INSTABILITY OF SOLITARY WAVE SOLUTIONS FOR DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION IN ENDPOINT CASE

CUI NING, MASAHITO OHTA, AND YIFEI WU

Abstract. We study the stability theory of solitary wave solutions for a type of the derivative nonlinear Schrödinger equation

$$i\partial_t u + \partial_x^2 u + i|u|^2\partial_x u + b|u|^4 u = 0.$$

The equation has a two-parameter family of solitary wave solutions of the form

$$e^{i\omega_0 t + i\frac{\omega_1}{2}(x-\omega_1 t) - \frac{i}{4}\int_{-\infty}^{x-\omega_1 t}|\varphi_\omega(\eta)|^2 d\eta}\varphi_\omega(x-\omega_1 t).$$

The stability theory in the frequency region of $|\omega_1| < 2\sqrt{\omega_0}$ was studied previously. In this paper, we prove the instability of the solitary wave solutions in the endpoint case $\omega_1 = 2\sqrt{\omega_0}$, in which the elliptic equation of φ_ω is "zero mass".

1. Introduction

In this paper, we study the stability theory of solitary wave solutions for the derivative nonlinear Schrödinger equation:

$$i\partial_t u + \partial_x^2 u + i|u|^2\partial_x u + b|u|^4 u = 0, \qquad t \in \mathbb{R}, x \in \mathbb{R}, \tag{1.1}$$

where $b > 0$. It describes an Alfvén wave and appears in plasma physics, nonlinear optics, and so on (see [16, 17]). When $b = 0$, by a suitable gauge transformation, (1.1) is transformed to the standard derivative nonlinear Schrödinger equation:

$$i\partial_t u + \partial_x^2 u + i\partial_x(|u|^2 u) = 0. \tag{1.2}$$

It was proved in [9, 10, 11, 19] that the Cauchy problem for (1.1) or (1.2) is locally well-posed in the energy space $H^1(\mathbb{R})$. See also [5, 22, 23, 20, 21, 1] for some of the previous or extended results. Furthermore, it was proved in [25] that (1.2) is globally well-posed in the energy space $H^1(\mathbb{R})$ when the initial data satisfies $\|u_0\|_{L^2} < 2\sqrt{\pi}$. See [3, 4, 7, 8, 11, 15, 19, 24] for the related results. See also [13, 14] for the stability results on the generalized derivative nonlinear Schrödinger equation.

The solution $u(t)$ of (1.1) satisfies three conservation laws

$$E(u(t)) = E(u_0), P(u(t)) = P(u_0), M(u(t)) = M(u_0)$$

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for all $t \in [0, T_{max})$, where

$$E(u(t)) = \frac{1}{2}\|\partial_x u\|_{L^2}^2 - \frac{1}{4}(i|u|^2\partial_x u, u)_{L^2} - \frac{b}{6}\|u\|_{L^6}^6,$$
$$P(u(t)) = \frac{1}{2}(i\partial_x u, u)_{L^2},$$
$$M(u(t)) = \frac{1}{2}\|u\|_{L^2}^2.$$

It is known (see for examples [6, 2, 25]) that (1.2) has a two-parameter family of solitary wave solutions of the form:

$$\widetilde{u}_\omega(t,x) = e^{i\omega_0 t + i\frac{\omega_1}{2}(x-\omega_1 t) - \frac{3}{4}i\int_{-\infty}^{x-\omega_1 t}|\widetilde{\varphi}_\omega(\eta)|^2 d\eta}\widetilde{\varphi}_\omega(x-\omega_1 t),$$

where $\omega = (\omega_0, \omega_1) \in \Omega := \{(\omega_0,\omega_1) \in \mathbb{R}^+ \times \mathbb{R} : \omega_1^2 \le 4\omega_0\}$, and $\widetilde{\varphi}_\omega$ is the solution of

$$-\partial_x^2\varphi + (\omega_0 - \frac{\omega_1^2}{4})\varphi + \frac{\omega_1}{2}|\varphi|^2\varphi - \frac{3}{16}|\varphi|^4\varphi = 0.$$

In [2], Colin and Ohta proved that $\widetilde{u}_\omega(t,x)$ is stable when $\omega_1^2 < 4\omega_0$. See also [6] for the case when $\omega_1 < 0$ and $\omega_1^2 < 4\omega_0$. The stability theory on the endpoint case $\omega_1^2 = 4\omega_0$ remains open.

When $b > 0$, (1.1) has a two-parameter family of solitary wave solutions of the form:

$$u_\omega(t,x) = e^{i\omega_0 t + i\frac{\omega_1}{2}(x-\omega_1 t) - \frac{i}{4}\int_{-\infty}^{x-\omega_1 t}|\varphi_\omega(\eta)|^2 d\eta}\varphi_\omega(x-\omega_1 t), \tag{1.3}$$

where $\omega \in \Omega,\ \gamma = 1 + \frac{16}{3}b$, and φ_ω is the solution of

$$-\partial_x^2\varphi + (\omega_0 - \frac{\omega_1^2}{4})\varphi + \frac{\omega_1}{2}|\varphi|^2\varphi - \frac{3}{16}\gamma|\varphi|^4\varphi = 0. \tag{1.4}$$

In [18], Ohta showed that there exists $\kappa \in (0,1)$ such that $u_\omega(t,x)$ is stable when $-2\sqrt{\omega_0} < \omega_1 < 2\kappa\sqrt{\omega_0}$, and unstable when $2\kappa\sqrt{\omega_0} < \omega_1 < 2\sqrt{\omega_0}$. After this work, the stability theory on the endpoint cases $\omega_1 = 2\kappa\sqrt{\omega_0}$ and $\omega_1^2 = 4\omega_0$ remain open. In particular, the case $\omega_1^2 = 4\omega_0$ is the "zero mass" case in (1.4).

In this paper, we settle the stability theory for (1.1) on the endpoint case $\omega_1 = 2\sqrt{\omega_0}$. We put $\omega_1 = c > 0$, $\omega_0 = c^2/4$, and denote the solitary wave solutions (1.3) for this case as follows:

$$R_c(t,x) = e^{i\frac{c^2}{4}t}\phi_c(x-ct),$$

where $c > 0$, and

$$\phi_c(x) = e^{i\frac{c}{2}x - \frac{i}{4}\int_{-\infty}^{x}|\varphi_c(\eta)|^2 d\eta}\varphi_c(x). \tag{1.5}$$

We note that $\phi_c(x)$ is a solution of

$$-\partial_x^2\phi + \frac{c^2}{4}\phi + ci\partial_x\phi - i|\phi|^2\partial_x\phi - b|\phi|^4\phi = 0, \tag{1.6}$$

and $\varphi_c(x)$ is a solution of

$$-\partial_x^2\varphi + \frac{c}{2}|\varphi|^2\varphi - \frac{3}{16}\gamma|\varphi|^4\varphi = 0, \quad \gamma = 1 + \frac{16}{3}b.$$

From Wu [25], the equation $-W_{xx}+\frac{1}{2}W^3-\frac{3}{16}W^5=0$ has a unique (up to some symmetries) positive solution $W(x)=2(x^2+1)^{-\frac{1}{2}}$. According to this, we have

$$\varphi_c(x)=\gamma^{-\frac{1}{4}}l^{\frac{1}{2}}W(lx), \tag{1.7}$$

where $l=c\gamma^{-\frac{1}{2}}$.

For $\theta=(\theta_0,\theta_1)\in\mathbb{R}^2$ and $u\in H^1(\mathbb{R})$, we define

$$T(\theta)u=e^{i\theta_0}u(x-\theta_1),\ \theta=(\theta_0,\theta_1)\in\mathbb{R}^2.$$

Especially, the solitary wave solution $R_c(t,x)$ can be written as $R_c(t,x)=T(\theta(t))\phi_c(x)$ for $\theta(t)=(\frac{c^2}{4}t,ct)$.

For $\varepsilon>0$, we define

$$U_\varepsilon(\phi_c)=\{u\in H^1(\mathbb{R}):\inf_{\theta\in\mathbb{R}^2}\|u-T(\theta)\phi_c\|_{H^1}<\varepsilon\}.$$

Definition 1. *We say that the solitary wave solution $R_c(t,x)$ of (1.1) is stable if for any $\varepsilon>0$ there exists $\delta>0$ such that if $u_0\in U_\delta(\phi_c)$, then the solution $u(t)$ of (1.1) with $u(0)=u_0$ exists for all $t>0$, and $u(t)\in U_\varepsilon(\phi_c)$ for all $t>0$. Otherwise, $R_c(t,x)$ is said to be unstable.*

Now we state the main result of this paper. In order to avoid the tedious calculation, we only consider the case when b is close to 0, in which the equation (1.1) can be regarded as the approximate form of (1.2).

Theorem 1. *Let $b\in(0,b_0)$ for some small $b_0>0$, then the solitary wave solution $R_c(t,x)$ of (1.1) is unstable.*

This paper is organized as follows. In Section 2, we give the definitions of some important functionals and some useful lemmas. In Section 3, we construct the negative direction. In Section 4, we prove the Theorem 1.

2. Preliminaries

2.1. **Notations.** We use $X\lesssim Y$ to denote an estimate of the form $X\le CY$ for some constant $C>0$. Similarly, we will write $X\sim Y$ to mean $X\lesssim Y$ and $Y\lesssim X$. And we denote $\langle x\rangle=\sqrt{1+x^2}$.

For $u,v\in L^2(\mathbb{R})=L^2(\mathbb{R},\mathbb{C})$, we define

$$(u,v)_{L^2}=\operatorname{Re}\int_{\mathbb{R}}u(x)\overline{v(x)}\,dx$$

and regard $L^2(\mathbb{R})$ as a real Hilbert space.

For a function $f(x)$, its L^q-norm $\|f\|_{L^q}=\Big(\int_{\mathbb{R}}|f(x)|^q dx\Big)^{\frac{1}{q}}$ and its H^1-norm $\|f\|_{H^1}=(\|f\|_{L^2}^2+\|\partial_x f\|_{L^2}^2)^{\frac{1}{2}}$.

From the definitions of E, P and M, we have

$$E'(u) = -\partial_x^2 u - i|u|^2\partial_x u - b|u|^4 u, \tag{2.1}$$

$$P'(u) = i\partial_x u, \tag{2.2}$$

$$M'(u) = u. \tag{2.3}$$

Now we define

$$\begin{aligned} S_c(u) &= E(u) + cP(u) + \frac{c^2}{4}M(u) \\ &= \frac{1}{2}\|\partial_x u\|_{L^2}^2 - \frac{1}{4}(i|u|^2\partial_x u, u)_{L^2} - \frac{b}{6}\|u\|_{L^6}^6 + \frac{c}{2}(i\partial_x u, u)_{L^2} + \frac{c^2}{8}\|u\|_{L^2}^2, \\ K_c(u) &= \|\partial_x u\|_{L^2}^2 - (i|u|^2\partial_x u, u)_{L^2} - b\|u\|_{L^6}^6 + c(i\partial_x u, u)_{L^2} + \frac{c^2}{4}\|u\|_{L^2}^2. \end{aligned}$$

Then we have

$$\begin{aligned} S_c'(u) &= E'(u) + cP'(u) + \frac{c^2}{4}M'(u) \\ &= -\partial_x^2 u - i|u|^2\partial_x u - b|u|^4 u + ci\partial_x u + \frac{c^2}{4}u, \end{aligned} \tag{2.4}$$

$$K_c'(u) = -2\partial_x^2 u - 4i|u|^2\partial_x u - 6b|u|^4 u + 2ci\partial_x u + \frac{c^2}{2}u. \tag{2.5}$$

Moreover, (1.6) is equivalent to $S_c'(\phi) = 0$, and

$$K_c(u) = \langle S_c'(u), u\rangle.$$

Hence for the solution ϕ_c to (1.6), we have $K_c(\phi_c) = 0$. We also need the following elementary formulas on these two functionals.

Lemma 1. *$S_c''(\phi_c)$ is self-adjoint, that is, for any $f, g \in H^1(\mathbb{R})$,*

$$\langle S_c''(\phi_c)f, g\rangle = \langle S_c''(\phi_c)g, f\rangle. \tag{2.6}$$

Moreover,

$$\begin{aligned} S_c''(\phi_c)\phi_c &= -2i|\phi_c|^2\partial_x\phi_c - 4b|\phi_c|^4\phi_c, \\ S_c''(\phi_c)i\partial_x\phi_c &= 4bi\phi_c^3\overline{\phi_c\partial_x\phi_c} - 2\phi_c|\partial_x\phi_c|^2, \\ K_c'(\phi_c) &= -2i|\phi_c|^2\partial_x\phi_c - 4b|\phi_c|^4\phi_c, \end{aligned}$$

Proof. First, noting that

$$\partial_t\partial_s S_c(\phi_c + sg + tf) = \partial_s\partial_t S_c(\phi_c + sg + tf),$$

then taking $t = s = 0$ above, we get (2.6). Moreover, by (2.4), we have

$$\begin{aligned} S_c''(\phi_c)f = &- \partial_x^2 f + ci\partial_x f + \frac{c^2}{4}f \\ &- i|\phi_c|^2\partial_x f - 2i\partial_x\phi_c\mathrm{Re}\big(\phi_c\overline{f}\big) - b(3|\phi_c|^4 f + 2\phi_c^3\overline{\phi_c}\overline{f}). \end{aligned} \tag{2.7}$$

Then the rest formulas follow from the formula above, (2.5) and a direct computation. □

2.2. **Useful Lemmas.** From (1.5), (1.7) and a direct computation, we have

Lemma 2. *Let $b>0$ and $\gamma = 1+\frac{16}{3}b$. Then we have*

$$\begin{aligned} P(\phi_c) &= -c\pi(\gamma-1)\gamma^{-\frac{3}{2}}, \\ M(\phi_c) &= 2\pi\gamma^{-\frac{1}{2}}, \\ \partial_c P(\phi_c) &= -\pi(\gamma-1)\gamma^{-\frac{3}{2}}, \\ \partial_c M(\phi_c) &= 0. \end{aligned}$$

Next, we consider the following minimization problem:

$$\mu(c) = \inf\{S_c(u) : u \in H^1(\mathbb{R}) \setminus \{0\}, K_c(u) = 0\}. \tag{2.8}$$

Let $\mathscr{M}_c$ be the set of all minimizations for (2.8), i.e.

$$\mathscr{M}_c = \{\phi \in H^1(\mathbb{R}) \setminus \{0\} : S_c(\phi) = \mu(c), K_c(\phi) = 0\}.$$

Let $\mathscr{G}_c$ be the set of all critical points of S_c, so

$$\mathscr{G}_c = \{\phi \in H^1(\mathbb{R}) \setminus \{0\} : S_c'(\phi) = 0\}.$$

Now we give a lemma about the relation of two sets, which was proved in Lemma 3 of [12].

Lemma 3. *$\mathscr{G}_c = \{T(\theta)\phi_c : \ \theta \in \mathbb{R}^2\}$, and $\mathscr{M}_c = \mathscr{G}_c$. In particular, if $v \in H^1(\mathbb{R})$ satisfies $K_c(v) = 0$ and $v \neq 0$, then $S_c(\phi_c) \leq S_c(v)$.*

Lemma 4. *Let $b>0$. Then $\langle S_c''(\phi_c)\phi_c, \phi_c\rangle < 0$.*

Proof. We write the function

$$(0,\infty) \ni \lambda \mapsto S_c(\lambda\phi_c) = \frac{\lambda^2}{2}L_c(\phi_c) - \frac{\lambda^4}{4}(i|\phi_c|^2\partial_x\phi_c, \phi_c)_{L^2} - \frac{\lambda^6}{6}b\|\phi_c\|_{L^6}^6,$$

here

$$L_c(u) = \|\partial_x u\|_{L^2}^2 + \frac{c^2}{4}\|u\|_{L^2}^2 + c(i\partial_x u, u)_{L^2}.$$

Note that $L_c(u) \geq 0$ for any $u \in H^1(\mathbb{R})$. Then

$$\frac{d}{d\lambda}S_c(\lambda\phi_c) = \lambda L_c(\phi_c) - \lambda^3(i|\phi_c|^2\partial_x\phi_c, \phi_c)_{L^2} - \lambda^5 b\|\phi_c\|_{L^6}^6.$$

When $\lambda = 1$,

$$\begin{aligned} \frac{d}{d\lambda}S_c(\lambda\phi_c) &= L_c(\phi_c) - (i|\phi_c|^2\partial_x\phi_c, \phi_c)_{L^2} - b\|\phi_c\|_{L^6}^6 \\ &= K_c(\phi_c) = 0; \end{aligned}$$

when $0 < \lambda < 1$,

$$\begin{aligned} \frac{d}{d\lambda}S_c(\lambda\phi_c) &= \lambda L_c(\phi_c) - \lambda^3(i|\phi_c|^2\partial_x\phi_c, \phi_c)_{L^2} - \lambda^5 b\|\phi_c\|_{L^6}^6 \\ &> \lambda^3 L_c(\phi_c) - \lambda^3(i|\phi_c|^2\partial_x\phi_c, \phi_c)_{L^2} - \lambda^3 b\|\phi_c\|_{L^6}^6 \\ &= \lambda^3 K_c(\phi_c) = 0; \end{aligned}$$

when $\lambda > 1$, similarly, we have

$$\frac{d}{d\lambda} S_c(\lambda\phi_c) < \lambda^3 K_c(\phi_c) = 0.$$

Hence $S_c(\lambda\phi_c)$ has a strictly local maximum at $\lambda = 1$, so

$$0 \geq \frac{d^2}{d\lambda^2} S_c(\lambda\phi_c) \mid_{\lambda=1} = \langle S_c''(\phi_c)\phi_c, \phi_c\rangle.$$

According to the expression of ϕ_c, we have $\frac{d^2}{d\lambda^2} S_c(\lambda\phi_c) \neq 0$. Therefore we complete the proof of the lemma. $\square$

3. Negative direction and modulation

Denote $\gamma_0 = 1 + \frac{16}{3} b_0$. For $R > 0$, let $\chi_R(x) = \chi(\frac{x}{R})$, where χ is a smooth cutoff function such that $\chi(x) = 1$ when $|x| \leq 1$; $\chi(x) = 0$ when $|x| \geq 2$. The localization technique is employed here, because $\partial_c \phi_c$ does not belong to $L^2(\mathbb{R})$, as will be seen in the proof of the following lemma, which is the key to construct the negative direction.

Lemma 5. *Suppose $f \in H^1(\mathbb{R})$ satisfies*

(i) $|\langle S_c''(\phi_c) f, f\rangle| \lesssim 1$,

(ii) *for some positive constants c_0, c_1, C_0, C_1,*

$$c_0 \leq |\langle P'(\phi_c), f\rangle| \leq C_0, \;\; c_1 \leq |\langle M'(\phi_c), f\rangle| \leq C_1.$$

Then there exist $\mu = \mu(\gamma)$, $\nu = \nu(\gamma)$ and $R = R(\gamma)$ such that for the function $\psi = \phi_c + \mu\chi_R \partial_c \phi_c + \nu f \in H^1(\mathbb{R})$, the following properties hold:

(1) $\langle P'(\phi_c), \psi\rangle = \langle M'(\phi_c), \psi\rangle = 0$,

(2) $\mu(\gamma) \to \infty$ *as* $\gamma \to 1$; $|\nu(\gamma)| \lesssim 1$ *for any* $\gamma \in (1, \gamma_0]$,

(3) $\langle S_c''(\phi_c)\psi, \psi\rangle < 0$ *for any* $\gamma \in (1, \gamma_0]$.

Proof. (1) It is sufficient to find μ, ν such that

$$\begin{cases} \langle P'(\phi_c), \phi_c + \mu\chi_R\partial_c\phi_c + \nu f\rangle = 0, \\ \langle M'(\phi_c), \phi_c + \mu\chi_R\partial_c\phi_c + \nu f\rangle = 0. \end{cases}$$

By (2.2), (2.3) and Lemma 2, we have

$$\begin{cases} 2P(\phi_c) + \dfrac{1}{2}\mu\partial_c \mathrm{Im}\displaystyle\int \chi_R \partial_x\phi_c \overline{\phi_c}\, dx + \nu\langle P'(\phi_c), f\rangle = 0, \\ 2M(\phi_c) + \dfrac{1}{2}\mu\partial_c \displaystyle\int \chi_R |\phi_c|^2\, dx + \nu\langle M'(\phi_c), f\rangle = 0. \end{cases}$$

From the definitions (1.5) and (1.7), and a cumbersome but direct computation (see Appendix A.1 and A.2), we have

$$\begin{aligned}\partial_c \mathrm{Im}\int \chi_R\partial_x\phi_c\overline{\phi_c}\,dx =& 2\partial_c P(\phi_c)+\gamma^{-\frac12}\int\Big[\chi\Big(\frac{x}{lR}\Big)-1-\frac{x}{lR}\chi'\Big(\frac{x}{lR}\Big)\Big]\Big(\frac12 W^2-\frac14\gamma^{-1}W^4\Big)\,dx\\ =& 2\partial_c P(\phi_c)+O(\frac{1}{R}),\end{aligned}\tag{3.1}$$

$$\begin{aligned}\partial_c\int\chi_R|\phi_c|^2\,dx =& -c^{-1}\gamma^{-\frac12}\int\frac{x}{lR}\chi'\Big(\frac{x}{lR}\Big)W^2\,dx\\ =& O(\frac{1}{R}).\end{aligned}\tag{3.2}$$

Making use of (3.1) and (3.2), and choosing $R=\big[c_0(\gamma-1)\big]^{-1}$ for some suitable small constant $c_0>0$, then under the assumption (ii), we can solve μ,ν by

$$\begin{cases}\mu=\dfrac{-B}{\partial_c P(\phi_c)+O(R^{-1})}\sim\dfrac{B\gamma^{\frac32}}{\pi(\gamma-1)},\\ \nu=\dfrac{-\|\phi_c\|_{L^2}^2+O(|\mu|R^{-1})}{\langle M'(\phi_c),f\rangle}\sim\dfrac{-\|\phi_c\|_{L^2}^2}{\langle M'(\phi_c),f\rangle},\end{cases}$$

where we denote

$$B=2P(\phi_c)-\frac{\|\phi_c\|_{L^2}^2}{\langle M'(\phi_c),f\rangle}\langle P'(\phi_c),f\rangle.$$

(2) First we claim that under the assumption (ii), there exist some positive constants c_2,C_2 such that

$$c_2\le|B|\le C_2.\tag{3.3}$$

Indeed, from Lemma 2, we have

$$B=-2c\pi(\gamma-1)\gamma^{-\frac32}-4\pi\gamma^{-\frac12}\frac{\langle P'(\phi_c),f\rangle}{\langle M'(\phi_c),f\rangle}.$$

Note that the first term tends to 0 when $\gamma\to1$, the second term is upper controlled by $4\pi\gamma^{-\frac12}C_0c_1^{-1}$ and lower controlled by $4\pi\gamma^{-\frac12}c_0C_1^{-1}$. Hence we have (3.3).

Employing (3.3), we have

$$\mu\sim\frac{B\gamma^{\frac32}}{\pi(\gamma-1)}\to\infty\quad\text{as}\quad\gamma\to1.\tag{3.4}$$

Also, from assumption (ii), we have

$$|\nu|\le4\pi\gamma^{-\frac12}c_1^{-1}.$$

(3) Differentiating $S_c'(\phi_c)=0$ with respect to c, we have

$$S_c''(\phi_c)\partial_c\phi_c=-P'(\phi_c)-\frac{c}{2}M'(\phi_c).$$

Then from (1), for $\psi=\phi_c+\mu\chi_R\partial_c\phi_c+\nu f$ we have

$$\langle S_c''(\phi_c)\partial_c\phi_c,\psi\rangle=0.\tag{3.5}$$

By using Lemma 2, we can get

$$\begin{aligned}\langle S_c''(\phi_c)\partial_c\phi_c, \partial_c\phi_c\rangle &= -\partial_c P(\phi_c) - \frac{c}{2}\partial_c M(\phi_c)\\ &= \pi(\gamma-1)\gamma^{-\frac{3}{2}} > 0. \end{aligned} \tag{3.6}$$

Further,

$$\begin{aligned}\langle S_c''(\phi_c)\psi, f\rangle &= \langle S_c''(\phi_c)(\phi_c + \mu\chi_R\partial_c\phi_c + \nu f), f\rangle\\ &= \langle S_c''(\phi_c)\phi_c, f\rangle + \mu\langle S_c''(\phi_c)\chi_R\partial_c\phi_c, f\rangle + \nu\langle S_c''(\phi_c)f, f\rangle. \end{aligned} \tag{3.7}$$

Now by (2.6) and according to the selection of ψ, we expand

$$\begin{aligned}\langle S_c''(\phi_c)\phi_c, \phi_c\rangle =&\langle S_c''(\phi_c)(\psi - \mu\chi_R\partial_c\phi_c - \nu f), \psi - \mu\chi_R\partial_c\phi_c - \nu f\rangle\\ =&\langle S_c''(\phi_c)\psi, \psi\rangle - 2\mu\langle S_c''(\phi_c)\chi_R\partial_c\phi_c, \psi\rangle - 2\nu\langle S_c''(\phi_c)\psi, f\rangle\\ &+ \mu^2\langle S_c''(\phi_c)\chi_R\partial_c\phi_c, \chi_R\partial_c\phi_c\rangle + 2\mu\nu\langle S_c''(\phi_c)\chi_R\partial_c\phi_c, f\rangle\\ &+ \nu^2\langle S_c''(\phi_c)f, f\rangle. \end{aligned} \tag{3.8}$$

First, from (3.7) we reduce (3.8) to

$$\begin{aligned}&\langle S_c''(\phi_c)\psi, \psi\rangle - 2\mu\langle S_c''(\phi_c)\chi_R\partial_c\phi_c, \psi\rangle - 2\nu\langle S_c''(\phi_c)\phi_c, f\rangle\\ &\quad + \mu^2\langle S_c''(\phi_c)\chi_R\partial_c\phi_c, \chi_R\partial_c\phi_c\rangle - 2\mu\nu\langle S_c''(\phi_c)\chi_R\partial_c\phi_c, f\rangle - 2\nu^2\langle S_c''(\phi_c)f, f\rangle\\ &\quad + 2\mu\nu\langle S_c''(\phi_c)\chi_R\partial_c\phi_c, f\rangle + \nu^2\langle S_c''(\phi_c)f, f\rangle. \end{aligned}$$

Merging the same terms we lastly write $\langle S_c''(\phi_c)\phi_c, \phi_c\rangle$ as

$$\begin{aligned}&\langle S_c''(\phi_c)\psi, \psi\rangle - 2\mu\langle S_c''(\phi_c)\chi_R\partial_c\phi_c, \psi\rangle - 2\nu\langle S_c''(\phi_c)\phi_c, f\rangle\\ &\quad - \nu^2\langle S_c''(\phi_c)f, f\rangle + \mu^2\langle S_c''(\phi_c)\chi_R\partial_c\phi_c, \chi_R\partial_c\phi_c\rangle. \end{aligned} \tag{3.9}$$

Now we estimate the terms from the second to the fifth in (3.9). First, we claim that

$$|\langle S_c''(\phi_c)\chi_R\partial_c\phi_c, \psi\rangle| \lesssim c_0. \tag{3.10}$$

To prove (3.10), we use (3.5) and obtain that

$$\langle S_c''(\phi_c)\chi_R\partial_c\phi_c, \psi\rangle = -\langle S_c''(\phi_c)(1-\chi_R)\partial_c\phi_c, \psi\rangle. \tag{3.11}$$

We need the following estimate.

Lemma 6. *Let $R > 0$. Then*

$$|S_c''(\phi_c)(1-\chi_R)\partial_c\phi_c\,(x)| \lesssim \big(1 - \chi_{\frac{R}{2}}(x)\big)\langle x\rangle^{-2}.$$

Proof. From (1.5), we have

$$\partial_c\phi_c = e^{i\frac{c}{2}x - \frac{i}{4}\int_{-\infty}^x |\varphi_c(\eta)|^2 d\eta}\Big(\frac{i}{2}x\varphi_c - \frac{i}{2}\varphi_c\int_{-\infty}^x \varphi_c\partial_c\varphi_c\, d\eta + \partial_c\varphi_c\Big). \tag{3.12}$$

From the definition (1.7), we have

$$|x\varphi_c| \lesssim 1, \quad \varphi_c \lesssim \langle x\rangle^{-1}, \quad |\partial_x\varphi_c| \lesssim \langle x\rangle^{-2}, \quad \text{and} \quad |\partial_{xx}\varphi_c| \lesssim \langle x\rangle^{-3}, \tag{3.13}$$

and further

$$\partial_c\varphi_c \lesssim \langle x\rangle^{-1}, \quad \text{and} \quad |\partial_x\partial_c\varphi_c| \lesssim \langle x\rangle^{-2}. \tag{3.14}$$

Using (3.12) and (3.13), we get

$$|\partial_c\phi_c| \lesssim 1, \quad \text{and} \quad |\partial_x\partial_c\phi_c| \lesssim \langle x\rangle^{-1}. \tag{3.15}$$

(The proof of (3.13)–(3.15) can be found in Appendix A.3). Moreover, from the following identity for suitable function f,

$$\partial_x^2 f - ci\partial_x f - \frac{c^2}{4}f = e^{\frac{c}{2}ix}\partial_x^2\big(e^{-\frac{c}{2}ix}f\big),$$

and (2.7), we can write $S_c''(\phi_c)(1-\chi_R)\partial_c\phi_c$ as

$$\begin{aligned}
&- e^{\frac{c}{2}ix}\partial_{xx}\Big[e^{-\frac{i}{4}\int_{-\infty}^x|\varphi_c(\eta)|^2d\eta}(1-\chi_R)\Big(\frac{i}{2}x\varphi_c - \frac{i}{2}\varphi_c\int_{-\infty}^x \varphi_c\partial_c\varphi_c\,d\eta + \partial_c\varphi_c\Big)\Big]\\
&\quad - (1-\chi_R)\Big[i|\phi_c|^2\partial_x\partial_c\phi_c + 2i\partial_x\phi_c\mathrm{Re}\big(\overline{\phi_c}\partial_c\phi_c\big) + b(3|\phi_c|^4\partial_c\phi_c + 2\phi_c^3\overline{\phi_c}\,\overline{\partial_c\phi_c})\Big].
\end{aligned}$$

Now using (3.13)–(3.15), we find that every term in the expression above can be controlled by $\langle x\rangle^{-2}$. Thus, we obtain that

$$|S_c''(\phi_c)(1-\chi_R)\partial_c\phi_c\,(x)| \lesssim \langle x\rangle^{-2}.$$

Since the support of $S_c''(\phi_c)(1-\chi_R)\partial_c\phi_c$ is included in $[R,+\infty)$, we prove the lemma. □

Now we obtain from (3.11) that

$$|\langle S_c''(\phi_c)\chi_R\partial_c\phi_c,\psi\rangle| \lesssim \|S_c''(\phi_c)(1-\chi_R)\partial_c\phi_c\|_{L^1}\|\psi\|_{L^\infty}.$$

Note that $\|\psi\|_{L^\infty} \lesssim |\mu|$ and recall that $R = \big[c_0(\gamma-1)\big]^{-1}$, then by Lemma 6, we get

$$|\langle S_c''(\phi_c)\chi_R\partial_c\phi_c,\psi\rangle| \lesssim |\mu|R^{-1} \lesssim c_0.$$

This proves (3.10). From Lemma 1 and the boundedness of ν, we have

$$|2\nu\langle S_c''(\phi_c)\phi_c, f\rangle| \lesssim \|\phi_c\|_{H^1}^3\|f\|_{L^2} \lesssim \|f\|_{L^2}. \tag{3.16}$$

This gives the estimate of the third term in (3.9). From the assumption (i) and the conclusion (2), we know that

$$|\nu^2\langle S_c''(\phi_c)f,f\rangle| \lesssim 1. \tag{3.17}$$

This gives the estimate of the fourth term in (3.9). From (3.3), (3.4) and (3.6), we find that

$$\mu^2\langle S_c''(\phi_c)\partial_c\phi_c,\partial_c\phi_c\rangle \sim |\mu| \sim \frac{1}{\gamma-1}.$$

Further, by Lemma 6 and argued similarly as (3.10), we have

$$\mu^2|\langle S_c''(\phi_c)\chi_R\partial_c\phi_c,(1-\chi_R)\partial_c\phi_c\rangle| \lesssim c_0|\mu|,$$

and

$$\mu^2|\langle S_c''(\phi_c)\partial_c\phi_c,(1-\chi_R)\partial_c\phi_c\rangle| \lesssim c_0|\mu|.$$

Therefore, by choosing c_0 small enough, we have the estimate of the fifth term in (3.9) as follows,

$$\mu^2\langle S_c''(\phi_c)\chi_R\partial_c\phi_c,\chi_R\partial_c\phi_c\rangle \sim |\mu|. \tag{3.18}$$

Note that $|\mu| \to +\infty$, when $\gamma \to 1$. Hence, combining with the estimates (3.10), (3.16), (3.17) and (3.18), and choosing γ_0 suitably close to 1, the second, the third, and the fourth terms in (3.9) are dominated by the fifth term. Therefore, we obtain from (3.9) that

$$\langle S_c''(\phi_c)\phi_c, \phi_c\rangle > \langle S_c''(\phi_c)\psi, \psi\rangle.$$

Together with Lemma 4, we get that for any $\gamma \in (1, \gamma_0]$,

$$\langle S_c''(\phi_c)\psi, \psi\rangle < \langle S_c''(\phi_c)\phi_c, \phi_c\rangle < 0.$$

This finishes the proof of Lemma 5. □

Remark 1. *Note that $f = |\phi_c|^2\phi_c$ verifies the assumptions in Lemma 5.*

Corollary 1. *There exists a constant $\beta_0 > 0$ such that*

$$S_c(\phi_c + \beta\psi) < S_c(\phi_c),$$

for all $\beta \in (-\beta_0, 0) \cup (0, \beta_0)$.

Proof. By Taylor's expansion, for $\beta \in \mathbb{R}$, we have

$$\begin{aligned} S_c(\phi_c + \beta\psi) &= S_c(\phi_c) + \beta\langle S_c'(\phi_c), \psi\rangle + \beta^2 \int_0^1 (1-s)\langle S_c''(\phi_c + s\beta\psi)\psi, \psi\rangle ds \\ &= S_c(\phi_c) + \beta^2 \int_0^1 (1-s)\langle S_c''(\phi_c + s\beta\psi)\psi, \psi\rangle ds. \end{aligned}$$

Since $\langle S_c''(\phi_c)\psi, \psi\rangle < 0$, by the continuity of $\beta \mapsto \langle S_c''(\phi_c + \beta\psi)\psi, \psi\rangle$, there exists a constant $\beta_0 > 0$, such that

$$\langle S_c''(\phi_c + \beta\psi)\psi, \psi\rangle \le \frac{1}{2}\langle S_c''(\phi_c)\psi, \psi\rangle < 0, \quad \text{for any } \beta \in (-\beta_0, 0) \cup (0, \beta_0).$$

Thus, for any $\beta \in (-\beta_0, 0) \cup (0, \beta_0)$, we have

$$S_c(\phi_c + \beta\psi) \le S_c(\phi_c) + \frac{\beta^2}{4}\langle S_c''(\phi_c)\psi, \psi\rangle < S_c(\phi_c).$$

□

We denote $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Then we can get the following proposition.

Proposition 1. *There exist a constant $\varepsilon_0 > 0$ and a C^1-function $\theta = (\theta_0, \theta_1) : U_{\varepsilon_0}(\phi_c) \to \mathbb{T} \times \mathbb{R}$ such that $\theta(\phi_c) = 0$, and*

(1) $\langle iu, T(\theta)\phi_c\rangle = 0$, $\langle -\partial_x u, T(\theta)\phi_c\rangle = 0$,

(2) $\theta(T(\xi)u) = \theta(u) + \xi$ *for any* $u \in U_{\varepsilon_0}(\phi_c)$ *and* $\theta_0 \in \mathbb{T} \times \mathbb{R}$,

(3) $\|\partial_u \theta_j(u)\|_{H^1(\mathbb{R})} \le C$ *for any* $u \in U_{\varepsilon_0}(\phi_c)$, $j = 0, 1$.

Proof. (1) We define the function

$$F(u, \theta) = (F_0(u, \theta), F_1(u, \theta)),$$

where

$$F_0(u, \theta) = \langle iu, T(\theta)\phi_c\rangle, \; F_1(u, \theta) = \langle -\partial_x u, T(\theta)\phi_c\rangle.$$

Then $F_0(\phi_c, 0) = \langle i\phi_c, \phi_c\rangle = 0$ and $F_1(\phi_c, 0) = \langle -\partial_x\phi_c, \phi_c\rangle = 0$, that is,

$$F(\phi_c, 0) = (0, 0).$$

According to the definition of $F(u,\theta)$, we have

$$\partial_{\theta_0} F_0(u,\theta) = \langle iu, iT(\theta)\phi_c\rangle, \qquad \partial_{\theta_1} F_0(u,\theta) = \langle iu, -\partial_x T(\theta)\phi_c\rangle, \tag{3.19}$$

$$\partial_{\theta_0} F_1(u,\theta) = \langle -\partial_x u, iT(\theta)\phi_c\rangle, \qquad \partial_{\theta_1} F_1(u,\theta) = \langle -\partial_x u, -\partial_x T(\theta)\phi_c\rangle. \tag{3.20}$$

We denote

$$\partial_\theta F(u,\theta) = \begin{pmatrix} \partial_{\theta_0} F_0(u,\theta) & \partial_{\theta_1} F_0(u,\theta) \\ \partial_{\theta_0} F_1(u,\theta) & \partial_{\theta_1} F_1(u,\theta) \end{pmatrix}.$$

Since

$$\partial_{\theta_0} F_0(\phi_c,0) = \langle i\phi_c, i\phi_c\rangle = \|\phi_c\|_{L^2}^2 = 4\pi\gamma^{-\frac{1}{2}},$$

$$\partial_{\theta_1} F_0(\phi_c,0) = \langle i\phi_c, -\partial_x\phi_c\rangle = \operatorname{Im}\int_{\mathbb{R}} \phi_c\overline{\partial_x\phi_c}dx = -2\pi c\gamma^{-\frac{3}{2}}(\gamma-1),$$

$$\partial_{\theta_0} F_1(\phi_c,0) = \langle -\partial_x\phi_c, i\phi_c\rangle = \operatorname{Im}\int_{\mathbb{R}} \phi_c\overline{\partial_x\phi_c}dx = -2\pi c\gamma^{-\frac{3}{2}}(\gamma-1),$$

$$\partial_{\theta_1} F_1(\phi_c,0) = \langle -\partial_x\phi_c, -\partial_x\phi_c\rangle = \|\partial_x\phi_c\|_{L^2}^2 = \frac{3}{2}\pi c^2\gamma^{-\frac{5}{2}} - \frac{3}{2}\pi c^2\gamma^{-\frac{3}{2}} + \pi c^2\gamma^{-\frac{1}{2}},$$

the Jacobian

$$|\partial_\theta F(\phi_c,0)| = \|\phi_c\|_{L^2}^2\ \|\partial_x\phi_c\|_{L^2}^2 - (\operatorname{Im}\int_{\mathbb{R}} \phi_c\overline{\partial_x\phi_c}dx)^2 = 2\pi^2c^2\gamma^{-3}(\gamma+1) > 0. \tag{3.21}$$

Therefore by implicit function theorem, there exist a $\varepsilon_0 > 0$ and a unique $\mathbb{C}^1$-function

$$\theta(u) = (\theta_0(u), \theta_1(u)) : U_{\varepsilon_0}(\phi_c) \to \mathbb{T}\times\mathbb{R}, \quad \text{and} \quad \theta(\phi_c) = 0,$$

such that for any $u \in U_{\varepsilon_0}(\phi_c)$,

$$F(u,\theta(u)) = 0,$$

that is

$$\langle iu, T(\theta)\phi_c\rangle = 0,\ \langle -\partial_x u, T(\theta)\phi_c\rangle = 0.$$

(2) In particular, let $\widetilde{u} = T(\xi)u$, for (1), $\theta_{\widetilde{u}} = \theta(u) + \xi$ satisfies (1) for $\widetilde{u}$. Then by the uniqueness, we have $\theta(T(\xi)u) = \theta(u) + \xi$.

(3) From (3.21) and the continuity, $\partial_\theta F(u,\theta)$ is invertible for any $u \in U_{\varepsilon_0}(\phi_c)$, and

$$\partial_\theta F^{-1}(u,\theta) = \frac{1}{|\partial_\theta F(u,\theta)|}\begin{pmatrix} \partial_{\theta_1} F_1(u,\theta) & -\partial_{\theta_1} F_0(u,\theta) \\ -\partial_{\theta_0} F_1(u,\theta) & \partial_{\theta_0} F_0(u,\theta) \end{pmatrix}.$$

Differentiating $F(u,\theta(u)) = 0$ with u, then

$$\partial_u\theta = -\partial_\theta F^{-1}(u,\theta)\cdot F_u^T(u,\theta),$$

where $F_u^T(u,\theta) = (-iT(\theta)\phi_c, \partial_x T(\theta)\phi_c)^T$.
Then by a simple calculation, we can get

$$\partial_u\theta_0(u) = \frac{1}{|\partial_\theta F(u,\theta)|}(\partial_{\theta_1} F_1(u,\theta)\ iT(\theta)\phi_c + \partial_{\theta_1} F_0(u,\theta)\ \partial_x T(\theta)\phi_c),$$

$$\partial_u\theta_1(u) = -\frac{1}{|\partial_\theta F(u,\theta)|}(\partial_{\theta_0} F_1(u,\theta)\ iT(\theta)\phi_c + \partial_{\theta_0} F_0(u,\theta)\ \partial_x T(\theta)\phi_c).$$

From (3.19), (3.20), (3.21) and the continuity, we see that

$$\|\partial_u\theta_j(u)\|_{H^1(\mathbb{R})} \le C \quad \text{for any} \quad u \in U_{\varepsilon_0}(\phi_c), \quad j = 0\ ,1.$$

Then we complete the proof of the proposition. □

4. PROOF OF THEOREM 1

For $u \in U_{\varepsilon_0}(\phi_c)$, we define

$$A(u) = (iu, T(\theta(u))\psi)_{L^2},$$

$$q(u) = T(\theta(u))\psi + i(u, T(\theta(u))\psi)\partial_u\theta_0(u) + i(iu, -\partial_x T(\theta(u))\psi)\partial_u\theta_1(u).$$

Then we have

$$A'(u) = -iT(\theta(u))\psi + (u, T(\theta(u))\psi)\partial_u\theta_0(u) + (iu, -\partial_x T(\theta(u))\psi)\partial_u\theta_1(u) = -iq(u).$$

Lemma 7. *For $u \in U_{\varepsilon_0}(\phi_c)$,*

(1) *$A(T(\xi)u) = A(u)$ for all $\xi \in \mathbb{T} \times \mathbb{R}$,*

(2) *$q(u)$ is continuous from $U_{\varepsilon_0}(\phi_c)$ to $H^1(\mathbb{R})$ and $q(\phi_c) = \psi$,*

(3) *$\langle q(u), P'(u)\rangle = \langle q(u), M'(u)\rangle = 0$.*

Proof. (1) By Proposition 1 (2), we have

$$A(T(\xi)u) = (iT(\xi)u, T(\theta(T(\xi)u))\psi)_{L^2} = (iT(\xi)u, T(\xi)T(\theta(u))\psi)_{L^2} = A(u).$$

(2) By Lemma 5 (1),

$$\begin{aligned} q(\phi_c) &= \psi + (\phi_c, \psi)i\partial_u\theta_0(\phi_c) + (i\phi_c, -\partial_x\psi)i\partial_u\theta_1(\phi_c) \\ &= \psi + (\phi_c, \psi)i\partial_u\theta_0(\phi_c) + (i\partial_x\phi_c, \psi)i\partial_u\theta_1(\phi_c) \\ &= \psi. \end{aligned}$$

Moreover, from the definition we know that $q(u)$ is continuous from $U_{\varepsilon_0}(\phi_c)$ to $H^1(\mathbb{R})$.

(3) Differentiating $A(T(\xi)u) = A(u)$ with $\xi_j,\ j = 0, 1$, we have

$$0 = \partial_{\xi_0} A(T(\xi)u)|_{\xi=0} = \langle A'(T(\xi)u), iT(\xi)u\rangle|_{\xi=0} = \langle A'(u), iu\rangle = \langle -q(u), u\rangle,$$

$$0 = \partial_{\xi_1} A(T(\xi)u)|_{\xi=0} = \langle A'(T(\xi)u), -\partial_x T(\xi)u\rangle|_{\xi=0} = \langle A'(u), -\partial_x u\rangle = \langle -q(u), i\partial_x u\rangle.$$

That is,

$$\langle q(u), P'(u)\rangle = \langle q(u), M'(u)\rangle = 0.$$

□

Now, we prove Theorem 1.

Proof. Let $b \in (0, b_0)$. Let β_0 and ε_0 be the positive constants given in Corollary 1 and Proposition 1, respectively. Let $u_\beta(0) = \phi_c + \beta\psi$ and let $u_\beta(t)$ be the solution of (1.1) with the initial data $u_\beta(0)$. Suppose $R_c(t, x)$ is stable. Then for any fixed $\varepsilon_0 > 0$, there exists a small positive constant $\beta_0' < \beta_0$, such that for any $\beta \in (-\beta_0', 0) \cup (0, \beta_0')$, $u_\beta(t) \in U_{\varepsilon_0}(\phi_c)$ for any $t > 0$.

Now we consider the quantity $A(u_\beta(t))$. By Lemma 5 (3) and (2.4), we have

$$\begin{aligned} \partial_t A(u_\beta(t)) &= \langle A'(u_\beta), \partial_t u_\beta\rangle = \langle iA'(u_\beta), i\partial_t u_\beta\rangle = \langle q(u_\beta), E'(u_\beta)\rangle \\ &= \langle q(u_\beta), E'(u_\beta) + cP'(u_\beta) + \frac{c^2}{4}M'(u_\beta)\rangle = \langle q(u_\beta), S_c'(u_\beta)\rangle. \end{aligned}$$

So, we get that

$$\lambda\partial_t A(u_\beta(t)) = S_c(u_\beta+\lambda q(u_\beta))-S_c(u_\beta)-\lambda^2\int_0^1(1-s)\langle S_c''(\phi_c+s\lambda q(u_\beta))q(u_\beta), q(u_\beta)\rangle ds.$$

Now we claim that

$$\langle K_c'(\phi_c), \psi\rangle \neq 0. \tag{4.1}$$

To show this, we need the following lemma.

Lemma 8. *If $v \in H^1(\mathbb{R})$ satisfies $\langle K_c'(\phi_c), v\rangle = 0$, then $\langle S_c''(\phi_c)v, v\rangle \geq 0$.*

Proof. See Lemma 4 in [18] for the proof. □

By Lemma 5 (3) and Lemma 8, we have (4.1). Then applying the implicit functional theorem, we can find a $\lambda(u_\beta) \in (-\lambda_0, \lambda_0)\setminus\{0\}$, such that for any $u_\beta \in U_{\varepsilon_0}(\phi_c)$,

$$K_c(u_\beta + \lambda(u_\beta)q(u_\beta)) = 0.$$

Then by Lemma 3, we have

$$S_c(u_\beta + \lambda(u_\beta)q(u_\beta)) \geq S_c(\phi_c).$$

Without loss of generality, we assume $\lambda(u) > 0$. By the conservation laws, we have $S_c(u_\beta(t)) = S_c(u_\beta(0)) = S_c(\phi_c + \beta\psi)$. Then

$$\begin{aligned}&S_c(u_\beta + \lambda(u_\beta)q(u_\beta)) - S_c(u_\beta) - \lambda^2\int_0^1(1-s)\langle S_c''(\phi_c + s\lambda q(u_\beta))q(u_\beta), q(u_\beta)\rangle ds\\ &\geq S_c(\phi_c) - S_c(\phi_c+\beta\psi) - \frac{\lambda^2}{4}\langle S_c''(\phi_c)\psi,\psi\rangle\\ &\geq S_c(\phi_c) - S_c(\phi_c+\beta\psi) > 0.\end{aligned}$$

Hence

$$\lambda(u_\beta)\partial_t A(u_\beta(t)) \geq S_c(\phi_c) - S_c(\phi_c+\beta\psi).$$

From Corollary 1, $S_c(\phi_c) - S_c(\phi_c + \beta\psi) > 0$. Hence,

$$\partial_t A(u_\beta(t)) \geq \frac{1}{\lambda(u_\beta)}(S_c(\phi_c) - S_c(\phi_c+\beta\psi)) \geq \frac{1}{\lambda_0}(S_c(\phi_c) - S_c(\phi_c+\beta\psi)) > 0.$$

Therefore, we get that $\partial_t A(u_\beta(t)) \to +\infty$ as $t \to \infty$. On the other hand,

$$|\partial_t A(u_\beta(t))| \leq \|u_\beta\|_{L^2}\|\psi\|_{L^2} \leq C \quad \text{for any} \quad t > 0.$$

This is a contradiction. This finishes the proof of Theorem 1. □

Appendix A: Some element estimates

A.1. Proof of (3.1)**.** From (1.5), (1.7), and changing of variables,

$$\begin{aligned}
\partial_c \mathrm{Im}\int \chi_R \partial_x\phi_c\overline{\phi_c}\,dx =&\partial_c \mathrm{Im}\int \chi_R\Big(\frac{c}{2}i\varphi_c-\frac{1}{4}i\varphi_c^3+\partial_x\varphi_c\Big)\varphi_c\,dx\\
=&\partial_c\int \chi_R\Big(\frac{c}{2}\varphi_c^2-\frac14\varphi_c^4\Big)\,dx\\
=&\partial_c\int \chi_R\Big[\frac{c}{2}\gamma^{-\frac12}lW^2(lx)-\frac14\gamma^{-1}l^2W^4(lx)\Big]\,dx\\
=&\partial_c\int \chi_R\Big[\frac12 l^2W^2(lx)-\frac14\gamma^{-1}l^2W^4(lx)\Big]\,dx\\
=&\partial_c\Big(l\int \chi\Big(\frac{x}{lR}\Big)\Big[\frac12W^2-\frac14\gamma^{-1}W^4\Big]\,dx\Big)\\
=&\gamma^{-\frac12}\int \chi\Big(\frac{x}{lR}\Big)\Big[\frac12W^2-\frac14\gamma^{-1}W^4\Big]\,dx\\
&\quad -l\int \frac{x}{clR}\chi'\Big(\frac{x}{lR}\Big)\Big[\frac12W^2-\frac14\gamma^{-1}W^4\Big]\,dx\\
=&\gamma^{-\frac12}\Big(\frac12\|W\|_{L^2}^2-\frac14\gamma^{-1}\|W\|_{L^4}^4\Big)\\
&\quad +\gamma^{-\frac12}\int\Big(\chi\Big(\frac{x}{lR}\Big)-1-\frac{x}{lR}\chi'\Big(\frac{x}{lR}\Big)\Big[\frac12W^2-\frac14\gamma^{-1}W^4\Big]\,dx\\
=&2\partial_cP(\phi_c)+\gamma^{-\frac12}\int\Big[\chi\Big(\frac{x}{lR}\Big)-1-\frac{x}{lR}\chi'\Big(\frac{x}{lR}\Big)\Big]\Big(\frac12W^2-\frac14\gamma^{-1}W^4\Big)\,dx\\
=&2\partial_cP(\phi_c)+O(\frac{1}{R});
\end{aligned}$$

A.2. Proof of (3.2)**.** Arguing as above,

$$\begin{aligned}
\partial_c\int\chi_R|\phi_c|^2\,dx =&\partial_c\int \chi_R\,|\varphi_c|^2\,dx\\
=&\gamma^{-\frac12}\partial_c\Big(l\int\chi_R\,W^2(lx)\,dx\Big)\\
=&\gamma^{-\frac12}\partial_c\int\chi\Big(\frac{x}{lR}\Big)\,W^2\,dx\\
=&-c^{-1}\gamma^{-\frac12}\int\frac{x}{lR}\chi'\Big(\frac{x}{lR}\Big)W^2\,dx\\
=&O(\frac1R).
\end{aligned}$$

A.3. Proof of (3.13)–(3.15)**.** Recall that $\varphi_c(x)=\gamma^{-\frac14}l^{\frac12}W(lx)$, $l=c\gamma^{-\frac12}$. So we have $\varphi_c\lesssim\langle x\rangle^{-1}$ and $|x\varphi_c|\lesssim1$, here and in the following, the implicit constants are only dependent on c,γ. Moreover,

$$\partial_x\varphi_c(x)=\gamma^{-\frac14}l^{\frac32}W'(lx),\quad \partial_{xx}\varphi_c(x)=\gamma^{-\frac14}l^{\frac52}W''(lx).$$

Since $|W'| \lesssim \langle x\rangle^{-2}, |W''| \lesssim \langle x\rangle^{-3}$, we have

$$|\partial_x \varphi_c| \lesssim \langle x\rangle^{-2}, \quad \text{and} \quad |\partial_{xx}\varphi_c| \lesssim \langle x\rangle^{-3}.$$

Now we consider the estimates on $\partial_c\varphi_c$. By direct computations,

$$\partial_c\varphi_c(x) = \frac{1}{2}\gamma^{-\frac{3}{4}} l^{-\frac{1}{2}} W(lx) + \frac{1}{2}\gamma^{-\frac{3}{4}} l^{\frac{1}{2}} xW'(lx),$$

and

$$\partial_x\partial_c\varphi_c(x) = \gamma^{-\frac{3}{4}} l^{\frac{1}{2}} W'(lx) + \frac{1}{2}\gamma^{-\frac{3}{4}} l^{\frac{3}{2}} xW''(lx).$$

Since $W \lesssim \langle x\rangle^{-1}, |xW'| \lesssim \langle x\rangle^{-1}, |xW''| \lesssim \langle x\rangle^{-2}$, we have

$$|\partial_c\varphi_c| \lesssim \langle x\rangle^{-1}, \quad |\partial_x\partial_c\varphi_c| \lesssim \langle x\rangle^{-2}.$$

This proves (3.14).

Last, we give the estimates on ϕ_c. By (3.12),

$$\begin{aligned} |\partial_c\phi_c| &\lesssim |x\varphi_c| + \varphi_c \int_{-\infty}^{x} \varphi_c |\partial_c\varphi_c|\, d\eta + |\partial_c\varphi_c| \\ &\lesssim 1 + \langle x\rangle^{-1} \int_{-\infty}^{x} \langle \eta\rangle^{-2}\, d\eta + \langle x\rangle^{-1} \\ &\lesssim 1. \end{aligned}$$

Further,

$$\begin{aligned} \partial_x\partial_c\phi_c =& \left(\frac{c}{2} i - \frac{1}{4} i\varphi_c^2\right) \partial_c\phi_c + e^{i\frac{c}{2}x - \frac{i}{4}\int_{-\infty}^{x} |\varphi_c(\eta)|^2 d\eta} \Big(\frac{i}{2}\varphi_c + \frac{i}{2} x\partial_x\varphi_c \\ & - \frac{i}{2}\partial_x\varphi_c \int_{-\infty}^{x} \varphi_c\partial_c\varphi_c\, d\eta - \frac{i}{2}\varphi_c^2\partial_c\varphi_c + \partial_x\partial_c\varphi_c\Big). \end{aligned}$$

Similarly, using (3.13) and (3.14), we have

$$\begin{aligned} |\partial_x\partial_c\phi_c| \lesssim& \left|\frac{c}{2} + \varphi_c^2\right| |\partial_c\phi_c| + |\varphi_c| + |x\partial_x\varphi_c| + |\partial_x\varphi_c| \int_{-\infty}^{x} \varphi_c|\partial_c\varphi_c|\, d\eta + \varphi_c^2|\partial_c\varphi_c| + |\partial_x\partial_c\varphi_c| \\ \lesssim& \langle x\rangle^{-1}. \end{aligned}$$

This proves (3.15).

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School of Mathematics, South China University of Technology, Guangzhou, Guangdong 510640, P.R.China

E-mail address: ningcui2013@126.com

Department of Mathematics, Tokyo University of Science, 1–3 Kagurazaka, Shinjukuku, Tokyo 162–8601, Japan.

E-mail address: mohta@rs.tus.ac.jp

School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, P.R.China

E-mail address: yifei@bnu.edu.cn