# On the decomposition of random hypergraphs

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**Abstract**

For an $r$-uniform hypergraph $H$, let $f(H)$ be the minimum number of complete $r$-partite $r$-uniform subhypergraphs of $H$ whose edge sets partition the edge set of $H$. For a graph $G$, $f(G)$ is the bipartition number of $G$ which was introduced by Graham and Pollak in 1971. In 1988, Erdős conjectured that if $G \in G(n, 1/2)$, then with high probability $f(G) = n - \alpha(G)$, where $\alpha(G)$ is the independence number of $G$. This conjecture and its related problems have received a lot of attention recently. In this paper, we study the value of $f(H)$ for a typical $r$-uniform hypergraph $H$. More precisely, we prove that if $(\log n)^{2.001}/n \leq p \leq 1/2$ and $H \in H^{(r)}(n,p)$, then with high probability $f(H) = (1 - \pi(K_r^{(r-1)}) + o(1))\binom{n}{r-1}$, where $\pi(K_r^{(r-1)})$ is the Turán density of $K_r^{(r-1)}$.

## 1 Introduction

For a graph $G$, the *bipartition number* $\tau(G)$ is the minimum number of complete bipartite subgraphs of $G$ so that each edge of $G$ belongs to exactly one of them. This parameter of a graph was introduced by Graham and Pollak [12] in 1971. The famous Graham–Pollak [12] Theorem asserts $\tau(K_n) = n - 1$. Since its original proof using Sylvester's Law of Inertia, many other proofs have been discovered, see [16], [17], [18], [19], [20], [21].

Let $\alpha(G)$ be the independence number of $G$. It is easy to observe $\tau(G) \leq |V(G)| - \alpha(G)$. Erdős (see [13]) conjectured that the equality holds for almost all graphs. Namely, if $G \in G(n, 1/2)$, then $\tau(G) = n - \alpha(G)$ with high probability. Alon [2] disproved this conjecture by showing $\tau(G) \leq n - \alpha(G) - 1$ with high probability for most values of $n$. Improving Alon's result, Alon, Bohman, and Huang [3] proved that if $G \in G(n, 1/2)$, then with high probability $\tau(G) \leq n - (1 + c)\alpha(G)$ for some positive constant $c$. Chung and the author [6] proved that if $G \in G(n,p)$, $p$ is a constant, and $p \leq 1/2$, then with high probability we have $\tau(G) \geq n - \delta(\log_{1/p} n)^{3+\epsilon}$ for any constants $\delta$ and $\epsilon$. When $p$ satisfies $\frac{2}{n} \leq p \leq c$ for some absolute (small) constant $c$, Alon [2] showed that if $G \in G(n,p)$, then $\tau(G) = n - \Theta\left(\frac{\log(np)}{p}\right)$ with high probability.

The hypergraph analogue of the bipartition number is well-defined. For $r \geq 3$ and an $r$-uniform hypergraph $H$, let $f(H)$ be the minimum number of complete $r$-partite $r$-uniform subhypergraphs of $H$ whose edge sets partition the edge set of $H$. Aharoni and Linial (see [1]) first asked to determine the value of $f(K_n^{(r)})$ for $r \geq 3$, where $K_n^{(r)}$ is the complete $r$-uniform hypergraph with $n$ vertices. The value of $f(K_n^{(r)})$ is related to a perfect hashing problem from computer science. Alon [1] proved $f(K_n^{(3)}) = n - 2$ and $c_1(r)n^{\lfloor \frac{r}{2} \rfloor} \leq f(K_n^{(r)}) \leq c_2(r)n^{\lfloor \frac{r}{2} \rfloor}$ for $r \geq 4$. For improvements and variations, readers are referred to [7], [8], [9], [10], [14], and [15]. For each real $0 \leq p \leq 1$, let $H^{(r)}(n,p)$ denote the random $r$-uniform hypergraph in which each $r$-set $F \in \binom{[n]}{r}$ is selected as an edge with probability $p$ independently. In this

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paper, we examine the value of $f(H)$ for the random hypergraph $H \in H^{(r)}(n,p)$. To state our main theorem, we need a few more definitions.

For an $r$-uniform hypergraph $H$, the *Turán number* $\mathrm{ex}(n,H)$ is the maximum number of edges in an $n$-vertex $r$-uniform hypergraph which does not contain $H$ as a subhypergraph. We define the *Turán density* of $H$ as

$$\pi(H) = \lim_{n\to\infty} \frac{\mathrm{ex}(n,H)}{\binom{n}{r}}.$$

For each $r \geq 3$, we use $K_r^{(r-1)}$ to denote the compete $(r-1)$-uniform hypergraph with $r$ vertices.

By extending techniques from [2] and [6], we are able to prove the following theorem.

**Theorem 1** *For $r \geq 3$, if $(\log n)^{2.001}/n \leq p \leq 1/2$ and $H \in H^{(r)}(n,p)$, then with high probability we have*

$$f(H) = (1 - \pi(K_r^{(r-1)}) + o(1))\binom{n}{r-1}.$$

From this theorem, we can see the typical value of $f(H)$ has the order of magnitude $n^{r-1}$ while $f(K_n^{(r)})$ has the order of magnitude $n^{\lfloor \frac{r}{2} \rfloor}$. We note $\pi(K_3^{(2)}) = \frac{1}{2}$ while the value of $\pi(K_r^{(r-1)})$ is not known for $r \geq 4$. We remark here that our techniques also work for $p \leq 1-c$ for any small positive constant $c$. However, we restrict out attention to the case where $p \leq 1/2$ in this paper.

We will use the following notation throughout this paper. For each $r \geq 3$, we will use $[n]$ to denote the set $\{1,2,\ldots,n\}$ and $\binom{[n]}{r}$ to denote the collection of all $r$-subsets of $[n]$. If $A_1, A_2, \ldots, A_r$ are pairwise disjoint subsets of $[n]$, then we use $\prod_{i=1}^r A_i$ to denote those $r$-subsets $F$ of $[n]$ such that $|F \cap A_i| = 1$ for each $1 \leq i \leq r$. We may also write $A_1 \times A_2 \times \cdots \times A_r$ for $\prod_{i=1}^r A_i$ on some occasions. The complete $r$-partite $r$-uniform hypergraph whose vertex parts are $A_1, A_2, \ldots, A_r$ is the $r$-uniform hypergraph with the edge set $\prod_{i=1}^r A_i$.

Let $H$ be an $r$-uniform hypergraph with vertex set $[n]$ and edge set $E$. For pairwise disjoint subsets $A_1, A_2, \ldots, A_r \subset [n]$, we say $A_1, A_2, \ldots, A_r$ form a complete $r$-partite $r$-uniform hypergraph if $\prod_{i=1}^r A_i \subseteq E(H)$.

For an $r$-uniform hypergraph $H$, suppose $E(H) = \sqcup_{i=1}^q \prod_{j=1}^r A_j^i$ is a partition of the edge set of $H$. For each $1 \leq i \leq q$, the $i$-th complete $r$-partite $r$-uniform hypergraph $H_i$ has vertex parts $A_1^i, \ldots, A_r^i$. We always assume $|A_1^i| \leq \cdots \leq |A_r^i|$. We say $H_i$ is a trivial complete $r$-partite $r$-uniform hypergraph if $|A_1^i| = \cdots = |A_{r-1}^i| = 1$. Otherwise, we say $H_i$ is a nontrivial one. The *prefix* $P_i$ of $H_i$ is the set $\{A_1^i, \ldots, A_{r-1}^i\}$ and the prefix set $\mathcal{P}$ of the partition is $\{P_1, \ldots, P_q\}$.

We say an event $\mathcal{X}_n$ occurs with high probability if the probability that $\mathcal{X}_n$ holds goes to one as $n$ approaches infinity. All logarithms are in base 2, unless otherwise specified.

The outline of the proof for Theorem 1 is the following. For the upper bound, we will give an explicit construction such that each $r$-uniform hypergraph with $n$ vertices can be decomposed into at most $(1 - \pi(K_r^{(r-1)}) + o(1))\binom{n}{r-1}$ trivial complete $r$-partite $r$-uniform hypergraphs. For the lower bound, we will prove $f(H) \geq (1 - \pi(K_r^{r-1}) - \epsilon)\binom{n}{r-1}$ with high probability for any positive constant $\epsilon$. Equivalently, we will show with tiny probability $f(H) \leq (1 - \pi(K_r^{r-1}) - \epsilon)\binom{n}{r-1}$ holds. To do so, for a given prefix set $\mathcal{P} = \{P_1, \ldots, P_q\}$ with $q \leq (1 - \pi(K_r^{r-1}) - \epsilon)\binom{n}{r-1}$, let $\mathcal{P}_1 = \{P_i \in \mathcal{P} : |P_j^i| = 1 \text{ for each } 1 \leq j \leq r-1\}$ and $\mathcal{P}_2 = \mathcal{P} \setminus \mathcal{P}_1$. We will show that there are at least $c(\epsilon)n^r$ edges of $H \in H^{(r)}(n,p)$ which must be contained by some nontrivial complete $r$-partite $r$-uniform hypergraph with prefix from $\mathcal{P}_2$. Theorem 4 will tell us this probability is sufficiently small. We will prove an upper bound on the number of possible choices for $\mathcal{P}$ and apply the union bound to complete the proof.

The idea here for proving the lower bound is quite similar to the one in [6]. The difference is explained as follows. For random graphs, after we remove edges contained by stars (trivial complete bipartite graphs), we still get a random graph with a smaller number of vertices. This property indeed helped us to prove concentrations of related random variables. For random $r$-uniform hypergraph with $r \geq 3$, if we delete edges contained by trivial complete $r$-partite $r$-uniform hypergraphs, then we will not end up with a random hypergraph. This causes a lot trouble and we will find a new way to overcome this difficulty.

The rest of the paper is organized as follows. In Section 2, we will prove several necessary lemmas. In Section 3, we will present the proof of an auxiliary theorem which is the key ingredient in the proof of the main result. Theorem 1 will be proved in Section 4. Several concluding remarks will be mentioned in Section 5.

## 2 Lemmas

In this section, we will collect some necessary lemmas which are needed to prove the main theorem. We will use the following versions of Chernoff's inequality and Azuma's inequality.

**Theorem 2** *[5] Let $X_1, \ldots, X_n$ be independent random variables with*

$$\Pr(X_i = 1) = p_i, \qquad \Pr(X_i = 0) = 1 - p_i.$$

*We consider the sum $X = \sum_{i=1}^n X_i$ with expectation $\mathrm{E}(X) = \sum_{i=1}^n p_i$. Then we have*

$$\text{(Lower tail)} \qquad \Pr(X \leq \mathrm{E}(X) - \lambda) \leq e^{-\lambda^2/2\mathrm{E}(X)},$$

$$\text{(Upper tail)} \qquad \Pr(X \geq \mathrm{E}(X) + \lambda) \leq e^{-\frac{\lambda^2}{2(\mathrm{E}(X)+\lambda/3)}}.$$

**Theorem 3** *[4] Let $X$ be a random variable determined by $m$ trials $T_1, \ldots, T_m$, such that for each $i$, and any two possible sequences of outcomes $t_1, \ldots, t_{i-1}, t_i$ and $t_1, \ldots, t_{i-1}, t_i'$:*

$$|\mathrm{E}\left(X | T_1 = t_1, \ldots, T_i = t_i\right) - \mathrm{E}\left(X | T_1 = t_1, \ldots, T_{i-1} = t_{i-1}, T_i = t_i'\right)| \leq c_i,$$

*then*

$$\Pr\left(|X - \mathrm{E}(X)| \geq \lambda\right) \leq 2\exp\left(-\lambda^2/2\sum_{i=1}^m c_i^2\right).$$

Recall that if $A_1, \ldots, A_r$ form a complete $r$-partite $r$-uniform hypergraph, then we assume $|A_1| \leq |A_2| \leq \cdots \leq |A_r|$. To prove an upper bound on the number of choices for the prefix set, we will need the following lemma.

**Lemma 1** *For $H \in H^{(r)}(n,p)$ with $p \leq 1/2$, with high probability the vertex parts $A_1, A_2, \cdots, A_r$ of each complete $r$-partite $r$-uniform hypergraph in $H$ satisfy $\prod_{i=1}^{r-1} |A_i| < (r+1)\log n$.*

**Proof:** We need only to prove the lemma for $p = 1/2$. For a collection of pairwise disjoint sets $A_1, A_2, \ldots, A_r \subset [n]$, we assume $|A_i| = k_i$ for each $1 \leq i \leq r$ and $k_1 \leq k_2 \leq \cdots \leq k_r$. Fix a selection of $A_1, \ldots, A_r$, the probability that they form a complete $r$-partite $r$-uniform hypergraph in $H^{(r)}(n, 1/2)$ is $2^{-\prod_{i=1}^r k_i}$. For fixed $k_1, \ldots, k_r$, there are at most $\prod_{i=1}^r \binom{n}{k_i}$ choices for $A_1, A_2, \ldots, A_r$ such that $|A_i| = k_i$ for each $1 \leq i \leq r$. Therefore, for fixed $k_1, \ldots, k_r$ satisfying $\prod_{i=1}^{r-1} k_i \geq (r+1)\log n$ and $k_1 \leq \cdots \leq k_r$, the probability that there are pairwise disjoint sets $A_1, A_2, \ldots, A_r$ such that $|A_i| = k_i$ and they form a complete $r$-partite $r$-uniform hypergraph is at most

$$\begin{aligned}\prod_{i=1}^{r}\binom{n}{k_i}2^{-\prod_{i=1}^{r}k_i} &< 2^{(\sum_{i=1}^{r}k_i)\log n-\prod_{i=1}^{r}k_i}\\ &= 2^{k_r\left((\sum_{i=1}^{r-1}k_i/k_r+1)\log n-\prod_{i=1}^{r-1}k_i\right)}\\ &\le 2^{k_r(r\log n-\prod_{i=1}^{r-1}k_i)}\\ &< 2^{-k_r\log n}.\end{aligned}$$

Put $s = \prod_{i=1}^{r-1} k_i$. We next estimate how many choices of $k_1, \ldots, k_r$ such that $\prod_{i=1}^{r-1} k_i = s$ and $k_1 \le \cdots \le k_r$. Let $t = \sum_{i=1}^{r-1} k_i$. If $s \ge \log n$, then $t \le s + r < 2s$ and $k_r \ge k_{r-1} \ge s^{1/(r-1)}$. Thus the number of choices for $k_1, \ldots, k_{r-1}$ satisfying $\prod_{i=1}^{r-1} k_i = s$ and $k_1 \le \cdots \le k_{r-1}$ is less than the number of positive solutions to the equation $\sum_{i=1}^{r-1} k_i = t$, which is less than $2s\binom{2s}{r-2}$ as $t \le 2s$. We have at most $n$ choices for $k_r$ regardless the choices of $k_1, \ldots, k_{r-1}$. Therefore, the probability that there are $A_1, A_2, \ldots, A_r$ which satisfy $s = \prod_{i=1}^{r-1} |A_i| \ge (r+1)\log n$ and form a complete $r$-partite $r$-uniform hypergraph in $H^{(r)}(n, 1/2)$ is at most

$$\sum_{s=(r+1)\log n}^{n} 2sn\binom{2s}{r-2}2^{-k_r\log n} \le \sum_{s=(r+1)\log n}^{n} 2sn\binom{2s}{r-2}2^{-s^{1/(r-1)}\log n} = o(1),$$

here we used the assumption $s \ge (r+1)\log n$. Then the lemma follows from Markov's inequality. □

To estimate the number of edges covered by a family of nontrivial complete $r$-partite $r$-uniform hypergraphs, we need to introduce a new concept and prove the next lemma.

For an $r$-uniform hypergraph $H = (V, E)$ and a prefix $P = \{A_1, A_2, \ldots, A_{r-1}\}$, we define

$$V(H, P) = \{v : v \in V(H) \setminus (\cup_{i=1}^{r-1} A_i) \text{ and } F \in E(H) \text{ for each } F \in A_1 \times \cdots \times A_{r-1} \times \{v\}\}.$$

Figure 1 is an illustrative example for $v \in V(H, P)$. It follows that $A_1, A_2, \ldots, A_r$ form

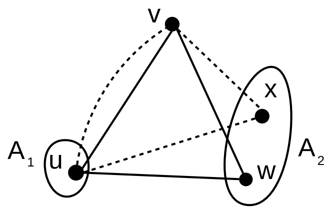


Figure 1: An example with $r = 3$, $P = \{A_1, A_2\}$, and $v \in V(H, P)$.

a complete $r$-partite $r$-uniform hypergraph if $A_r$ is contained in $V(H, P)$, namely, $A_r \subseteq V(H, P)$. We say an edge $F \in E(H)$ is covered by a complete $r$-partite $r$-uniform hypergraph with prefix $P$ if $F \in A_1 \times \cdots \times A_{r-1} \times V(H, P)$.

Let $\mathcal{P} = \{P_1, \ldots, P_q\}$ be a prefix set, where $P_i = \{A_1^i, \ldots, A_{r-1}^i\}$. We define $g(H, \mathcal{P})$ as the number of edges of $H$ which are covered by exactly one complete $r$-partite $r$-uniform hypergraph whose prefix is from $\mathcal{P}$. It is easy to see

$$g(H, \mathcal{P}) \le \sum_{i=1}^{q} g(H, P_i) \le \sum_{i=1}^{q} |V(H, P_i)| \prod_{j=1}^{r-1} |A_j^i|.$$

We have the following lemma on $g(H, \mathcal{P})$ for $H \in H^{(r)}(n, p)$.

**Lemma 2** *Assume $p \leq 1/2$ and $H \in H^{(r)}(n,p)$. Let $c(n)$ be a fixed function. Given a prefix set $\mathcal{P} = \{P_1, \ldots, P_q\}$, where $P_i = \{A_1^i, A_2^i, \ldots, A_{r-1}^i\}$ and $c(n) \leq \prod_{j=1}^{r-1} |A_j^i| < (r+1)\log n$ for each $1 \leq i \leq q$, then we have*

$$\Pr\left(g(H,\mathcal{P}) \geq qc(n)p^{c(n)}n + 2n^{r-0.3}\right) \leq 2\exp(-n^{r-0.8}).$$

**Proof:** We shall use Theorem 3 to prove this lemma. Let $m = \binom{n}{r}$ and we list all $r$-sets of $[n]$ as $F_1, F_2, \ldots, F_m$. For each $1 \leq i \leq m$, we consider $T_i \in \{\mathrm{H}, \mathrm{T}\}$, here $T_i = \mathrm{H}$ means $F_i$ is an edge and $T_i = \mathrm{T}$ means $F_i$ is a non-edge. To simplify the notation, we use $X$ to denote the random variable $g(H,\mathcal{P})$. We observe that $X$ is determined by $T_1, \ldots, T_m$. Fix the outcome $t_j$ of $T_j$ for each $1 \leq j \leq i-1$, we wish to show an upper bound for

$$|\mathrm{E}(X|T_1 = t_1, \ldots, T_{i-1} = t_{i-1}, T_i = \mathrm{H}) - \mathrm{E}(X|T_1 = t_1, \ldots, T_{i-1} = t_{i-1}, T_i = \mathrm{T})|. \quad (1)$$

If $T_i = \mathrm{H}$, then we assume $F_i$ is contained by some hypergraph whose prefix is $P_k$ for some $1 \leq k \leq q$. Otherwise changing the outcome of $T_i$ will not affect the value of $X$. Suppose $F_i = \{v_1, \ldots, v_r\}$, where $A_l^k \cap F_i = \{v_l\}$ for each $1 \leq l \leq r-1$ and $v_r \notin \cup_{l=1}^{r-1} A_l^k$. We next examine other edges which get covered because we change $F_i$ as an edge. These edges are from the family $A_1^k \times \cdots A_{r-1}^k \times \{v_r\}$. Therefore, $\prod_{l=1}^{r-1} |A_l^k|$ is an upper bound for (1). Recalling the assumption $\prod_{l=1}^{r-1} |A_l^k| < (r+1)\log n$, then (1) can be bounded from above by $(r+1)\log n$.

We note $\mathrm{E}(g(H,P_i)) \leq c(n)p^{c(n)}n$ as we assume $\prod_{j=1}^{r-1} |A_j^i| \geq c(n)$. We get

$$\mathrm{E}(X) \leq \sum_{i=1}^{q} \mathrm{E}(H, P_i) \leq c(n)qp^{c(n)}n.$$

Applying Theorem 3 with $\lambda = 2n^{r-0.3}$ and $c_i = (r+1)\log n$, we obtain

$$\begin{aligned}\Pr\left(X \geq c(n)qp^{c(n)}n + 2n^{r-0.3}\right) &\leq \Pr\left(X \geq \mathrm{E}(X) + 2n^{r-0.3}\right)\\ &\leq 2\exp\left(-4n^{2r-0.6}/(2m((r+1)\log n)^2)\right)\\ &\leq 2\exp(-n^{r-0.8}),\end{aligned}$$

here we used the fact $m < n^r$. □

We need a lemma that provides a lower bound for the number of edges not covered by a family of nontrivial complete $r$-partite $r$-uniform hypergraphs. Let $k(n)$ and $l(n)$ be given functions. Suppose $\mathcal{F} \subset \binom{[n]}{r}$ and $\mathcal{Q}$ is the power set of $\binom{[n]}{r} \setminus \mathcal{F}$. Consider a function $\mathcal{C} : \mathcal{F} \to \mathcal{Q}$ such that for each $F \in \mathcal{F}$ and each $R \in \mathcal{C}(F)$, we have $|R \cap F| = r-1$. Let $h(H,\mathcal{F},\mathcal{C})$ be the number of $F \in \mathcal{F}$ such that $F$ is an edge in $H \in H^{(r)}(n,p)$ and $R$ is not an edge in $H \in H^{(r)}(n,p)$ for all $R \in \mathcal{C}(F)$. We have the following lemma on $h(H,\mathcal{F},\mathcal{C})$.

**Lemma 3** *Suppose $p \leq 1/2$ and $\mathcal{F} \subset \binom{[n]}{r}$. Assume $H \in H^{(r)}(n,p)$, $|\mathcal{C}(F)| \leq k(n)$ for each $F \in \mathcal{F}$, and for each $R \in \cup_{F \in \mathcal{F}} \mathcal{C}(F)$, the number of $F \in \mathcal{F}$ satisfying $R \in \mathcal{C}(F)$ is at most $l(n)$, here $l(n)$ and $k(n)$ are some given functions. Then we have*

$$\Pr\left(h(H,\mathcal{F},\mathcal{C}) \leq |\mathcal{F}|p(1-p)^{k(n)} - 2n^{r-0.01}\right) \leq 2\exp(-n^{r-0.02}/l(n)^2).$$

**Proof:** To simplify the notation, we use $X$ to denote the random variable $h(H,\mathcal{F},\mathcal{C})$ again. We list all $r$-sets from $\mathcal{F} \cup_{F \in \mathcal{F}} \mathcal{C}(F)$ as $F_1, F_2, \ldots, F_m$, here $m \leq \binom{n}{r}$. For each $F_i$, we consider $T_i \in \{\mathrm{H}, \mathrm{T}\}$, here $T_i = \mathrm{H}$ means $F_i$ is an edge and $T_i = \mathrm{T}$ means $F_i$ is not an edge. Given the outcome $t_j$ of $T_j$ for each $1 \leq j \leq i-1$, we wish to establish an upper bound for

$$|\mathrm{E}(X|T_1 = t_1, \ldots, T_{i-1} = t_{i-1}, T_i = \mathrm{H}) - \mathrm{E}(X|T_1 = t_1, \ldots, T_{i-1} = t_{i-1}, T_i = \mathrm{T})|. \quad (2)$$

If $F_i \in \mathcal{F}$, then changing the outcome of $T_i$ can only affect (2) by one. If $F_i \in \cup_{F\in\mathcal{F}}\mathcal{C}(F)$, then changing the outcome of $T_i$ can affect (2) by at most $l(n)$ since $F_i \in \mathcal{C}(F)$ for at most $l(n)$ $r$-set $F$. Therefore, the expression (2) can be bounded from above by $l(n)$. Applying Theorem 3 with $\lambda = 2n^{r-0.01}$ and $c_i = l(n)$, we get

$$\Pr\left(|X - \mathrm{E}(X)| \geq 2n^{r-0.01}\right) \leq 2\exp\left(-4n^{2r-0.02}/2\sum_{i=1}^{m} c_i^2\right) \leq 2\exp(-n^{r-0.02}/l(n)^2),$$

here we used $m \leq \binom{n}{r}$. We note $\mathrm{E}(X) = \sum_{F\in\mathcal{F}} p(1-p)^{|\mathcal{C}(F)|} \geq |\mathcal{F}|p(1-p)^{k(n)}$ as $|\mathcal{C}(F)| \leq k(n)$. Therefore,

$$\begin{aligned}\Pr\left(h(H,\mathcal{F},\mathcal{C}) \leq |\mathcal{F}|p(1-p)^{k(n)} - 2n^{r-0.01}\right) &\leq \Pr\left(|X - \mathrm{E}(X)| \geq 2n^{r-0.01}\right) \\ &\leq 2\exp(-n^{r-0.02}/l(n)^2).\end{aligned}$$

We proved the lemma. □

When $p \leq 1/\log\log\log\log n$, we adapt the approach in [2]. The following two lemmas are the hypergraph version of Lemma 3.1 and Lemma 3.2 in [2]. Before we state them, we need one additional definition.

For positive integers $m \geq \log n$ and $r \geq 3$, let $\mathcal{T}_m$ be the set of tuples $(a_1, a_2, \ldots, a_r)$ satisfying the following properties:

1: $a_i$ is a positive integer for each $1 \leq i \leq r$;

2: $1 \leq a_1 \leq a_2 \leq \cdots \leq a_r$;

3: $a_1 \cdots a_r = m$;

4: $a_{r-1} \geq 2$.

**Lemma 4** *For any constant $c$, if $p$ satisfies $(\log n)^{2.001}/n \leq p \leq 1/\log\log\log\log n$, then the following holds for $n$ large enough. For every integer $m$ satisfying*

$$\frac{pcn}{16} \leq m \leq \frac{pcn}{4},$$

*we have*

$$\sum_{(a_1,\ldots,a_r)\in\mathcal{T}_m} \binom{n}{a_1}\binom{n-a_1}{a_2}\cdots\binom{n-\sum_{i=1}^{r-1} a_i}{a_r} p^m \leq 2^{-0.3\log(1/p)m}.$$

Recall that a complete $r$-partite $r$-uniform hypergraph whose vertex parts $A_1, \ldots, A_r$ satisfying $|A_1| \leq |A_2| \leq \cdots \leq |A_r|$ is nontrivial if $\prod_{i=1}^{r-1} |A_i| \geq 2$.

**Lemma 5** *For any constant $c$, if $p$ satisfies $(\log n)^{2.001}/n \leq p \leq 1/\log\log\log\log n$, then the probability that $H \in H^r(n,p)$ contains a set of at most $2n^{r-1}$ nontrivial complete $r$-partite $r$-uniform hypergraphs which cover at least $pcn^r/4$ edges is at most $2^{-0.05pc\log(1/p)n^r}$.*

As proofs of the two lemmas above go the same lines as those for proving Lemma 3.1 and Lemma 3.2 in [2], they are omitted here.

## 3 An auxiliary theorem

Let $\mathcal{F} \subset \binom{[n]}{r}$ with $|\mathcal{F}| \geq cn^r$ for some positive constant $c$. Suppose the probability $p$ satisfies $1/\log\log\log\log n \leq p \leq 1/2$. We shall prove if $H \in H^{(r)}(n,p)$, then with small probability that there are a few nontrivial complete $r$-partite $r$-uniform hypergraphs such that each edge $F \in E(H) \cap \mathcal{F}$ is in exactly one of them.

**Theorem 4** *Assume $\mathcal{F} \subset \binom{[n]}{r}$ with $|\mathcal{F}| \geq cn^r$ for some positive constant $c$. Let $\mathcal{P} = \{P_1, \ldots, P_t\}$ be a given prefix set, where $t = |\mathcal{P}| \leq n^{r-1}$ and $P_i = \{A_1^i, \ldots, A_{r-1}^i\}$ satisfying $2 \leq \prod_{j=1}^{r-1} |A_j^i| < (r+1)\log n$ for each $1 \leq i \leq t$. If $1/\log\log\log\log n \leq p \leq 1/2$ and $H \in H^{(r)}(n,p)$, then with probability at most $3\exp(-n^{r-0.92})$ there are $t$ nontrivial complete $r$-partite $r$-uniform hypergraphs such that its prefix set is $\mathcal{P}$ and each edge $F \in E(H) \cap \mathcal{F}$ is in exactly one of these hypergraphs.*

Suppose $H \in H^{(r)}(n,p)$ and

$$E(H) \cap \mathcal{F} \subseteq \bigsqcup_{i=1}^{t} \prod_{j=1}^{r} A_j^i,$$

where '$\sqcup$' denotes the disjoint union. For each $1 \leq i \leq t$, we assume $A_1^i, A_2^i, \ldots, A_r^i$ form a nontrivial complete $r$-partite $r$-uniform hypergraph. We fix a constant $K = \frac{4}{c}$ and a function $q(n) = \log\log\log\log n$. For each $0 \leq i \leq q(n) - 1$, we define $f_i = K^i 2^{q(n)}$. Let $\mathcal{P}_0 = \{P_k \in \mathcal{P} : \prod_{j=1}^{r-1} |A_j^k| < f_1\}$ and

$$\mathcal{P}_i = \left\{ P_k \in \mathcal{P} : f_i \leq \prod_{j=1}^{r-1} |A_j^k| < f_{i+1} \right\}$$

for each $1 \leq i \leq q(n) - 1$.

**Lemma 6** *There is some $1 \leq i \leq q(n) - 1$ such that $|\mathcal{P}_i| \leq \frac{t}{q(n)}$.*

The proof of this lemma is simple and it is omitted here. Let $1 \leq i_0 \leq q(n) - 1$ be the smallest integer satisfying the statement of Lemma 6. We consider

$$\mathcal{P}' = \mathcal{P}_0 \cup \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_{i_0}.$$

The idea for proving Theorem 4 is the following. We will show that there are many edges left if we delete edges covered by complete $r$-partite $r$-uniform hypergraphs with prefix from $\mathcal{P}'$. Thus these leftover edges must be covered by complete $r$-partite $r$-uniform hypergraphs with prefix from $\mathcal{P} \setminus \mathcal{P}'$. Together with Lemma 2, we can show $|\mathcal{P} \setminus \mathcal{P}'|$ should be larger than what we assumed, which leads to a contradiction. We note that the definition of the function $q(n)$ comes from the inequality (5) and the assumption on the probability $p$.

For an $r$-set $F = \{v_1, v_2, \ldots, v_r\} \in \mathcal{F}$ and each $v_j \in F$, we define

$$N_{\mathcal{P}',F}(v_j) = \left\{ P_i \in \mathcal{P}' : v_j \notin \cup_{s=1}^{r-1} A_s^i \text{ and } |F \cap A_s^i| = 1 \text{ for each } 1 \leq s \leq r-1 \right\}.$$

Figure 2 is an example for $P \in N_{\mathcal{P}',F}(v_j)$.

Roughly speaking, each $P_i \in N_{\mathcal{P}',F}(v_j)$ could be the prefix of a nontrivial complete $r$-partite $r$-uniform hypergraph that may contain $F$.

We note that $N_{\mathcal{P}',F}(v_j)$ and $N_{\mathcal{P}',F}(v_k)$ are disjoint if $j \neq k$. Let $N_{\mathcal{P}'}(F) = \cup_{j=1}^{r} N_{\mathcal{P}',F}(v_j)$ and $d_{\mathcal{P}'}(F) = |N_{\mathcal{P}'}(F)| = \sum_{j=1}^{r} |N_{\mathcal{P}',F}(v_j)|$.

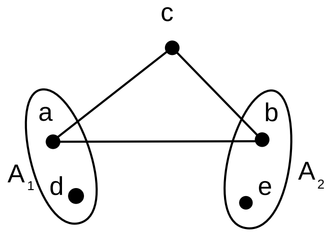


Figure 2: An example with $r = 3$, $F = \{a, b, c\}$, and $P = \{A_1, A_2\} \in N_{\mathcal{P}',F}(c)$.

**Lemma 7** *Assume* $|\mathcal{P}' \setminus \mathcal{P}_{i_0}| = xn^{r-1}$ *with* $x \geq 0.01c$. *Let* $\mathcal{F}' = \{F \in \mathcal{F} : d_{\mathcal{P}'}(F) \leq \frac{3}{c} x f_{i_0}\}$. *We have*

$$|\mathcal{F}'| \geq \frac{cn^r}{3}.$$

**Proof:** We observe that each $P_i = \{A_1^i, \ldots, A_{r-1}^i\} \in \mathcal{P}'$ contributes one to $d_{\mathcal{P}'}(F)$ for at most $n \prod_{j=1}^{r-1} |A_j^i|$ $r$-sets $F \in \mathcal{F}$. Recall the definition of $\mathcal{P}_i$ and Lemma 6. For $n$ large enough, we have

$$\begin{aligned}
\sum_{F \in \mathcal{F}} d_{\mathcal{P}'}(F) &\leq \sum_{P_i \in \mathcal{P}'} n \prod_{j=1}^{r-1} |A_j^i| \\
&= \sum_{P_i \in \mathcal{P}' \setminus \mathcal{P}_{i_0}} n \prod_{j=1}^{r-1} |A_j^i| + \sum_{P_i \in \mathcal{P}_{i_0}} n \prod_{j=1}^{r-1} |A_j^i| \\
&\leq n f_{i_0} |\mathcal{P}' \setminus \mathcal{P}_{i_0}| + n f_{i_0+1} |\mathcal{P}_{i_0}| \\
&\leq x f_{i_0} n^r + \frac{t n f_{i_0+1}}{q(n)} \\
&\leq 2x f_{i_0} n^r.
\end{aligned}$$

We note $\frac{tnf_{i_0+1}}{q(n)} \leq x f_{i_0} n^r$ since $t \leq n^{r-1}$ and $x \geq 0.01c$ as well as the definition of $i_0$. We get the following inequality

$$\frac{3x}{c} f_{i_0} |\mathcal{F} \setminus \mathcal{F}'| \leq \sum_{F \in \mathcal{F} \setminus \mathcal{F}'} d_{\mathcal{P}'}(F) \leq \sum_{F \in \mathcal{F}} d_{\mathcal{P}'}(F) \leq 2x f_{i_0} n^r.$$

Clearly, the inequality above implies $|\mathcal{F} \setminus \mathcal{F}'| \leq \frac{2cn^r}{3}$. Equivalently, $|\mathcal{F}'| \geq \frac{cn^r}{3}$. □

The number of uncovered edges from $\mathcal{F}'$ highly depends on the structure of $\mathcal{F}'$. To help us to control the concentration of related random variables, we will work on a subfamily $\mathcal{W}$ of $\mathcal{F}'$ that satisfies certain properties. Namely, for each $F \in \mathcal{W}$, we will associate with $F$ a set of $r$-sets $\mathcal{C}(F)$. The role of $\mathcal{C}(F)$ is to forbid all possibilities that $F$ is contained in a complete $r$-partite $r$-uniform hypergraph with prefix from $\mathcal{P}'$. In other words, the set $\mathcal{C}(F)$ makes $F$ uncovered. As illustrated by Figure 3, for an edge $F$ and for each complete $r$-partite $r$-uniform hypergraph that may contain $F$, we select an $r$-set $C$ (see the dashed one) and put it in $\mathcal{C}(F)$. If $C$ is not an edge for each $C \in \mathcal{C}(F)$ , then the edge $F$ is not covered by any complete $r$-partite $r$-uniform hypergraph with prefix from $\mathcal{P}'$. We have the following lemma on the set $\mathcal{W}$.

**Lemma 8** *Let* $\mathcal{F}'$ *be the subfamily of* $\mathcal{F}$ *given by Lemma 7. There is a subset* $\mathcal{W} \subseteq \mathcal{F}'$ *and a collection of* $r$*-sets* $\mathcal{C}(F) \subset \binom{[n]}{r} \setminus \mathcal{W}$ *associated with each* $F \in \mathcal{W}$ *which satisfy the following:*

1. $|\mathcal{W}| \geq \frac{c^2 n^r}{10 x f_{i_0}}$,

2. $|\mathcal{C}(F)| \leq \frac{3}{c} x f_{i_0}$ *for each* $F \in \mathcal{W}$,

3. *For each* $F = \{v_1, \ldots, v_r\} \in \mathcal{W}$ *and each* $1 \leq i \leq r$, *if* $P = \{A_1, \ldots, A_{r-1}\} \in \mathcal{P}'$ *and* $P \in N_{\mathcal{P}',F}(v_i)$, *then there is* $w \in A_{r-1} \setminus F$ *such that* $(F \setminus u) \cup w \in \mathcal{C}(F)$, *where* $u = F \cap A_{r-1}$. *The dashed edge in Figure 3 is an example for* $(F \setminus u) \cup w$.

**Proof:** To define $\mathcal{W}$, we first give a linear ordering of $r$-sets in $\mathcal{F}'$ and consider the following algorithm. We will define sets $\mathcal{F}_i$ recursively and build the set $\mathcal{W}$ step by step. Initially, let $\mathcal{F}_0 = \mathcal{F}'$ and $\mathcal{W}_0 = \emptyset$.

For each $i \geq 1$, if $\mathcal{F}_{i-1} \neq \emptyset$, then let $F_i = \{v_1, v_2, \ldots, v_r\}$ be the first $r$-set in $\mathcal{F}_{i-1}$. We will run the following process for each $P \in N_{\mathcal{P}'}(F_i)$. We note $N_{\mathcal{P}'}(F_i) = \cup_{j=1}^{r} N_{\mathcal{P}',F_i}(v_j)$. For each $1 \leq j \leq r$ and each $P = \{A_1, \ldots, A_{r-1}\} \in N_{\mathcal{P}',F_i}(v_j)$, we notice $|F_i \cap A_s| = 1$ for each $1 \leq s \leq r-1$ and $v_j \notin \cup_{s=1}^{r-1} A_s$ by the definition of $P \in N_{\mathcal{P}',F_i}(v_j)$. Suppose $F_i \cap A_{r-1} = u$. We have $|A_{r-1}| \geq 2$ as $P$ is the prefix of a nontrivial complete $r$-partite $r$-uniform hypergraph. We have two cases.
**Case 1:** There is some $w \in A_{r-1}$ such that $(F_i \setminus u) \cup w \notin \mathcal{F}_{i-1} \cup \mathcal{W}_{i-1}$, see the dashed edge in Figure 3. We do not do anything in this case.
**Case 2:** $(F_i \setminus u) \cup v \in \mathcal{F}_{i-1} \cup \mathcal{W}_{i-1}$ for each $v \in A_{r-1}$. We claim actually $(F_i \setminus u) \cup v \in \mathcal{F}_{i-1}$ for each $v \in A_{r-1}$. We proceed with the algorithm by assuming this claim. We choose an arbitrary $w \in A_{r-1} \setminus u$ and delete $(F_i \setminus u) \cup w$ from $\mathcal{F}_{i-1}$.

After we complete this process for all $P \in N_{\mathcal{P}'}(F)$, we let $\mathcal{W}_i = \mathcal{W}_{i-1} \cup \{F_i\}$ by moving $F_i$ from $\mathcal{F}_{i-1}$ to $\mathcal{W}_i$ and $\mathcal{F}_i$ be the resulting subset of $\mathcal{F}_{i-1}$. We mention here that $\mathcal{F}_i \subset \mathcal{F}_j$, $\mathcal{W}_j \subset \mathcal{W}_i$, and $\mathcal{F}_i \cup \mathcal{W}_i \subseteq \mathcal{F}_j \cup \mathcal{W}_j$ for $j < i$.

Now we prove the claim. Suppose $(F_i \setminus u) \cup v \in \mathcal{W}_{i-1}$ for some $v \in A_{r-1} \setminus u$. We pick such a vertex $v$ so that the $r$-set $F' = (F_i \setminus u) \cup v$ is the smallest one in $\mathcal{W}_{i-1}$ under the linear ordering. Suppose $F'$ was added to $\mathcal{W}_j$ at step $j$ with $j < i$. We examine the moment that $F'$ was moved to $\mathcal{W}_j$. If we are in the first case, i.e., there is some $s \in A_{r-1} \setminus v$ such that $(F' \setminus v) \cup s \notin \mathcal{F}_{j-1} \cup \mathcal{W}_{j-1}$, then $s \neq u$. Otherwise, $s = u$ implies $(F' \setminus v) \cup s = F_i \notin \mathcal{F}_{i-1}$. Here we notice $(\mathcal{F}_{i-1} \cup \mathcal{W}_{i-1}) \subseteq (\mathcal{F}_{j-1} \cup \mathcal{W}_{j-1})$ as $j < i$. Therefore, $(F' \setminus v) \cup s = (F_i \setminus u) \cup s \notin \mathcal{F}_{i-1} \cup \mathcal{W}_{i-1}$ and we are in the first case, which is a contradiction. Thus we are in the second case when we are examining $F'$. Since we chose $F'$ as the first one from $\mathcal{W}_{i-1}$ and of the form $(F_i \setminus u) \cup v$, we get $F'$ must satisfy the statement of the claim. Therefore, by the algorithm, there was a vertex $w \in A_{r-1} \setminus \{u, v\}$ such that $(F' \setminus v) \cup w = (F_i \setminus u) \cup w$ was deleted from $\mathcal{F}_{j-1}$ when we were moving $F'$ from $\mathcal{F}_{j-1}$ to $\mathcal{W}_j$. We note $(F_i \setminus u) \cup w \notin \mathcal{F}_{i-1} \cup \mathcal{W}_{i-1}$ and we are in the first case for $F_i$, which is a contradiction. We proved the claim.

If $\mathcal{F}_{i-1} = \emptyset$, then we stop and output $\mathcal{W} = \mathcal{W}_{i-1}$. We point out here that when we were examining $F_i$, the $r$-set $(F_i \setminus u) \cup w$ is not in $\mathcal{W}$ in either case.

Recall the definition of $\mathcal{F}'$, i.e., $d_{\mathcal{P}'}(F) \leq \frac{3}{c} x f_{i_0}$ for each $F \in \mathcal{F}'$. We get that each $F \in \mathcal{F}'$ can make at most $\frac{3}{c} x f_{i_0}$ other $r$-sets in $\mathcal{F}_{i-1}$ deleted from $\mathcal{F}_{i-1}$ if $F$ is added to $\mathcal{W}_i$ at time $i$. Recall $|\mathcal{F}'| \geq \frac{cn^r}{3}$. Thus

$$|\mathcal{W}| \geq \frac{|\mathcal{F}'|}{\frac{3}{c} x f_{i_0} + 1} \geq \frac{c^2 n^r}{10 x f_{i_0}}.$$

For each $F \in \mathcal{W}$, we next associate with $F$ a set of $r$-sets $\mathcal{C}(F) \subset \binom{[n]}{r} \setminus \mathcal{W}$. Assume $F = \{v_1, \ldots, v_r\}$. For each $1 \leq i \leq r$ and each $\{A_1, \ldots, A_{r-1}\} \in N_{\mathcal{P}',F}(v_i)$, let $F \cap A_{r-1} = u$. By the construction of $\mathcal{W}$, there is some $w \in A_{r-1} \setminus u$ such that $(F \setminus u) \cup w \notin \mathcal{W}$. The desired vertex $w$ exists by considering when $F$ is moved to $\mathcal{W}$. If $(F \setminus u) \cup w$ is not an edge, then it excludes the possibility that $F$ get covered by the complete $r$-partite $r$-uniform hypergraph with prefix $\{A_1, \ldots, A_{r-1}\}$. We put the $r$-set $(F \setminus u) \cup w$ in $\mathcal{C}(F)$. For an example, see Figure 3. We will call each $R \in \mathcal{C}(F)$ a *certificate* for $F$. We note that if $R \in \mathcal{C}(F)$, then $|F \cap R| = r - 1$ and the symmetric difference $F \triangle R$ is in $A_{r-1}$. We have $|\mathcal{C}(F)| \leq \frac{3}{c} x f_{i_0}$ as

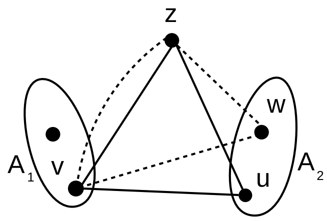


Figure 3: An example with $r = 3$, $F = \{u, v, z\}$, $P = \{A_1, A_2\} \in N_{\mathcal{P}'}(F)$, and $\{v, w, z\} \in \mathcal{C}(F)$.

the assumption for $|N_{\mathcal{P}'}(F)|$ for each $F \in \mathcal{F}'$. The lemma is proved. □

We next use Lemma 3 to show with high probability the number of $r$-sets $F \in \mathcal{W}$ such that $F$ is an edge in $H \in H^{(r)}(n,p)$ and $F$ is not contained in any nontrivial complete $r$-partite $r$-uniform hypergraph with prefix from $\mathcal{P}'$ is large. Since we will apply the union bound, we require the error probability to be sufficiently tiny. We observe that an $r$-set $C$ could serve as the certificate for many $r$-sets from $\mathcal{W}$. Therefore, change the outcome of the trial of this kind of $C$ will greatly change the value of the random variable. This is troublesome when we apply Lemma 3. In the next lemma, we will find a way around this difficulty.

**Lemma 9** *Assume* $1/\log\log\log\log n \leq p \leq 1/2$, $|\mathcal{P}' \setminus \mathcal{P}_{i_0}| = xn^{r-1}$ *with* $x \geq 0.01c$, $H \in H^{(r)}(n,p)$, *and Lemma 1 holds. With probability at least* $1 - 2\exp(-n^{r-0.92})$, *the number of edges in* $E(H) \cap \mathcal{F}$ *which is not contained in any complete* $r$*-partite* $r$*-uniform hypergraph with prefix from* $\mathcal{P}'$ *is greater than*

$$\frac{c^2 n^r p (1-p)^{\frac{3}{c} x f_{i_0}}}{12 x f_{i_0}}.$$

**Proof:** We will work on the collection of $r$-sets $\mathcal{W}$ given by Lemma 8. Let $Y$ be the number of $r$-sets from $\mathcal{W}$ which is an edge in $H \in H^{(r)}(n,p)$ and is not covered by any complete $r$-partite $r$-uniform hypergraph with prefix from $\mathcal{P}'$. For each $F \in \mathcal{W}$ and $R \in \mathcal{C}(F)$, we recall that $R$ is a certificate for $F$. We remark that an $r$-set $R$ could be a certificate for more than one $r$-set $F \in \mathcal{W}$.

Let $\mathcal{C} = \cup_{F \in \mathcal{W}} \mathcal{C}(F)$. For an $r$-set $R \in \mathcal{C}$, if $R \in \mathcal{C}(F)$ for more than $n^{0.45}$ sets $F \in \mathcal{W}$, then we call $R$ a *bad certificate*. Let $\mathcal{C}_1$ be the collection of bad certificates. For each $F \in \mathcal{W}$, we set $\mathcal{C}'(F) = \mathcal{C}(F) \setminus \mathcal{C}_1$. We fix the selection of $\mathcal{W}$, $\mathcal{C}'(F)$ for each $F \in \mathcal{W}$, and the collection of bad certificates $\mathcal{C}_1$. We sample all possible edges and let $X_F$ be the indicator random variable for the event that $F$ is an edge in $H \in H^{(r)}(n,p)$ and $R$ is not a edge in $H \in H^{(r)}(n,p)$ for each $R \in \mathcal{C}'(F)$. We note $X_F = 1$ indicates that $F$ is not covered by any nontrivial complete $r$-partite $r$-uniform hypergraphs with the prefix from $\mathcal{P}'$ and containing no bad certificate. To see this, suppose $F$ is covered by some $G$ with vertex parts $A_1, \ldots, A_{r-1}, A_r$ and $G$ dose not contain any bad certificate. By the definition of $\mathcal{C}'(F)$, there is some $F' \in \mathcal{C}'(F) \cap \prod_{i=1}^r A_i$. Since $X_F = 1$, we get $F'$ is not an edge. Thus $A_1, \ldots, A_r$ do not form a complete $r$-partite $r$-uniform hypergraph, which is a contradiction.

We define $X = \sum_{F \in \mathcal{W}} X_F$. Applying Lemma 3 with $\mathcal{F} = \mathcal{W}$, $k(n) = \frac{3}{c} x f_{i_0}$, and $l(n) = n^{0.45}$, we obtain with probability at least $1 - 2\exp(-n^{r-0.92})$,

$$X \geq \frac{c^2 n^r p (1-p)^{\frac{3}{c} x f_{i_0}}}{11 x f_{i_0}}.$$

We note $n^{r-0.01}$ in Lemma 3 is a lower term as the definition of $f_{i_0}$ and the assumption for $p$. We use $\mathcal{F}''$ to denote those $r$-sets $F \in \mathcal{W}$ such that $X_F = 1$. The argument above gives that with probability at least $1 - 2\exp(-n^{r-0.92})$, we have

$$|\mathcal{F}''| \geq \frac{c^2 n^r p (1-p)^{\frac{3}{c} x f_{i_0}}}{11 x f_{i_0}}.$$

Let us condition on this.

We note that an edge in $\mathcal{F}''$ could be covered by some complete $r$-partite $r$-uniform hypergraph which contains a bad certificate. We next prove an upper bound on the number of such edges. This upper bound works for all samplings of edges.

Let $A_1, \ldots, A_r$ be the vertex parts of such a complete $r$-partite $r$-uniform hypergraph $G$. Suppose $\{A_1, \ldots, A_{r-1}\} \in \mathcal{P}'$. We define

$$A'_r = \{v_r \in A_r : \text{ there are } v_1 \in A_1, \ldots, v_{r-1} \in A_{r-1} \text{ such that } \{v_1, \ldots, v_r\} \in \mathcal{F}''\}.$$

The number of edges from $\mathcal{F}''$ covered by $G$ is at most $|A'_r| \prod_{i=1}^{r-1} |A_i|$. We next relate the number of bad certificates contained in $G$ to the size of $A'_r$.

For each $w \in A'_r$, by the definition of $A'_r$, there is some $F = \{v_1, \ldots, v_{r-1}, w\} \in A_1 \times \cdots \times A_{r-1} \times \{w\}$ such that $F \in \mathcal{F}''$. We observe $\{A_1, \ldots, A_{r-1}\} \in N_{\mathcal{P}'}(F)$. Let $\{v_1, \ldots, v_{r-2}, z, w\}$ be the certificate of $F$ associated with $\{A_1, \ldots, A_{r-1}\}$, where $z \in A_{r-1}$. We notice $\{v_1, \ldots, v_{r-2}, z, w\}$ must be a bad certificate. Otherwise, as $F \in \mathcal{F}''$, we get $\{v_1, \ldots, v_{r-2}, z, w\}$ is a non-edge. Then $A_1, \ldots, A_r$ do not form a complete $r$-partite $r$-uniform hypergraph which is a contradiction. Therefore, each $w \in A'_r$ gives at least one bad certificate from $A_1 \times \cdots \times A_{r-1} \times \{w\}$ and these bad certificates are distinct for different $w \in A'_r$. We obtain that the number of bad certificates in $G$ is at least $|A'_r|$.

We divide those hypergraphs which contain a bad certificate into two subsets $\mathcal{H}_1$ and $\mathcal{H}_2$, where $\mathcal{H}_1 = \{G : |A'_r| \leq n^{0.9}\}$ and $\mathcal{H}_2 = \{G : |A'_r| > n^{0.9}\}$. We note that each $G \in \mathcal{H}_2$ contains at least $n^{0.9}$ bad certificates as the analysis above. We next prove absolute upper bounds for the number of edges from $\mathcal{F}''$ which are covered by $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. We observe that each $H \in \mathcal{H}_1$ can cover at most $|A'_r| \prod_{j=1}^{r-1} |A_j| \leq (r+1) n^{0.9} \log n$ edges from $\mathcal{F}''$ as we assume Lemma 1 holds. There are at most $t < n^{r-1}$ of them as assumptions in Theorem 4. Therefore, $\mathcal{H}_1$ covers at most $(r+1) n^{r-0.1} \log n$ edges from $\mathcal{F}''$.

We need an upper bound for the number of bad certificates in total. We consider pairs $(F, R)$ such that $F \in \mathcal{W}$ and $R \in \mathcal{C}(F)$. As $|\mathcal{C}(F)| \leq \frac{3}{c} x f_{i_0}$ for each $F \in \mathcal{W}$, the number of such pairs is less than

$$|\mathcal{W}| \frac{3}{c} x f_{i_0} < \frac{3}{c} x f_{i_0} n^r < n^r \log n,$$

here we used the fact $|\mathcal{W}| < n^r$ and the definition of $f_{i_0}$. The definition of a bad certificate together with a simple double counting method yield that the number of bad certificates is at most $n^{r-0.55} \log n$. Since each bad certificate (viewed as an edge) is contained in at most one $G \in \mathcal{H}_2$ (we are considering the partition of edges) and each $G \in \mathcal{H}_2$ contains at least $n^{0.9}$ bad certificates, we have $|\mathcal{H}_2| \leq n^{r-1.45} \log n$. The number of edges contained in each $G \in \mathcal{H}_2$ has an absolute upper bound $(r+1) n \log n$. Therefore, the number of edges from $\mathcal{F}''$ which are covered by $\mathcal{H}_2$ is at most $(r+1) n^{r-0.45} \log^2 n$.

Thus those complete $r$-partite $r$-uniform hypergraphs containing a bad certificate cover at most $(r+1)n^{r-0.1}\log n + (r+1)n^{r-0.45}\log^2 n$ edges from $\mathcal{F}''$. Therefore, we have

$$\begin{aligned} Y &\geq \frac{c^2 n^r p(1-p)^{\frac{3}{c}xf_{i_0}}}{11xf_{i_0}} - (r+1)n^{r-0.1}\log n - (r+1)n^{r-0.45}\log^2 n \\ &> \frac{c^2 n^r p(1-p)^{\frac{3}{c}xf_{i_0}}}{12xf_{i_0}}, \end{aligned}$$

the proof of this lemma is complete. □

We are now ready to prove Theorem 4.

**Proof of Theorem 4:** To simplify the notation, we define the following prefix sets:

$$\mathcal{Q}_1 = \mathcal{P}' \setminus \mathcal{P}_{i_0} = \left\{ P_i \in \mathcal{P} : \prod_{j=1}^{r-1} |A_j^i| < f_{i_0} \right\},$$

$$\mathcal{Q}_2 = (\mathcal{P} \setminus \mathcal{P}') \cup \mathcal{P}_{i_0} = \left\{ P_i \in \mathcal{P} : \prod_{j=1}^{r-1} |A_j^i| \geq f_{i_0} \right\},$$

$$\mathcal{Q}_3 = \mathcal{P} \setminus \mathcal{P}' = \left\{ P_i \in \mathcal{P} : \prod_{j=1}^{r-1} |A_j^i| \geq f_{i_0+1} \right\}.$$

Let $c_1(n) = 2, c_2(n) = f_{i_0}$, and $c_3(n) = f_{i_0+1}$. For $H \in H^{(r)}(n,p)$ and each $i \in \{1,2,3\}$, let $\mathcal{Z}_i$ be the event that $g(H, \mathcal{Q}_i) \leq |\mathcal{Q}_i| c_i(n) p^{c_i(n)} n + 2n^{r-0.3}$. Lemma 2 implies that with probability at least $1 - 6\exp(-n^{r-0.8})$, all events $\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3$ hold simultaneously. We condition on these three events. We note that for each $i \in \{1,2,3\}$, the number of edges from $\mathcal{F}$ which are covered by complete $r$-partite $r$-uniform hypergraphs with prefix from $\mathcal{Q}_i$ is bounded above by the function $g(H, \mathcal{Q}_i)$.

We proceed to prove $|\mathcal{Q}_1| \geq 0.01cn^{r-1}$. Suppose not. Because the event $\mathcal{Z}_1$ occurs, the number of edges from $\mathcal{F}$ covered by those complete $r$-partite $r$-uniform hypergraphs with prefix $P \in \mathcal{Q}_1$ is at most $(2+o(1))p^2 n|\mathcal{Q}_1| \leq \left(0.02cp^2 + o(1)\right) n^r$, here $2n^{r-0.2}$ is a lower term as we assume $p \geq 1/\log\log\log\log n$. A simple application of Theorem 2 yields that with probability at least $1 - \exp(-cpn^r/8)$ the number of $r$-sets in $\mathcal{F}$ being an edge in $H \in H^{(r)}(n,p)$ is at least $\frac{cpn^r}{2}$. Therefore, the number of edges covered by those complete $r$-partite $r$-uniform hypergraphs with prefix from $\mathcal{Q}_2$ is at least $\frac{cpn^r}{4}$. As the event $\mathcal{Z}_2$, we get

$$|\mathcal{Q}_2| \geq \frac{(\frac{pc}{4} + o(1))n^r}{f_{i_0} p^{f_{i_0}} n} > n^{r-1}$$

when $n$ is large enough. This is a contradiction to the assumption $|\mathcal{P}| \leq n^{r-1}$. Therefore, as long as events $\mathcal{Z}_1$ and $\mathcal{Z}_2$ as well as the lower bound for the number of edges from $\mathcal{F}$ hold, we have $|\mathcal{Q}_1| \geq 0.01cn^{r-1}$ which is one of the assumptions in Lemma 9.

Recall Lemma 9. Those uncovered edges given by Lemma 9 must be covered by complete $r$-partite $r$-uniform hypergraphs with prefix from $\mathcal{Q}_3$. As the event $\mathcal{Z}_3$, we get

$$|\mathcal{Q}_3| \geq \frac{c^2 n^{r-1} p(1-p)^{\frac{3}{c}xf_{i_0}}}{13xf_{i_0} f_{i_0+1} p^{f_{i_0+1}}},$$

we note $n^{r-0.3}$ is a lower order term. Recall $1/\log\log\log\log n \le p \le 1/2$ and $f_i = K^i 2^{q(n)}$. We get

$$
\begin{aligned}
|\mathcal{P}| = |\mathcal{P}'| + |\mathcal{Q}_3| &\ge xn^{r-1} + \frac{c^2 n^{r-1} p(1-p)^{\frac{3}{c}xf_{i_0}}}{13xf_{i_0}f_{i_0+1}p^{f_{i_0+1}}} \\
&\ge \frac{c^2 n^{r-1} p(1-p)^{\frac{3}{c}xf_{i_0}}}{13xf_{i_0}f_{i_0+1}p^{f_{i_0+1}}} \\
&\ge \frac{c^2 n^{r-1} p 2^{f_{i_0+1}-\frac{3}{c}xf_{i_0}}}{13xf_{i_0}f_{i_0+1}} \qquad (3) \\
&\ge \frac{c^2 n^{r-1} p 2^{f_{i_0+1}-\frac{3}{c}f_{i_0}}}{13f_{i_0}f_{i_0+1}} \qquad (4) \\
&= \frac{c^2 n^{r-1} p 2^{K^{i_0+1}2^{q(n)}-\frac{3}{c}K^{i_0}2^{q(n)}}}{13K^{2i_0+1}2^{2q(n)}} \\
&= \frac{c^2 n^{r-1} p 2^{\frac{1}{c}K^{i_0}2^{q(n)}}}{13K^{2i_0+1}2^{2q(n)}} \qquad (5) \\
&> n^{r-1},
\end{aligned}
$$

when $n$ is large enough. We used $p \le 1/2$ to get inequality (3), $x \le 1$ to get inequality (4), and $K = \frac{4}{c}$ to get inequality (5).

Therefore, as long as events $\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3$ occur, the lower bound for the number of edges in $\mathcal{F} \cap E(H)$ holds, and Lemma 9 holds, we get a contradiction. With probability at most $2\exp(-n^{r-0.92}) + 6\exp(-n^{r-0.8}) + \exp(-cpn^r/8) \le 3\exp(-n^{r-0.92})$, one of them does not hold, this completes the proof of the theorem. □

# 4 Proof of Theorem 1

Before we prove the main theorem, we need to show an upper bound on the number of choices for the prefix set $\mathcal{P}$.

**Lemma 10** *Suppose $\mathcal{P} = \{P_1, \ldots, P_q\}$, where $P_i = \{A_1^i, \ldots, A_{r-1}^i\}$ and $1 \le \prod_{j=1}^{r-1} |A_j^i| < (r+1)\log n$ for each $1 \le i \le q$. The number of choices for $\mathcal{P}$ with $|\mathcal{P}| \le n^{r-1}$ is bounded from above by $n^{(r+3)n^{r-1}\log n}$ when $n$ is large enough.*

**Proof:** We shall show the desired upper bound step by step. We have at most $n^{r-1}$ choices for the size of $\mathcal{P}$. First, we fix the size of $\mathcal{P}$. We will establish an absolute upper bound on the number of choices for each element $P_i$ of $\mathcal{P}$. For each $P_i = \{A_1^i, \ldots, A_{r-1}^i\} \in \mathcal{P}$, we have $t_i = |\cup_{j=1}^{r-1} A_j^i| \le (r+1)\log n + r < (r+2)\log n$ as $\prod_{j=1}^{r-1} |A_j^i| \le (r+1)\log n$. Therefore, $\cup_{j=1}^{r-1} A_j^i \in \binom{[n]}{\le (r+2)\log n}$, which implies that the number of choices for $\cup_{j=1}^{r-1} A_j^i$ is at most $n^{(r+2)\log n}$. We fix the selection of $\cup_{j=1}^{r-1} A_j^i$ and wish to partition it into $r-1$ disjoint parts $A_j^i$. Let $a_j = |A_j^i|$ for $1 \le j \le r-1$. Then we have $a_1 + \ldots + a_{r-1} = t_i$. The number of choices for the size of $a_1, \ldots, a_{j-1}$ equals the number of solutions to the equation $a_1 + \ldots + a_{j-1} = t_i$. Since $a_j \ge 1$, we have at most $\binom{t_i}{r-1}$ choices for $a_1, \ldots, a_{j-1}$ , which can be bounded from above by $((r+2)\log n)^{r-1}$ as $t_i \le (r+2)\log n$. If we fix the size of each $A_j^i$, then the number of ways to partition $\cup_{j=1}^{r-1} A_j^i$ into $A_1^i, \ldots, A_{r-1}^i$ equals $\binom{t_i}{a_1, \ldots, a_{j-1}}$, which is at most $t_i! \le ((r+2)\log n)^{(r+2)\log n}$. Therefore, the number of choices for $P_i$ is at most

$$n^{(r+2)\log n}\left((r+2)\log n\right)^{(r+2)\log n + r - 1}.$$

Recall the assumption $|\mathcal{P}| \leq n^{r-1}$. We get that the number of choices for $\mathcal{P}$ is at most

$$n^{r-1}\left(n^{(r+2)\log n}\left((r+2)\log n\right)^{(r+2)\log n+r-1}\right)^{|\mathcal{P}|} < n^{(r+3)n^{r-1}\log n},$$

when $n$ is sufficiently large. □

We are ready to prove Theorem 1.

**Proof of the upper bound:** We shall exhibit an explicit decomposition of each $r$-uniform hypergraph with $n$ vertices using at most $(1-\pi(K_r^{(r-1)})+o(1))\binom{n}{r-1}$ trivial complete $r$-partite $r$-uniform hypergraphs.

For each $r \geq 3$, let $G=([n],E)$ be an $(r-1)$-uniform hypergraph which has $\mathrm{ex}(n, K_r^{(r-1)})$ edges and does not contain $K_r^{(r-1)}$ as a subhypergraph. Obviously, $G$ is well-defined. Let $G'$ be the complement of $G$. Therefore, $E(G')=\binom{[n]}{r-1}\setminus E(G)$. We observe that an independent set of size $r$ in $G'$ will be a $K_r^{(r-1)}$ in $G$. As $G$ does not contain $K_r^{(r-1)}$, we get each $F \in \binom{[n]}{r}$ contains at least one edge of $G'$.

Suppose $q=|E(G')|$ and we list edges in $G'$ as $e_1,\ldots,e_q$. For each $r$-uniform hypergraph $H$ with $n$ vertices, we will show that $H$ can be decomposed into at most $q$ trivial complete $r$-partite $r$-uniform hypergraphs as follows.

Let $H_0=H$ and we will define a sequence of complete $r$-partite $r$-uniform hypergraphs recursively. For each $1 \leq i \leq q$, we assume the edge $e_i$ in $G'$ is $\{v_1,v_2,\ldots,v_{r-1}\}$. The key observation is the following. For an edge $F \in E(H)$, if $F$ is contained in a trivial complete $r$-partite $r$-uniform hypergraph with vertex parts $\{v_1\}\times\cdots\times\{v_{r-1}\}\times V_r$, then the set $\{v_1,\ldots,v_{r-1}\}$ must be a subset of $F$. We define $\mathcal{F}_i=\{F\in E(H_{i-1}) : e_i \subset F\}$ and $A_r=\cup_{F\in\mathcal{F}_i} F\setminus e_i$. If $A_r\neq\emptyset$, then the $i$-th complete $r$-partite $r$-uniform hypergraph $H_i'$ will have vertex parts $\{v_1\},\ldots,\{v_{r-1}\},A_r$. If the set $\mathcal{F}_i$ is empty, then we do not define $H_i'$. We set $E(H_i)=E(H_{i-1})\setminus E(H_i')$ for each $1\leq i\leq q-1$.

The definition of $H_i$'s ensures that each edge in $H$ is in exactly one of these trivial complete $r$-partite $r$-uniform hypergraphs. Clearly, for sufficiently large $n$, we have $q=(1-\pi(K_r^{(r-1)})+o(1))\binom{n}{r-1}$. Since the decomposition above applies to all $H$, it also works for the random hypergraph $H\in H^{(r)}(n,p)$.

**Proof of the lower bound:** We assume Lemma 1 holds. Thus each complete $r$-partite $r$-uniform hypergraph with vertex parts $A_1,A_2,\ldots,A_r$ satisfies $\prod_{i=1}^{r-1}|A_i| < (r+1)\log n$ provided $|A_1|\leq|A_2|\leq\ldots\leq|A_r|$. For any fixed small positive constant $\epsilon$, we shall show that the probability $f(H)\leq(1-\pi(K_r^{(r-1)})-\epsilon)\binom{n}{r-1}$ is small, where $H\in H^{(r)}(n,p)$.

Fix a prefix set $\mathcal{P}=\{P_1,\ldots,P_t\}$, where $P_i=\{A_1^i,\ldots,A_{r-1}^i\}$, $\prod_{j=1}^{r-1}|A_j^i|<(r+1)\log n$ for each $1\leq i\leq t$ and $t\leq(1-\pi(K_r^{(r-1)})-\epsilon)\binom{n}{r-1}$. Let $\mathcal{X}$ denote the event that there are $t$ sets $A_r^1,\ldots,A_r^t$ such that

$$E(H)=\bigsqcup_{i=1}^{t}\prod_{j=1}^{r}A_j^i,$$

provided $H\in H^{(r)}(n,p)$. Here $A_1^i,\ldots,A_r^i$ form a complete $r$-partite $r$-uniform hypergraph for each $1\leq i\leq t$. We assume the first $s$ of them are trivial complete $r$-partite $r$-uniform hypergraphs, i.e., $|A_1^i|=\cdots=|A_{r-1}^i|=1$ for each $1\leq i\leq s$.

As we did for proving the upper bound, we define an $(r-1)$-uniform hypergraph $G$ such that $V(G)=[n]$ and $E(G)=\binom{[n]}{r-1}\setminus(\cup_{i=1}^{s}\prod_{j=1}^{r-1}A_j^i)$. We note $|A_j^i|=1$ for each $1\leq i\leq s$ and $1\leq j\leq r-1$. As $s\leq t\leq(1-\pi(K_r^{(r-1)})-\epsilon)\binom{n}{r-1}$, we get $|E(G)|\geq(\pi(K_r^{(r-1)})+\epsilon)\binom{n}{r-1}$. By the supersaturation result for hypergraphs (see Theorem 1 in [11]), we get that there are at least $c(\epsilon)n^r$ copies of $K_r^{(r-1)}$ in $G$. Let $G'$ be the complement of $G$ and $\mathcal{F}$ be the collection of independent sets with size $r$ in $G'$. We have $|\mathcal{F}|\geq c(\epsilon)n^r$. We observe that if $H\in H^{(r)}(n,p)$, then edges in $\mathcal{F} \cap E(H)$ must be covered by those nontrivial complete $r$-partite $r$-uniform hypergraphs in the partition. Let $\mathcal{Y}$ be the event that each $F \in \mathcal{F} \cap E(H)$ is contained in exactly one of the last $t-s$ nontrivial complete $r$-partite $r$-uniform hypegraphs. We have two cases depending on the range of the probability $p$.

**Case 1:** $1/\log\log\log\log n \le p \le 1/2$. Applying Theorem 4 with $\mathcal{P}' = \{P_{s+1}, \dots, P_t\}$, we get that $\mathcal{Y}$ holds with probability at most $3\exp(-n^{r-0.92})$. This implies that the event $\mathcal{X}$ occurs with probability at most $3\exp(-n^{r-0.92})$. By Lemma 10, there are at most $n^{(r+3)n^{r-1}\log n}$ choices for $\mathcal{P}$ satisfying the desired properties. Applying the union bound, we get that the probability $f(H) \le (1-\pi(K_r^{(r-1)})-\epsilon)\binom{n}{r-1}$ is at most $3\exp(-n^{r-0.92})n^{(r+3)n^{r-1}\log n} < \exp(-n^{r-0.94})$ for any positive constant $\epsilon$.

**Case 2:** $(\log n)^{2.001}/n \le p \le 1/\log\log\log\log n$. We observe that the set $\mathcal{F}$ is determined by the prefix set $\mathcal{P}$. Therefore, Lemma 10 also gives an upper bound on the number of possible choices of $\mathcal{F}$. A simple application of Theorem 2 yields that with high probability $|\mathcal{F} \cap E(H)| \ge \frac{pcn^r}{4}$ for all $\mathcal{F}$ with $|\mathcal{F}| \ge n^r/\log\log n$. Edges in $\mathcal{F} \cap E(H)$ must be covered by the last $t-s$ nontrivial complete $r$-partite $r$-uniform hypergraphs. Since $t-s \le \binom{n}{r-1}$, Lemma 5 tells us that the event $\mathcal{Y}$ occurs with probability at most $2^{-0.05pc\log(1/p)n^r}$. This also implies that the event $\mathcal{X}$ occurs with probability at most $2^{-0.05pc\log(1/p)n^r}$. By Lemma 5 and the union bound, we get the probability $f(H) \le (1-\pi(K_r^{(r-1)})-\epsilon)\binom{n}{r-1}$ is at most $2^{-0.05pc\log(1/p)n^r} n^{(r+3)n^{r-1}\log n} \le 2^{-0.04pc\log(1/p)n^r}$ as $np \ge (\log n)^{2.001}$ and $c$ is a constant.

The proof of the theorem is finished. □

# 5 Concluding remarks

In this paper, we studied the problem of partitioning the edge set of a random $r$-uniform hypergraph into edge sets of complete $r$-partite $r$-uniform hypergraphs. We were able to show if $(\log n)^{2.001}/n \le p \le 1/2$ and $H \in H^{(r)}(n,p)$, then with high probability $f(H) = (1-\pi(K_r^{(r-1)})+o(1))\binom{n}{r-1}$. For the case of $r=2$, results from [2] and [6] assert that if $p$ is a constant, $p \le 1/2$, and $G \in G(n,p)$, then with high probability $n - o((\log n)^{3+\epsilon}) \le f(G) \le (2+o(1))\log_{1/(1-p)} n$ for any positive constant $\epsilon$. For $G \in G(n, 1/2)$, authors of [3] proved a better upper bound for $f(G)$. For sparse random graphs, Alon [2] determined the order of magnitude of the second term of $f(G)$. However, we do not have any information on the second order term of $f(H)$ for $r \ge 3$. This leads to the following question.

**Problem 1:** Determine the magnitude of the second order term of $f(H)$ for $H \in H^{(r)}(n,p)$ and $r \ge 3$.

We note that we were only able to determine the leading coefficient of $f(H)$ for $p \ge (\log n)^{2.001}/n$ and $H \in H^{(r)}(n,p)$. A natural question is to prove similar results for other range of the probability $p$.

We recall that for a graph $G$, the *strong bipartition number* $\mathrm{bp}'(G)$ of $G$ is the minimum number of nontrivial complete bipartite subgraphs (which are not stars) of $G$ such that each edge of $G$ is in exactly one of them. This parameter was introduced by Chung and the author in [6] when they were studying the bipartition number of random graphs. In particular, they proved that if $p$ is a constant, $p \le 1/2$, and $G \in G(n,p)$, then $\mathrm{bp}'(G) \ge 1.0001n$ with high probability. For sparse random graphs, Alon [2] proved a better lower bound. Namely, he showed with high probability $\mathrm{bp}'(G) \ge 2n$ if $G \in G(n,p)$. We remark here that our methods for proving Theorem 4 implicitly yield the following theorem.

**Theorem 5** *If $p$ is a constant, $p \leq 1/2$, and $G \in G(n,p)$, then with high probability*

$$\frac{\mathrm{bp}'(G)}{n} \to \infty \text{ as } n \to \infty.$$

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