STABILIZATION OF REGIME-SWITCHING PROCESSES BY FEEDBACK CONTROL BASED ON DISCRETE TIME OBSERVATIONS*

JINGHAI SHAO†

Abstract. This work aims to extend X.R. Mao's work [*Automatica J. IFAC*, 49 (2013), pp. 3677–3681] on stabilization of hybrid stochastic differential equations by discrete-time feedback control. In X.R. Mao's work, the feedback control depends on discrete-time observation of the state process but on continuous-time observation of the switching process, while, in this work, we study the feedback control depending on discrete-time observations of the state process and the switching process. Our criteria depend explicitly on the regular conditions of the coefficients of the stochastic differential equation and on the stationary distribution of the switching process. The sharpness of our criteria is shown through studying the stability of linear systems, which also shows explicitly that the stability of hybrid stochastic differential equations depends essentially on the long time behavior of the switching process.

Key words. stability, regime-switching, feedback control, discrete-time observation

AMS subject classifications. 60H10, 93D15, 60J10

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1. Introduction. Regime-switching processes have received much attention lately, and they can provide more realistic models in many fields such as biology, mathematical finance, etc. See, for example, [6, 8, 16, 27]. The regime-switching process consists of two components $(X(t), \Lambda(t))$. Usually, $(X(t))$ describes the continuous dynamics and $(\Lambda(t))$ describes the random switching device. We call $(X(t))$ the state process and $(\Lambda(t))$ the switching process. Due to the complexity of the system, many properties of regime-switching processes such as recurrence, stability, and the strong Feller property show special characteristics compared with diffusion processes or Markov chains. Refer to, for instance, [4, 17, 18, 20, 21, 27] for the study of recurrent property, and [2, 11, 19, 26, 28] for the study of stability. In particular, [16] has studied the stability and asymptotic stability of functional SDEs (FSDEs) with Markovian regime switching, which is closely related to the topic of this work.

This paper investigates the stabilization of regime-switching diffusion processes by feedback control based on discrete-time observations of the state process and the switching process. Given an unstable regime-switching diffusion process of the form (1.1) with $b = 0$, it is required to find a feedback control $b(X(t), \Lambda(t))$ so that the controlled system

$$\mathrm{d}X(t) = \big(a(X(t), \Lambda(t)) - b(X(t), \Lambda(t))\big)\mathrm{d}t + \sigma(X(t), \Lambda(t))\mathrm{d}W(t) \tag{1.1}$$

becomes stable. Here $(\Lambda(t))$ is a continuous time Markov chain on a finite or infinite countable state space $\mathcal{S} = \{1, 2, \ldots, N\}$, $2 \le N \le \infty$, and independent of the Brownian motion $(W(t))$. Such a feedback control needs continuous observation of the state $X(t)$ and the switching process $\Lambda(t)$ for all $t \ge 0$. For the sake of saving costs and

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†Tianjin University and School of Mathematical Sciences, Beijing Normal University, Beijing, China (shaojh@bnu.edu.cn).

being more realistic, Mao in [12] studied the stabilization problem by discrete-time feedback controls, which develops the corresponding studies for deterministic differential equations (see, e.g., [1, 7]). Namely, design a discrete-time feedback control $b(X([t/\tau]\tau),\Lambda(t))$, where $[t/\tau]$ denotes the integer part of t/τ, so that the controlled system

$$\mathrm{d}X(t)=\big(a(X(t),\Lambda(t))-b(X([t/\tau]\tau),\Lambda(t))\big)\mathrm{d}t+\sigma(X(t),\Lambda(t))\mathrm{d}W(t) \tag{1.2}$$

becomes stable. Here τ is a positive constant standing for the time interval between two consecutive observations. Mao [12] showed that when τ is less than some upper bound τ^*, which depends on the Lipschitz constants and bounds of coefficients a, b, σ, the controlled system (1.2) is mean-square stable. Mao et al. in [13] provided a better bound of τ, especially an explicit bound for linear hybrid SDEs, and in [25] You et al. weakened the global Lipschitz assumption on coefficients and investigated further the asymptotic stabilization of such controlled system.

A natural and important question left by the works [12, 13, 25] is to stabilize the given system by a feedback control $b(X([t/\tau]\tau),\Lambda([t/\tau]\tau))$ depending only on the discrete-time observations of the state $(X(t))$ and switching process $(\Lambda(t))$. Geromel and Gabriel [5] pointed out the necessity to design the feedback control based on $X([t/\tau]\tau)$ and $\Lambda([t/\tau]\tau)$ from the numerical point of view when studying the state feedback sampled-data control design problem of Markov jump linear systems. Also, there exists an essential difference in method between stabilizing a given system based on $X([t/\tau]\tau)$ or based on $\Lambda([t/\tau]\tau)$. Heuristically, since $X([t/\tau]\tau)$ tends to $X(t)$ as τ tends to 0, the effect caused by $X([t/\tau]\tau)$ can be controlled in terms of $X(t)$ and $X(t)-X([t/\tau]\tau)$. While $(\Lambda(t))$ is a jumping process, which yields that $\Lambda(t-)$ and $\Lambda(t+)$ may be at a different state of $\mathcal{S}$. As a consequence, it is useless to expect that the effect caused by $\Lambda([t/\tau]\tau)$ is similar to that caused by $\Lambda(t)$ no matter how small τ is.

Precisely, we study the stability of the following system:

$$\begin{aligned}&\mathrm{d}X(t)=[a(X(t),\Lambda(t))-b(\Lambda(\delta(t)))X(\delta(t))]\mathrm{d}t+\sigma(X(t),\Lambda(t))\mathrm{d}W(t),\\ &X(0)=x(0)\in\mathbb{R}^d,\end{aligned} \tag{1.3}$$

where $\delta(t)=[t/\tau]\tau$, $\tau>0$, is a constant, and $(\Lambda(t))$ is a Markov chain on a finite or infinite countable state space $\mathcal{S}=\{1,2,\dots,N\}$ $(2\le N\le\infty)$ with a regular irreducible Q-matrix (q_{ij}). $(W(t))$ is a Wiener process in $\mathbb{R}^d$ and independent of $(\Lambda(t))$. When $N=\infty$, the process (Λ_t) is assumed to be nonexplosive throughout this work. The coefficients $a:\mathbb{R}^d\times\mathcal{S}\to\mathbb{R}^d$, $b:\mathcal{S}\to[0,\infty)$, $\sigma:\mathbb{R}^d\times\mathcal{S}\to\mathbb{R}^{d\times d}$ satisfy the following conditions:

(H1) There exist nonnegative functions $C(\cdot)$ and $c(\cdot)$ on $\mathcal{S}$ such that

$$c(i)|x|^2\le 2\langle a(x,i),x\rangle+\|\sigma(x,i)\|_{\mathrm{HS}}^2\le C(i)|x|^2,\quad (x,i)\in\mathbb{R}^d\times\mathcal{S},$$

where $\|\sigma(x,i)\|_{\mathrm{HS}}^2=\mathrm{trace}(\sigma\sigma^*)(x,i)$ with σ^* denoting the transpose of the matrix σ.

(H2) There exists a positive constant $\bar{K}$ such that

$$|a(x,i)-a(y,i)|+\|\sigma(x,i)-\sigma(y,i)\|_{\mathrm{HS}}\le\bar{K}|x-y|,\ \ x,y\in\mathbb{R}^d,\ \ i\in\mathcal{S}.$$

(H3) There exists a positive constant M_a such that $|a(x,i)|\le M_a|x|$ for all $(x,i)\in\mathbb{R}^d\times\mathcal{S}$.

Conditions (H1) and (H2) help to guarantee the existence and uniqueness of a strong solution of the FSDE (1.3), which will be discussed in section 2. Our main result is the following theorem.

THEOREM 1.1. *Let* $(X(t),\Lambda(t))$ *satisfy* (1.3). *Assume* (H1)–(H3) *hold, and* $\bar C = \max_{i\in\mathcal S} C(i) < \infty$, $\bar b = \max_{i\in\mathcal S} b(i) < \infty$. *Set*

$$K(\tau) = 2(2\bar C + 3M_a + \bar b^2)\tau e^{(2\bar C + M_a + 1)\tau}. \tag{1.4}$$

Assume τ *is sufficiently small so that* $K(\tau) < 1$. *Assume that* $(\Lambda(t))$ *is ergodic and denote by* $(\pi_i)_{i\in\mathcal S}$ *its stationary distribution.*

(i) *If*

$$\sum_{i\in\mathcal S}\pi_i\left(C(i) - 2\left(1 - \left(\frac{K(\tau)}{1-K(\tau)}\right)^{\frac12}\right)b(i)\right) < 0, \tag{1.5}$$

then $\lim_{t\to\infty}\mathbb E|X(t)|^2 = 0$.

(ii) *If*

$$\sum_{i\in\mathcal S}\pi_i\left(c(i) - 2\left(1 + \left(\frac{K(\tau)}{1-K(\tau)}\right)^{\frac12}\right)b(i)\right) > 0, \tag{1.6}$$

then $\lim_{t\to\infty}\mathbb E|X(t)|^2 = \infty$.

To see the sharpness of the criteria in Theorem 1.1, we consider the following linear system on the line:

$$\begin{aligned} &\mathrm dX(t) = \big(a(\Lambda(t))X(t) - b(\Lambda([t/\tau]\tau))X([t/\tau]\tau)\big)\mathrm dt + \sigma(\Lambda(t))X(t)\mathrm dW(t),\\ &X(0) = x\in\mathbb R, \end{aligned} \tag{1.7}$$

where $(\Lambda(t))$ is a continuous time Markov chain on $\mathcal S$ with irreducible Q-matrix $(q_{ij})_{i,j\in\mathcal S}$. Assume $(\Lambda(t))$ is ergodic, and denote by $(\pi_i)_{i\in\mathcal S}$ its stationary distribution.

COROLLARY 1.2. *Suppose that* $\bar C = \max_{i\in\mathcal S}(2a(i)+\sigma(i)^2) < \infty$, $\bar b = \max_{i\in\mathcal S} b(i) < \infty$, $M_a = \max_{i\in\mathcal S}|a(i)| < \infty$. *Set*

$$K(\tau) = 2(2\bar C + 2M_a^2 + \bar b^2 + 1)\tau e^{(2\bar C + 2M_a^2 + 1)\tau}.$$

If

$$\sum_{i\in\mathcal S}\pi_i\left(2a(i) + \sigma(i)^2 - 2\left(1 - \left(\frac{K(\tau)}{1-K(\tau)}\right)^{\frac12}\right)b(i)\right) < 0, \tag{1.8}$$

then $\lim_{t\to\infty}\mathbb E X(t)^2 = 0$. *If*

$$\sum_{i\in\mathcal S}\pi_i\left(2a(i) + \sigma(i)^2 - 2\left(1 + \left(\frac{K(\tau)}{1-K(\tau)}\right)^{\frac12}\right)b(i)\right) > 0, \tag{1.9}$$

then $\lim_{t\to\infty}\mathbb E X(t)^2 = \infty$.

The results of Corollary 1.2 show that our criteria given in Theorem 1.1 are quite sharp. Moreover, these results also explicitly show that the stability of FSDEs with regime switching does depend on the long time behavior of the switching process $(\Lambda(t))$, which is represented by the appearance of the stationary distribution (π_i) of $(\Lambda(t))$. There is no quantity such as (π_i) which appeared in the criteria established in [12, 13, 25].

The present work is organized as follows. In section 2, we prove the existence and uniqueness of the solution of an FSDE (1.3) with state-dependent regime switching. Here, we consider the equation with a more general switching, state-dependent switching, as a preparation for further work, which is of independent interest. In section 3, we provide the proofs of Theorem 1.1 and Corollary 1.2, where, in particular, we need to apply the property of skeleton processes of continuous-time Markov chains. Besides, we also provide a more general result than Theorem 1.1 there. At the end, as an application, we study the stabilization of linear regime-switching diffusion processes on $\mathbb{R}^d$ in section 4.

2. Existence and uniqueness of a strong solution. Equation (1.3) is an FSDE with regime switching. As it is not a standard FSDE with regime switching, we first study the existence and uniqueness of the strong solution of this kind of FSDE with regime switching. We shall consider a more general case, i.e., an FSDE with state-dependent regime switching in an infinite state space.

The following stochastic differential delay equation is well studied by many works (cf. e.g., [9, 15]):

$$\mathrm{d}X(t) = b(X(t), X(t-\tau), t)\mathrm{d}t + \sigma(X(t), X(t-\tau), t)\mathrm{d}W(t),\ X_0 = \xi \in C([-\tau, 0]) \tag{2.1}$$

or

$$\begin{aligned}\mathrm{d}X(t) &= b\left(X(t), \int_{t-\tau}^{t} X(s)\mathrm{d}s, t\right)\mathrm{d}t + \sigma\left(X(t), \int_{t-\tau}^{t} X(s)\mathrm{d}s, t\right)\mathrm{d}W(t),\\ X_0 &= \xi \in C([-\tau, 0]),\end{aligned} \tag{2.2}$$

where τ is a fixed positive constant, $(W(t))$ is a Brownian motion,

$$C([-\tau, 0]) = \{\ell : [-\tau, 0] \to \mathbb{R}^d;\ \text{continuous}\},$$

and $X_t \in C([-\tau, 0])$ is defined by $X_t(\theta) = X(t + \theta)$ for $\theta \in [-\tau, 0]$. The common character of (2.1) and (2.2) is that their coefficients can be viewed as functionals on the product space $\mathbb{R}^d \times C([-\tau, 0]) \times [0, \infty)$. However, the term $X(\delta(t))$ of (1.3) cannot be viewed as a functional on the previous product space. So, a little more care should be paid on the existence and uniqueness of the solution of (1.3). To this aim, let us first consider the following general FSDE without regime switching, then go to deal with the case with regime switching:

$$\mathrm{d}X(t) = b(X(t), X([t/\tau]\tau), t)\mathrm{d}t + \sigma(X(t), X([t/\tau]\tau), t)\mathrm{d}W(t), \quad X(0) = x \in \mathbb{R}^d. \tag{2.3}$$

Under the following conditions, $\forall\, x, y \in \mathbb{R}^d,\ t \in [0, \infty)$,

$$\begin{aligned}|b(x, y, t) - b(x_1, y_1, t)| + \|\sigma(x, y, t) - \sigma(x_1, y_1, t)\|_{\mathrm{HS}} &\le K(|x - x_1| + |y - y_1|),\\ |b(x, y, t)| + \|\sigma(x, y, t)\|_{\mathrm{HS}} &\le K(1 + |x| + |y|)\end{aligned}$$

for some constant $K > 0$; one can prove that FSDE (2.3) admits a unique nonexplosive solution by using the Picard iterations following the line of [10, Chapter 5, Theorem 2.2].

Next, we study a more general FSDE with regime switching than FSDE (1.3). Consider

$$\mathrm{d}X(t)=\big[a(X(t),\Lambda(t))+b(X(\delta(t)),\Lambda(\delta(t)))\big]\mathrm{d}t+\sigma(X(t),\Lambda(t))\mathrm{d}W(t), \tag{2.4}$$

with $X(0)=x$, $\Lambda(0)=i$, $\delta(t)=[t/\tau]\tau$ for some fixed constant $\tau>0$. Here $(\Lambda(t))$ is a jumping process on the state space $\mathcal{S}=\{1,2,\ldots,N\}$, $2\le N\le\infty$, satisfying

$$\mathbb{P}(\Lambda(t+\delta)=l\big|\Lambda(t)=k,X(t)=x)=\begin{cases}q_{kl}(x)\delta+o(\delta) & \text{if } k\neq l,\\ 1+q_{kk}(x)\delta+o(\delta) & \text{if } k=l\end{cases} \tag{2.5}$$

for $\delta>0$. In the rest of this section, for each $x\in\mathbb{R}^d$, we assume $Q_x=(q_{kl}(x))_{kl}$ is irreducible and conservative (i.e., $q_i(x)=-q_{ii}(x)=\sum_{j\neq i}q_{ij}(x)\ \forall\, i\in\mathcal{S}$). When $(q_{kl}(x))$ is independent of x and $(\Lambda(t))$ is independent of $(W(t))$, $(X(t),\Lambda(t))$ is said to be a state-independent regime-switching process; otherwise, it is called a state-dependent one.

Now, we collect the assumptions on the coefficients used later. Here K is a positive constant.

- (A1) $2\langle a(x,i),x\rangle+\|\sigma(x,i)\|_{\mathrm{HS}}^2\le K(1+|x|^2)$, $|b(x,i)|^2\le K(1+|x|^2)\ \ \forall\, x\in\mathbb{R}^d$, $i\in\mathcal{S}$;
- (A2) $|a(x,i)-a(y,i)|+|b(x,i)-b(y,i)|+\|\sigma(x,i)-\sigma(y,i)\|_{\mathrm{HS}}\le K|x-y|\ \ \forall\, x,y\in\mathbb{R}^d$, $i\in\mathcal{S}$;
- (A3) $q_i(x)\le K(i+|x|)\quad\forall\, x\in\mathbb{R}^d$, $i\in\mathcal{S}$;
- (A4) there exist positive constants κ and c_q such that for each fixed $x\in\mathbb{R}^d$ and $i\in\mathcal{S}$, $q_{ij}(x)=0$ for any $j\in\mathcal{S}$ with $|i-j|>\kappa$, and $|q_{kl}(y)-q_{kl}(z)|\le c_q|y-z|$ for all $y,z\in\mathbb{R}^d$, $k,l\in\mathcal{S}$.

We shall use the representation of $(\Lambda(t))$ in terms of the Poisson random measure (cf. [22, Chapter II-2.1] or [21]). Precisely, for each $x\in\mathbb{R}^d$, construct a family of intervals $\{\Gamma_{ij}(x);\, i,j\in\mathcal{S}\}$ on the real line in the following manner:

$$\Gamma_{12}(x)=[0,q_{12}(x)),\ \Gamma_{13}(x)=[q_{12}(x),q_{12}(x)+q_{13}(x)),\ldots$$
$$\Gamma_{21}(x)=[q_1(x),q_1(x)+q_{21}(x)),\ \Gamma_{23}(x)=[q_1(x)+q_{21}(x),q_1(x)+q_{21}(x)+q_{23}(x)),$$

and so on. For convenience of notation, set $\Gamma_{ii}(x)=\emptyset$ and $\Gamma_{ij}(x)=\emptyset$ if $q_{ij}(x)=0$ for $i\neq j$. Then for each fixed $x\in\mathbb{R}^d$, $(\Gamma_{ij}(x))_{ij}$ is a family of disjoint intervals and the length of $\Gamma_{ij}(x)$ equals $q_{ij}(x)$. Define a function $h:\mathbb{R}^d\times\mathcal{S}\times\mathbb{R}\to\mathbb{R}$ by

$$h(x,i,z)=\begin{cases}j-i & \text{if } z\in\Gamma_{ij}(x),\\ 0 & \text{otherwize.}\end{cases} \tag{2.6}$$

Then $(\Lambda(t))$ can be expressed by

$$\Lambda(t)=\Lambda(0)+\int_0^t\int_{\mathbb{R}}h(X(s),\Lambda(s-),z)N(\mathrm{d}s,\mathrm{d}z), \tag{2.7}$$

where $N(\mathrm{d}t,\mathrm{d}z)$ is a Poisson random measure with intensity $\mathrm{d}t\times\mathrm{d}z$, and independent of $(W(t))$. Set $\tilde{N}(\mathrm{d}t,\mathrm{d}z)=N(\mathrm{d}t,\mathrm{d}z)-\mathrm{d}t\mathrm{d}z$.

Proposition 2.1 (a priori estimate). *Let* $(X(t),\Lambda(t))$ *satisfy* (2.4) *and* (2.5). *Assume* (A1)–(A4) *hold. Then for every* $T>0$*, it holds*

$$\mathbb{E}\big[\|X\|_T^2+\|\Lambda\|_T^2\big]\le\left(\frac{4}{3}|x|^2+4i^2\right)e^{\frac{8}{3}(1+K+C_1^2K+3(3+TK)\kappa^2K)T},$$

where C_1 is a universal constant induced by the Burkholder–Davis–Gundy inequality, $\|X\|_t = \sup_{s\le t}|X(s)|$, $\|\Lambda\|_t = \sup_{s\le t}\Lambda(s)$, $t>0$.

Proof. Set $\tau_M = \inf\{t>0; |X(t)|+\Lambda(t)>M\}$ for positive constant $M > |X(0)|+\Lambda(0)$. Using Itô's formula, we get

$$
\begin{aligned}
|X(t\wedge\tau_M)|^2 = |X(0)|^2 &+ \int_0^{t\wedge\tau_M} 2\langle X(s), a(X(s),\Lambda(s)) + b(X(\delta(s)),\Lambda(\delta(s)))\rangle \mathrm{d}s \\
&+ \int_0^{t\wedge\tau_M}\|\sigma(X(s),\Lambda(s))\|_{\mathrm{HS}}^2\mathrm{d}s + 2\int_0^{t\wedge\tau_M}\langle X(s),\sigma(X(s),\Lambda(s))\mathrm{d}W(s)\rangle
\end{aligned} \tag{2.8}
$$

and

$$
\begin{aligned}
\Lambda(t\wedge\tau_M) &= i + \int_0^{t\wedge\tau_M}\int_{\mathbb{R}} h(X(s),\Lambda(s-),z)N(\mathrm{d}s,\mathrm{d}z) \\
&\le i + \int_0^{t\wedge\tau_M}\int_{\mathbb{R}} h(X(s),\Lambda(s-),z)\tilde N(\mathrm{d}s,\mathrm{d}z) + \kappa\int_0^{t\wedge\tau_M} q_{\Lambda(s-)}(X(s))\mathrm{d}s.
\end{aligned} \tag{2.9}
$$

Since

$$
\mathbb{E}\int_0^{t\wedge\tau_M}|X(s)|^2\|\sigma(X(s),\Lambda(s))\|_{\mathrm{HS}}^2\mathrm{d}s \le 2M^2\mathbb{E}\int_0^t 1+M^2\mathrm{d}s < \infty
$$

and

$$
\begin{aligned}
&\mathbb{E}\int_0^{t\wedge\tau_M}\int_{\mathbb{R}} h(X(s),\Lambda(s-),z)^2\mathrm{d}s\mathrm{d}z \\
&\quad\le \kappa^2\mathbb{E}\int_0^{t\wedge\tau_M} q_{\Lambda(s-)}(X(s))\mathrm{d}s \\
&\quad\le \kappa^2\mathbb{E}\int_0^{t\wedge\tau_M} K(\Lambda(s-)+|X(s)|)\mathrm{d}s \le \kappa^2 KMt < \infty,
\end{aligned}
$$

we can obtain that the processes $\{\int_0^{t\wedge\tau_M}\langle X(s),\sigma(X(s),\Lambda(s))\mathrm{d}W(s)\rangle;\ t\in[0,T]\}$ and $\{\int_0^{t\wedge\tau_M}\int_{\mathbb{R}} h(X(s),\Lambda(s-),z)\tilde N(\mathrm{d}s,\mathrm{d}z);\ t\in[0,T]\}$ are martingales. By the Burkholder–Davis–Gundy inequality, we obtain

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t\le T}\left|\int_0^{T\wedge\tau_M}\langle X(s),\sigma(X(s),\Lambda(s))\mathrm{d}W(s)\rangle\right|\right] \\
&\quad\le C_1\mathbb{E}\sqrt{\int_0^{T\wedge\tau_M}|X(s)|^2\|\sigma(X(s),\Lambda(s))\|_{\mathrm{HS}}^2\mathrm{d}s} \\
&\quad\le C_1\mathbb{E}\left[\|X\|_{T\wedge\tau_M}\sqrt{\int_0^{T\wedge\tau_M}\|\sigma(X(s),\Lambda(s))\|_{\mathrm{HS}}^2\mathrm{d}s}\right] \\
&\quad\le \frac{1}{4}\mathbb{E}\|X\|_{T\wedge\tau_M}^2 + 2C_1^2K\mathbb{E}\int_0^{T\wedge\tau_M}\big(1+\|X\|_s^2\big)\mathrm{d}s
\end{aligned} \tag{2.10}
$$

and

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\le T}\left|\int_0^{t\wedge\tau_M}\int_{\mathbb{R}} h(X(s),\Lambda(s-),z)\tilde N(\mathrm{d}s,\mathrm{d}z)\right|^2\right]\\
&\quad\le 4\mathbb{E}\int_0^{T\wedge\tau_M}\int_{\mathbb{R}} h^2(X(s),\Lambda(s-),z)\mathrm{d}s\mathrm{d}z\\
&\quad\le 4\kappa^2\mathbb{E}\int_0^{T\wedge\tau_M} q_{\Lambda(s-)}(X(s))\mathrm{d}s\\
&\quad\le 4\kappa^2K\mathbb{E}\int_0^{T\wedge\tau_M}\|\Lambda\|_s+\|X\|_s\mathrm{d}s.
\end{aligned}\tag{2.11}$$

Invoking (2.9) and (2.11), we have

$$\begin{aligned}
&\mathbb{E}\big[\|\Lambda\|_{T\wedge\tau_M}^2\big]\\
&\quad\le 3i^2+3\mathbb{E}\left[\left(\sup_{t\le T}\int_0^{t\wedge\tau_M}\int_{\mathbb{R}} h(X(s),\Lambda(s-),z)\tilde N(\mathrm{d}s,\mathrm{d}z)\right)^2\right]\\
&\qquad+3\mathbb{E}\left[\left(\int_0^{T\wedge\tau_M}\int_{\mathbb{R}} h(X(s),\Lambda(s-),z)\mathrm{d}s\mathrm{d}z\right)^2\right]\\
&\quad\le 3i^2+12\kappa^2K\mathbb{E}\int_0^{T\wedge\tau_M}\|\Lambda\|_s+\|X\|_s\,\mathrm{d}s+6\kappa^2T\mathbb{E}\int_0^{T\wedge\tau_M}K^2(\|\Lambda\|_s^2+\|X\|_s^2)\,\mathrm{d}s\\
&\quad\le 3i^2+12\kappa^2K\mathbb{E}\int_0^{T\wedge\tau_M}\|\Lambda\|_s+\frac12(\|\Lambda\|_s^2+\|X\|_s^2)\,\mathrm{d}s\\
&\qquad+6\kappa^2K^2T\mathbb{E}\int_0^{T\wedge\tau_M}\|\Lambda\|_s^2+\|X\|_s^2\,\mathrm{d}s\\
&\quad\le 3i^2+6(3+TK)\kappa^2K\mathbb{E}\int_0^{T\wedge\tau_M}\|\Lambda\|_s^2+\|X\|_s^2\,\mathrm{d}s.
\end{aligned}\tag{2.12}$$

Consequently, by (2.8), (2.12), and (A1), we have

$$\begin{aligned}
&\mathbb{E}\big[\|X\|_{T\wedge\tau_M}^2+\|\Lambda\|_{T\wedge\tau_M}^2\big]\\
&\quad\le |x|^2+(1+2K)\mathbb{E}\int_0^{T\wedge\tau_M}\big(1+\|X\|_s^2\big)\,\mathrm{d}s\\
&\qquad+\frac14\mathbb{E}\|X\|_{T\wedge\tau_M}^2+2C_1^2K\mathbb{E}\int_0^{T\wedge\tau_M}\big(1+\|X\|_s^2\big)\,\mathrm{d}s\\
&\qquad+3i^2+6(3+TK)\kappa^2K\mathbb{E}\int_0^{T\wedge\tau_M}\|\Lambda\|_s^2+\|X\|_s^2\,\mathrm{d}s,
\end{aligned}$$

which yields that

$$\mathbb{E}\big[\|X\|_{T\wedge\tau_M}^2+\|\Lambda\|_{T\wedge\tau_M}^2\big]\le\left(\frac43|x|^2+4i^2\right)e^{\frac83(1+K+C_1^2K+3(3+TK)\kappa^2K)T}$$

by Gronwall's inequality. The proof is completed. □

Recall a basic lemma on the formula (2.6) proved in [21, Lemma 2.1].

LEMMA 2.2 (see [21]). *Under the condition* (A3), *it holds that for* $0 \le p \le 1$,

$$\int_{\mathbb{R}} |h(x,i,z) - h(y,i,z)|^p \mathrm{d}z \le 2\kappa^{p+1}(\kappa + 2i)c_q|x-y| \quad \forall\, x,y \in \mathbb{R}^d,\ i \in \mathcal{S}.$$

Similarly to [24], we set $0 < \tau_1 < \tau_2 < \cdots < \tau_n < \cdots$ to be the jumping times of the stationary point process $(p(t))$ corresponding to the above Poisson random measure $N(\mathrm{d}t, \mathrm{d}z)$. Set

$$\tilde{\tau} = \lim_{n\to\infty} \tau_n.$$

THEOREM 2.3. *Suppose that* (A1)–(A4) *hold. Then there exists a unique nonexplosive strong solution of* (2.4) *and* (2.5) *up to* $\tilde{\tau}$ *with* $(X(0), \Lambda(0)) = (x,i)$.

Proof. According to (2.7), we can rewrite (2.4) and (2.5) in the following form:

$$\begin{aligned}\mathrm{d}\begin{pmatrix} X(t) \\ \Lambda(t)\end{pmatrix} &= \begin{pmatrix} a(X(t),\Lambda(t)) + b(X(\delta(t)),\Lambda(\delta(t))) \\ 0 \end{pmatrix}\mathrm{d}t \\ &\quad + \int_{\mathbb{R}} \begin{pmatrix} 0 \\ h(X(t),\Lambda(t-),z)\end{pmatrix} N(\mathrm{d}t,\mathrm{d}z) + \begin{pmatrix}\sigma(X(t),\Lambda(t)) \\ 0\end{pmatrix}\mathrm{d}W(t).\end{aligned}$$

Then, under the conditions (A1)–(A4), the existence of a weak solution is clear (cf. [23], [29, Theorem 1.3], and note that on $[0,\tau]$, this equation becomes an SDE without delay as $(X(\delta(t)), \Lambda(\delta(t)) = (x,i))$. Then we can proceed with the argument on $[\tau, 2\tau]$, $[2\tau, 3\tau]$, etc., to obtain a solution. Refer to [16, p. 274] for more details on this technique.). Due to the Yamada–Watanabe principle, we only need to show the pathwise uniqueness of (2.4) and (2.5).

Assume that $(X(t), \Lambda(t))$ and $(Y(t), \Lambda'(t))$ are two solutions of the FSDE (2.4) and (2.5) with the same initial condition. For any $t \in [0, \tau_1)$, (2.4) is equivalent to

$$\mathrm{d}X^{(i)}(t) = [a(X^{(i)}(t), i) + b(X^{(i)}(\delta(t)), i)]\mathrm{d}t + \sigma(X^{(i)}(t), i)\mathrm{d}W(t), \quad X^{(i)}(0) = x,$$

which has a unique nonexplosive strong solution $(X^{(i)}(t))$. So

$$(X(t), \Lambda(t)) = (X^{(i)}(t), i) = (Y(t), \Lambda'(t)) \quad \text{for } t \in [0, \tau_1).$$

As

$$\begin{aligned}\Lambda(t) &= i + \int_0^t \int_{\mathbb{R}} h(X(s), \Lambda(s-), z) N(\mathrm{d}s, \mathrm{d}z), \\ \Lambda'(t) &= i + \int_0^t \int_{\mathbb{R}} h(Y(s), \Lambda'(s-), z) N(\mathrm{d}s, \mathrm{d}z),\end{aligned}$$

this yields $\Lambda(\tau_1) = \Lambda'(\tau_1)$. Combining this with the continuity of the paths of $(X(t))$ and $(Y(t))$, we have $(X(t), \Lambda(t)) = (Y(t), \Lambda'(t))$ for $t \in [0, \tau_1]$.

Next, we consider the time interval $[\tau_1, \tau_2)$. Denote $\Lambda(\tau_1) = \Lambda'(\tau_1) = k \in \mathcal{S}$. Then $(X(t))$ and $(Y(t))$ both satisfy the equation

$$\begin{aligned}\mathrm{d}X^{(k)}(t) = {} & a(X^{(k)}(t), k)\mathrm{d}t + \sigma(X^{(k)}(t), k)\mathrm{d}W(t) + b\big(X^{(i)}(\delta(t))\mathbf{1}_{[0,\tau_1)}(\delta(t)) \\ & + X^{(k)}(\delta(t))\mathbf{1}_{[\tau_1,\tau_2)}(\delta(t)), i\mathbf{1}_{[0,\tau_1)}(\delta(t)) + k\mathbf{1}_{[\tau_1,\tau_2)}(\delta(t))\big)\mathrm{d}t\end{aligned}$$

which still has a unique nonexplosive strong solution. Therefore,

$$(X(t), \Lambda(t)) = (Y(t), \Lambda'(t)) \text{ for } t \in [\tau_1, \tau_2).$$

Applying formula (2.7) again, we get $\Lambda(\tau_2) = \Lambda'(\tau_2)$ and, further, that $(X(t), \Lambda(t)) = (Y(t), \Lambda'(t))$ for $t \in [\tau_1, \tau_2]$. Continuing this procedure inductively, we can show

$$(X(t), \Lambda(t)) = (Y(t), \Lambda'(t)) \text{ for } t \in [0, \tilde{\tau}).$$

Thus, we prove the existence and uniqueness of the strong solution of (2.4) and (2.5) up to $\tilde{\tau}$. The nonexplosiveness of $(X(t), \Lambda(t))$ follows from Proposition 2.1. ∎

Remark 2.4. When $\mathcal{S}$ is a finite state space, similar to [24], we can modify the definition of $N(\mathrm{d}t, \mathrm{d}z)$ under the assumption that $q_i(x) \le K$ for all $i \in \mathcal{S}$ and $x \in \mathbb{R}^d$ so that $\tilde{\tau}$ defined in Theorem 2.3 is almost surely finite. When $\mathcal{S}$ is an infinite countable space but $(q_{ij}(x))$ is independent of x, similarly to [16, Theorem 7.12], Theorem 2.3 still holds for $\tilde{\tau} = \infty$ almost surely in this case.

3. Proof of the main result. To make the idea clear, we provide a concise construction of the probability space. Let

$$\Omega_1 = \{\omega;\, \omega : [0, \infty) \to \mathbb{R}^d \text{ continuous with } \omega(0) = 0\},$$

which is endowed with locally uniform topology and the Wiener measure $\mathbb{P}_1$ so that the coordinate process $W(t)(\omega) := \omega(t)$ is a standard d-dimensional Brownian motion. Set

$$\Omega_2 = \left\{\omega;\ \omega = \sum_{i=1}^{n} \delta_{t_i, u_i},\, n \in \mathbb{N} \cup \{\infty\}, (t_i, u_i) \in [0, \infty) \times [0, \infty)\right\},$$

which is equipped with the Skorokhod topology and a probability measure $\mathbb{P}_2$ such that the coordinate process $N(\mathrm{d}t, \mathrm{d}u, \omega) := \omega(\mathrm{d}t, \mathrm{d}u)$ is a Poisson random measure with the intensity $\mathrm{d}t \times \mathrm{d}u$. Define

$$(\Omega, \mathscr{F}, \mathbb{P}) = (\Omega_1 \times \Omega_2, \mathscr{B}(\Omega_1) \times \mathscr{B}(\Omega_2), \mathbb{P}_1 \times \mathbb{P}_2).$$

Then, under the probability measure $\mathbb{P} = \mathbb{P}_1 \times \mathbb{P}_2$, for $\omega = (\omega_1, \omega_2) \in \Omega$, $\omega_1(\cdot)$ is a Brownian motion, and $\omega_2(\cdot)$ is a Poisson random measure with the intensity $\mathrm{d}t \times \mathrm{d}u$. As a consequence, (1.3) and (2.7) can be rewritten in the form

$$\begin{aligned} \mathrm{d}X(t) &= \big[a(X(t), \Lambda(t)) - b(\Lambda(\delta(t)))X(\delta(t))\big]\mathrm{d}t + \sigma(X(t), \Lambda(t))\mathrm{d}\omega_1(t), \\ \Lambda(t) &= \Lambda(0) + \int_0^t \int_{\mathbb{R}} h(X(s), \Lambda(s-), z)\omega_2(\mathrm{d}s, \mathrm{d}z), \end{aligned}$$

and note that $h(x, i, z)$ is independent of x in the present situation because the transition rate matrix (q_{ij}) of $(\Lambda(t))$ is independent of x. So $(\Lambda(t))$ is completely determined by $(\omega_2(t))$. In the rest of this work, we shall work on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ constructed above. Denote $\mathbb{E}^\Lambda[\,\cdot\,]$ as the conditional expectation with respect to the σ-algebra $\sigma\{\omega_2(t); 0 \le t < \infty\}$.

LEMMA 3.1. *Let* $(X(t), \Lambda(t))$ *be the solution of* (1.3). *Under the same conditions as that of Theorem* 1.1, *then*

$$\mathbb{E}^\Lambda |X(t) - X(\delta(t))|^2(\omega_2) \le \frac{K(\tau)}{1 - K(\tau)} \mathbb{E}^\Lambda |X(t)|^2(\omega_2), \quad \mathbb{P}_2\text{-}a.s. \tag{3.1}$$

Proof. Due to Itô's formula,

$$
\begin{aligned}
&|X(t)-X(\delta(t))|^2\\
&\quad=\int_{\delta(t)}^t 2\langle X(s)-X(\delta(s)),a(X(s),\Lambda(s))-b(\Lambda(\delta(s)))X(\delta(s))\rangle\mathrm{d}s\\
&\qquad+\int_{\delta(t)}^t\|\sigma(X(s),\Lambda(s))\|_{\mathrm{HS}}^2\mathrm{d}s+\int_{\delta(t)}^t 2\langle X(s)-X(\delta(s)),\sigma(X(s),\Lambda(s))\mathrm{d}\omega_1(s)\rangle.
\end{aligned}
$$

By the mutual independence of $(\Lambda(t))$ and $(\omega_1(t))$ and (H1), (H2), we get

$$
\begin{aligned}
&\mathbb{E}^\Lambda|X(t)-X(\delta(t))|^2(\omega_2)\\
&\quad=\mathbb{E}^\Lambda\left[\int_{\delta(t)}^t 2\langle X(s)-X(\delta(s)),a(X(s),\Lambda(s))\rangle+\|\sigma(X(s),\Lambda(s))\|_{\mathrm{HS}}^2\mathrm{d}s\right](\omega_2)\\
&\qquad-\mathbb{E}^\Lambda\left[\int_{\delta(t)}^t 2\langle X(s)-X(\delta(s)),b(\Lambda(\delta(s)))X(\delta(s))\rangle\mathrm{d}s\right](\omega_2)\\
&\quad\le\int_{\delta(t)}^t C(\Lambda(s))\mathbb{E}^\Lambda\big[|X(s)|^2\big](\omega_2)+2M_a\mathbb{E}^\Lambda\big[|X(s)||X(\delta(s))|\big](\omega_2)\mathrm{d}s\\
&\qquad+\int_{\delta(t)}^t 2b(\Lambda(\delta(s)))\mathbb{E}^\Lambda\big[|X(s)-X(\delta(s))||X(\delta(s))|\big](\omega_2)\mathrm{d}s\\
&\quad\le\int_{\delta(t)}^t 2C(\Lambda(s))\mathbb{E}^\Lambda\big[|X(s)-X(\delta(s))|^2+|X(\delta(s))|^2\big](\omega_2)\\
&\qquad+M_a\mathbb{E}^\Lambda\big[|X(s)-X(\delta(s))|^2+3|X(\delta(s))|^2\big](\omega_2)\mathrm{d}s\\
&\qquad+\int_{\delta(t)}^t\mathbb{E}^\Lambda\big[|X(s)-X(\delta(s))|^2+b(\Lambda(\delta(s)))^2|X(\delta(s))|^2\big](\omega_2)\mathrm{d}s\\
&\quad\le\int_{\delta(t)}^t(2C(\Lambda(s))+M_a+1)\mathbb{E}^\Lambda\big[|X(s)-X(\delta(s))|^2\big](\omega_2)\\
&\qquad+(2C(\Lambda(s))+3M_a)\mathbb{E}^\Lambda|X(\delta(s))|^2(\omega_2)\mathrm{d}s\\
&\qquad+(b(\Lambda(\delta(t)))^2)\tau\mathbb{E}^\Lambda|X(\delta(t))|^2(\omega_2)\\
&\quad\le\int_{\delta(t)}^t(2C(\Lambda(s))+M_a+1)\mathbb{E}^\Lambda[|X(s)-X(\delta(s))|^2](\omega_2)\mathrm{d}s\\
&\qquad+(2\bar{C}+3M_a+\bar{b}^2)\tau\mathbb{E}^\Lambda|X(\delta(t))|^2(\omega_2).
\end{aligned}
$$

Applying Gronwall's inequality, we obtain that

$$
\begin{aligned}
&\mathbb{E}^\Lambda|X(t)-X(\delta(t))|^2(\omega_2)\\
&\quad\le(2\bar{C}+3M_a+\bar{b}^2)\tau\mathbb{E}^\Lambda|X(\delta(t))|^2(\omega_2)\mathrm{e}^{\int_{\delta(t)}^t(2C(\Lambda(s))+M_a+1)\mathrm{d}s}\\
&\quad\le 2(2\bar{C}+3M_a+\bar{b}^2)\tau\big\{\mathbb{E}^\Lambda[|X(t)-X(\delta(t))|^2](\omega_2)\\
&\qquad\qquad+\mathbb{E}^\Lambda|X(t)|^2(\omega_2)\big\}\mathrm{e}^{\int_{\delta(t)}^t(2C(\Lambda(s))+M_a+1)\mathrm{d}s},
\end{aligned}
$$

which leads to

$$
\mathbb{E}^\Lambda|X(t)-X(\delta(t))|^2(\omega_2)\le\frac{K(\tau)}{1-K(\tau)}\mathbb{E}^\Lambda|X(t)|^2(\omega_2),
$$

by invoking the definition of $K(\tau)$ and the condition $K(\tau)<1$. We conclude the proof. ∎

Lemma 3.2. *Under the same conditions as that of Theorem* 1.1, *the following estimates hold:*

$$\mathbb{E}|X(t)|^2 \le |x(0)|^2\mathbb{E}\left[\mathrm{e}^{\int_0^t C(\Lambda(r))-\left(2-2\sqrt{\frac{K(\tau)}{1-K(\tau)}}\right)b(\Lambda(\delta(r)))\mathrm{d}r}\right],\quad t>0, \tag{3.2}$$

and

$$\mathbb{E}|X(t)|^2 \ge |x(0)|^2\mathbb{E}\left[\mathrm{e}^{\int_0^t c(\Lambda(r))-\left(2+2\sqrt{\frac{K(\tau)}{1-K(\tau)}}\right)b(\Lambda(\delta(r)))\mathrm{d}r}\right],\quad t>0. \tag{3.3}$$

Proof. Applying Itô's formula and (H1), we get

$$\begin{aligned}\mathrm{d}|X(t)|^2 &= [2\langle X(t),a(X(t),\Lambda(t))\rangle+\|\sigma(X(t),\Lambda(t))\|_{\mathrm{HS}}^2]\mathrm{d}t\\ &\quad-2b(\Lambda(\delta(t)))\langle X(t),X(\delta(t))\rangle\mathrm{d}t+2\langle X(t),\sigma(X(t),\Lambda(t))\mathrm{d}\omega_1(t)\rangle\\ &\le \big(C(\Lambda(t))-(2-\varepsilon)b(\Lambda(\delta(t)))\big)|X(t)|^2\mathrm{d}t\\ &\quad+\frac{1}{\varepsilon}b(\Lambda(\delta(t)))|X(t)-X(\delta(t))|^2\mathrm{d}t\\ &\quad+2\langle X(t),\sigma(X(t),\Lambda(t))\mathrm{d}\omega_1(t),\quad \varepsilon>0.\end{aligned}\tag{3.4}$$

By Lemma 3.1, for $0\le s<t$,

$$\begin{aligned}&\mathbb{E}^\Lambda|X(t)|^2(\omega_2)-\mathbb{E}^\Lambda|X(s)|^2(\omega_2)\\ &\quad\le\int_s^t\Big(\big(C(\Lambda(r))-(2-\varepsilon)b(\Lambda(\delta(r)))\big)\mathbb{E}^\Lambda|X(r)|^2(\omega_2)\\ &\qquad\quad+\frac{1}{\varepsilon}b(\Lambda(\delta(r)))\mathbb{E}^\Lambda[|X(r)-X(\delta(r))|^2](\omega_2)\Big)\,\mathrm{d}r\\ &\quad\le\int_s^t\left[C(\Lambda(r))-\left(2-\varepsilon-\frac{K(\tau)}{\varepsilon(1-K(\tau))}\right)b(\Lambda(\delta(r)))\right]\mathbb{E}^\Lambda|X(r)|^2(\omega_2)\mathrm{d}r.\end{aligned}$$

Set $u(t)(\omega_2)=\mathbb{E}^\Lambda|X(t)|^2(\omega_2)$, then the previous inequality can be rewritten as

$$u(t)(\omega_2)-u(s)(\omega_2)\le\int_s^t h(\Lambda(r),\Lambda(\delta(r)))u(r)(\omega_2)\mathrm{d}r,\quad 0\le s<t, \tag{3.5}$$

where

$$h(\Lambda(r),\Lambda(\delta(r)))=C(\Lambda(r))-\left(2-\varepsilon-\frac{K(\tau)}{\varepsilon(1-K(\tau))}\right)b(\Lambda(\delta(r))). \tag{3.6}$$

As $(X(t))$ is the unique nonexplosive strong solution of FSDE (1.3), it is clear that for $\mathbb{P}_2$-a.e. $\omega_2\in\Omega_2$, $t\mapsto u(t)(\omega_2)=\mathbb{E}^\Lambda|X(t)|^2(\omega_2)$ is differentiable on $[0,\infty)$. However, in order to get an estimate on $u'(t)(\omega_2)$ from (3.5), we also need to consider the continuity of the function $r\mapsto h(\Lambda(r),\Lambda(\delta(r)))$. It is well known that the probability that $(\Lambda(t))$ owns a finite number of jumps during a finite interval equals 1. Let us focus on the intervals $I_n:=[n\tau,(n+1)\tau)$ for $n\in\{0,1,\dots\}$.

Case 1. If $(\Lambda(t)(\omega_2))$ has no jump during I_n for some $n\in\mathbb{N}$, then $t\mapsto h(\Lambda(r),\Lambda(\delta(r)))$ is continuous on I_n. Therefore, dividing (3.5) by $t-s$ and letting $s\uparrow t$ yields

$$u'(t)(\omega_2)\le h(\Lambda(t)(\omega_2),\Lambda(\delta(t))(\omega_2))u(t)(\omega_2),\quad t\in(n\tau,(n+1)\tau), \tag{3.7}$$

which implies

$$u(t)(\omega_2)\le u(s)(\omega_2)\mathrm{e}^{\int_s^t h(\Lambda(r)(\omega_2),\Lambda(\delta(r))(\omega_2))\mathrm{d}r},\quad s,t\in(n\tau,(n+1)\tau). \tag{3.8}$$

Letting $s\downarrow n\tau$, we get

$$u(t)(\omega_2)\le u(n\tau)(\omega_2)\mathrm{e}^{\int_{n\tau}^t h(\Lambda(r)(\omega_2),\Lambda(\delta(r))(\omega_2))\mathrm{d}r},\quad t\in I_n. \tag{3.9}$$

Case 2. If $(\Lambda(t)(\omega_2))$ owns jumps in I_n for some $n\in\mathbb{N}$, then we denote its jumping time by $n\tau=\tau_0<\tau_1<\cdots<\tau_m<\tau_{m+1}=(n+1)\tau$. Then $r\mapsto h(\Lambda(r)(\omega_2),\Lambda(\delta(r))(\omega_2))$ is continuous on (τ_{i-1},τ_i) for $i=1,\dots,m+1$. Applying (3.9) inductively for $t=\tau_i$, $s=\tau_{i-1}$, $i=1,\dots,m+1$, yields that

$$\begin{aligned}
u(\tau_1-)(\omega_2)&\le u(\tau_0)(\omega_2)\mathrm{e}^{\int_{\tau_0}^{\tau_1} h(\Lambda(r)(\omega_2),\Lambda(\delta(r))(\omega_2))\mathrm{d}r},\\
u(\tau_2-)(\omega_2)&\le u(\tau_1+)(\omega_2)\mathrm{e}^{\int_{\tau_1}^{\tau_2} h(\Lambda(r)(\omega_2),\Lambda(\delta(r))(\omega_2))\mathrm{d}r},\\
&\ \ \vdots\\
u(t)(\omega_2)&\le u(\tau_i+)(\omega_2)\mathrm{e}^{\int_{\tau_i}^{t} h(\Lambda(r)(\omega_2),\Lambda(\delta(r))(\omega_2))\mathrm{d}r},\quad t\in(\tau_i,\tau_{i+1}).
\end{aligned}$$

Combining the previous inequalities together and using the continuity of $t\mapsto u(t)(\omega_2)$, we get

$$u(t)(\omega_2)\le u(n\tau)\mathrm{e}^{\int_{n\tau}^t h(\Lambda(r)(\omega_2),\Lambda(\delta(r))(\omega_2))\mathrm{d}r},\quad t\in I_n. \tag{3.10}$$

In view of the arguments in cases 1 and 2, (3.10) is valid when $t\in I_n$ and Λ has no jump or a finite number of jumps in the interval I_n. Now for any $t>0$, we can find an n such that $t\in I_n$. Either Λ has no jump or a finite number of jumps in the interval I_n. In any case, (3.10) holds. Then one can use the same argument to obtain

$$\begin{aligned}
u(t)(\omega_2)&\le u(n\tau)(\omega_2)\mathrm{e}^{\int_{n\tau}^t h(\Lambda(r)(\omega_2),\Lambda(\delta(r))(\omega_2))\mathrm{d}r}\\
&\le u((n-1)\tau)(\omega_2)\mathrm{e}^{\int_{(n-1)\tau}^t h(\Lambda(r)(\omega_2),\Lambda(\delta(r))(\omega_2))\mathrm{d}r}\\
&\le\cdots\\
&\le u(0)(\omega_2)\mathrm{e}^{\int_0^t h(\Lambda(r)(\omega_2),\Lambda(\delta(r))(\omega_2))\mathrm{d}r}.
\end{aligned} \tag{3.11}$$

Taking the expectation w.r.t. the probability measure $\mathbb{P}_2$ on both sides of (3.11) yields that

$$\mathbb{E}|X(t)|^2\le|x(0)|^2\mathbb{E}\Big[\mathrm{e}^{\int_0^t h(\Lambda(r),\Lambda(\delta(r)))\mathrm{d}r}\Big],\quad t>0, \tag{3.12}$$

which implies the desired upper bound (3.2) after taking the optimal choice of $\varepsilon=\sqrt{\frac{K(\tau)}{1-K(\tau)}}$.

Next, we proceed to the lower bound estimate. Applying Itô's formula and (H1) again, we get

$$\begin{aligned}
\mathrm{d}|X(t)|^2\ge&\left[\big(c(\Lambda(t))-(2+\varepsilon)b(\Lambda(\delta(t)))\big)|X(t)|^2-\frac{1}{\varepsilon}b(\Lambda(\delta(t)))|X(t)-X(\delta(t))|^2\right]\mathrm{d}t\\
&+2\langle X(t),\sigma(X(t),\Lambda(t))\mathrm{d}\omega_1(t)\rangle.
\end{aligned}$$

Due to Lemma 3.1, for $0 \le s < t$,

$$\begin{aligned}&\mathbb{E}^\Lambda |X(t)|^2(\omega_2) - \mathbb{E}^\Lambda |X(s)|^2(\omega_2)\\ &\ge \int_s^t \Big(c(\Lambda(r)) - \Big(2+\varepsilon+\frac{K(\tau)}{\varepsilon(1-K(\tau))}\Big) b(\Lambda(\delta(r)))\Big)\, \mathbb{E}^\Lambda |X(r)|^2(\omega_2)\mathrm{d}r.\end{aligned}$$

Analogously to the discussion above for the upper bound (3.2), we can obtain the lower bound (3.3) whose details are omitted. □

In view of the estimates provided by Lemma 3.2, the mean-square stability of $(X(t))$ is determined by the long time behavior of two terms in the form separately,

$$\int_0^t f(\Lambda(s))\mathrm{d}s, \qquad \int_0^t g(\Lambda(\delta(s)))\mathrm{d}s, \quad f, g \text{ bounded functions on } \mathcal{S}.$$

The long time behavior of the first term can be determined by using the strong ergodic theorem for $(\Lambda(t))$ which has been assumed to be ergodic. However, to deal with long time behavior of the second term, we need to study further a skeleton process of the Markov chain $(\Lambda(t))$.

Set $Y_n = \Lambda(n\tau)$ for $n \in \mathbb{N}$. Then (Y_n) is a discrete-time homogeneous Markov chain on $\mathcal{S}$. As a skeleton process of $(\Lambda(t))$, the transition matrix $(P_{ij})_{i,j\in\mathcal{S}}$ of (Y_n) can be deduced from the transition rate of $(\Lambda(t))$. Precisely,

$$P_{ij} = f_{ij}(\tau), \quad i, j \in \mathcal{S},$$

where $f_{ij}(t) = \mathbb{P}(\Lambda(t) = j|\Lambda(0) = i)$, $i, j \in \mathcal{S}$, $t > 0$. It is known that $f_{ij}(t)$ is a solution of the following equation

$$f_{ij}(t) = e^{q_i t}\delta_{ij} + \sum_{k\neq j}\int_0^t f_{ik}(t-s)q_{kj}e^{-q_j s}\mathrm{d}s,$$

where $q_i = \sum_{j\neq i} q_{ij}$, $\delta_{ij} = 1$ if $i = j$; otherwise, $\delta_{ij} = 0$. Since $(\Lambda(t))$ is an irreducible, ergodic Markov chain on $\mathcal{S}$, this implies that $P_{ij} > 0$ for all $i, j \in \mathcal{S}$, and (Y_n) is also ergodic with the same stationary distribution $(\pi_i)_{i\in\mathcal{S}}$ as that of $(\Lambda(t))$. See, e.g., [3, Chapter 4] for more details.

Proof of Theorem 1.1. Set

$$\begin{aligned}\Theta(t,\omega_2) &= \int_0^t C(\Lambda(r)(\omega_2)) - \Big(2 - 2\sqrt{\frac{K(\tau)}{1-K(\tau)}}\Big) b(\Lambda(\delta(r))(\omega_2))\mathrm{d}r,\\ \theta(t,\omega_2) &= \int_0^t c(\Lambda(r)(\omega_2)) - \Big(2 + 2\sqrt{\frac{K(\tau)}{1-K(\tau)}}\Big) b(\Lambda(\delta(r))(\omega_2))\mathrm{d}r.\end{aligned}$$

By the strong ergodic theorem

(3.13)
$$\begin{aligned}\lim_{t\to\infty}\frac{\Theta(t,\omega_2)}{t} &= \lim_{t\to\infty}\frac{1}{t}\int_0^t C(\Lambda(r)(\omega_2))\mathrm{d}r-\left(2-2\sqrt{\frac{K(\tau)}{1-K(\tau)}}\right)\lim_{t\to\infty}\frac{1}{t}\int_0^t b(\Lambda(\delta(r))(\omega_2))\mathrm{d}r\\ &=\sum_{i\in\mathcal{S}}\pi_i C(i)-\left(2-2\sqrt{\frac{K(\tau)}{1-K(\tau)}}\right)\lim_{t\to\infty}\frac{[t/\tau]\tau}{t}\cdot\frac{1}{[t/\tau]\tau}\sum_{n=0}^{[t/\tau]}b(\Lambda(n\tau)(\omega_2))\tau\\ &=\sum_{i\in\mathcal{S}}\pi_i\left[C(i)-\left(2-2\sqrt{\frac{K(\tau)}{1-K(\tau)}}\right)b(i)\right]=:\alpha,\quad \mathbb{P}_2\text{-a.s.}\end{aligned}$$

Similarly,

$$\lim_{t\to\infty}\frac{\theta(t,\omega_2)}{t}=\sum_{i\in\mathcal{S}}\pi_i\left[c(i)-\left(2+2\sqrt{\frac{K(\tau)}{1-K(\tau)}}\right)b(i)\right]=:\beta,\quad \mathbb{P}_2\text{-a.s.} \tag{3.14}$$

Therefore, if $\alpha<0$, due to (3.2), (3.13), and applying Fatou's lemma, we get

$$\limsup_{t\to\infty}\mathbb{E}|X(t)|^2\le\limsup_{t\to\infty}|x(0)|^2\mathbb{E}\mathrm{e}^{\Theta(t)}\le|x(0)|^2\mathbb{E}\limsup_{t\to\infty}\mathrm{e}^{\alpha t}=0,$$

which implies that $\lim_{t\to\infty}\mathbb{E}|X(t)|^2=0$.

If $\beta>0$, by (3.3), (3.14), and Fatou's lemma, we get

$$\liminf_{t\to\infty}\mathbb{E}|X(t)|^2\ge|x(0)|^2\mathbb{E}\liminf_{t\to\infty}\mathrm{e}^{\beta t}=\infty.$$

Hence, $(X(t))$ is not mean-square stable, and the proof is completed. ∎

Remark 3.3. In view of [25, Theorem 3.4] or [14, Theorem 2.1], one may expect to obtain the asymptotic stability of $(X(t))$ (i.e., $\lim_{t\to\infty}X(t)=0$ a.s.) under the condition that $\alpha<0$. If we use their method to study the asymptotic stability for $(X(t))$ given by (1.3), the main difficulty is for us is to verify

$$\mathbb{E}\int_0^\infty|X(t)|^2\mathrm{d}t\le\mathbb{E}\int_0^\infty\mathrm{e}^{\int_0^t\Theta(r)\mathrm{d}r}\mathrm{d}t<\infty \tag{3.15}$$

under the condition $\alpha<0$. Precisely, to show

$$\mathbb{E}\int_0^\infty\mathrm{e}^{\int_0^t f(\Lambda(s))\mathrm{d}s}\mathrm{d}t<\infty\text{ and }\mathbb{E}\int_0^\infty\mathrm{e}^{\int_0^t g(\Lambda(\delta(s)))\mathrm{d}s}\mathrm{d}t<\infty$$

for bounded functions f, g on $\mathcal{S}$. The previous two terms are difficult to check for general bounded functions f and g, especially the second term.

Proof of Corollary 1.2. It is easy to check that $C(i)=c(i)=2a(i)+\sigma(i)^2$ under (H1) for (1.7). Then the desired conclusions that $\lim_{t\to\infty}\mathbb{E}X(t)^2=0$ under (1.8) and $\lim_{t\to\infty}\mathbb{E}X(t)^2=\infty$ under (1.9) follow immediately from Theorem 1.1. ∎

4. Application to feedback control of linear SDEs with regime switching. Linear SDEs with regime switching have been extensively investigated in the past. It is also well known as linear hybrid SDEs. Especially, in [13], the issue to stabilize a given linear hybrid SDE based on discrete-time observations of the state process has been studied. In this section, we shall apply Theorem 1.1 to stabilize a given linear hybrid SDE based on discrete-time observations of the state process and the switching process.

Precisely, consider the d-dimensional linear hybrid SDE

$$\mathrm{d}X(t) = A(\Lambda(t))X(t)\mathrm{d}t + \sum_{k=1}^{d} B_k(\Lambda(t))X(t)\mathrm{d}w_k(t),\ \ X(0) = x_0 \in \mathbb{R}^d, \tag{4.1}$$

where A, $B_k : \mathcal{S} \to \mathbb{R}^{d\times d}$, $W(t) = (w_1(t), w_2(t), \ldots, w_d(t))$ is a d-dimensional Brownian motion, and $(\Lambda(t))$ is an irreducible conservative Markov chain on the state space $\mathcal{S} = \{1, \ldots, N\}$, $2 \le N \le \infty$. Assume throughout this section that $(\Lambda(t))$ is ergodic and denote (π_i) as its stationary distribution. Suppose (4.1) is not mean-square stable, and we want to find a feedback control $D(\Lambda(\delta(t)))X(\delta(t))$ with $\delta(t) = [t/\tau]\tau$ for some $\tau > 0$ such that the controlled system

$$\mathrm{d}X(t) = [A(\Lambda(t))X(t) - D(\Lambda(\delta(t)))X(\delta(t))]\mathrm{d}t + \sum_{k=1}^{d} B_k(\Lambda(t))X(t)\mathrm{d}w_k(t) \tag{4.2}$$

will be mean-square stable. Here $D : \mathcal{S} \to \mathbb{R}^{d\times d}$. Let us introduce some notation. Set $\sigma(x, i) = (B_1(i)x, \ldots, B_d(i)x) \in \mathbb{R}^{d\times d}$ for $(x, i) \in \mathbb{R}^d \times \mathcal{S}$. Set

$$\begin{aligned}
C(i) &= \sup_{|x|=1} \big(2\langle A(i)x, x\rangle + \|\sigma(x, i)\|_{\mathrm{HS}}^2\big),\\
c(i) &= \inf_{|x|=1} \big(2\langle A(i)x, x\rangle + \|\sigma(x, i)\|_{\mathrm{HS}}^2\big),\\
\xi_{\max}^i &= \sup_{|x|=1} |A(i)x|, \quad \xi_{\min}^i = \inf_{|x|=1} |A(i)x|,\\
\eta_{\max}^i &= \sup_{|x|=1} |D(i)x|, \quad \eta_{\min}^i = \inf_{|x|=1} |D(i)x|,\ \ i \in \mathcal{S}.
\end{aligned}$$

PROPOSITION 4.1. *Let $(X(t), \Lambda(t))$ be the solution of (4.2). Assume $\bar{C} = \sup_{i\in\mathcal{S}} C(i) < \infty$, $\bar{\xi} = \sup_{i\in\mathcal{S}} \xi_{\max}(i) < \infty$, $\bar{\eta} = \sup_{i\in\mathcal{S}} \eta_{\max}(i) < \infty$. Set*

$$\widetilde{K}(\tau) = 2\left(2\bar{C} + 2\bar{\xi}^2 + \bar{\eta}^2 + 1\right)\tau \mathrm{e}^{(2\bar{C}+2\bar{\xi}^2+1)\tau}.$$

Suppose τ is sufficiently small so that $\widetilde{K}(\tau) < 1$.

(i) *If*

$$\sum_{i\in\mathcal{S}} \pi_i \left(C(i) - 2\left(\eta_{\min}^i - \eta_{\max}^i \left(\frac{\widetilde{K}(\tau)}{1 - \widetilde{K}(\tau)} \right)^{\frac{1}{2}} \right) \right) < 0,$$

then $\lim_{t\to\infty} \mathbb{E}|X(t)|^2 = 0$.

(ii) *If*

$$\sum_{i\in\mathcal{S}} \pi_i \left(c(i) - 2\left(1 + \left(\frac{\widetilde{K}(\tau)}{1 - \widetilde{K}(\tau)} \right)^{\frac{1}{2}} \right) \eta_{\max}^i \right) > 0,$$

then $\lim_{t\to\infty} \mathbb{E}|X(t)|^2 = \infty$.

Proof. The proof of this proposition is completely analogous to that of Theorem 1.1. But, we shall pay attention to the replacement of $b : \mathcal{S} \to [0, \infty)$ in (1.3) with $D : \mathcal{S} \to \mathbb{R}^{d\times d}$ in (4.2), which causes some differences in the estimates in the proof. In the present situation, we shall use the estimate

$$\eta^i_{\min}|x||y| \le \langle D(i)x, y\rangle \le \eta^i_{\max}|x||y|, \quad i \in \mathcal{S},\ x,\, y \in \mathbb{R}^d.$$

To be more precise, similarly to Lemma 3.1, it holds

$$\mathbb{E}^\Lambda|X(t) - X(\delta(t))|^2(\omega_2) \le \frac{\widetilde{K}(\tau)}{1-\widetilde{K}(\tau)}\mathbb{E}^\Lambda|X(t)|^2(\omega_2), \quad \mathbb{P}_2\text{-a.s.} \tag{4.3}$$

Then by Itô's formula, for any $\varepsilon > 0$

$$\begin{aligned}
\mathrm{d}|X(t)|^2 &= \big(2\langle A(\Lambda(t))X(t), X(t)\rangle + \|\sigma(X(t),\Lambda(t))\|^2_{\mathrm{HS}}\big)\mathrm{d}t \\
&\quad - 2\langle X(t), D(\Lambda(\delta(t)))X(\delta(t))\rangle\mathrm{d}t + 2\langle X(t), \sigma(X(t),\Lambda(t))\mathrm{d}W(t)\rangle \\
&\le \big(C(\Lambda(t)) - 2\eta^{\Lambda(\delta(t))}_{\min}\big)|X(t)|^2\mathrm{d}t - 2\langle X(t), D(\Lambda(\delta(t)))(X(\delta(t)) - X(t))\rangle\mathrm{d}t \\
&\quad + 2\langle X(t), \sigma(X(t),\Lambda(t))\mathrm{d}W(t)\rangle \\
&\le \big(C(\Lambda(t)) - 2\eta^{\Lambda(\delta(t))}_{\min}\big)|X(t)|^2\mathrm{d}t + \varepsilon|X(t)|^2\mathrm{d}t \\
&\quad + \frac{1}{\varepsilon}\eta^{\Lambda(\delta(t))}_{\max}|X(\delta(t)) - X(t)|^2\mathrm{d}t + 2\langle X(t), \sigma(X(t),\Lambda(t))\mathrm{d}W(t)\rangle.
\end{aligned}$$

Invoking the estimate (4.3) and optimizing the choice of $\varepsilon > 0$, we can get

$$\begin{aligned}
&\mathbb{E}^\Lambda|X(t)|^2(\omega_2) - \mathbb{E}^\Lambda|X(s)|^2(\omega_2) \\
&\le \int_s^t \left(C(\Lambda(r)) - 2\eta^{\Lambda(\delta(r))}_{\min} + 2\eta^{\Lambda(\delta(r))}_{\max}\left(\frac{\widetilde{K}(\tau)}{1-\widetilde{K}(\tau)}\right)^{\frac{1}{2}}\right)\mathbb{E}^\Lambda|X(r)|^2(\omega_2)\mathrm{d}r.
\end{aligned}$$

Then, similarly to the argument of Lemma 3.2, we can obtain an upper estimate of

$$\mathbb{E}|X(t)|^2 \le |x_0|^2\mathbb{E}\left[\mathrm{e}^{\int_0^t C(\Lambda(r)) - 2\eta^{\Lambda(\delta(r))}_{\min} + 2\eta^{\Lambda(\delta(r))}_{\max}\left(\frac{\widetilde{K}(\tau)}{1-\widetilde{K}(\tau)}\right)^{\frac{1}{2}}\mathrm{d}r}\right].$$

Consequently, this implies the assertion (i) of this proposition by using the argument of Theorem 1.1. The statement (ii) can be proved in a similar way, and the details are omitted. ∎

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