Spectral analogues of Moon-Moser's theorem on Hamilton paths in bipartite graphs

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Abstract: In 1962, Erdős proved a theorem on the existence of Hamilton cycles in graphs with given minimum degree and number of edges. Significantly strengthening in case of balanced bipartite graphs, Moon and Moser proved a corresponding theorem in 1963. In this paper we establish several spectral analogues of Moon and Moser's theorem on Hamilton paths in balanced bipartite graphs and nearly balanced bipartite graphs. One main ingredient of our proofs is a structural result of its own interest, involving Hamilton paths in balanced bipartite graphs with given minimum degree and number of edges.

Keywords: Spectral analogues; Moon-Moser's theorem; Hamilton path; Balanced bipartite graphs; Nearly balanced bipartite graphs

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1 Introduction

This is a sequel to our previous paper [12]. In this paper, we are interested in establishing tight spectral sufficient conditions for Hamilton paths in balanced bipartite graphs and nearly balanced bipartite graphs. Throughout this paper, a bipartite graph with the bipartition $\{X, Y\}$ is called *balanced* if $|X| = |Y|$; and is called *nearly balanced* if $|X| - |Y| = 1$ (by the symmetry). A graph G is called *Hamiltonian* if it contains a spanning cycle, and is called *traceable* if it contains a spanning path.

The topic of Hamiltonicity of graphs has a long history. In 1961, Ore [24] proved that every graph on n vertices has a Hamilton cycle if $e(G) > \binom{n-1}{2} + 1$. One year later, Erdős [6] generalized Ore's theorem by introducing the minimum degree of a graph as a new parameter. More precisely, Erdős proved that

Theorem 1.1 (Erdős [6])**.** *Let G be a graph on n vertices, with minimum degree $\delta(G)$. If $n/2 > \delta(G) \geq k \geq 1$, and*

$$e(G) > \max\left\{\binom{n-k}{2} + k^2, \binom{n - \lfloor \frac{n-1}{2} \rfloor}{2} + \left\lfloor \frac{n-1}{2} \right\rfloor^2\right\},$$

then G is Hamiltonian.

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Motivated by Erdős' work [6], Moon and Moser [16] presented some corresponding results for balanced bipartite graphs. We state one of their theorems as follows, which is the starting point of our present paper.

Theorem 1.2 (Moon and Moser [16])**.** *Let G be a balanced bipartite graph on $2n$ vertices, with minimum degree $\delta(G) \geq k$, where $1 \leq k \leq n/2$. If*

$$e(G) > \max\left\{n(n-k)+k^2, n\left(n-\left\lfloor\frac{n}{2}\right\rfloor\right)+\left\lfloor\frac{n}{2}\right\rfloor^2\right\},$$

then G is Hamiltonian.

Compared with the number of edges of graphs, eigenvalues of graphs are also very powerful for describing the structure of graphs. There are several well known examples, such as the spectral proof of Friendship Theorem [7]. For Hamiltonicity of graphs, early pioneer work include those of van den Heuvel [10], Krivelevich and Sudakov [11], Butler and Chung [5], etc.

Recently, spectral extremal graph theory has rapidly developed, where extremal properties of graphs are studied by means of eigenvalues of associated matrices of graphs. In this area, many beautiful and deep results have been proved, such as a spectral Turán theorem [17], a spectral Erdős-Stone-Bollobás theorem [18], a spectral version of Zarankiewicz problem [19], spectral sufficient conditions for paths and cycles [20, 25, 26], etc. For an excellent survey on recent development of spectral extremal graph theory, we refer the reader to Nikiforov [21]. In particular, on the topic of Hamiltonicity, Fielder and Nikiforov [9] gave spectral analogues of Ore's theorem [24]. More work in this vein can be found in Zhou [27], Lu, Liu and Tian [14], as well as Liu, Shiu and Xue [13]. Towards finding spectral analogues of Erdős' theorem, the first attempt was made by the second author and Ge [22], and finally was completed by the present authors in [12]. In the meantime, the authors obtained some spectral analogues of Moon-Moser's theorem for Hamilton cycles in balanced bipartite graphs [12].

One may look for spectral conditions for Hamilton paths in bipartite graphs. The situation seems a little more complicated. The main reason is that every traceable bipartite graph must be balanced, or nearly balanced. That is, there are two situations for us to explore.

In fact, we need to consider the following two Brualdi-Solheid-Turán-type problems. Here we use $\widehat{G}$ to denote the *quasi-complement* of a bipartite graph G with the bipartition $\{X, Y\}$, i.e., one with vertex set $V(\widehat{G}) = V(G)$ and for any $x \in X$ and $y \in Y$, $xy \in E(\widehat{G})$ if and only if $xy \notin E(G)$; and we use $\rho(G)$ and $q(G)$ to denote the spectral radius and signless Laplacian spectral radius of G, respectively.

Problem 1. Among all non-traceable balanced bipartite graphs G on $2n$ vertices, with $\delta(G) \geq k$, determine $\max \rho(G), \min \rho(\widehat{G}), \max q(G)$ and $\min q(\widehat{G})$, respectively.

Problem 2. Among all non-traceable nearly balanced bipartite graphs G on $2n-1$ vertices, with $\delta(G) \geq k$, determine $\max \rho(G), \min \rho(\widehat{G}), \max q(G)$ and $\min q(\widehat{G})$, respectively.

In this paper, we solve the above problems for graphs of sufficiently large order. The main theorems and related notation are given in Section 2.

In order to solve these problems, we need to use several spectral inequalities and convert the original problems into new ones involving the number of edges. We also use spectral inequalities to characterize the extremal graphs. In particular, we prove spectral inequalities to compare the (signless Laplacian) spectral radii of certain types of graphs. These are given in Section 3.

The proofs of our main theorems also need detailed structural analysis. We need to use the closure theory of Hamilton cycles in balanced bipartite graph due to Bondy and Chvatál [4]. With the help of this theory, we need to use an analogous theorem for Hamilton paths in balanced bipartite graphs. We establish a theorem on the existence of a complete bipartite subgraph with large order in a balanced bipartite graph with sufficiently many edges. We also prove a theorem on the existence of Hamilton paths in a balanced bipartite graph with given number of edges. All these structural lemmas and proofs are given in Section 4.

In Section 5, we prove our main theorems. Finally, in Section 6, we conclude the paper with some remarks and problems.

2 Main theorems

2.1 Notation

To describe all extremal graphs in our coming theorems, we introduce some terminology and notation. We use $\mathcal{G}_{m,n}$ to denote the set of bipartite graphs with partition sets of sizes m and n. As usual, $K_{m,n}$ denotes the complete bipartite graph, and we set $\Phi_{m,n} = \widehat{K_{m,n}}$. In this paper, when we mention a bipartite graph, we always fix its partition sets, e.g., $\Phi_{m,n}$ and $\Phi_{n,m}$ are considered as different bipartite graphs, unless $m = n$ (although they are both the empty graphs of order $m + n$).

Let G_1, G_2 be two bipartite graphs, with the bipartition $\{X_1, Y_1\}$ and $\{X_2, Y_2\}$, respectively. We use $G_1 \sqcup G_2$ to denote the graph obtained from $G_1 \cup G_2$ by adding all possible edges between X_1 and Y_2 and all possible edges between Y_1 and X_2. We set

$$B_n^k = K_{k,n-k} \sqcup \Phi_{n-k,k} \text{ and } \mathcal{B}_n^k = \{H \sqcup \Phi_{n-k,k} : H \in \mathcal{G}_{k,n-k}\} \ (1 \le k \le n/2).$$

The graphs B_n^k play a crucial role in the proofs of results in [12]. Notice that B_n^k is the graph in $\mathcal{B}_n^k$ with the largest number of edges. We remark that for any (spanning) subgraph G of B_n^k, $\rho(\widehat{G}) = \rho(\widehat{B_n^k})$ ($q(\widehat{G}) = q(\widehat{B_n^k})$) if and only if $G \in \mathcal{B}_n^k$.

We define some classes of graphs as follows:

$$\begin{aligned}
Q_n^k &= K_{k,n-k-1} \sqcup \Phi_{n-k,k+1} && (0 \le k \le (n-1)/2),\\
R_n^k &= K_{k,k} \cup K_{n-k,n-k} && (1 \le k \le n/2),\\
S_n^k &= K_{k,n-k-1} \sqcup \Phi_{n-k,k} && (1 \le k \le (n-1)/2),\\
\mathcal{S}_n^k &= \{H \sqcup \Phi_{k,n-k} : H \in \mathcal{G}_{n-k-1,k}\} && (1 \le k \le (n-1)/2),\\
T_n^k &= K_{k,n-k-1} \sqcup \Phi_{n-k-1,k+1} && (0 \le k \le n/2-1),\\
\mathcal{T}_n^k &= \{H \sqcup \Phi_{k+1,n-k-1} : H \in \mathcal{G}_{n-k-1,k}\} && (0 \le k \le n/2-1).
\end{aligned}$$

Additionally, let $\Gamma_n^0 = K_{n-2,n} \cup K_{1,0}$ and let L be the graph in Fig. 1.

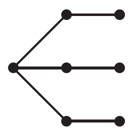


Fig. 1. The graph L.

Note that S_n^k is the graph in $\mathcal{S}_n^k$ with the largest number of edges, and T_n^k is the graph in $\mathcal{T}_n^k$ with the largest number of edges. Similarly, we remark that for any (spanning) subgraph G of S_n^k, $\rho(\widehat{G}) = \rho(\widehat{S_n^k})$ $(q(\widehat{G}) = q(\widehat{S_n^k}))$ if and only if $G \in \mathcal{S}_n^k$; and for any (spanning) subgraph G of T_n^k, $\rho(\widehat{G}) = \rho(\widehat{T_n^k})$ $(q(\widehat{G}) = q(\widehat{T_n^k}))$ if and only if $G \in \mathcal{T}_n^k$.

2.2 Main results

In this subsection, we state all our main theorems. Since we consider the classes of balanced bipartite graphs and nearly balanced bipartite graphs, and for each class of graphs, we consider sufficient conditions in terms of (signless Laplacian) spectral radii of graphs or the complements, we obtain eight theorems as follows.

For balanced bipartite graphs, we have

Theorem 2.1. *Let G be a balanced bipartite graph on $2n$ vertices, with minimum degree $\delta(G) \geq k$, where $k \geq 0$ and $n \geq (k+2)^2$.*
(1) If $k \neq 1$ and $\rho(G) \geq \rho(Q_n^k)$, then G is traceable unless $G = Q_n^k$.
(2) If $k = 1$ and $\rho(G) \geq \rho(R_n^1)$, then G is traceable unless $G = R_n^1$.

Theorem 2.2. *Let G be a balanced bipartite graph on $2n$ vertices, with minimum degree $\delta(G) \geq k$, where $k \geq 0$ and $n \geq (k+2)^2$. If $q(G) \geq q(Q_n^k)$, then G is traceable unless $G = Q_n^k$.*

Theorem 2.3. *Let G be a balanced bipartite graph on $2n$ vertices, with minimum degree $\delta(G) \geq k$, where $k \geq 0$ and $n \geq 2k$.*
(1) If $k \geq 1$ and $\rho(\widehat{G}) \leq \rho(\widehat{R_n^k})$, then G is traceable unless $G = R_n^k$.
(2) If $k = 0$ and $\rho(\widehat{G}) \geq \rho(\widehat{Q_n^0})$, then G is traceable unless $G = Q_n^0$.

Theorem 2.4. *Let G be a balanced bipartite graph on $2n$ vertices. If $q(\widehat{G}) \leq n$, then G is traceable unless $G \in \{R_n^k : 1 \leq k \leq \lfloor n/2 \rfloor\}$.*

Remark 1. Our Theorem 2.1 generalizes Theorem 2.10 in [13] due to Liu et al.

For nearly balanced bipartite graphs, we have

Theorem 2.5. *Let G be a nearly balanced bipartite graph on $2n-1$ vertices, with minimum degree $\delta(G) \geq k$, where $k \geq 0$ and $n \geq (k+1)^2$.*
(1) If $k \geq 1$ and $\rho(G) \geq \rho(S_n^k)$, then G is traceable unless $G = S_n^k$.
(2) If $k = 0$ and $\rho(G) \geq \rho(T_n^0)$, then G is traceable unless $G = T_n^0$.

Theorem 2.6. *Let G be a nearly balanced bipartite graph on $2n-1$ vertices, with minimum degree $\delta(G) \geq k$, where $k \geq 0$ and $n \geq (k+1)^2$.*
(1) If $k \geq 1$ and $q(G) \geq q(S_n^k)$, then G is traceable unless $G = S_n^k$.
(2) If $k = 0$ and $q(G) \geq q(S_n^1)$, then G is traceable unless $G = S_n^1$.

Theorem 2.7. *Let G be a nearly balanced bipartite graph on $2n-1$ vertices, with minimum degree $\delta(G) \geq k$, where $k \geq 0$ and $n \geq 2k+1$.*
(1) If $k \geq 1$ and $\rho(\widehat{G}) \leq \rho(\widehat{S_n^k})$, then G is traceable unless $G \in \mathcal{S}_n^k$.
(2) If $k = 0$ and $\rho(\widehat{G}) \leq \rho(\widehat{T_n^0})$, then G is traceable unless $G \in \mathcal{S}_n^1 \cup \{T_n^0\}$.

Theorem 2.8. *Let G be a nearly balanced bipartite graph on $2n-1$ vertices. If $q(\widehat{G}) \leq n$, then G is traceable unless $G \in (\bigcup_{k=1}^{\lfloor (n-1)/2 \rfloor} \mathcal{S}_n^k) \cup (\bigcup_{k=0}^{\lfloor n/2 \rfloor - 1} \mathcal{T}_n^k)$, or $n = 4$ and $G = L$.*

3 Spectral inequalities

We will use the following spectral inequalities for graphs and bipartite graphs, respectively. The first theorem is a direct corollary of a result of Nosal [23]. (See also [3].)

Theorem 3.1 (Nosal [23], Bhattacharya, Friedland and Peled [3])**.** *Let G be a bipartite graph. Then*

$$\rho(G) \le \sqrt{e(G)}.$$

The next theorem has been proved in [12], with the help of a result due to Feng and Yu [8, Lemma 2.4], which can be traced back to Merris [15].

Theorem 3.2 (Li and Ning [12])**.** *Let G be a balanced bipartite graph on $2n$ vertices. Then*

$$q(G) \le \frac{e(G)}{n} + n.$$

The following two theorems can be proved similarly as Lemma 2.1 in [2] and Theorem 2 in [1], respectively. We omit the proofs.

Theorem 3.3. *Let G be a graph with non-empty edge set. Then*

$$\rho(G) \ge \min\{\sqrt{d(u)d(v)} : uv \in E(G)\}.$$

Moreover, if G is connected, then equality holds if and only if G is regular or semi-regular bipartite.

Theorem 3.4. *Let G be a graph with non-empty edge set. Then*

$$q(G) \ge \min\{d(u) + d(v) : uv \in E(G)\}.$$

Moreover, if G is connected, then the equality holds if and only if G is regular or semi-regular bipartite.

Lemma 1. *For $k \ge 1$, $n \ge 2k+1$, we have*

$\rho(Q_n^k) > \rho(K_{n,n-k-1}) = \sqrt{n(n-k-1)}$; $\rho(S_n^k) > \rho(K_{n,n-k-1}) = \sqrt{n(n-k-1)}$;
$q(Q_n^k) > q(K_{n,n-k-1}) = 2n-k-1$; $q(S_n^k) > q(K_{n,n-k-1}) = 2n-k-1$;
$\rho(\widehat{R_n^k}) = \rho(K_{k,n-k}) = \sqrt{k(n-k)}$; $\rho(\widehat{S_n^k}) = \rho(K_{n-k,k}) = \sqrt{k(n-k)}$;
$q(\widehat{R_n^k}) = q(K_{k,n-k}) = n$; $q(\widehat{S_n^k}) = q(K_{n-k,k}) = q(\widehat{T_n^k}) = q(K_{n-k-1,k+1}) = n$.

Proof. Since $K_{n,n-k-1}$ is a proper subgraph of Q_n^k or S_n^k, the first four inequalities follow from the Perron-Frobenius Theorem. The others can be checked easily. □

Lemma 2. *(1) For $n \ge 3$, $\rho(S_n^1) < \rho(Q_n^1) \le \rho(R_n^1) = \rho(T_n^0) = n-1$, where the second inequality becomes equality only if $n = 3$.*
(2) For $n \ge 3$, $2n-1 = q(Q_n^0) > q(Q_n^1) > q(R_n^1) = 2n-2$.

Proof. (1) First, note that S_n^1 is a proper subgraph of Q_n^1. Thus, $\rho(S_n^1) < \rho(Q_n^1)$.

Next, we show that $\rho(Q_n^1) \le n-1$. Recall that $Q_n^1 = K_{1,n-2} \sqcup \Phi_{n-1,2}$. Let $\{X_1, Y_1\}$ be the bipartition of $K_{1,n-2}$, where $|X_1| = 1, |Y_1| = n-2$. Let $\{X_2, Y_2\}$ be the bipartition of $\Phi_{n-1,2}$, where $|X_2| = n-1$, $|Y_2| = 2$. Set $\rho = \rho(Q_n^1)$. Let $X = (x_1, x_2, \ldots, x_n)$ be a positive unit eigenvector of Q_n^1 corresponding to ρ. Since any pair of vertices in the same partite set, say v_1, v_2, have the same neighborhood, we know $x_{v_1} = x_{v_2}$. Thus, we can assume that

$$\begin{aligned} x &:= x_v, v \in X_1; \\ y &:= x_v, v \in Y_1; \\ z &:= x_v, v \in X_2; \\ t &:= x_v, v \in Y_2. \end{aligned}$$

The eigenvalue equations can be reduced to the following four ones:

$$\rho x = (n-2)y + 2t, \tag{1}$$
$$\rho y = x + (n-1)z, \tag{2}$$
$$\rho z = (n-2)y, \tag{3}$$
$$\rho t = x. \tag{4}$$

Multiplying the two sides of (2) by ρ, and putting (3) into it, we have

$$\rho^2 y = \rho x + (n-1)(n-2)y,$$

that is,

$$[\rho^2 - (n-1)(n-2)]y = \rho x. \tag{5}$$

Similarly, multiplying the two sides of (1) by ρ, and eliminating t, we obtain

$$(\rho^2 - 2)x = (n-2)\rho y. \tag{6}$$

Combining (5) and (6), and cancelling xy yields

$$\rho^4 - (n^2 - 2n + 2)\rho^2 + 2(n-1)(n-2) = 0. \tag{7}$$

By solving Equation (7), we obtain

$$\rho^2 = \frac{(n^2 - 2n + 2) + \sqrt{(n^2 - 2n + 2)^2 - 8(n-1)(n-2)}}{2}.$$

By simple algebra, we get $\rho^2 < (n-1)^2$ when $n \geq 4$ and $\rho = n - 1$ when $n = 3$.

(2) Since Q_n^1 contains $K_{n,n-2}$ as its proper subgraph, from the Perron-Frobenius Theorem, we can see $q(Q_n^1) > q(K_{n,n-2}) = 2n - 2 = q(R_n^1)$. On the other hand, by Theorem 3.2, we have

$$q(Q_n^1) \leq \frac{e(Q_n^1)}{n} + n = \frac{n(n-2) + 2}{n} + n = 2n - 2 + \frac{2}{n} < 2n - 1$$

when $n \geq 3$. This proves the statement (2). □

4 Structural lemmas

In this section, we state some known structural theorems and prove some new ones.

The first tool we need is the closure theory of Hamilton cycles in balanced bipartite graphs introduced by Bondy and Chvátal [4]. Let G be a balanced bipartite graph on $2n$ vertices. The *bipartite closure* (or briefly, *B-closure*) of G, denoted by $cl_B(G)$, is the graph obtained from G by recursively joining pairs of nonadjacent vertices in different partition sets whose degree sum is at least $n+1$ until no such pair remains. A balanced bipartite graph G on $2n$ vertices is *B-closed* if $G = cl_B(G)$, i.e., if every two nonadjacent vertices in distinct partition sets of G have degree sum at most n.

Theorem 4.1 (Bondy and Chvátal [4]). *A balanced bipartite graph G is Hamiltonian if and only if $cl_B(G)$ is Hamiltonian.*

Lemma 3. [1] *A balanced bipartite graph G is traceable if and only if $cl_B(G)$ is traceable.*

Proof. Clearly G being traceable implies that $cl_B(G)$ being traceable. Now we assume that $cl_B(G)$ is traceable. If $cl_B(G)$ is Hamiltonian, then G is Hamiltonian by Theorem 4.1. Now we assume that $cl_B(G)$ has a Hamilton path P but no Hamilton cycle. Let x, y be the two end-vertices of P. Then $xy \notin E(cl_B(G))$.

Let $G' = G + xy$. Then $cl_B(G) + xy \subseteq cl_B(G')$. Thus, $cl_B(G')$ is Hamiltonian, and G' is Hamiltonian by Theorem 4.1. So, G is traceable. □

We need two theorems proved in [12].

Theorem 4.2 (Li and Ning [12]). *Let G be a B-closed balanced bipartite graph on $2n$ vertices. If $n \geq 2k+1$ for some $k \geq 1$ and*

$$e(G) > n(n-k-1) + (k+1)^2,$$

then G contains a complete bipartite subgraph of order $2n-k$. Furthermore, if $\delta(G) \geq k$, then $K_{n,n-k} \subseteq G$.

Theorem 4.3 (Li and Ning [12]). *Let G be a balanced bipartite graph on $2n$ vertices. If $\delta(G) \geq k \geq 1$, $n \geq 2k+1$ and*

$$e(G) > n(n-k-1) + (k+1)^2,$$

then G is Hamiltonian unless $G \subseteq B_n^k$.

Using the above two theorems, we prove the following corresponding lemmas for the existence of Hamilton paths and complete bipartite subgraphs in balanced bipartite graphs, respectively.

Lemma 4. *Let G be a B-closed balanced bipartite graph on $2n$ vertices. If $n \geq 2k+3$ for some $k \geq 1$ and*

$$e(G) > n(n-k-2) + (k+2)^2,$$

then G contains a complete bipartite subgraph on $2n-k-1$ vertices. Furthermore, if $\delta(G) \geq k$, then $K_{n,n-k-1} \subseteq G$, or $k = 1$ and $K_{n-1,n-1} \subseteq G$.

[1] This result may have appeared in some early reference, but we could not find any. We include its short proof here to keep our paper self-contained.

Proof. The existence of a complete bipartite subgraph on $2n-k-1$ vertices can be deduced from Theorem 4.2. Let X, Y be the partition sets of G, and $X' \subseteq X$, $Y' \subset Y$ such that $G[X' \cup Y'] = K_{s,t}$, where $s+t \geq 2n-k-1$ and $s \geq t$. We choose s, t such that s is as large as possible.

Now suppose that $\delta(G) \geq k$. If $K_{n,n-k-1} \not\subseteq G$, then $n-k \leq t \leq s \leq n-1$. Note that every vertex in $X \backslash X'$ has degree at least k and every vertex in Y' has degree at least s. If $s+k \geq n+1$, then every vertex in $X \backslash X'$ and every vertex in Y' are adjacent, and $K_{n,n-k} \subseteq G$, a contradiction. This implies that $s+k \leq n$, i.e., $s \leq n-k$. Hence we have $s = t = n-k$. Recall that $s+t \geq 2n-k-1$. We have $k=1$ and $s=t=n-1$. Thus $K_{n-1,n-1} \subseteq G$. □

Lemma 5. *Let G be a balanced bipartite graph on $2n$ vertices. If $\delta(G) \geq k \geq 1$, $n \geq 2k+3$ and*

$$e(G) > n(n-k-2) + (k+2)^2,$$

then G is traceable unless $G \subseteq Q_n^k$, or $k=1$ and $G \subseteq R_n^1$.

Proof. Let $G' = cl_B(G)$. If G' is traceable, then so is G by Lemma 3. Now we assume that G' is not traceable. We first deal with the case $k=1$. Note that $\delta(G') \geq \delta(G)$ and $e(G') \geq e(G)$. By Lemma 4, either $K_{n,n-2} \subseteq G'$ or $K_{n-1,n-1} \subseteq G'$. Recall that $\delta(G') \geq 1$. It is easy to check the only non-traceable balanced bipartite graphs of order $2n$ without isolated vertices containing $K_{n,n-2}$ or $K_{n-1,n-1}$ are Q_n^1 and R_n^1, respectively. Thus $G' = Q_n^1$ or R_n^1, and this implies that $G \subseteq Q_n^1$ or $G \subseteq R_n^1$.

Now assume that $k \geq 2$. By Lemma 4, $K_{n,n-k-1} \subseteq G'$. Let t be the largest integer such that $K_{n,t} \subseteq G$. Clearly $n-k-1 \leq t < n$. Let X, Y be the partition sets of G, and $Y' \subset Y$ such that $G[X \cup Y'] = K_{n,t}$.

We first claim that $t = n-k-1$. If $t \geq n-k+1$, then every vertex of X has degree at least $n-k+1$ in G' and every vertex in Y has degree at least k in G', implying that G' is complete bipartite. Thus G' is traceable, a contradiction. Suppose now that $t = n-k$. If some vertex in $Y \backslash Y'$ has degree at least $k+1$ in G', then it will be adjacent to every vertex in X in G', a contradiction. So we conclude that every vertex in $Y \backslash Y'$ has degree exactly k. If a vertex $x \in X$ is adjacent to some vertex in $Y \backslash Y'$, then $d_{G'}(x) \geq n-k+1$ and x will be adjacent to every vertex in $Y \backslash Y'$. This implies that all the vertices in $Y \backslash Y'$ are adjacent to k common vertices in X, i.e., $G' = B_n^k$. Note that B_n^k is traceable, a contradiction. Thus $t = n-k-1$, as we claimed.

Next we show that every vertex of $Y \backslash Y'$ has degree exactly k. Suppose that there is a vertex $y \in Y \backslash Y'$ which has degree at least $k+1$ in G'. If $d_{G'}(y) \geq k+2$, then since $d_{G'}(x) \geq n-k-1$ for every $x \in X$, y will be adjacent to every vertex of X, a contradiction. So we have $d_{G'}(y) = k+1$. Let X' be the set of $n-k-1$ vertices in X nonadjacent to y. Then for every $x \in X'$, x is nonadjacent to any vertex of $Y \backslash Y'$; otherwise $d_{G'}(x) \geq n-k$, implying that $xy \in E(G)$. Now consider the subgraph $H = G'[X \backslash X', Y \backslash Y']$. Note that for every $y' \in Y \backslash Y'$, $d_H(y') \geq k$ and for every $x' \in X \backslash X'$, $d_H(x') \geq 1$. If every vertex in $X \backslash X'$ has degree at least 2 in H, then $cl_B(H)$ is complete and bipartite, implying that H is traceable; if there is a vertex, say x in $X \backslash X'$, with degree 1 in H, i.e., x has only one neighbor y in H, then $H - \{x, y\}$ is complete and bipartite, also implying that H is traceable. Note that $G'[X, Y']$ is complete. So G' is traceable, a contradiction. Thus we conclude that every vertex of $Y \backslash Y'$ has degree exactly k.

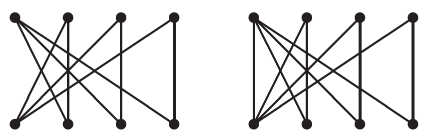
Let x be an arbitrary vertex in X. If x is adjacent to at least two vertices in $Y\backslash Y'$, then $d(x) \geq n-k+1$, implying that x is adjacent to all vertices in $Y\backslash Y'$. Thus we conclude that every vertex in X is adjacent to either no vertices, or only one vertex, or all vertices in $Y\backslash Y'$. We call the vertex x a *simple* (*frontier*, *saturated*, resp.) vertex if x is adjacent to no (one, every, resp.) vertex in $Y\backslash Y'$.

If every vertex in $Y\backslash Y'$ is adjacent to at least two frontier vertices, then we can take $k+1$ vertex-disjoint P_3's such that every vertex in $Y\backslash Y'$ is the center of a P_3. Since $G'[X, Y']$ is complete and bipartite, it is easy to check that G' is traceable, a contradiction. If every vertex in $Y\backslash Y'$ is adjacent to exactly one frontier vertex, implying that there are $k-1$ saturated vertices. (Note that every vertex in $Y\backslash Y'$ is adjacent to the same number of frontier vertices.) In this case, there are $k-1$ vertex-disjoint P_3's with the centers in $Y\backslash Y'$ and two additional independent edges incident to vertices in $Y\backslash Y'$. Since $G'[X, Y']$ is complete and bipartite, it is easy to check that G' is traceable, a contradiction.

Now assume that there are no frontier vertices. Thus every vertex in $Y\backslash Y'$ is adjacent to (the common) k saturated vertices. In this case $G' = Q_n^k$ and $G \subseteq Q_n^k$. □

Finally, we recall two theorems proved in [12].

Theorem 4.4 (Li and Ning [12]). *Let G be a balanced bipartite graph on $2n$ vertices, with minimum degree $\delta(G) \geq k \geq 1$.*
(1) If $n \geq (k+1)^2$ and $\rho(G) \geq \rho(B_n^k)$, then G is Hamiltonian unless $G = B_n^k$.
(2) If $n \geq (k+1)^2$ and $q(G) \geq q(B_n^k)$, then G is Hamiltonian unless $G = B_n^k$.
(3) If $n \geq 2k$ and $\rho(\widehat{G}) \leq \rho(\widehat{B_n^k})$, then G is Hamiltonian unless $G \in \mathcal{B}_n^k$, or $k=2$, $n=4$ and $G = L_1$ or L_2 (see Fig. 2).



L_1 $\qquad$ L_2

Fig. 2. The graphs L_1 and L_2.

Theorem 4.5 (Li and Ning [12]). *Let G be a balanced bipartite graph on $2n$ vertices. If $q(\widehat{G}) \leq n$, then G is Hamiltonian unless $G \in \bigcup_{k=1}^{\lfloor n/2 \rfloor} \mathcal{B}_n^k$, or $n=4$ and $G = L_1$ or L_2 (see Fig. 2).*

5 Proofs

In this section, we prove our main theorems.

Proof of Theorem 2.1. Suppose that G is not traceable. If $k \geq 1$, then by Lemmas 1, 2(1) and Theorem 3.1,

$$\sqrt{e(G)} \geq \rho(G) \geq \rho(Q_n^k) > \sqrt{n(n-k-1)}.$$

Thus, we have

$$e(G) > n(n-k-1) \geq n(n-k-2) + (k+2)^2$$

when $n \geq (k+2)^2$. Since $n \geq (k+2)^2 > 2k+3$, by Lemma 5, $G \subseteq Q_n^k$ or $k=1$ and $G \subseteq R_n^1$. If $k \geq 2$, then $G \subseteq Q_n^k$. But if $G \subsetneq Q_n^k$, then $\rho(G) < \rho(Q_n^k)$, a contradiction. Thus $G = Q_n^k$. If $k=1$, then $G \subseteq Q_n^1$ or $G \subseteq R_n^1$. But if $G \subseteq Q_n^1$ or $G \subsetneq R_n^1$, then by Lemma 2(1), we get $\rho(G) < \rho(R_n^1)$, a contradiction. Thus $G = R_n^1$.

Now assume that $k=0$. If G has no isolated vertex, i.e., $\delta(G) \geq 1$, then by the above analysis,

$$\rho(G) \leq \rho(R_n^1) = n-1 < \rho(Q_n^0) = \sqrt{n(n-1)},$$

a contradiction. Thus G has an isolated vertex and $G \subseteq Q_n^0$. But if $G \subsetneq Q_n^0$, then $\rho(G) < \rho(Q_n^0)$, a contradiction. Thus $G = Q_n^0$. □

Proof of Theorem 2.2. Suppose that G is not traceable. If $k \geq 1$, then by Lemma 1 and Theorem 3.2, we have

$$\frac{e(G)}{n} + n \geq q(G) \geq q(Q_n^k) > 2n-k-1.$$

Thus, we have

$$e(G) > n(n-k-1) \geq n(n-k-2) + (k+2)^2$$

when $n \geq (k+2)^2$. By Lemma 5, $G \subseteq Q_n^k$ or $k=1$ and $G \subseteq R_n^1$. If $k \geq 2$, then $G \subseteq Q_n^k$. But if $G \subsetneq Q_n^k$, then $q(G) < q(Q_n^k)$, a contradiction. Thus $G = Q_n^k$. If $k=1$, then $G \subseteq Q_n^1$ or $G \subseteq R_n^1$. But if $G \subsetneq Q_n^1$ or $G \subseteq R_n^1$, then by Lemma 2(2), we obtain $q(G) < q(Q_n^1)$, a contradiction. Thus $G = Q_n^1$.

Now assume that $k=0$. If G has no isolated vertex, i.e., $\delta(G) \geq 1$, then by the analysis above and Lemma 2(2), we obtain

$$q(G) \leq q(Q_n^1) < q(Q_n^0) = 2n-1,$$

a contradiction. Thus G has an isolated vertex and $G \subseteq Q_n^0$. But if $G \subsetneq Q_n^0$, then $q(G) < q(Q_n^0)$, a contradiction. Thus $G = Q_n^0$. □

Proof of Theorem 2.3. Suppose that G is not traceable. Then G is not hamiltonian. If $k \geq 1$, then

$$\rho(\widehat{G}) \leq \rho(\widehat{R_n^k}) = \rho(\widehat{B_n^k}) = \sqrt{k(n-k)}.$$

By Theorem 4.4, $G \in \mathcal{B}_n^k$, or $k=2$, $n=4$ and $G = L_1$ or L_2. But if $G \in \mathcal{B}_n^k \backslash \{R_n^k\}$, or $G = L_1$ or L_2, then G is traceable, a contradiction. Thus we conclude $G = R_n^k$.

Now assume that $k=0$. If G has no isolated vertex, i.e., $\delta(G) \geq 1$, then by the above analysis,

$$\rho(\widehat{G}) \leq \rho(\widehat{R_n^1}) = \sqrt{n-1} < \rho(\widehat{Q_n^0}) = \sqrt{n},$$

a contradiction. This implies that G has an isolated vertex and $G \subseteq Q_n^0$. But if $G \subsetneq Q_n^0$, then $\rho(\widehat{G}) > \rho(\widehat{Q_n^0})$, a contradiction. Thus $G = Q_n^0$. □

Proof of Theorem 2.4. Suppose that G is not traceable. Then G is not hamiltonian. By Theorem 4.5, $G \in \bigcup_{k=1}^{\lfloor n/2 \rfloor} \mathcal{B}_n^k$, or $n=4$ and $G = L_1$ or L_2. But if $G \in \bigcup_{k=1}^{\lfloor n/2 \rfloor} (\mathcal{B}_n^k \backslash \{R_n^k\})$, or $G = L_1$ or L_2, then G is traceable, a contradiction. Thus we conclude $G \in \{R_n^k : 1 \leq k \leq \lfloor n/2 \rfloor\}$. □

Proof of Theorem 2.5. Let $\{X, Y\}$ be the partition of $V(G)$ such that $|X| = n-1$ and $|Y| = n$. Let G' be the graph obtained from G by adding one new vertex x' and connecting x' to every vertex in Y by an edge. Clearly G is traceable if and only if G' is Hamiltonian.

If $k \geq 1$, then by Lemma 1 and Theorem 3.1,

$$\sqrt{e(G)} \geq \rho(G) \geq \rho(S_n^k) > \sqrt{n(n-k-1)}.$$

Thus, we have

$$e(G) > n(n-k-1) \geq n(n-k-2) + (k+1)^2$$

when $n \geq (k+1)^2$. This implies that $e(G') > n(n-k-1) + (k+1)^2$. Note that $\delta(G') \geq \delta(G) \geq k$. By Theorem 4.3, G' is Hamiltonian or $G' \subseteq B_n^k$. Thus, G is traceable or $G \subseteq S_n^k$ or $G \subseteq T_n^{k-1}$. But if $G \subsetneq S_n^k$, then $\rho(G) < \rho(S_n^k)$; if $G \subseteq T_n^{k-1}$, then $\delta(G) \leq k-1$. Thus $G = S_n^k$.

Now assume that $k = 0$. If G has no isolated vertex, then $\delta(G) \geq 1$. If $n = 2$, then clearly G is traceable. So we may assume that $n \geq 3$. By the above analysis, and by Lemma 2(1),

$$\rho(G) \leq \rho(S_n^1) < \rho(T_n^0) = n-1,$$

a contradiction. This implies that G has an isolated vertex, and $G \subsetneq T_n^0$ or $G \subseteq \Gamma_n^0$. But if $G \subsetneq T_n^0$ or $G \subseteq \Gamma_n^0$, then $\rho(G) < \rho(T_n^0)$, a contradiction. Thus $G = T_n^0$. □

Proof of Theorem 2.6. Let G' be defined as in the proof of Theorem 2.5. If $k \geq 1$, then by Lemma 1 and Theorem 3.2, we have

$$\frac{e(G)}{n} + n \geq q(G) > 2n - k - 1.$$

Note that here we consider G as a balanced bipartite graph having an isolated vertex. Thus

$$e(G) > n(n-k-1) \geq n(n-k-2) + (k+1)^2$$

when $n \geq (k+1)^2$. This implies that $e(G') > n(n-k-1) + (k+1)^2$. Note that $\delta(G') \geq \delta(G) \geq k$. By Theorem 4.3, G' is Hamiltonian or $G' \subseteq B_n^k$. Thus G is traceable or $G \subseteq S_n^k$ or $G \subseteq T_n^{k-1}$. But if $G \subsetneq S_n^k$, then $q(G) < q(S_n^k)$; if $G \subseteq T_n^{k-1}$, then $\delta(G) \leq k-1$. Thus $G = S_n^k$.

Now assume that $k = 0$. If G has an isolated vertex, then $G \subseteq T_n^0$ or $G \subseteq \Gamma_n^0$. But if $G \subsetneq T_n^0$ or $G \subseteq \Gamma_n^0$, then $q(G) < q(S_n^1)$, a contradiction. Here notice that $q(T_n^0) = q(\Gamma_n^0) = 2n-2$, and $K_{n,n-2} \subsetneq S_n^1$. So we assume that G has no isolated vertex, i.e., $\delta(G) \geq 1$. By the above analysis, G is traceable unless $G = S_n^1$. □

Proof of Theorem 2.7. We suppose first that $k \geq 1$. Let G' be defined as in the proof of Theorem 2.5. Note that $\delta(G') \geq \delta(G) \geq k$ and $\rho(\widehat{G'}) = \rho(\widehat{G}) \leq \rho(\widehat{S_n^k}) = \rho(\widehat{B_n^k})$. By Theorem 4.4, G' is Hamiltonian unless $G' \in \mathcal{B}_n^k$, or $k = 2$, $n = 4$ and $G' = L_1$ or L_2. Thus G is traceable unless $G \in \mathcal{S}_n^k$ or $G \in \mathcal{T}_n^{k-1}$, or $n = 4$, $k = 2$ and $G = L$. But if $G \in \mathcal{T}_n^{k-1}$, or $n = 4$, $k = 2$ and $G = L$, then $\delta(G) \leq k-1$, a contradiction. Thus $G \in \mathcal{S}_n^k$.

Now assume that $k = 0$. Then $\rho(\widehat{G}) \leq \rho(\widehat{T_n^0}) = \rho(\widehat{S_n^1})$. If G has no isolated vertex, then $\delta(G) \geq 1$ and by the above analysis, G is traceable unless $G \in \mathcal{S}_n^1$. If G has an isolated vertex, then $G \subseteq \Gamma_n^0$ or $G \subseteq T_n^0$. But if $G \subseteq \Gamma_n^0$ or $G \subsetneq T_n^0$, then $\rho(\widehat{G}) > \rho(\widehat{T_n^0})$, a contradiction. Thus $G \in \mathcal{S}_n^1 \cup \{T_n^0\}$. □

Proof of Theorem 2.8. Let G' be defined as in the proof of Theorem 2.5. Note that $\delta(G') \geq \delta(G) \geq k$, $q(\widehat{G'}) = q(\widehat{G}) \leq n$. By Theorem 4.5, G' is Hamiltonian unless $G' \in \bigcup_{k=1}^{\lfloor n/2 \rfloor} \mathcal{B}_n^k$, or $n = 4$ and $G' = L_1$ or L_2. Thus G is traceable unless $G \in (\bigcup_{k=1}^{\lfloor (n-1)/2 \rfloor} \mathcal{S}_n^k) \cup (\bigcup_{k=0}^{\lfloor n/2 \rfloor - 1} \mathcal{T}_n^k)$, or $n = 4$ and $G = Ł$. The proof is complete. □

6 Concluding remarks

In fact, during our proofs of main theorems, we have actually proved the following theorems. All these results maybe stimulate our further study.

Theorem 6.1. *Let G be a balanced bipartite graph on $2n$ vertices, with minimum degree $\delta(G) \geq k$, where $k \geq 1$ and $n \geq (k+2)^2$. If $\rho(G) \geq \sqrt{n(n-k-1)}$, then G is traceable unless $G \subseteq Q_n^k$ or $k = 1$ and $G \subseteq R_n^1$.*

Theorem 6.2. *Let G be a balanced bipartite graph on $2n$ vertices, with minimum degree $\delta(G) \geq k$, where $k \geq 1$ and $n \geq (k+2)^2$. If $q(G) \geq 2n - k - 1$, then G is traceable unless $G \subseteq Q_n^k$ or $k = 1$ and $G \subseteq R_n^1$.*

Theorem 6.3. *Let G be a nearly balanced bipartite graph on $2n-1$ vertices, with minimum degree $\delta(G) \geq k$, where $k \geq 1$ and $n \geq (k+1)^2$. If $\rho(G) \geq \sqrt{n(n-k-1)}$, then G is traceable unless $G \subseteq S_n^k$.*

Theorem 6.4. *Let G be a nearly balanced bipartite graph on $2n-1$ vertices, with minimum degree $\delta(G) \geq k$, where $k \geq 1$ and $n \geq (k+1)^2$. If $q(G) \geq 2n - k - 1$, then G is traceable unless $G \subseteq S_n^k$.*

On the other hand, notice that in Theorem 1.2, the order of a graph is required to be linear multiple of the minimum degree of a graph. But in our Theorems 2.1, 2.2, 2.5 and 2.6, the order of a graph is required to be at least square multiple of minimum degree of a graph. It is natural to ask whether the required order could be improved to linear multiple of minimum degree of the graph. Till now, we cannot solve this problem.

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