}(\tau)^2\eta_{50,25}(\tau)^3}{\eta_{10,5}(\tau)^4\eta_{50,10}(\tau)^2\eta_{50,15}(\tau)^3} + \frac{4\eta_{5,0}(\tau)\eta_{10,0}(\tau)\eta_{50,20}(\tau)}{\eta_{50,0}(\tau)\eta_{10,5}(\tau)^2\eta_{50,5}(\tau)\eta_{50,10}(\tau)\eta_{50,15}(\tau)}$$

$$+ \frac{8\eta_{10,0}(\tau)^2\eta_{50,0}(\tau)\eta_{50,20}(\tau)^5\eta_{50,25}(\tau)}{\eta_{5,0}(\tau)^2\eta_{50,10}(\tau)\eta_{50,15}(\tau)^6} + \frac{32\eta_{5,0}(\tau)\eta_{50,5}(\tau)^3\eta_{50,20}(\tau)}{\eta_{10,5}(\tau)^4\eta_{50,10}(\tau)\eta_{50,15}(\tau)^2\eta_{50,25}(\tau)}$$

$$+ \frac{8\eta_{5,0}(\tau)\eta_{10,0}(\tau)\eta_{50,20}(\tau)}{\eta_{50,0}(\tau)\eta_{10,5}(\tau)^2\eta_{50,5}(\tau)\eta_{50,10}(\tau)\eta_{50,25}(\tau)} - \frac{4\eta_{50,0}(\tau)\eta_{50,5}(\tau)^3\eta_{50,20}(\tau)^5\eta_{50,25}(\tau)^4}{\eta_{10,5}(\tau)^4\eta_{50,10}(\tau)\eta_{50,15}(\tau)^2}$$

$$+ \frac{16\eta_{10,0}(\tau)\eta_{50,20}(\tau)}{\eta_{10,5}(\tau)^2\eta_{50,10}(\tau)\eta_{50,15}(\tau)^4\eta_{50,25}(\tau)} + \frac{32\eta_{10,0}(\tau)^2\eta_{50,0}(\tau)\eta_{50,10}(\tau)^2\eta_{50,20}(\tau)^2}{\eta_{5,0}(\tau)^2\eta_{50,5}(\tau)\eta_{50,15}(\tau)^2\eta_{50,25}(\tau)^2}$$

$$\left. - \frac{32\eta_{5,0}(\tau)\eta_{50,5}(\tau)^4\eta_{50,20}(\tau)^2}{\eta_{10,5}(\tau)^4\eta_{50,10}(\tau)^2\eta_{50,15}(\tau)^3\eta_{50,25}(\tau)}\right\}.$$

It can be verified that each term of the above identity is a modular function with respect to $\Gamma_1(50)$. Using the algorithm in [13], we can determine the constant B in (3.21) which is equal to -155. From Theorem 3.8, it suffices to verify the identity in the q-expansion past q^{155}. Thus, we complete the proof of Lemma 3.7. ∎

4 Proofs of Theorem 1.4 and Theorem 1.5

In this section, we give proofs of Theorem 1.4 and Theorem 1.5. A key ingredient of the proofs is the following theorem due to Liaw [24].

Theorem 4.1 (Liaw). *If p and r are positive integers with $p \geq 2$ and $r < p$, define*

$$\sum_{n=0}^{\infty} b_{p,r}(n)q^n := \frac{(q^p;q^p)_\infty}{(q^r;q^p)_\infty (q^{p-r};q^p)_\infty},$$

then $b_{p,r}(n) \geq 0$ for all n.

We are now in a position to show Theorem 1.4.

Proof of Theorem 1.4. (1) We first show (1.7) and (1.8). Comparing the coefficients of q^{3n} of (1.4) in Theorem 1.1, we find that

$$\sum_{n=0}^{\infty} (\overline{N}(0,6,3n) + \overline{N}(1,6,3n) - \overline{N}(2,6,3n) - \overline{N}(3,6,3n))q^n = \frac{J_6^3 J_{3,6}}{J_{1,6}^2 J_2}. \tag{4.1}$$

Using the identity (1.2) in [22, Theorem 1.1],

$$\sum_{n=0}^{\infty} (\overline{N}(0,3,3n) - \overline{N}(1,3,3n))q^n = -1 + \frac{(q^3;q^3)_\infty^2 (-q;q)_\infty}{(q;q)_\infty (-q^3;q^3)_\infty^2} = -1 + \frac{J_6^3 J_{3,6}}{J_{1,6}^2 J_2},$$

we deduce that for $n \geq 1$,

$$\begin{aligned} &\overline{N}(0,6,3n) + \overline{N}(1,6,3n) - \overline{N}(2,6,3n) - \overline{N}(3,6,3n) \\ &= \overline{N}(0,3,3n) - \overline{N}(1,3,3n). \end{aligned} \tag{4.2}$$

Observe that

$$\overline{N}(s,\ell,n) = \overline{N}(s,2\ell,n) + \overline{N}(\ell+s,2\ell,n) = \overline{N}(s,2\ell,n) + \overline{N}(\ell-s,2\ell,n), \tag{4.3}$$

so (4.2) is equivalent to

$$\begin{aligned} &\overline{N}(0,6,3n) + \overline{N}(1,6,3n) - \overline{N}(2,6,3n) - \overline{N}(3,6,3n) \\ &= \overline{N}(0,6,3n) + \overline{N}(3,6,3n) - \overline{N}(1,6,3n) - \overline{N}(2,6,3n), \end{aligned}$$

which implies that for $n \geq 1$

$$\overline{N}(1,6,3n) = \overline{N}(3,6,3n),$$

and hence (1.7) is obtained.

We turn to show (1.8). It suffices to show the coefficients of q^n on the right-hand side of (4.1) are nonnegative. First, observe that

$$\frac{J_6^3 J_{3,6}}{J_{1,6}^2 J_2} = \frac{(q^3;q^6)_\infty^2 (q^6;q^6)_\infty}{(q,q^5;q^6)_\infty^2 (q^2,q^4;q^6)_\infty} = \frac{(q^3;q^3)_\infty}{(q,q^2;q^3)_\infty} \frac{(q^3;q^6)_\infty}{(q,q^5;q^6)_\infty} = \frac{(q^3;q^3)_\infty}{(q,q^2;q^3)_\infty} \frac{(q^3;q^6)_\infty^2}{(q;q^2)_\infty},$$

and

$$\frac{(q^3;q^6)_\infty^2}{(q;q^2)_\infty} = \frac{(-q;q)_\infty}{(-q^3;q^3)_\infty^2} = \frac{(-q;q^3)_\infty(-q^2;q^3)_\infty}{(-q^3;q^3)_\infty} = \frac{1}{(q^6;q^6)_\infty} \sum_{n=-\infty}^{\infty} q^{(3n^2-n)/2},$$

where the last summation is by Jacobi's triple product [2]. Thus (4.1) can be written as

$$\begin{aligned}
&\sum_{n=0}^{\infty} (\overline{N}(0,6,3n) + \overline{N}(1,6,3n) - \overline{N}(2,6,3n) - \overline{N}(3,6,3n)) q^n \\
&\quad = \frac{(q^3;q^3)_\infty}{(q,q^2;q^3)_\infty} \frac{1}{(q^6;q^6)_\infty} \sum_{n=-\infty}^{\infty} q^{(3n^2-n)/2} \\
&\quad = \frac{1}{(q^6;q^6)_\infty} \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} \frac{q^{2m+(3n^2-n)/2}}{(q^{3m+1};q^3)_\infty (q^3;q^3)_m}.
\end{aligned}$$

The $m = n = 0$ term is

$$\frac{1}{(q;q^3)_\infty (q^6;q^6)_\infty},$$

which each term in the above identity has strictly positive coefficients. Then, we deduce that for $n \geq 0$

$$\overline{N}(0,6,3n) + \overline{N}(1,6,3n) - \overline{N}(2,6,3n) - \overline{N}(3,6,3n) > 0.$$

Together with (1.7), we obtain (1.8).

(2) We next show (1.9) and (1.10). Comparing the coefficients of q^{3n+1} of (1.4) in Theorem 1.1, we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty} (\overline{N}(0,6,3n+1) + \overline{N}(1,6,3n+1) - \overline{N}(2,6,3n+1) - \overline{N}(3,6,3n+1)) q^n \\
&\quad = \frac{2J_6^3}{J_{1,6} J_2}. \qquad (4.4)
\end{aligned}$$

By the identity (1.3) in [22, Theorem 1.1],

$$\sum_{n=0}^{\infty}(\overline{N}(0,3,3n+1)-\overline{N}(1,3,3n+1))q^n=\frac{2(q^3;q^3)_\infty(q^6;q^6)_\infty}{(q;q)_\infty}=\frac{2J_6^3}{J_{1,6}J_2},$$

we deduce that for $n\geq 0$,

$$\begin{aligned}&\overline{N}(0,6,3n+1)+\overline{N}(1,6,3n+1)-\overline{N}(2,6,3n+1)-\overline{N}(3,6,3n+1)\\&=\overline{N}(0,3,3n+1)-\overline{N}(1,3,3n+1).\end{aligned}\tag{4.5}$$

Using (4.3), we see that (4.5) is equivalent to

$$\begin{aligned}&\overline{N}(0,6,3n+1)+\overline{N}(1,6,3n+1)-\overline{N}(2,6,3n+1)-\overline{N}(3,6,3n+1)\\&=\overline{N}(0,6,3n+1)+\overline{N}(3,6,3n+1)-\overline{N}(1,6,3n+1)-\overline{N}(2,6,3n+1),\end{aligned}$$

which implies that for $n\geq 0$

$$\overline{N}(1,6,3n+1)=\overline{N}(3,6,3n+1),$$

so (1.9) is verified.

To show (1.10), by (1.9), we find that (4.4) can be written as

$$\sum_{n=0}^{\infty}(\overline{N}(0,6,3n+1)-\overline{N}(2,6,3n+1))q^n=\frac{2J_6^3}{J_{1,6}J_2}=\frac{(q^6;q^6)_\infty}{(q^2,q^4;q^6)_\infty}\frac{2}{(q,q^5;q^6)_\infty}.\tag{4.6}$$

From Theorem 4.1, we see that the coefficients of q^n in

$$\frac{(q^6;q^6)_\infty}{(q^2,q^4;q^6)_\infty}$$

is nonnegative for $n\geq 0$. In particular, when $n=0$, the coefficient is equal to one. Hence, from (4.6), we deduce that $\overline{N}(0,6,3n+1)>\overline{N}(2,6,3n+1)$, and so (1.10) is verified.

(3) To show (1.11), we first establish the generating function of $\overline{N}(0,6,3n+2)-\overline{N}(2,6,3n+2)$. We aim to show that

$$\begin{aligned}&\sum_{n=0}^{\infty}(\overline{N}(0,6,3n+2)-\overline{N}(2,6,3n+2))q^n\\&=-\frac{2(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{3n^2+6n+1}}{1-q^{6n+2}}.\end{aligned}\tag{4.7}$$

By Theorem 1.1, we derive

$$\sum_{n=0}^{\infty}(\overline{N}(0,6,3n+2)+\overline{N}(1,6,3n+2)-\overline{N}(2,6,3n+2)-\overline{N}(3,6,3n+2))q^n$$

$$= \frac{4J_6^3}{J_2J_{3,6}} - \frac{2}{J_{3,6}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n^2+3n}}{1+q^{3n+1}}. \tag{4.8}$$

Together with the identity (1.4) in [22]

$$\sum_{n=0}^{\infty} (\overline{N}(0,3,3n+2) - \overline{N}(1,3,3n+2))q^n$$
$$= \frac{4(-q^3;q^3)_\infty^2 (q^6;q^6)_\infty^2}{(q^2;q^2)_\infty} - \frac{6(-q^3;q^3)_\infty}{(q^3;q^3)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n^2+3n}}{1-q^{3n+1}}$$
$$= \frac{4J_6^3}{J_2J_{3,6}} - \frac{6}{J_{3,6}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n^2+3n}}{1-q^{3n+1}}, \tag{4.9}$$

and by (4.3), we obtain

$$\sum_{n=0}^{\infty} (\overline{N}(0,6,3n+2) - \overline{N}(2,6,3n+2))q^n$$
$$= \frac{4J_6^3}{J_2J_{3,6}} - \frac{3}{J_{3,6}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n^2+3n}}{1-q^{3n+1}} - \frac{1}{J_{3,6}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n^2+3n}}{1+q^{3n+1}}. \tag{4.10}$$

Then the identity (4.7) can be derived from (4.10) by using the following identity in [26, p.1],

$$\frac{J_1^3}{j(z;q)} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\binom{n+1}{2}}}{1-zq^n}. \tag{4.11}$$

Replacing $q \to q^6$ and setting $z = q^2$ in (4.11), we have

$$\frac{J_6^3}{J_2} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n^2+3n}}{1-q^{6n+2}}$$
$$= \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n^2+3n}}{1+q^{3n+1}} + \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n^2+3n}}{1-q^{3n+1}}. \tag{4.12}$$

Plug (4.12) into (4.10) to get

$$\sum_{n=0}^{\infty} (\overline{N}(0,6,3n+2) - \overline{N}(2,6,3n+2))q^n$$
$$= \frac{1}{J_{3,6}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n^2+3n}}{1+q^{3n+1}} - \frac{1}{J_{3,6}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n^2+3n}}{1-q^{3n+1}}$$
$$= -\frac{2(-q^3;q^3)_\infty}{(q^3;q^3)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n^2+6n+1}}{1-q^{6n+2}},$$

which is the desired generating function (4.7) of $\overline{N}(0,6,3n+2)-\overline{N}(2,6,3n+2)$.

We now consider the non-positivity of (4.7). Note that

$$\frac{(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{3n^2+6n+1}}{1-q^{6n+2}}$$

$$=\frac{(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}\left(\sum_{n=0}^{\infty}\frac{(-1)^nq^{3n^2+6n+1}}{1-q^{6n+2}}+\sum_{n=0}^{\infty}\frac{(-1)^nq^{3n^2+6n+2}}{1-q^{6n+4}}\right)$$

$$=\frac{(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}\left(\sum_{n=0}^{\infty}\frac{q^{12n^2+12n+1}}{1-q^{12n+2}}-\sum_{n=0}^{\infty}\frac{q^{12n^2+24n+10}}{1-q^{12n+8}}+\sum_{n=0}^{\infty}\frac{q^{12n^2+12n+2}}{1-q^{12n+4}}-\sum_{n=0}^{\infty}\frac{q^{12n^2+24n+11}}{1-q^{12n+10}}\right)$$

$$=\frac{(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}\sum_{n=0}^{\infty}\frac{q^{12n^2+12n+1}}{(1-q^{12n+2})(1-q^{12n+8})}\left((1-q^{12n+8})(1-q^{12n+9})+q^{24n+11}(1-q^6)\right)$$

$$+\frac{(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}\sum_{n=0}^{\infty}\frac{q^{12n^2+12n+2}}{(1-q^{12n+4})(1-q^{12n+10})}\left((1-q^{12n+9})(1-q^{12n+10})+q^{24n+13}(1-q^6)\right)$$

$$=\frac{(-q^3;q^3)_\infty}{(q^6;q^3)_\infty}\left(\sum_{n=0}^{\infty}\frac{q^{12n^2+12n+1}}{1-q^{12n+2}}\frac{1-q^{12n+9}}{1-q^3}+\sum_{n=0}^{\infty}\frac{q^{12n^2+36n+12}(1+q^3)}{(1-q^{12n+2})(1-q^{12n+8})}\right.$$

$$\left.+\sum_{n=0}^{\infty}\frac{q^{12n^2+12n+2}}{1-q^{12n+4}}\frac{1-q^{12n+9}}{1-q^3}+\sum_{n=0}^{\infty}\frac{q^{12n^2+36n+15}(1+q^3)}{(1-q^{12n+4})(1-q^{12n+10})}\right)$$

$$=\frac{(-q^3;q^3)_\infty}{(q^6;q^3)_\infty}\left(\sum_{n=0}^{\infty}\frac{q^{12n^2+12n+1}}{1-q^{12n+2}}\sum_{m=0}^{4n+2}q^{3m}+\sum_{n=0}^{\infty}\frac{q^{12n^2+36n+12}(1+q^3)}{(1-q^{12n+2})(1-q^{12n+8})}\right.$$

$$\left.+\sum_{n=0}^{\infty}\frac{q^{12n^2+12n+2}}{1-q^{12n+4}}\sum_{m=0}^{4n+2}q^{3m}+\sum_{n=0}^{\infty}\frac{q^{12n^2+36n+15}(1+q^3)}{(1-q^{12n+4})(1-q^{12n+10})}\right).$$

It is easy to check that each term in the above identity has positive coefficients. It follows from (4.7) that $\overline{N}(0,6,3n+2)<\overline{N}(2,6,3n+2)$, and so (1.11) is verified.

We finish the proof of Theorem 1.4 by showing (1.12). To this end, we aim to show that for $n\geq 0$

$$\overline{N}(0,6,3n+2)+\overline{N}(1,6,3n+2)>\overline{N}(2,6,3n+2)+\overline{N}(3,6,3n+2). \tag{4.13}$$

Then the inequality (1.12) can be derived by subtracting (1.11) from (4.13).

Combining (4.8) and (4.12), we find that

$$\sum_{n=0}^{\infty}(\overline{N}(0,6,3n+2)+\overline{N}(1,6,3n+2)-\overline{N}(2,6,3n+2)-\overline{N}(3,6,3n+2))q^n$$

$$=\frac{2(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{3n^2+3n}}{1-q^{3n+1}}. \tag{4.14}$$

We now investigate the nonnegativity of (4.14). Observe that

$$\begin{aligned}
&\frac{(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3n^2+3n}}{1-q^{3n+1}}\\
&=\frac{(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}\left(\sum_{n=-\infty}^{\infty}\frac{q^{12n^2+6n}}{1-q^{6n+1}}-\sum_{n=-\infty}^{\infty}\frac{q^{12n^2+18n+6}}{1-q^{6n+4}}\right)\\
&=\frac{(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}\left(\sum_{n=0}^{\infty}\frac{q^{12n^2+6n}}{1-q^{6n+1}}-\sum_{n=0}^{\infty}\frac{q^{12n^2+24n+11}}{1-q^{6n+5}}-\sum_{n=0}^{\infty}\frac{q^{12n^2+18n+6}}{1-q^{6n+4}}+\sum_{n=0}^{\infty}\frac{q^{12n^2+12n+2}}{1-q^{6n+2}}\right)\\
&=\frac{(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}\left(\sum_{n=0}^{\infty}\frac{q^{12n^2+6n}}{(1-q^{6n+1})(1-q^{6n+4})}((1-q^{6n+4})(1-q^{12n+6})+q^{18n+7}(1-q^3))\right.\\
&\qquad\left.+\sum_{n=0}^{\infty}\frac{q^{12n^2+12n+2}}{(1-q^{6n+2})(1-q^{6n+5})}((1-q^{6n+5})(1-q^{12n+9})+q^{18n+11}(1-q^3))\right)\\
&=\frac{(-q^3;q^3)_\infty}{(q^6;q^3)_\infty}\left(\sum_{n=0}^{\infty}\frac{q^{12n^2+6n}}{1-q^{6n+1}}\sum_{m=0}^{4n+1}q^{3m}+\sum_{n=0}^{\infty}\frac{q^{12n^2+24n+7}}{(1-q^{6n+1})(1-q^{6n+4})}\right)\\
&\quad+\frac{(-q^3;q^3)_\infty}{(q^6;q^3)_\infty}\left(\sum_{n=0}^{\infty}\frac{q^{12n^2+12n+2}}{1-q^{6n+2}}\sum_{m=0}^{4n+2}q^{3m}+\sum_{n=0}^{\infty}\frac{q^{12n^2+30n+13}}{(1-q^{6n+2})(1-q^{6n+5})}\right).
\end{aligned}$$

It is easy to see that in the first series, from $n=0$ and $m=0$ we get the term $1/(1-q)$ which gives strictly positive coefficients of q^n. Hence we derive (4.13) and so (1.12) is proved. Thus we complete the proof of Theorem 1.4. ∎

We conclude this section by showing Theorem 1.5.

Proof of Theorems 1.5. From Theorem 1.2, we derive that

$$\begin{aligned}
&\sum_{n=0}^{\infty}(\overline{N}(0,10,5n)+\overline{N}(1,10,5n)-\overline{N}(4,10,5n)-\overline{N}(5,10,5n))q^n\\
&=\frac{J_{2,10}^2J_{3,10}^2J_{5,10}^4}{J_{1,2}^3J_{4,10}J_{10}^3}+\frac{8q^2J_{1,10}J_{10}^3}{J_{1,2}^2J_{4,10}}\\
&=\frac{(q^5;q^5)_\infty^2}{(q,q^4;q^5)_\infty^2}\frac{1}{(q^{10};q^{10})_\infty(q,q^9;q^{10})_\infty^2(q,q^4;q^5)_\infty^2(q^2,q^3;q^5)_\infty(q^3,q^7;q^{10})_\infty^3}\\
&\quad+\frac{(q^{10};q^{10})_\infty}{(q^3,q^7;q^{10})_\infty}\frac{8q^2}{(q,q^9;q^{10})_\infty^3(q^2,q^8;q^{10})_\infty^2(q^3,q^7;q^{10})_\infty^3(q^4,q^6;q^{10})_\infty^3(q^5;q^{10})_\infty^4}.
\end{aligned}$$

By Theorem 4.1, we see that the coefficients of q^n in

$$\frac{(q^5;q^5)_\infty^2}{(q,q^4;q^5)_\infty^2}\quad\text{and}\quad\frac{(q^{10};q^{10})_\infty}{(q^3,q^7;q^{10})_\infty}$$

are nonnegative for $n \geq 0$ respectively. Especially, the constant terms are equal to one. Hence the inequality (1.13) follows. This completes the proof of Theorem 1.5. ∎

5 Proofs of Theorem 1.8 and Theorem 1.9

This section is devoted to showing the relations between the rank differences of overpartitions and mock theta functions stated in Theorem 1.8 and Theorem 1.9. To this end, we express the third order mock theta functions $\omega(q)$ and $\rho(q)$ and the tenth order mock theta functions $\phi(q)$ and $\psi(q)$ in terms of the Appell-Lerch sum $m(x,q,z)$. Recall that the Appell-Lerch sum is defined as

$$m(x,q,z) := \frac{1}{j(z;q)} \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{\binom{r}{2}} z^r}{1-q^{r-1}xz}, \tag{5.1}$$

where $x, z \in \mathbb{C}^*$ with neither z nor xz an integral power of q.

Hickerson and Mortenson [17] showed that

$$\begin{aligned} \omega(q) &= -2q^{-1}m(q,q^6,q^2) + \frac{J_6^3}{J_2 J_{3,6}}, &(5.2)\\ \rho(q) &= q^{-1}m(q,q^6,-q), &(5.3)\\ \phi(q) &= -2q^{-1}m(q,q^{10},q^2) + \frac{J_5 J_{10} J_{4,10}}{J_{2,5} J_{2,10}}, &(5.4)\\ \psi(q) &= -2m(q^3,q^{10},q) - \frac{qJ_5 J_{10} J_{2,10}}{J_{1,5} J_{4,10}}. &(5.5)\end{aligned}$$

The universal mock theta function $g_2(x,q)$ is defined by [14]

$$g_2(x,q) := \frac{1}{J_{1,2}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)}}{1-xq^n}.$$

Hickerson and Mortenson [17] showed $g_2(x,q)$ and $m(x,q,z)$ have the following relation,

$$g_2(x,q) = -x^{-1}m(x^{-2}q,q^2,x). \tag{5.6}$$

We are ready to show Theorem 1.8 and Theorem 1.9. Let us begin with Theorem 1.8.

Proof of Theorem 1.8. From Theorem 1.1, we have

$$\begin{aligned}&\sum_{n=0}^{\infty}(\overline{N}(0,6,3n+2)+\overline{N}(1,6,3n+2)-\overline{N}(2,6,3n+2)-\overline{N}(3,6,3n+2))q^n\\ &= \frac{4J_6^3}{J_2J_{3,6}} - \frac{2}{J_{3,6}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{3n^2+3n}}{1+q^{3n+1}} = \frac{4J_6^3}{J_2J_{3,6}} - 2g_2(-q,q^3). \end{aligned} \tag{5.7}$$

Replacing q by q^3 in (5.6) and setting $x=-q$, we have

$$g_2(-q,q^3)=q^{-1}m(q,q^6,-q), \tag{5.8}$$

and by (5.3), we deduce that

$$\rho(q)=g_2(-q,q^3). \tag{5.9}$$

Using the identity [29]

$$\omega(q)+2\rho(q)=\frac{3J_6^3}{J_2J_{3,6}}, \tag{5.10}$$

hence (1.16) follows upon substituting (5.9) and (5.10) into (5.7). Thus we complete the proof of Theorem 1.8. ∎

We finish this paper with the proof of Theorem 1.9.

Proof of Theorem 1.9. (1) We first show (1.17). By Theorem 1.2, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}(\overline{N}(0,10,5n+1)+\overline{N}(1,10,5n+1)-\overline{N}(4,10,5n+1)-\overline{N}(5,10,5n+1))q^n\\
&=\frac{2J_{4,10}J_{5,10}^4J_{10}^3}{J_{1,2}^4J_{2,10}^2J_{3,10}}-8q\frac{J_{1,10}^4J_{4,10}^3J_{5,10}^4}{J_{1,2}^5J_{2,10}^2J_{10}^3}-16q^3\frac{J_{2,10}^4J_{3,10}^2J_{5,10}J_{10}^3}{J_{1,2}^4J_{4,10}^5}\\
&\qquad+\frac{2q}{J_{5,10}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{5n^2+5n}}{1+q^{5n+2}}.
\end{aligned} \tag{5.11}$$

We aim to show that

$$\frac{1}{J_{5,10}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{5n^2+5n}}{1+q^{5n+2}}=-\frac{1}{2}q^{-1}\phi(q)+\frac{J_5J_{10}J_{4,10}}{2qJ_{2,5}J_{2,10}}+\frac{J_{10}^3\overline{J}_{0,10}\overline{J}_{5,10}}{J_{2,10}\overline{J}_{2,10}J_{3,10}\overline{J}_{3,10}}. \tag{5.12}$$

Then the identity (1.17) can be derived by plugging (5.12) into (5.11), namely

$$\begin{aligned}
&\sum_{n=0}^{\infty}(\overline{N}(0,10,5n+1)+\overline{N}(1,10,5n+1)-\overline{N}(4,10,5n+1)-\overline{N}(5,10,5n+1))q^n\\
&=-\phi(q)+M_1(q),
\end{aligned}$$

where

$$\begin{aligned}
M_1(q)=&\frac{J_5J_{10}J_{4,10}}{J_{2,5}J_{2,10}}+2q\frac{J_{10}^3\overline{J}_{0,10}\overline{J}_{5,10}}{J_{2,10}\overline{J}_{2,10}J_{3,10}\overline{J}_{3,10}}+\frac{2J_{4,10}J_{5,10}^4J_{10}^3}{J_{1,2}^4J_{2,10}^2J_{3,10}}\\
&-8q\frac{J_{1,10}^4J_{4,10}^3J_{5,10}^4}{J_{1,2}^5J_{2,10}^2J_{10}^3}-16q^3\frac{J_{2,10}^4J_{3,10}^2J_{5,10}J_{10}^3}{J_{1,2}^4J_{4,10}^5}.
\end{aligned}$$

By the definition of $g_2(x,q)$, we find that

$$\frac{1}{J_{5,10}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{5n^2+5n}}{1+q^{5n+2}}=g_2(-q^2,q^5). \tag{5.13}$$

Letting $q \to q^5$ and setting $x = -q^2$ in (5.6) yields

$$g_2(-q^2, q^5) = q^{-2} m(q, q^{10}, -q^2). \tag{5.14}$$

Replacing q by q^{10}, putting $z_1 = -q^2$, $z_0 = q^2$ and $x = q$ in (**??**), it follows that

$$m(q, q^{10}, -q^2) - m(q, q^{10}, q^2) = \frac{q^2 J_{10}^3 \overline{J}_{0,10} \overline{J}_{5,10}}{J_{2,10} \overline{J}_{2,10} J_{3,10} \overline{J}_{3,10}}. \tag{5.15}$$

Substituting (5.15) into (5.14) and by (5.4), we derive that

$$g_2(-q^2, q^5) = -\frac{1}{2} q^{-1} \phi(q) + \frac{J_5 J_{10} J_{4,10}}{2q J_{2,5} J_{2,10}} + \frac{J_{10}^3 \overline{J}_{0,10} \overline{J}_{5,10}}{J_{2,10} \overline{J}_{2,10} J_{3,10} \overline{J}_{3,10}}. \tag{5.16}$$

Hence, combining (5.16) and (5.13), we obtain (5.12). This completes the proof of (1.17).

(2) Analogue to the above process, we will show (1.18). By Theorem 1.3, we have

$$\begin{aligned} &\sum_{n=0}^{\infty} (\overline{N}(1, 10, 5n+1) + \overline{N}(2, 10, 5n+1) - \overline{N}(3, 10, 5n+1) - \overline{N}(4, 10, 5n+1)) q^n \\ &= \frac{8q J_1^8 J_{10}^7 J_{5,10}}{J_{1,2}^6 J_{2,10}^4 J_{4,10}^5} - \frac{8q^2 J_{10}^3 J_{1,10}^2 J_{5,10}^2}{J_{1,2}^4 J_{2,10} J_{3,10}} + \frac{16 q^3 J_{10}^6 J_{2,10}^6}{J_1^6 J_{1,10}^2 J_{4,10}^3} - \frac{2q}{J_{5,10}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{5n^2+5n}}{1+q^{5n+2}}. \end{aligned} \tag{5.17}$$

Then the desired identity (1.18) can be immediately obtained when substituting (5.12) into (5.17), namely,

$$\begin{aligned} &\sum_{n=0}^{\infty} (\overline{N}(1, 10, 5n+1) + \overline{N}(2, 10, 5n+1) - \overline{N}(3, 10, 5n+1) - \overline{N}(4, 10, 5n+1)) q^n \\ &= \phi(q) + M_2(q), \end{aligned}$$

where

$$\begin{aligned} M_2(q) = &-\frac{J_5 J_{10} J_{4,10}}{J_{2,5} J_{2,10}} - 2q \frac{J_{10}^3 \overline{J}_{0,10} \overline{J}_{5,10}}{J_{2,10} \overline{J}_{2,10} J_{3,10} \overline{J}_{3,10}} + \frac{8q J_1^8 J_{10}^7 J_{5,10}}{J_{1,2}^6 J_{2,10}^4 J_{4,10}^5} \\ &- \frac{8q^2 J_{10}^3 J_{1,10}^2 J_{5,10}^2}{J_{1,2}^4 J_{2,10} J_{3,10}} + \frac{16 q^3 J_{10}^6 J_{2,10}^6}{J_1^6 J_{1,10}^2 J_{4,10}^3}. \end{aligned}$$

(3) Finally, we show (1.19). Similarly, using Theorem 1.3, we derive

$$\begin{aligned} &\sum_{n=0}^{\infty} (\overline{N}(1, 10, 5n+4) + \overline{N}(2, 10, 5n+4) - \overline{N}(3, 10, 5n+4) - \overline{N}(4, 10, 5n+4)) q^n \\ &= \frac{8 J_{10}^3 J_{3,10}^2}{J_{1,2}^2 J_{1,10} J_{4,10}} - \frac{4 J_{1,10}^3 J_{4,10}^3 J_{5,10}^5}{J_1^2 J_{1,2}^4 J_{10}^4} + \frac{16 q^2 J_{10}^8}{J_1^2 J_{1,2}^2 J_{3,10}^2 J_{4,10}} + \frac{32 q^2 J_{10}^6 J_{2,10}^3}{J_1^6 J_{1,10} J_{5,10}} - \frac{32 q^3 J_{10}^3 J_{1,10}^4}{J_{1,2}^4 J_{2,10} J_{3,10}} \end{aligned}$$

$$-\frac{2}{J_{5,10}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{5n^2+5n}}{1+q^{5n+4}}. \tag{5.18}$$

To prove (1.19), it is necessary to show that

$$\frac{1}{J_{5,10}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{5n^2+5n}}{1+q^{5n+4}} = -\frac{1}{2}q^{-1}\psi(q) - \frac{J_5J_{10}J_{2,10}}{2J_{1,5}J_{4,10}} + \frac{J_{10}^3\overline{J}_{0,10}\overline{J}_{5,10}}{J_{1,10}\overline{J}_{1,10}J_{4,10}\overline{J}_{4,10}}. \tag{5.19}$$

We then obtain (1.19) upon substituting (5.19) into (5.18), that is,

$$\sum_{n=0}^{\infty}(\overline{N}(1,10,5n+4)+\overline{N}(2,10,5n+4)-\overline{N}(3,10,5n+4)-\overline{N}(4,10,5n+4))q^n$$
$$= q^{-1}\psi(q) + M_3(q),$$

where

$$M_3(q) = \frac{J_5J_{10}J_{2,10}}{J_{1,5}J_{4,10}} - \frac{2J_{10}^3\overline{J}_{0,10}\overline{J}_{5,10}}{J_{1,10}\overline{J}_{1,10}J_{4,10}\overline{J}_{4,10}} + \frac{8J_{10}^3J_{3,10}^2}{J_{1,2}^2J_{1,10}J_{4,10}} - \frac{4J_{1,10}^3J_{4,10}^3J_{5,10}^5}{J_1^2J_{1,2}^4J_{10}^4}$$
$$+ \frac{16q^2J_{10}^8}{J_1^2J_{1,2}^2J_{3,10}^2J_{4,10}} + \frac{32q^2J_{10}^6J_{2,10}^3}{J_1^6J_{1,10}J_{5,10}} - \frac{32q^3J_{10}^3J_{1,10}^4}{J_{1,2}^4J_{2,10}J_{3,10}}.$$

From the definition of $g_2(x,q)$, we note that

$$\frac{1}{J_{5,10}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{5n^2+5n}}{1+q^{5n+4}} = g_2(-q^4,q^5). \tag{5.20}$$

Letting $q\to q^5$ and setting $x=-q^4$ in (5.6) yields

$$g_2(-q^4,q^5) = q^{-4}m(q^{-3},q^{10},-q^4). \tag{5.21}$$

On the other hand, we need the following relation

$$m(q^3,q^{10},-q^{-4}) = q^{-3}m(q^{-3},q^{10},-q^4), \tag{5.22}$$

which can be derived from the following identity in [30] by letting $q\to q^{10}$, $x=q^3$ and $z=-q^{-4}$,

$$m(x,q,z) = x^{-1}m(x^{-1},q,z^{-1}).$$

Combining (5.21) and (5.22) yields

$$g_2(-q^4,q^5) = q^{-1}m(q^3,q^{10},-q^{-4}). \tag{5.23}$$

Replacing q by q^{10}, putting $z_1=-q^{-4}$, $z_0=q$ and $x=q^3$ in (**??**), it follows that

$$m(q^3,q^{10},-q^{-4}) - m(q^3,q^{10},q) = \frac{qJ_{10}^3\overline{J}_{0,10}\overline{J}_{5,10}}{J_{1,10}\overline{J}_{1,10}J_{4,10}\overline{J}_{4,10}}. \tag{5.24}$$

Substituting (5.24) into (5.23) and by (5.5), we derive that

$$g_2(-q^4,q^5) = -\frac{1}{2}q^{-1}\psi(q) - \frac{J_5J_{10}J_{2,10}}{2J_{1,5}J_{4,10}} + \frac{J_{10}^3\overline{J}_{0,10}\overline{J}_{5,10}}{J_{1,10}\overline{J}_{1,10}J_{4,10}\overline{J}_{4,10}}. \tag{5.25}$$

Thus (5.19) follows from (5.20) and (5.25). This completes the proof of Theorem 1.9. ∎

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