STABILITY OF MULTI-SOLITONS FOR THE DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION

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Abstract. The nonlinear Schrödinger equation with derivative cubic nonlinearity admits a family of solitons, which are orbitally stable in the energy space. In this work, we prove the orbital stability of multi-solitons configurations in the energy space, under suitable assumptions on the speeds and frequencies of the composing solitons. The main ingredients of the proof are modulation theory, energy coercivity and monotonicity properties.

Contents

1. Introduction

We consider the nonlinear Schrödinger equation with derivative nonlinearity:

$$iu_t + u_{xx} + i|u|^2 u_x = 0. \tag{dNLS}$$

The unknown function u is a complex-valued function of time $t \in \mathbb{R}$ and space $x \in \mathbb{R}$.

The derivative nonlinear Schrödinger equation was originally introduced in Plasma Physics as a simplified model for Alfvén waves propagation, see [40, 46]. Since then, it has attracted a lot of attention from the mathematical community. Let us give a few examples. It was first studied using integrable methods by Kaup and Nevel [25]. Later on, Hayashi and Ozawa [21, 22] obtained local well-posedness in the energy space $H^1(\mathbb{R})$. Local well-posedness in low regularity spaces $H^s(\mathbb{R})$, $s \geqslant 1/2$ was investigated by Takaoka [47]. The problem of global well-posedness for small mass

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initial data in low regularity spaces has attracted the attention of a number of authors, see e.g. [8, 9, 20, 39]. When considered on the torus, local existence in $H^s(\mathbb{T})$ was proved by Herr [23] for $s \geqslant 1/2$. Results for $s < 1/2$ were recently obtained by Takaoka [48]. A probabilistic approach to local existence was initiated by Thomann and Tzvetkov [49]. Recently, Wu [52, 53] proved global existence in $H^1(\mathbb{R})$ for initial data u_0 having mass $M(u_0) = \frac{1}{2}\|u_0\|_{L^2}^2$ less than threshold 2π. The method introduced by Wu was extend to the torus by Mosincat and Oh [41]. Despite the amount of studies devoted to (dNLS), existence of blowing up solutions remains a totally open problem. Global existence was recently investigated using integrability techniques by Liu, Perry and Sulem [31, 32] and by Pelinovsky and Shimabukuro [43]. Analysis of singular profiles in a supercritical version of (dNLS) was performed by Cher, Simpson and Sulem [6].

Before presenting our results, we start with some preliminaries.

Under gauge transformations, (dNLS) may take various (equivalent) forms. In particular, if

$$v(t,x) = \exp\left(-\frac{i}{2}\int_{-\infty}^{x} |u(t,y)|^2 dy\right) u(t,x),$$

then v solves

$$iv_t + v_{xx} + i(|v|^2 v)_x = 0. \tag{1}$$

Alternatively, setting

$$w(t,x) = \exp\left(\frac{i}{4}\int_{-\infty}^{x} |u(t,y)|^2 dy\right) u(t,x),$$

then w solves

$$iw_t + w_{xx} - iw^2 \bar{w}_x + \frac{1}{2}|w|^4 w = 0. \tag{2}$$

Under the form (dNLS), the derivative nonlinear Schrödinger equation is sometimes referred to as the Chen-Liu-Lee equation [5]. The form (1) might be called the Kaup-Newell equation [25]. The form (2) is the Gerdzhikov-Ivanov equation [18]. Yet another notable (but apparently not christened) form of (dNLS) is obtained setting

$$z(t,x) = \exp\left(\frac{i}{2}\int_{-\infty}^{x} |u(t,y)|^2 dy\right) u(t,x).$$

Then z solves

$$iz_t + z_{xx} - \frac{i}{2}|z|^2 z_x + \frac{i}{2} z^2 \bar{z}_x + \frac{3}{16}|z|^4 z = 0. \tag{3}$$

The equation (3) played a central role in the papers on global well-posedness [52, 53]. Since all these derivative nonlinear Schrödinger equations are related via gauge transformations, any result on one of the forms can a priori be transfered to the other forms. Depending on the aim, some form usually turns out to be much easier to work with than the others. In this paper, we will mostly use the form (dNLS).

Interestingly, given a solution u of (dNLS) and $\lambda > 0$, then

$$u_\lambda(t,x) = \frac{1}{\sqrt{\lambda}} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right)$$

is also a solution of (dNLS). In particular, (dNLS) is L^2-critical. However, its behavior widely differs from the one of its celebrated power-type counter part, the $1d$ quintic nonlinear Schrödinger equation. In particular, (dNLS) is not invariant by the pseudo-conformal transformation and no explicit (and in fact not at all) blow-up solution is known for (dNLS). Another main difference between (dNLS) and quintic NLS is that the former one is *not* invariant by a Galilean transform.

The equation (dNLS) can be written in Hamiltonian form as

$$iu_t = E'(u),$$

where the *Hamiltonian* (or *energy*) is given by

$$E(u) = \frac{1}{2}\|u_x\|_{L^2}^2 + \frac{1}{4}\mathcal{I}m\int_{\mathbb{R}} |u|^2\bar{u}u_x dx.$$

At least formally, the Hamiltonian E is conserved along the flow of (dNLS). In addition, two other quantities are conserved: the mass and the momentum, defined by

$$M(u) = \frac{1}{2}\|u\|_{L^2}^2, \quad P(u) = \frac{1}{2}\mathcal{I}m\int_{\mathbb{R}} u\bar{u}_x dx.$$

Note that (dNLS) is an integrable equation (see e.g. [25]) and there exists in fact an infinity of conservation laws (see e.g. [50]). However, our goal here is to study the properties of the solutions of (dNLS) with a robust method not relying on its algebraic peculiarities. Our approach may be applied to other similar non-integrable equations, e.g. the generalized derivative nonlinear Schrödinger equations considered in [1, 16, 33, 34, 42].

As is now well-known, given real parameters $\omega > 0$ and $-2\sqrt{\omega} < c \leqslant 2\sqrt{\omega}$, there exist traveling waves solutions of (dNLS) of the form

$$R_{\omega,c}(t,x) = e^{i\omega t}\phi_{\omega,c}(x-ct).$$

The profiles $\phi_{\omega,c}$ are unique up to phase shifts and translations (see e.g. [7]) and are given by an explicit formula (see Section 2 for details). The stability of the solitary waves of (dNLS) was considered in [7, 19, 26]. In particular, in [7], Colin and Ohta proved that for any $(\omega, c) \in \mathbb{R}^2$ with $c^2 < 4\omega$, the solitary wave $R_{\omega,c}$ is orbitally stable. The orbital stability of the lump soliton in the case $c = 2\sqrt{\omega}$ was considered very recently by Kwon and Wu in [26]. The stability theory for a more general version of (dNLS) was developed in [34] by Liu, Simpson and Sulem.

Multi-solitons are solutions to (dNLS) that behave at large time like a sum of solitons. They can be proved to exist by inverse scattering transform (see [25] for (dNLS) or [54] for the cubic nonlinear Schrödinger equation), using energy methods (see [11, 12, 35] for the nonlinear Schrödinger equation or e.g. [14, 24] for Schrödinger systems) or with fixed point arguments (see [11, 28, 29] for the nonlinear Schrödinger equation).

As each individual solitary wave of (dNLS) is stable, it is reasonable to investigate the stability of a sum of solitary waves; this will be our goal in this paper. Precisely, we want to show that, under some conditions, if the initial data is close to a sum of solitons profiles, then the associated solution of (dNLS) will behave for any positive time as a sum of solitons. To our knowledge, no information was available so far on the stability of these multi-solitons configurations.

To study the stability of multi-solitons, several approaches are possible. One can work in weighted spaces and get asymptotic stability results for multi-solitons configurations [44, 45]. An alternative approach, when the underlying equation is integrable, is to take advantage of the integrable structure to obtain stability (in a relatively restricted class of functions), see e.g. [17]. Finally, one can work in the energy space, and this is what will be done in this paper. We will prove a result in the same family as the results obtained by Martel, Merle and Tsai for the Kortweg-de Vries equation [36] and the twisted nonlinear Schrödinger equation [37]. This approach was later extended to the Gross-Pitaevskii equation by Bethuel, Gravejat and Smets [3] and to the Landau-Lifshitz equation by de Laire and Gravejat [13]. Instability results are available in [10, 11].

Our main result is the following.

Theorem 1.1. *Let $N \in \mathbb{N}$. For $j = 1, \dots, N$ let $\omega_j \in (0,\infty)$, $c_j \in (-2\sqrt{\omega_j}, 2\sqrt{\omega_j})$, $x_j \in \mathbb{R}$ and $\theta_j \in \mathbb{R}$. Let $(\phi_j) = (\phi_{\omega_j,c_j})$ be the corresponding solitary wave profiles given by the explicit formula (5). For $j = 2, \dots, N$, let*

$$\sigma_j = 2\frac{\omega_j - \omega_{j-1}}{c_j - c_{j-1}}.$$

Assume that for $j = 2, \dots, N$ we have

$$\sigma_j > 0 \quad \text{and} \quad c_{j-1} < \sigma_j < c_j. \tag{4}$$

Then there exist $\alpha > 0$, $L_0 > 0$, $A_0 > 0$ and $\delta_0 > 0$ such that for any $u_0 \in H^1(\mathbb{R})$, $L > L_0$ and $0 < \delta < \delta_0$, the following property is satisfied. If

$$\left\| u_0 - \sum_{j=1}^{N} e^{i\theta_j}\phi_j(\cdot - x_j) \right\|_{H^1} \leqslant \delta$$

and if for all $j = 1, \dots, N-1$,

$$x_{j+1} - x_j > L,$$

the solution u of (dNLS) with $u(0) = u_0$ is globally defined in $H^1(\mathbb{R})$ for $t \geqslant 0$, and there exist functions $\tilde{x}_1(t), \dots, \tilde{x}_N(t) \in \mathbb{R}$, and $\tilde{\theta}_1(t), \dots, \tilde{\theta}_N(t) \in \mathbb{R}$, such that for all $t \geqslant 0$,

$$\left\| u(t) - \sum_{j=1}^{N} e^{i\tilde{\theta}_j(t)}\phi_j(\cdot - \tilde{x}_j(t)) \right\|_{H^1} \leqslant A_0 \left(\delta + e^{-\alpha L}\right).$$

Remark 1.2. We can further describe the behavior of the functions $\tilde{\theta}_j$ and $\tilde{x}_j$, see Proposition 4.1. In particular, they are of class $\mathcal{C}^1$ and verify the dynamical laws

$$\partial_t \tilde{x}_j \sim c_j, \quad \partial_t \tilde{\theta}_j \sim \omega_j.$$

Hence the behavior of $e^{i\tilde{\theta}_j(t)}\phi_j(\cdot - \tilde{x}_j(t))$ is close to the one of $R_{\omega_j,c_j}(t)$.

Remark 1.3. As we are trying to prove a stability result for the multi-solitons configuration, it natural to assume that we start with well ordered solitary waves, i.e.

$$c_1 < c_2 < \dots < c_N, \quad \text{and} \quad x_1 < x_2 < \dots < x_N.$$

This prevents the crossing of solitary waves at a later time.

As in [37], the stronger condition (4) that we impose on the parameters of the waves is needed for technical purposes.

A consequence of (4) is that

$$\omega_1 < \omega_2 < \dots < \omega_N.$$

In other words, the larger the amplitude is, the faster the soliton should travel. This is somehow reminiscent from what happens in the context of the Korteweg-de Vries (KdV) equation, where speed and amplitude are controlled by the same parameter [36]. Another similitude with the KdV equation, consequence of (4), is that our solitons should all be traveling to the right (i.e. with positive speeds), except for the first one, which is allowed to travel to the left (i.e. with negative speed).

The speed/frequency ratio condition is different from the equivalent one in [37, condition (A3)]. In particular, [37, condition (A3)] allows for the solitary waves to have equal frequencies, in which case the only condition for the speeds is to be strictly increasing. This is ruled out by (4).

An example of a range of parameters verifying (4) is given by

$$\omega_j = j^2 + 1, \quad c_j = 2j, \quad j = 1, \dots, N.$$

It is not hard to construct many other examples.

Remark 1.4. The constant α in Theorem 1.1 can be made explicit: $\alpha = \frac{1}{32}\omega_\star$, where $\omega_\star$ is the minimal decay rate of the solitons defined in (17).

Remark 1.5. Our theorem does not cover the case where one of the solitons is a lump soliton, i.e. when $c_j = 2\sqrt{\omega_j}$ for some $j = 1, \dots, N$. Indeed, lump solitons are significantly different from the other solitons (algebraic decay instead of exponential decay, weaker stability, etc.), which prevent to include them in our analysis.

Remark 1.6. The main technical differences between (NLS) and (dNLS) are that the later one is not any more Galilean invariant and there is no scaling between solitons. Nevertheless the proof of our result is largely inspired by the proof of [37, Theorem 1]. As far as possible, we have tried to keep the same (or similar) notations.

Our strategy for the proof of Theorem 1.1 is, as in [36, 37], the following. We use a bootstrap argument, which goes as follows. Assume that an initial data u_0 is located close enough to a sum of soliton profiles, and that the associated solution u to (dNLS) stays until some time T in a neighborhood of size ε of a sum of (modulated) solitons profiles. The bootstrap argument tells us that, in fact, u stays until the same time T in *a neighborhood of size* $\varepsilon/2$ of a sum of (modulated) solitons profiles. This allows us to extend the time T up to ∞ and proves the stability of the configuration. To obtain the bootstrap result, we rely on several ingredients. First, we need a modulation result around the soliton profiles. As usual, modulation is obtained via the Implicit Function Theorem, with here the particularity that we rely on explicit calculations to prove invertibility of the Jacobian. This is a specific feature of (dNLS) which allows us to modulate in a more natural way than in [37]. The second ingredient is a coercivity property for a linearized action functional. The functional is based on the linearized action around each soliton, which we prove to be coercive (up to some orthogonality conditions). The third ingredient is a series of monotonicity properties for suitably localized mass/momentum functionals. These monotonicity properties are involved in a crucial way in the control of the modulation parameters.

The rest of the paper is organized as follows. In Section 2 we review and develop the stability theory for a single solitary wave. In particular, we obtain a coercivity property for the linearized action functional around a single soliton using variational characterizations. In Section 3 we state the bootstrap argument and prove Theorem 1.1. Section 4 is devoted to the modulation result. In section 5 we derive the monotonicity properties of localized mass/momentum functionals. Section 6 deals with the construction of the action-like linearized functional for the sum of solitons and its coercivity properties. In Section 7 we control the modulation parameters using the monotonicity properties. Finally, in Section 8 we prove the bootstrap result. The Appendix A contains explicit formulas that we use at several occasions in the paper.

Notation. *The space $L^2(\mathbb{R})$ is considered as a real Hilbert space with the scalar product*

$$(u,v)_2 = \mathcal{Re}\int_{\mathbb{R}} u\bar{v}dx.$$

Whenever an inequality is true up to a positive constant, we use the notation $\gtrsim$ or $\lesssim$. Throughout the paper, the letter C will denote various positive constants whose exact value may vary from line to line but is of no importance in the analysis.

2. Solitary Waves and Stability Theory

We will need for the study of the stability of a sum of solitary waves some tools coming from the stability theory of a single solitary wave. These tools are however not immediately available in the literature and we need to introduce them ourselves in this section. We believe that the results presented in this section are of independent interest and may be useful for further studies of (dNLS).

Recall that given parameters $\omega > 0$ and $|c| < 2\sqrt{\omega}$, there exist traveling waves solutions of (dNLS) of the form

$$R_{\omega,c}(t,x) = e^{i\omega t}\phi_{\omega,c}(x-ct).$$

The profile $\phi_{\omega,c}$ is unique up to phase shifts and translations (see e.g. [7]) and is given by the explicit formula

$$\phi_{\omega,c}(x) = \varphi_{\omega,c}(x)\exp\left(\frac{c}{2}ix - \frac{i}{4}\int_{-\infty}^{x}|\varphi_{\omega,c}(\xi)|^2 d\xi\right) \tag{5}$$

where

$$\varphi_{\omega,c}(x) = \left(\frac{\sqrt{\omega}}{4\omega - c^2}\left(\cosh\left(x\sqrt{4\omega-c^2}\right) - \frac{c}{2\sqrt{\omega}}\right)\right)^{-\frac{1}{2}}. \tag{6}$$

The profile is also the unique (up to phase shifts and translations) solution to the elliptic ordinary differential equation

$$-\phi_{xx} - i|\phi|^2\phi_x + \omega\phi + ic\phi_x = 0, \tag{7}$$

and it is also a critical point of the action functional S,

$$S = S_{\omega,c} = E + \omega M + cP.$$

For future reference, note that the function $\varphi_{\omega,c}$ verifies the equation

$$-\varphi_{xx} + \left(\omega - \frac{c^2}{4}\right)\varphi - \frac{1}{2}\mathcal{Im}\left(\varphi\bar{\varphi}_x\right)\varphi + \frac{c}{2}|\varphi|^2\varphi - \frac{3}{16}|\varphi|^4\varphi = 0,$$

or, since $\varphi_{\omega,c}$ is real,

$$-\varphi_{xx} + \left(\omega - \frac{c^2}{4}\right)\varphi + \frac{c}{2}|\varphi|^2\varphi - \frac{3}{16}|\varphi|^4\varphi = 0. \tag{8}$$

From the explicit formula (5), we see that $\phi_{\omega,c}$ is exponentially decaying. Precisely, for any $\alpha < 1$ we have

$$|\partial_x\phi_{\omega,c}(x)| + |\phi_{\omega,c}(x)| \leqslant C_\alpha e^{-\alpha\sqrt{\omega - \frac{c^2}{4}}|x|}. \tag{9}$$

Note that this decay is not affected by a Gauge transform or a Galilean transform.

The main result of this section is the following coercivity property.

Proposition 2.1 (Coercivity for one solitary wave)**.** *For any $\omega, c \in \mathbb{R}$ with $4\omega > c^2$, there exists $\mu \in \mathbb{R}$ such that for any $\varepsilon \in H^1(\mathbb{R})$ verifying the orthogonality conditions*

$$(\varepsilon, i\phi_{\omega,c})_2 = (\varepsilon, \partial_x \phi_{\omega,c})_2 = (\varepsilon, \phi_{\omega,c} + i\mu\partial_x\phi_{\omega,c})_2 = 0,$$

we have

$$H_{\omega,c}(\varepsilon) := \left\langle S''_{\omega,c}(\phi_{\omega,c})\varepsilon, \varepsilon \right\rangle \gtrsim \|\varepsilon\|^2_{H^1}.$$

Remark 2.2. If $c < 0$, then we can choose $\mu = 0$.

Remark 2.3. Following the classical approach by Weinstein [51] (see e.g. [27] for an introduction), one obtains as a corollary of Proposition 2.1 the orbital stability of solitary waves.

Notation. *Many quantities defined in this paper will depend on ω and c. For the sake of clarity in notation, we shall very often drop the subscript ω, c and dependency in ω and c will be only understood.*

For future reference, we give the explicit expression of $S''(\phi)\varepsilon$:

$$S''(\phi)\varepsilon = -\partial_{xx}\varepsilon - i|\phi|^2\partial_x\varepsilon - 2i\mathcal{R}e(\phi\bar{\varepsilon})\phi_x + \omega\varepsilon + ic\partial_x\varepsilon, \tag{10}$$

and the explicit expression of H:

$$H(\varepsilon) = \|\varepsilon_x\|^2_{L^2} + \mathcal{I}m\int_{\mathbb{R}} |\phi|^2\bar{\varepsilon}\varepsilon_x dx + 2\int_{\mathbb{R}} \mathcal{R}e(\phi\bar{\varepsilon})\mathcal{I}m(\phi_x\bar{\varepsilon})dx + \omega\|\varepsilon\|^2_{L^2} + c\mathcal{I}m\int_{\mathbb{R}} \varepsilon\bar{\varepsilon}_x dx.$$

The proof of Proposition 2.1 makes use of the following minimization result. Define for $v \in H^1(\mathbb{R})$ the Nehari functional corresponding to (7) by

$$I(v) = I_{\omega,c}(v) = \|v_x\|^2_{L^2} + \mathcal{I}m\int_{\mathbb{R}} |v|^2\bar{v}v_x dx + \omega\|v\|^2_{L^2} + c\mathcal{I}m\int_{\mathbb{R}} v\bar{v}_x dx.$$

Remark that $I(v) = \frac{\partial}{\partial t}S(tv)_{|t=0}$. Define the minimum of the action S on the Nehari manifold by

$$m_{\mathcal{N}} = \inf\{S(v);\, v \in H^1(\mathbb{R}) \setminus \{0\},\, I(v) = 0\}$$

and let the set of minimizers be denoted by

$$\mathcal{G}_{\mathcal{N}} := \{v \in H^1(\mathbb{R}) \setminus \{0\};\, I(v) = 0,\, S(v) = m_{\mathcal{N}}\}.$$

Proposition 2.4. *Let $\omega, c \in \mathbb{R}$ be such that $4\omega > c^2$. Then $m_{\mathcal{N}} > 0$ and $\phi_{\omega,c}$ is up to phase shift and translation the unique minimizer for $m_{\mathcal{N}}$, that is*

$$\mathcal{G}_{\mathcal{N}} = \{e^{i\theta}\phi_{\omega,c}(x-y);\, \theta, y \in \mathbb{R}\}.$$

Proposition 2.4 was first obtained by Colin and Ohta [7, Lemma 10]. For the sake of completeness, we reproduce here the proof.

We will make use of the following two classical lemmas (see [4, 15, 30]).

Lemma 2.5. *Let (v_n) be a bounded sequence in $H^1(\mathbb{R})$. Assume that there exists $p \in (2, \infty)$ such that*

$$\limsup_{n\to\infty} \|v_n\|_{L^p} > 0.$$

Then there exist $(y_n) \subset \mathbb{R}$ and $v_\infty \in H^1(\mathbb{R}) \setminus \{0\}$ such that $(v_n(\cdot - y_n))$ has a convergent subsequence to v_∞ weakly in $H^1(\mathbb{R})$.

Lemma 2.6. *Let $2 \leqslant p < \infty$ and (v_n) be a bounded sequence in $L^p(\mathbb{R})$. Assume that $v_n \to v_\infty$ a.e. in $\mathbb{R}$. Then*

$$\|v_n\|_{L^p}^p - \|v_n - v_\infty\|_{L^p}^p - \|v_\infty\|_{L^p}^p \to 0, \qquad \text{as } n \to \infty.$$

Proof of Proposition 2.4. Step 1. We show that $m_{\mathcal{N}} > 0$. Let $v \in H^1(\mathbb{R}) \setminus \{0\}$ be such that $I(v) = 0$. Introduce the notation

$$L(v) = L_{\omega,c}(v) = \|v_x\|_{L^2}^2 + \omega\|v\|_{L^2}^2 + c\mathcal{Im}\int_{\mathbb{R}} v\bar{v}_x dx$$

and remark that

$$L(v) = \left\|v_x - \frac{ic}{2}v\right\|_{L^2}^2 + \left(\omega - \frac{c^2}{4}\right)\|v\|_{L^2}^2.$$

Using $I(v) = 0$, we have

$$\|v\|_{H^1}^2 \lesssim \left\|v_x - \frac{ic}{2}v\right\|_{L^2}^2 + \left(\omega - \frac{c^2}{4}\right)\|v\|_{L^2}^2 = -\mathcal{Im}\int_{\mathbb{R}} |v|^2\bar{v}v_x dx \lesssim \|v\|_{H^1}^4.$$

Therefore there exists $\delta > 0$ independant of v such that

$$\|v\|_{H^1}^2 > \delta.$$

In addition, we have

$$S(v) = S(v) - \frac{1}{4}I(v) = \frac{1}{4}L(v) \gtrsim \|v\|_{H^1}^2 > \delta > 0.$$

Therefore

$$m_{\mathcal{N}} > 0.$$

Step 2. We show that

$$m_{\mathcal{N}} = \frac{1}{4}\inf\{L(v);\, v \in H^1(\mathbb{R}) \setminus \{0\},\, I(v) \leqslant 0\}.$$

Indeed, let $v \in H^1(\mathbb{R}) \setminus \{0\}$ be such that $I(v) < 0$. Then there exists $\lambda \in (0,1)$ such that $I(\lambda v) = 0$. Moreover,

$$m_{\mathcal{N}} \leqslant S(\lambda v) = S(\lambda v) - \frac{1}{4}I(\lambda v) = \frac{1}{4}L(\lambda v) = \frac{\lambda^2}{4}L(v) < \frac{1}{4}L(v).$$

Step 3. We show convergence of the minimizing sequences. Let $(v_n) \subset H^1(\mathbb{R}) \setminus \{0\}$ be such that $I(v_n) = 0$ for all $n \in \mathbb{N}$ and $S(v_n) \to m_{\mathcal{N}}$ as $n \to \infty$. In the sequel, all statements will be true up to the extraction of a subsequence. The sequence (v_n) is bounded from above and below in $H^1(\mathbb{R})$. Moreover, we claim that

$$\limsup_{n\to\infty}\|v_n\|_{L^6} > 0.$$

Indeed, assume by contradiction that $\lim_{n\to\infty}\|v_n\|_{L^6} = 0$. Then from $I(v_n) = 0$ and by Cauchy-Schwartz inequality, we have

$$\|v_n\|_{H^1}^2 \lesssim L(v_n) = \left|-\mathcal{Im}\int_{\mathbb{R}} |v_n|^2\bar{v}_n\partial_x v_n dx\right| \lesssim \|v_n\|_{L^6}^3\|\partial_x v_n\|_{L^2} \to 0.$$

It is a contradiction with the boundedness from below of (v_n) in $H^1(\mathbb{R})$. Therefore $\limsup_{n\to\infty}\|v_n\|_{L^6} > 0$ and we can apply Lemma 2.5 to obtain the existence of $v_\infty \in H^1(\mathbb{R}) \setminus \{0\}$ and $(y_n) \subset \mathbb{R}$ such that

$$v_n(\cdot - y_n) \rightharpoonup v_\infty \text{ weakly in } H^1(\mathbb{R}), \quad v_n(\cdot - y_n) \to v_\infty \text{ a.e.}$$

From now on, we replace v_n by $v_n(\cdot - y_n)$. By weak convergence we have

$$L(v_n) - L(v_n - v_\infty) - L(v_\infty) \to 0. \tag{11}$$

By Lemma 2.6 we have

$$\|v_n\|_{L^4}^4 - \|v_n - v_\infty\|_{L^4}^4 - \|v_\infty\|_{L^4}^4 \to 0, \tag{12}$$

$$\|v_n\|_{L^6}^6 - \|v_n - v_\infty\|_{L^6}^6 - \|v_\infty\|_{L^6}^6 \to 0. \tag{13}$$

Remark that for any $v \in H^1(\mathbb{R})$ we can rewrite $I(v)$ as

$$I(v) = \left\| v_x - \frac{ic}{2}v + \frac{i}{2}|v|^2 v \right\|_{L^2}^2 + \left(\omega - \frac{c^2}{4}\right)\|v\|_{L^2}^2 + \frac{c}{2}\|v\|_{L^4}^4 - \frac{1}{4}\|v\|_{L^6}^6. \tag{14}$$

Introduce the functions w_n and w_∞ defined by

$$w_n = \partial_x v_n - \frac{ic}{2}v_n + \frac{i}{2}|v_n|^2 v_n, \quad w_\infty = \partial_x v_\infty - \frac{ic}{2}v_\infty + \frac{i}{2}|v_\infty|^2 v_\infty.$$

Then $w_n \rightharpoonup w_\infty$ in $L^2(\mathbb{R})$ and we have

$$\|w_n\|_{L^2}^2 - \|w_n - w_\infty\|_{L^2}^2 - \|w_\infty\|_{L^2}^2 \to 0.$$

Combined with (12),(13) and (14), this gives

$$I(v_n) - I(v_n - v_\infty) - I(v_\infty) \to 0. \tag{15}$$

Step 4. We show that the minimal element v_∞ verifies $I(v_\infty) \leqslant 0$. Assume by contradiction that $I(v_\infty) > 0$. From (15) and since $I(v_n) = 0$, we have

$$\lim_{n\to\infty} I(v_n - v_\infty) = -I(v_\infty) < 0.$$

Therefore, by Step 2, $L(v_n - v_\infty) > 4m_{\mathcal{N}}$. Combining this with $\lim_{n\to\infty} L(v_n) = 4m_{\mathcal{N}}$ and (11), we obtain

$$L(v_\infty) = \lim_{n\to\infty} (L(v_n) - L(v_n - v_\infty)) \leqslant 0.$$

However $v_\infty \neq 0$ and thus $L(v_\infty) > 0$, which is a contradiction. Therefore $I(v_\infty) \leqslant 0$.

Step 5. We prove that v_∞ achieves the minimum for $m_{\mathcal{N}}$. We have

$$4m_{\mathcal{N}} \leqslant L(v_\infty) \leqslant \lim_{n\to\infty} L(v_n) = 4m_{\mathcal{N}}.$$

Therefore

$$L(v_\infty) = \lim_{n\to\infty} L(v_n)$$

and thus $v_n \to v_\infty$ strongly in $H^1(\mathbb{R})$. This implies $I(v_\infty) = 0$ and therefore

$$S(v_\infty) = m_{\mathcal{N}}.$$

Step 6. Conclusion. The last step of the proof consists in proving that v_∞ is in fact a solution of (7). As v_∞ is a minimizer for $m_{\mathcal{N}}$, there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that

$$S'(v_\infty) = \lambda I'(v_\infty).$$

Therefore

$$0 = I(v_\infty) = \langle S'(v_\infty), v_\infty \rangle = \lambda \langle I'(v_\infty), v_\infty \rangle.$$

Since

$$\langle I'(v_\infty), v_\infty \rangle = 2L(v_\infty) + 4\mathcal{I}m \int |v_\infty|^2 \bar{v}_\infty \partial_x v_\infty dx = -2L(v_\infty) < 0,$$

we necessarily have $\lambda = 0$. This implies that v_∞ is a solution of (7). Since by uniqueness any solution of (7) can be written for some $\theta, y \in \mathbb{R}$ as

$$v_\infty = e^{i\theta}\phi_{\omega,c}(\cdot - y),$$

this concludes the proof of Proposition 2.4. □

With Proposition 2.4 in hand, we can now proceed to the proof of Proposition 2.1.

Proof of Proposition 2.1. For simplicity in notation we drop the subscript ω, c in $\phi_{\omega,c}$ and simply write ϕ instead. We shall follow more or less the scheme of proof already used in [2]. We start by rewriting $S''(\phi)$ (see (10)) as a two by two matrix operator acting on $(\mathcal{Re}(\varepsilon), \mathcal{Im}(\varepsilon))$:

$$S''(\phi) = \begin{pmatrix} -\partial_{xx} + \omega + 2\mathcal{Re}(\phi)\mathcal{Im}(\phi_x) & 2\mathcal{Im}(\phi)\mathcal{Im}(\phi_x) + |\phi|^2\partial_x - c\partial_x \\ -2\mathcal{Re}(\phi)\mathcal{Re}(\phi_x) - |\phi|^2\partial_x + c\partial_x & -\partial_{xx} + \omega - 2\mathcal{Im}(\phi)\mathcal{Re}(\phi_x) \end{pmatrix}.$$

Step 1. Global spectral picture. Since ϕ is exponentially localized, $S''(\phi)$ can be considered as compact perturbation of

$$\begin{pmatrix} -\partial_{xx} + \omega & -c\partial_x \\ c\partial_x & -\partial_{xx} + \omega \end{pmatrix}.$$

Therefore its essential spectrum is $[\omega - \frac{c^2}{4}, \infty)$ and by Weyl's Theorem its spectrum in $(-\infty, \omega - \frac{c^2}{4})$ consists of isolated eigenvalues. Due to the variational characterization Proposition 2.4, $S''(\phi)$ admits at most one negative eigenvalue. Using that ϕ satisfies to the Nehari constraint $I(\phi) = 0$, we have

$$\langle S''(\phi)\phi, \phi\rangle = 2\mathcal{Im}\int_{\mathbb{R}} |\phi|^2\bar{\phi}\phi_x dx = -2L(\phi) < 0.$$

This implies that the operator $S''(\phi)$ has exactly one negative eigenvalue.

Step 2. Non-degeneracy of the kernel. We claim that $\ker(S''(\phi)) = \operatorname{span}\{i\phi, \phi_x\}$. Write

$$\varepsilon = \exp\left(i\left(\frac{c}{2}x - \frac{1}{4}\int_{-\infty}^x |\phi|^2 dy\right)\right)\left(k - \frac{i\varphi}{2}\int_{-\infty}^x \varphi\mathcal{Re}(k)dy\right),$$

where $\varphi = \varphi_{\omega,c}$ is the real part of ϕ given explicitly in formula (6). Then

$$\begin{aligned} S''(\phi)\varepsilon = -k_{xx} + \left(\omega - \frac{c^2}{4}\right)k + i\frac{1}{2}\varphi\varphi_x\mathcal{Im}(k) - i\frac{1}{2}\varphi^2\mathcal{Im}(k_x) \\ + \frac{c}{2}\varphi^2 k + c\varphi^2\mathcal{Re}(k) - \frac{3}{16}\varphi^4 k - \frac{12}{16}\varphi^4\mathcal{Re}(k). \end{aligned}$$

Separating in real and imaginary part, we obtain

$$S''(\phi)\varepsilon = L_+\mathcal{Re}(k) + iL_-\mathcal{Im}(k),$$

where

$$\begin{aligned} L_+ &= -\partial_{xx} + \left(\omega - \frac{c^2}{4}\right) + \frac{3c}{2}\varphi^2 - \frac{15}{16}\varphi^4, \\ L_- &= -\partial_{xx} + \left(\omega - \frac{c^2}{4}\right) + \frac{c}{2}\varphi^2 - \frac{3}{16}\varphi^4 + \frac{\varphi\varphi_x}{2} - \frac{\varphi^2}{2}\partial_x. \end{aligned}$$

Hence proving non-degeneracy for $S''(\phi)$ amounts to proving non-degeneracy of L_+ and L_-. That is, we want to prove that

$$\ker(L_+) = \operatorname{span}\{\varphi_x\}, \quad \ker(L_-) = \operatorname{span}\{\varphi\}.$$

Since φ satisfies (8), it is clear that $L_-\varphi = 0$. Let $v \in H^2(\mathbb{R}) \setminus \{0\}$ be such that $L_- v = 0$. Consider the Wronskian of φ and v:

$$W = \varphi_x v - \varphi v_x.$$

It verifies the equation

$$W' = -\frac{\varphi^2}{2} W$$

and therefore it is of the form

$$W(x) = Ce^{-\frac{1}{2}\int_0^x \varphi^2(y)dy}.$$

Since $\varphi, v \in H^2(\mathbb{R})$, we have

$$\lim_{x\to\pm\infty} \varphi(x) = \lim_{x\to\pm\infty} \varphi_x(x) = \lim_{x\to\pm\infty} v(x) = \lim_{x\to\pm\infty} v_x(x) = 0.$$

Therefore,

$$\lim_{x\to\pm\infty} W(x) = 0,$$

which is possible only if $W \equiv 0$. Therefore $v \in \operatorname{span}\{\varphi\}$ and this proves non-degeneracy of L_-. The non-degeneracy of L_+ follows from similar arguments.

Step 3. Construction of a negative direction. Differentiating (7) with respect to ω and c, we observe that

$$S''(\phi)\partial_\omega\phi = -\phi, \quad S''(\phi)\partial_c\phi = -i\phi_x.$$

Let $\mu \in \mathbb{R}$ to be chosen later and define

$$\psi = \partial_\omega\phi + \mu\partial_c\phi.$$

We have

$$\langle S''(\phi)\psi, \psi\rangle = -\langle \phi, \partial_\omega\phi\rangle - \mu\langle \phi, \partial_c\phi\rangle - \mu\langle i\phi_x, \partial_\omega\phi\rangle - \mu^2\langle i\phi_x, \partial_c\phi\rangle.$$

Moreover, using (70)-(72), we get

$$\langle \phi, \partial_\omega\phi\rangle = \frac{\partial}{\partial\omega}M(\phi_{\omega,c}) = \frac{-\frac{c}{\omega}}{\sqrt{4\omega - c^2}}, \qquad \langle \phi, \partial_c\phi\rangle = \frac{\partial}{\partial c}M(\phi_{\omega,c}) = \frac{2}{\sqrt{4\omega - c^2}},$$

$$\langle i\phi_x, \partial_\omega\phi\rangle = \frac{\partial}{\partial\omega}P(\phi_{\omega,c}) = \frac{2}{\sqrt{4\omega - c^2}}, \quad \langle i\phi_x, \partial_c\phi\rangle = \frac{\partial}{\partial c}P(\phi_{\omega,c}) = \frac{-c}{\sqrt{4\omega - c^2}}.$$

This gives

$$\langle S''(\phi)\psi, \psi\rangle = \left(\frac{c}{\omega} - 4\mu + c\mu^2\right)\frac{1}{\sqrt{4\omega - c^2}}. \tag{16}$$

Therefore, since $4\omega - c^2 > 0$, there always exists μ such that

$$\langle S''(\phi)\psi, \psi\rangle < 0.$$

Let such a μ be fixed now. If $c < 0$, we can choose $\mu = 0$. If $c = 0$, we can choose $\mu = 1$ and if $c > 0$ we can choose $\mu = \frac{2}{c}$.

Step 4. Positivity. Let us now denote by $-\lambda$ and ξ the negative eigenvalue of $S''(\phi)$ and its corresponding normalized eigenvector, i.e.

$$S''(\phi)\xi = -\lambda\xi, \quad \|\xi\|_{L^2} = 1.$$

We write the decomposition of ψ along the spectrum of $S''(\phi)$:

$$\psi = \alpha\xi + \zeta + \eta,$$

with $\alpha \in \mathbb{R}$, $\alpha \neq 0$, $\zeta \in \ker(S''(\phi))$ and η in the positive eigenspace of $S''(\phi)$. In particular, we have

$$\langle S''(\phi)\eta, \eta\rangle \gtrsim \|\eta\|_{L^2}^2.$$

Take $\varepsilon \in H^1(\mathbb{R}) \setminus \{0\}$ such that the following orthogonality conditions hold

$$(\varepsilon, i\phi)_2 = (\varepsilon, \partial_x \phi)_2 = (\varepsilon, \phi + i\mu\phi_x)_2 = 0.$$

We also write the decomposition of ε along the spectrum of $S''(\phi)$:

$$\varepsilon = \beta\xi + \rho,$$

with ρ in the positive eigenspace of $S''(\phi)$. Since

$$-\phi - i\mu\phi_x = S''(\phi)\psi = -\alpha\lambda\xi + S''(\phi)\eta,$$

we have

$$0 = -(\phi + i\mu\phi_x, \varepsilon)_2 = \langle S''(\phi)\psi, \varepsilon\rangle = -\alpha\beta\lambda + \langle S''(\phi)\eta, \rho\rangle,$$

thus

$$\langle S''(\phi)\eta, \rho\rangle = \alpha\beta\lambda.$$

From Cauchy-Schwartz inequality, we have

$$(\alpha\beta\lambda)^2 = \langle S''(\phi)\eta, \rho\rangle^2 \leqslant \langle S''(\phi)\eta, \eta\rangle \langle S''(\phi)\rho, \rho\rangle.$$

In addition, since $\langle S''(\phi)\psi, \psi\rangle < 0$, we have

$$\langle S''(\phi)\eta, \eta\rangle < \alpha^2\lambda.$$

Therefore,

$$\langle S''(\phi)\varepsilon, \varepsilon\rangle = -\beta^2\lambda + \langle S''(\phi)\rho, \rho\rangle \geqslant -\beta^2\lambda + \frac{(\alpha\beta\lambda)^2}{\langle S''(\phi)\eta, \eta\rangle} > -\beta^2\lambda + \frac{(\alpha\beta\lambda)^2}{\alpha^2\lambda} = 0.$$

Step 5. Coercivity. To obtain the desired coercivity property, we argue by contradiction. Let ε_n be such that $\|\varepsilon_n\|_{H^1} = 1$ and

$$\lim_{n\to\infty} \langle S''(\phi)\varepsilon_n, \varepsilon_n\rangle = 0.$$

By boundedness, there exists $\varepsilon_\infty \in H^1(\mathbb{R})$ such that

$$\varepsilon_n \rightharpoonup \varepsilon_\infty \text{ weakly in } H^1(\mathbb{R}).$$

On one hand, by weak convergence, ε_∞ verifies the orthogonality conditions

$$(\varepsilon_\infty, i\phi_{\omega,c})_2 = (\varepsilon_\infty, \partial_x\phi_{\omega,c})_2 = (\varepsilon_\infty, \phi_{\omega,c} + i\mu\phi_x)_2 = 0.$$

In particular, if $\varepsilon_\infty \not\equiv 0$, then, by Step 4, we have

$$\langle S''(\phi)\varepsilon_\infty, \varepsilon_\infty\rangle > 0.$$

On the other hand, we remark that $\langle S''(\phi)\varepsilon, \varepsilon\rangle = H(\varepsilon)$ can be written

$$H(\varepsilon) = \left\|\varepsilon_x - i\frac{c}{2}\varepsilon\right\|_{L^2}^2 + \left(\omega - \frac{c^2}{4}\right)\|\varepsilon\|_{L^2}^2 + \mathcal{Im}\int_{\mathbb{R}} |\phi|^2\bar{\varepsilon}\varepsilon_x dx + 2\int_{\mathbb{R}} \mathcal{Re}(\phi\bar{\varepsilon})\mathcal{Im}(\phi_x\bar{\varepsilon})dx.$$

By weak convergence of ε_n, exponential localization of ϕ and ϕ_x and compactness of the injection of H^1 into L^2 for bounded domain, we have

$$\langle S''(\phi)\varepsilon_\infty, \varepsilon_\infty\rangle \leqslant \lim_{n\to\infty} \langle S''(\phi)\varepsilon_n, \varepsilon_n\rangle = 0.$$

Therefore we must have $\varepsilon_\infty = 0$. Since $\|\varepsilon_n\|_{H^1} = 1$, there exists $\delta > 0$ such that

$$\left\|\partial_x\varepsilon_n - i\frac{c}{2}\varepsilon_n\right\|_{L^2}^2 + \left(\omega - \frac{c^2}{4}\right)\|\varepsilon_n\|_{L^2}^2 > \delta.$$

Moreover, since $\lim_{n\to\infty}\langle S''(\phi)\varepsilon_n,\varepsilon_n\rangle = 0$, we have

$$\mathcal{I}m\int_{\mathbb{R}}|\phi|^2\varepsilon_\infty(\bar{\varepsilon}_\infty)_x dx + 2\int_{\mathbb{R}}\mathcal{R}e(\phi\bar{\varepsilon}_\infty)\mathcal{I}m(\bar{\phi}_x\varepsilon_\infty)dx < -\delta < 0.$$

However, it is a contradiction with $\varepsilon_\infty = 0$. Hence the coercivity result Proposition 2.1 holds. □

3. The Bootstrap Argument

This section is devoted to the proof of Theorem 1.1 using a bootstrap argument which will be proved later.

Let $c_1 < \cdots < c_N$ and $0 < \omega_1 < \cdots < \omega_N$ be such that $c_j^2 < 4\omega_j$ for $j = 1, \dots, N$ and the speed-frequency ratio assumption (4) is verified. Let $(\phi_j) = (\phi_{\omega_j,c_j})$ be the corresponding solitons profiles. We define the minimal decay rate of the profiles by

$$\omega_\star = \min\left\{\sqrt{\omega_j - \frac{c_j^2}{4}};\ j = 1, \dots, N\right\}. \tag{17}$$

Define also the *minimal relative speed* $c_\star$ by

$$c_\star = \min\{|c_j - c_k|;\ j,k = 1,\dots,N,\ j \neq k\}. \tag{18}$$

Given $A_0, L, \delta > 0$, define a tubular neighborhood of the N-soliton profiles by

$$\mathcal{V}(\delta, L, A_0) = \left\{u \in H^1(\mathbb{R});\ \inf_{\substack{x_j > x_{j-1}+L\\ \theta_j\in\mathbb{R}}}\left\|u - \sum_{j=1}^N e^{i\theta_j}\phi_j(\cdot - x_j)\right\|_{H^1} < A_0\left(\delta + e^{-\frac{1}{32}\omega_\star L}\right)\right\}.$$

Theorem 1.1 is a straightforward consequence of the following bootstrap result.

Proposition 3.1 (Bootstrap). *There exist $A_0 > 1$, fixed, and $L_0 > 0$ and $\delta_0 > 0$ such that for all $L > L_0$, $0 < \delta < \delta_0$ the following property is satisfied. If $u_0 \in H^1(\mathbb{R})$ verifies*

$$u_0 \in \mathcal{V}(\delta, L, 1),$$

and if $t^\star > 0$ is such that for all $t \in [0, t^\star]$ the solution u of (dNLS) with $u(0) = u_0$ verifies

$$u(t) \in \mathcal{V}(\delta, L, A_0),$$

then for all $t \in [0, t^\star]$ we have

$$u(t) \in \mathcal{V}\left(\delta, L, \frac{A_0}{2}\right).$$

Being performing the proof of Proposition 3.1, let us indicate how it implies Theorem 1.1.

Proof of Theorem 1.1. Since $u_0 \in \mathcal{V}(\delta, L, 1)$, and u is continuous in $H^1(\mathbb{R})$, there exists a maximal time $t^\star \in (0, \infty]$ such that for all $t \in [0, t^\star)$ we have

$$u(t) \in \mathcal{V}(\delta, L, A_0).$$

Arguing by contradiction, we assume that $t^\star < \infty$. By Proposition 3.1, for all $t \in [0, t^\star)$ we have

$$u(t) \in \mathcal{V}\left(\delta, L, \frac{A_0}{2}\right).$$

By continuity of u in $H^1(\mathbb{R})$, there must exist $t^{\star\star} > t^\star$ such that for all $t \in [0, t^{\star\star})$ we have

$$u(t) \in \mathcal{V}(\delta, L, A_0).$$

This however contradicts the maximality of $t^\star$. Hence $t^\star = \infty$. This concludes the proof. □

The rest of this paper is devoted to the proof of Proposition 3.1. From now on, we assume that we are given $A_0 > 1$, $L > L_0 = L_0(A_0) > 0$ and $0 < \delta < \delta_0 = \delta_0(A_0)$ such that

$$u_0 \in \mathcal{V}(\delta, L, 1),$$

and there exists $t^\star > 0$ such that for all $t \in [0, t^\star]$ the solution u of (dNLS) with $u(0) = u_0$ verifies

$$u(t) \in \mathcal{V}(\delta, L, A_0).$$

In the sequel, we shall always assume $t \in [0, t^\star]$.

4. Modulation

We first explain how to decompose u close to the sum of solitons. Roughly speaking, we project u on the manifold of the sum of soliton profiles modulated in phase, speed, space and scaling. Since we impose the modulated speed and scaling to have the same ratio as the original speed and scaling, we in fact modulate on a family of $3N$ parameters.

Proposition 4.1 (Modulation)**.** *For δ and $1/L$ small enough, the following property is verified. For $j = 1, \dots, N$ there exist (unique) $\mathcal{C}^1$-functions*

$$\tilde{\theta}_j : [0, t^\star] \to \mathbb{R}, \quad \tilde{\omega}_j : [0, t^\star] \to (0, \infty), \quad \tilde{x}_j : [0, t^\star] \to \mathbb{R}, \quad \tilde{c}_j : [0, t^\star] \to \mathbb{R}$$

such that if we define modulated solitons $\tilde{R}_j$ and ε by

$$\tilde{R}_j(t) = e^{i\tilde{\theta}(t)} \phi_{\tilde{\omega}_j(t), \tilde{c}_j(t)}(\cdot - \tilde{x}_j(t)), \quad \varepsilon(t) = u(t) - \sum_{j=1}^{N} \tilde{R}_j(t),$$

then ε satisfies for all $t \in [0, t^\star]$ the orthogonality conditions (the constants μ_j are given by Proposition 2.1)

$$\left(\varepsilon, i\tilde{R}_j\right)_2 = \left(\varepsilon, \partial_x \tilde{R}_j\right)_2 = \left(\varepsilon, \tilde{R}_j + \mu_j i \partial_x \tilde{R}_j\right)_2 = 0, \quad j = 1, \dots, N. \tag{19}$$

The scaling and speed parameters verify for all $t \in [0, t^\star]$ and for any $j = 1, \dots, N$ the relationship

$$\tilde{c}_j(t) - c_j = \mu_j(\tilde{\omega}_j(t) - \omega_j). \tag{20}$$

Moreover, there exists $\tilde{C} > 0$ such that for all $t \in [0, t^\star]$ we have

$$\|\varepsilon(t)\|_{H^1} + \sum_{j=1}^{N} \left(|\tilde{\omega}_j(t) - \omega_j| + |\tilde{c}_j(t) - c_j|\right) \leqslant \tilde{C} A_0 \left(\delta + e^{-\frac{1}{4}\omega_\star L}\right), \tag{21}$$

$$\tilde{x}_{j+1}(t) - \tilde{x}_j(t) > \frac{L}{2}, \quad \textit{for } j = 1, \dots, N-1, \tag{22}$$

and the derivatives in time verify

$$\sum_{j=1}^{N} |\partial_t \tilde{c}_j| + |\partial_t \tilde{\omega}_j| + \left|\partial_t \tilde{\theta}_j - \tilde{\omega}_j\right| + |\partial_t \tilde{x}_j - \tilde{c}_j| \leqslant \tilde{C} \|\varepsilon(t)\|_{H^1} + \tilde{C} e^{-\frac{1}{4}\omega_\star L}. \tag{23}$$

Finally, at $t = 0$ the estimate does not depend on A_0 and we have

$$\|\varepsilon(0)\|_{H^1} + \sum_{j=1}^{N} \left(|\tilde{\omega}_j(0) - \omega_j| + |\tilde{c}_j(0) - c_j|\right) \leqslant \tilde{C} \left(\delta + e^{-\frac{1}{4}\omega_\star L}\right). \tag{24}$$

Remark 4.2. We modulate here in a different way as in [37]. First, we (artificially) modulate also in speed, whereas in [37] modulation was only on phase, position and scaling. Second, we modulate in position on the full profile $\phi_{\omega,c}$, whereas in [37] modulation in position was done only on the modulus of the profile (the equivalent of $\varphi_{\omega,c}$ in our setting). The way we modulate is more natural, but it introduces a technical difficulty in the proof of the modulation result. Precisely, the Jacobian given by (26) is not diagonal and its invertibility is not obvious. We are able to overcome this difficulty in our setting thanks to our knowledge of the explicit expressions of the profiles.

Proof of Proposition 4.1. The existence and regularity of the functions $(\tilde{\theta}_j, \tilde{x}_j, \tilde{\omega}_j, \tilde{c}_j)$ follow from classical arguments involving the Implicit Function Theorem. The main difficulty here is that we have a non-diagonal Jacobian, and proving its invertibility requires an additional argument compare to the usual setting. The modulation equations (23) are obtained via the combination of the equation verified by ε with the orthogonality conditions (19). We only give the important steps of the proof.

Let

$$q = \left(\tilde{\theta}_1, \dots, \tilde{\theta}_N; \tilde{x}_1, \dots, \tilde{x}_N; \tilde{\omega}_1, \dots, \tilde{\omega}_N; v\right) \in \mathbb{R}^{3N} \times H^1(\mathbb{R})$$

and

$$q_0 = \left(\theta_1, \dots, \theta_N; x_1, \dots, x_N; \omega_1, \dots, \omega_N; \sum_{j=1}^{N} e^{i\theta_j}\phi_j(\cdot - x_j)\right).$$

For a given q, define the speeds $\tilde{c}_j$ by

$$\tilde{c}_j = c_j + \mu_j(\tilde{\omega}_j - \omega_j). \tag{25}$$

Define also the modulated profiles $\tilde{R}_j$ and the difference ε by

$$\tilde{R}_j = e^{i\tilde{\theta}}\phi_{\tilde{\omega}_j,\tilde{c}_j}(\cdot - \tilde{x}_j), \quad \varepsilon = v - \sum_{j=1}^{N} \tilde{R}_j,$$

Consider the function $\Phi : \mathbb{R}^{3N} \times H^1(\mathbb{R}) \to \mathbb{R}^{3N}$ defined by $\Phi = (\Phi^1, \Phi^2, \Phi^3)$ with

$$\Phi_j^1 = \left(\varepsilon, i\tilde{R}_j\right)_2, \quad \Phi_j^2 = \left(\varepsilon, \partial_x\tilde{R}_j\right)_2, \quad \Phi_j^3 = \left(\varepsilon, \tilde{R}_j + i\mu_j\partial_x\tilde{R}_j\right)_2.$$

Note that

$$\frac{\partial\varepsilon}{\partial\tilde{\theta}_j}\Big|_{q=q_0} = -iR_j, \quad \frac{\partial\varepsilon}{\partial\tilde{x}_j}\Big|_{q=q_0} = \partial_x R_j,$$

and by the relationship (25), we have

$$\frac{\partial\varepsilon}{\partial\tilde{\omega}_j}\Big|_{q=q_0} = -\left(\partial_\omega R_j + \mu_j\partial_c R_j\right).$$

Then,

$$\frac{d\Phi_j^1}{d\tilde{\theta}_j}\Big|_{q=q_0} = -\|R_j\|_{L^2}^2 = -8\arctan\left(\sqrt{\frac{2\sqrt{\omega_j}+c_j}{2\sqrt{\omega_j}-c_j}}\right),$$
$$\frac{d\Phi_j^2}{d\tilde{x}_j}\Big|_{q=q_0} = \|\partial_x R_j\|_{L^2}^2 = 8\omega_j\arctan\left(\sqrt{\frac{2\sqrt{\omega_j}+c_j}{2\sqrt{\omega_j}-c_j}}\right),$$

where the explicit values come from the formulas in (63) and (66). Using

$$R_j + i\mu_j\partial_x R_j = M'(R_j) + \mu_j P'(R_j),$$

and (70)–(72), we have

$$\begin{aligned}\frac{d\Phi_j^3}{d\tilde{\omega}_j}\Big|_{q=q_0} &= -\left(\partial_\omega R_j + \mu_j\partial_c R_j, R_j + i\mu_j\partial_x R_j\right)_2\\ &= -\big(\partial_\omega M(R_j) + \mu_j\left(\partial_c M(R_j) + \partial_\omega P(R_j)\right) + \mu_j^2\partial_c P(R_j)\big)\\ &= \left(\frac{c_j}{\omega_j} - 4\mu_j + c_j\mu_j^2\right)\frac{1}{\sqrt{4\omega_j - c_j^2}} < 0,\end{aligned}$$

where at the last inequality we recalled (16) and the choice of μ_j. Moreover, since $x_{j+1} - x_j > L$, when $j \neq k$, for $l = 1, 2, 3$ we have

$$\frac{d\Phi_j^l}{d\tilde{\theta}_k}\Big|_{q=q_0} = \frac{d\Phi_j^l}{d\tilde{x}_k}\Big|_{q=q_0} = \frac{d\Phi_j^l}{d\tilde{\omega}_k}\Big|_{q=q_0} = O(e^{-\frac{1}{2}\omega_\star L}).$$

We easily verify that

$$\left(\frac{d\Phi_j^3}{d\tilde{\theta}_j}\Big|_{q=q_0}\right) = \left(-iR_j, R_j + i\mu_j\partial_x R_j\right)_2 = 0,$$
$$\left(\frac{d\Phi_j^3}{d\tilde{x}_j}\Big|_{q=q_0}\right) = \left(\partial_x R_j, R_j + i\mu_j\partial_x R_j\right)_2 = 0.$$

Furthermore, using the explicit expression of ϕ_j as well as the formulas (63)–(64), we have

$$\begin{aligned}\frac{d\Phi_j^1}{d\tilde{x}_j}\Big|_{q=q_0} &= -\frac{d\Phi_j^2}{d\tilde{\theta}_j}\Big|_{q=q_0} = \left(\partial_x R_j, iR_j\right)_2\\ &= \left(\partial_x\phi_j, i\phi_j\right)_2 = \left(\partial_x\varphi_j + \frac{ic}{2}\varphi_j - \frac{i}{4}|\varphi_j|^2\varphi_j, i\varphi_j\right)_2 = \frac{c}{2}\|\varphi_j\|_{L^2}^2 - \frac{1}{4}\|\varphi_j\|_{L^4}^4\\ &= -2\sqrt{4\omega_j - c_j^2}.\end{aligned}$$

The Jacobian matrix of the derivative of the function $q \mapsto \Phi(q)$ with respect to $(\tilde{\theta}_j, \tilde{x}_j, \tilde{\omega}_j)$ is the 3×3 block matrix

$$D\Phi = \left.\begin{pmatrix} \frac{d\Phi_j^1}{d\tilde{\theta}_k} & \frac{d\Phi_j^1}{d\tilde{x}_k} & \frac{d\Phi_j^1}{d\tilde{\omega}_k}\\ \frac{d\Phi_j^2}{d\tilde{\theta}_k} & \frac{d\Phi_j^2}{d\tilde{x}_k} & \frac{d\Phi_j^2}{d\tilde{\omega}_k}\\ \frac{d\Phi_j^3}{d\tilde{\theta}_k} & \frac{d\Phi_j^3}{d\tilde{x}_k} & \frac{d\Phi_j^3}{d\tilde{\omega}_k}\end{pmatrix}\right|_{q=q_0}. \tag{26}$$

Each block is diagonal up to $O(e^{-\frac{1}{2}\omega_\star L})$. All terms on the main diagonal are non-zero and have been explicitly computed. The block terms at $(3,1)$ and $(3,2)$ are of order $O(e^{-\frac{1}{2}\omega_\star L})$. Therefore, the determinant of the matrix is

$$\det(D\Phi) = \prod_{j=1}^{N} \frac{\left(\frac{c_j}{\omega_j} - 4\mu_j + c_j\mu_j^2\right)}{\sqrt{4\omega_j - c_j^2}} \cdot$$

$$\cdot \prod_{j=1}^{N} \left(-64\omega_j \left(\arctan\left(\sqrt{\frac{2\sqrt{\omega_j}+c_j}{2\sqrt{\omega_j}-c_j}} \right) \right)^2 + \left(2\sqrt{4\omega_j - c_j^2} \right)^2 \right) + O(e^{-\frac{1}{2}\omega_\star L}).$$

Consider the function $f(\omega,c)$ defined by

$$f(\omega,c) = 8\sqrt{\omega}\arctan\left(\sqrt{\frac{2\sqrt{\omega}+c}{2\sqrt{\omega}-c}} \right) - 2\sqrt{4\omega - c^2}.$$

By explicit calculations, we have

$$\partial_c f(\omega,c) = \frac{2(c+2\sqrt{\omega})}{\sqrt{4\omega - c^2}} > 0.$$

Moreover, at $c = -2\sqrt{\omega}$ the function starts with $f(\omega, -2\sqrt{\omega}) = 0$. This implies that $f(\omega,c) > 0$ for all $(\omega,c) \in \mathbb{R}^2$ with $4\omega > c^2$. As a consequence, for any $j = 1,\dots,N$ we have

$$64\omega_j \arctan\left(\sqrt{\frac{2\sqrt{\omega_j}+c_j}{2\sqrt{\omega_j}-c_j}} \right) > \left(2\sqrt{4\omega_j - c_j^2} \right)^2, \tag{27}$$

and we infer that

$$\det(D\Phi) \neq 0.$$

Hence we can apply the Implicit Function Theorem to Φ to obtain the existence of a function $\tilde{q}_1 = (\tilde{\theta}_j, \tilde{x}_j, \tilde{\omega}_j)$ from $[0,t^\star]$ to $\mathbb{R}^{3N}$ such that for any $t \in [0,t^\star]$ we have

$$\Phi(\tilde{q}_1(t), u(t)) = 0.$$

We refer the reader to [37] for the regularity of the modulation parameters. This concludes the first part of the proof.

We now want to obtain the modulation equations (23). Recall first that u is a solution of (dNLS), hence it verifies

$$iu_t = E'(u).$$

Since $u = \sum_{j=1}^{N} \tilde{R}_j + \varepsilon$, the equation verified by ε is

$$i\varepsilon_t + \sum_{j=1}^{N} \left(-\partial_t\tilde{\theta}_j \tilde{R}_j - i\partial_t\tilde{x}_j \partial_x \tilde{R}_j + i\partial_t\tilde{\omega}_j \left(\partial_\omega \tilde{R}_j + i\mu_j \partial_c \tilde{R}_j \right) \right) = E'\left(\sum_{j=1}^{N} \tilde{R}_j + \varepsilon \right).$$

By exponential localization of the solitons and (22), we have

$$E'\left(\sum_{j=1}^{N} \tilde{R}_j + \varepsilon \right) = \sum_{j=1}^{N} \left(E'\left(\tilde{R}_j\right) + E''\left(\tilde{R}_j\right)\varepsilon \right) + O\left(e^{-\frac{1}{4}\omega_\star L}\right) + O\left(\|\varepsilon\|_{H^1}^2\right).$$

Recall that each $\tilde{R}_j$ verifies the equation

$$E'(\tilde{R}_j) + \tilde{\omega}_j M'(R_j) + \tilde{c}_j P'(R_j) = 0.$$

Therefore, the equation for ε can be written as

$$i\varepsilon_t+\sum_{j=1}^{N}\left((\tilde{\omega}_j-\partial_t\tilde{\theta}_j)\tilde{R}_j+(\tilde{c}_j-\partial_t\tilde{x}_j)i\partial_x\tilde{R}_j+i\partial_t\tilde{\omega}_j\left(\partial_\omega\tilde{R}_j+\mu_j\partial_c\tilde{R}_j\right)\right)$$
$$=\sum_{j=1}^{N}E''\left(\tilde{R}_j\right)\varepsilon+O\left(e^{-\frac{1}{4}\omega_\star L}\right)+O\left(\|\varepsilon\|_{H^1}^2\right).$$

One can see already the modulation equations appearing. For convenience, we denote by $\mathrm{Mod}(t)$ the vector of modulation equations, i.e.

$$\mathrm{Mod}(t)=\left(\tilde{\omega}_j-\partial_t\tilde{\theta}_j,\tilde{c}_j-\partial_t\tilde{x}_j,i\partial_t\tilde{\omega}_j\right)_{j=1,\dots,N}.$$

Differentiating with respect to time the orthogonality conditions (19), we obtain

$$0=-\left(i\varepsilon_t,\tilde{R}_j\right)_2+\left(\varepsilon,i\partial_t\tilde{R}_j\right)_2, \tag{28}$$

$$0=\left(i\varepsilon_t,i\partial_x\tilde{R}_j\right)_2+\left(\varepsilon,\partial_t\partial_x\tilde{R}_j\right)_2, \tag{29}$$

$$0=\left(i\varepsilon_t,i\tilde{R}_j-\mu_j\partial_x\tilde{R}_j\right)_2+\left(\varepsilon,\partial_t(\tilde{R}_j+i\mu_j\partial_x\tilde{R}_j)\right)_2. \tag{30}$$

We have

$$\left|\left(\varepsilon,i\partial_t\tilde{R}_j\right)_2\right|+\left|\left(\varepsilon,\partial_t\partial_x\tilde{R}_j\right)_2\right|+\left|\left(\varepsilon,\partial_t(\tilde{R}_j+i\mu_j\partial_x\tilde{R}_j)\right)_2\right|$$
$$\lesssim(1+|\mathrm{Mod}(t)|)\,\|\varepsilon\|_{L^2}.$$

Using the equation for ε, for the first term in the left hand side of (28), we get

$$-\left(i\varepsilon_t,\tilde{R}_j\right)_2=(\tilde{\omega}_j-\partial_t\tilde{\theta}_j)\|\tilde{R}_j\|_{L^2}^2+(\tilde{c}_j-\partial_t\tilde{x}_j)\left(i\partial_x\tilde{R}_j,\tilde{R}_j\right)_2$$
$$+\partial_t\tilde{\omega}_j\left(i(\partial_\omega\tilde{R}_j+\mu_j\partial_c\tilde{R}_j),\tilde{R}_j\right)_2+(1+|\mathrm{Mod}(t)|)O\left(e^{-\frac{1}{4}\omega_\star L}\right)+O\left(\|\varepsilon\|_{H^1}\right).$$

For the first term in the left hand side of (29), we get

$$-\left(i\varepsilon_t,i\partial_x\tilde{R}_j\right)_2=(\tilde{\omega}_j-\partial_t\tilde{\theta}_j)\left(\tilde{R}_j,i\partial_x\tilde{R}_j\right)_2+(\tilde{c}_j-\partial_t\tilde{x}_j)\|\partial_x\tilde{R}_j\|_{L^2}^2$$
$$+\partial_t\tilde{\omega}_j\left(i(\partial_\omega\tilde{R}_j+\mu_j\partial_c\tilde{R}_j),i\partial_x\tilde{R}_j\right)_2+(1+|\mathrm{Mod}(t)|)O\left(e^{-\frac{1}{4}\omega_\star L}\right)+O\left(\|\varepsilon\|_{H^1}\right).$$

For the first term in the left hand side of (30), there are some cancellations, and we get

$$-\left(i\varepsilon_t,i\tilde{R}_j-\mu_j\partial_x\tilde{R}_j\right)_2=\partial_t\tilde{\omega}_j\left(i(\partial_\omega\tilde{R}_j+\mu_j\partial_c\tilde{R}_j),i\tilde{R}_j-\mu_j\partial_x\tilde{R}_j\right)_2$$
$$+(1+|\mathrm{Mod}(t)|)O\left(e^{-\frac{1}{4}\omega_\star L}\right)+O\left(\|\varepsilon\|_{H^1}\right).$$

Remark that, in contrary to what happens in [37], the ε term is still of order 1 and not of order 2 (unless $\mu=0$, but this is ruled out by our assumptions on the (c_j)). The attentive reader will have noticed the appearance of the same elements as in the matrix $D\Phi$. We have indeed

$$M\begin{pmatrix}\tilde{\omega}_j-\partial_t\tilde{\theta}_j\\ \tilde{c}_j-\partial_t\tilde{x}_j\\ \partial_t\tilde{\omega}_j\end{pmatrix}=(1+\mathrm{Mod}(t))\left(O\left(\|\varepsilon\|_{L^2}\right)+O\left(e^{-\frac{1}{4}\omega_\star L}\right)\right),$$

for $M = (m_{kl})$, where

$$m_{11} = \|\tilde{R}_j\|_{L^2}^2, \qquad m_{12} = \left(i\partial_x \tilde{R}_j, \tilde{R}_j\right)_2,$$
$$m_{21} = \left(\tilde{R}_j, i\partial_x \tilde{R}_j\right)_2, \qquad m_{22} = \|\partial_x \tilde{R}_j\|_{L^2}^2,$$
$$m_{31} = \left(i\tilde{R}_j - \mu_j\partial_x\tilde{R}_j, \tilde{R}_j\right)_2, \qquad m_{32} = \left(i\tilde{R}_j - \mu_j\partial_x\tilde{R}_j, i\partial_x\tilde{R}_j\right)_2,$$
$$m_{13} = \left(i(\partial_\omega \tilde{R}_j + \mu_j\partial_c\tilde{R}_j), \tilde{R}_j\right)_2, \qquad m_{23} = \left(i(\partial_\omega \tilde{R}_j + \mu_j\partial_c\tilde{R}_j), i\partial_x\tilde{R}_j\right)_2,$$
$$m_{33} = \left(i(\partial_\omega \tilde{R}_j + \mu_j\partial_c\tilde{R}_j), i\tilde{R}_j - \mu_j\partial_x\tilde{R}_j\right)_2.$$

Using the explicit values of the coefficients (m_{kl}) (see above calculations for $D\Phi$), we obtain

$$M = \begin{pmatrix} 8\arctan\left(\sqrt{\frac{2\sqrt{\omega_j}+c_j}{2\sqrt{\omega_j}-c_j}}\right) & 2\sqrt{4\omega_j - c_j^2} & * \\ 2\sqrt{4\omega_j - c_j^2} & 8\omega_j\arctan\left(\sqrt{\frac{2\sqrt{\omega_j}+c_j}{2\sqrt{\omega_j}-c_j}}\right) & * \\ 0 & 0 & \frac{-\left(\frac{\tilde{c}_j}{\tilde{\omega}_j} - 4\mu_j + \tilde{c}_j\mu_j^2\right)}{\sqrt{4\tilde{\omega}_j - \tilde{c}_j^2}} \end{pmatrix}.$$

Hence M is invertible using the same arguments as to prove that $D\Phi$ is, see in particular (27), and we can infer that

$$\sum_{j=1}^N \left|\tilde{\omega}_j - \partial_t\tilde{\theta}_j\right| + |\tilde{c}_j - \partial_t\tilde{x}_j| + |\partial_t\tilde{\omega}_j| \lesssim \|\varepsilon\|_{H^1} + e^{-\frac{1}{4}\omega_\star L}.$$

This concludes the proof. □

We complete this section by giving estimates on the interaction between $\tilde{R}_j$ and $\tilde{R}_k$ when $j \neq k$.

Lemma 4.3 (Interaction estimates One)**.** *There exists a function $g \in L_t^\infty L_x^1(\mathbb{R}, \mathbb{R}^d) \cap L_t^\infty L_x^\infty(\mathbb{R}, \mathbb{R}^d)$ such that for all $j, k = 1, \dots, N$ such that $j \neq k$, we have*

$$|\tilde{R}_j| + |\partial_x\tilde{R}_j| + |\partial_\omega\tilde{R}_j| + |\partial_c\tilde{R}_j| \lesssim e^{-\omega_\star|x-\tilde{x}_j|}.$$
$$e^{-\omega_\star|x-\tilde{x}_j|}e^{-\omega_\star|x-\tilde{x}_k|} \leqslant e^{-\frac{1}{2}\omega_\star|\tilde{x}_j-\tilde{x}_k|}g(t,x).$$

Moreover, if (21)-(23) hold, then

$$|\tilde{x}_j - \tilde{x}_k| \geqslant \frac{1}{2}(L + c_\star t).$$

Proof. We first remark that, by the exponential decay of the soliton profiles (9) and the definition (17) of $\omega_\star$, for $j = 1, \dots, N$, we have

$$|\tilde{R}_j| + |\partial_x\tilde{R}_j| + |\partial_\omega\tilde{R}_j| + |\partial_c\tilde{R}_j| \lesssim e^{-\omega_\star|x-\tilde{x}_j|}.$$

There exists $g \in L_t^\infty L_x^1(\mathbb{R}, \mathbb{R}^d) \cap L_t^\infty L_x^\infty(\mathbb{R}, \mathbb{R}^d)$ such that

$$e^{-\omega_\star|x-\tilde{x}_j|}e^{-\omega_\star|x-\tilde{x}_k|} \leqslant e^{-\frac{1}{2}\omega_\star|\tilde{x}_j-\tilde{x}_k|}g(t,x),$$

Indeed, let

$$g_{jk}(t,x) = e^{-\frac{1}{2}\omega_\star(|x-\tilde{x}_j|+|x-\tilde{x}_k|)}.$$

One can then take

$$g(t,x) = \sum_{j\neq k} g_{j,k}(t,x).$$

In view of (21)-(23), we can chose δ and $1/L$ small enough such that for any $j \neq k$ we have

$$|\tilde{x}_j - \tilde{x}_k| \geqslant \frac{L}{2} + \frac{1}{2}c_\star t.$$

This concludes the proof. □

5. Monotonicity of Localized Conservations Laws

We are using an energy technique to control the main difference ε between u and the sum of modulated solitons $\sum \tilde{R}_j$. The energy technique consists in using the coercivity of a linearized action functional related to the conservation laws and the solitons. It can be viewed as a generalization of the method used to prove stability of a single soliton. The main difference when considering a sum of solitons is that we need to introduce a localization procedure around each soliton. We recover this way the desired coercivity property, but the price to pay is that the quantities involved are no longer conserved. Controlling their variations in time becomes a main issue, which is dealt with using monotonicity properties.

The localization procedure is the following. Let $\psi : \mathbb{R} \to [0,1]$ be a smooth cut-off function such that

$$\psi(x) = 0 \text{ for } x \leqslant -1, \quad \psi(x) = 1 \text{ for } x \geqslant 1, \quad \psi'(x) > 0 \text{ for } x \in (-1,1),$$
$$(\psi'(x))^2 \lesssim \psi(x), \quad (\psi''(x))^2 \lesssim \psi'(x), \quad \text{for all } x \in \mathbb{R}.$$

For $j = 2, \dots, N$, set

$$\tilde{\sigma}_j = 2\frac{\tilde{\omega}_j(0) - \tilde{\omega}_{j-1}(0)}{\tilde{c}_j(0) - \tilde{c}_{j-1}(0)}. \tag{31}$$

Recall that (ω_j) and (c_j) verify the speed-frequency ratio assumption (4). Therefore, for δ and $1/L$ small enough and by the estimate on the modulation parameters at initial time (24) we have

$$\max(c_{j-1}, \tilde{c}_{j-1}(0)) < \tilde{\sigma}_j < \min(c_j, \tilde{c}_j(0)), \quad j = 2, \dots, N.$$

Set

$$x_j^\sigma = \frac{\tilde{x}_{j-1}(0) + \tilde{x}_j(0)}{2}, \quad a = \frac{L^2}{64},$$

and define

$$\psi_1 \equiv 1, \quad \psi_j(t,x) = \psi\left(\frac{x - x_j^\sigma - \tilde{\sigma}_j t}{\sqrt{t+a}}\right), \ j = 2, \dots, N, \quad \psi_{N+1} \equiv 0.$$

We define the cut-off functions around the j-th solitary wave by

$$\chi_j(t,x) = \psi_j(t,x) - \psi_{j+1}(t,x), \quad j = 1, \dots, N.$$

The reason for the introduction of cut-off functions of this form will become clear in the proof of the monotonicity properties.

We define the following functional, which is made by the combination of localized masses and momenta around each solitary wave, weighted with the corresponding modulated parameters $\tilde{\omega}_j(0)$ and $\tilde{c}_j(0)$ at $t = 0$:

$$\mathcal{I}(t) = \frac{1}{2}\sum_{j=1}^{N}\int_{\mathbb{R}} \left(\tilde{\omega}_j(0)|u|^2 + \tilde{c}_j(0)\mathcal{I}m(u\bar{u}_x)\right)\chi_j dx. \tag{32}$$

The following monotonicity property for $\mathcal{I}$ will be a key feature of the proof of Theorem 1.1.

Proposition 5.1 (Monotonicity One)**.** *If δ and $1/L$ are small enough, then for all $t \in [0, t^\star]$, we have*

$$\mathcal{I}(t) - \mathcal{I}(0) \lesssim \frac{1}{L} \sup_{s \in [0,t]} \|\varepsilon(s)\|_{L^2}^2 + e^{-\frac{1}{16}\omega_\star (c_\star^\sigma t + L)}.$$

To prove Proposition 5.1, it is convenient to rewrite $\mathcal{I}$ using the functionals defined for $j = 2, \dots, N$ by

$$\mathcal{I}_j(t) = \frac{1}{2} \int_{\mathbb{R}} \left(\frac{\tilde{\sigma}_j}{2} |u|^2 + \mathcal{I}m(u \bar{u}_x) \right) \psi_j dx.$$

Lemma 5.2 (Decomposition of the functional $\mathcal{I}$)**.** *We have*

$$\mathcal{I}(t) = \tilde{\omega}_1(0) M(u) + \tilde{c}_1(0) P(u) + \sum_{j=2}^{N} (\tilde{c}_j(0) - \tilde{c}_{j-1}(0)) \mathcal{I}_j(t).$$

The proof of Lemma 5.2 consists in a simple rearrangement of the sum in the definition (32) of $\mathcal{I}$ using the definition (31) of $\tilde{\sigma}_j$. We omit the details.

Proposition 5.1 is a consequence of Lemma 5.2, the conservation of mass and momentum, and the following monotonicity result for each of the functionals $\mathcal{I}_j$.

Proposition 5.3 (Monotonicity Two)**.** *If δ and $1/L$ are small enough, then for any $j = 2, \dots, N$ and $t \in [0, t^\star]$ we have*

$$\mathcal{I}_j(t) - \mathcal{I}_j(0) \lesssim \frac{1}{L} \sup_{s \in [0,t]} \|\varepsilon(s)\|_{L^2}^2 + e^{-\frac{1}{16}\omega_\star (c_\star^\sigma t + L)}.$$

Proof of Proposition 5.3. Fix $j \in \{2, \dots, N\}$. To express the time derivative of $\mathcal{I}_j$ in a form to which we can give a sign, we will use a Galilean transformation. We define v by

$$u(t,x) = e^{i \frac{\tilde{\sigma}_j}{2} \left(x - x_j^\sigma - \frac{\tilde{\sigma}_j}{2} t \right)} v(t, x - x_j^\sigma - \tilde{\sigma}_j t).$$

We insist on the fact that since (dNLS) is not Galilean invariant, v *is not a solution of* (dNLS) *anymore*. It satisfies the modified equation

$$i v_t + v_{xx} + i |v|^2 v_x - \frac{\tilde{\sigma}_j}{2} |v|^2 v = 0,$$

and we have

$$\mathcal{I}_j(t) = \frac{1}{2} \int_{\mathbb{R}} \mathcal{I}m(v \bar{v}_x) \psi \left(\frac{x}{\sqrt{t+a}} \right) dx.$$

One realizes that the advantage of introducing $\mathcal{I}_j$ is that there is now no mass factor in the expression of $\mathcal{I}_j$ in terms of v. Computing the time derivative, we obtain

$$\begin{aligned}
\frac{\partial}{\partial t} \mathcal{I}_j(t) = &- \frac{1}{\sqrt{t+a}} \int_{\mathbb{R}} \left(|v_x|^2 + \frac{1}{2} \mathcal{I}m \left(|v|^2 \bar{v} v_x \right) + \frac{\tilde{\sigma}_j}{8} |v|^4 \right) \psi_x \left(\frac{x}{\sqrt{t+a}} \right) dx \\
&+ \frac{1}{4(t+a)^{\frac{3}{2}}} \int_{\mathbb{R}} |v|^2 \psi_{xxx} \left(\frac{x}{\sqrt{t+a}} \right) dx \\
&- \frac{1}{4(t+a)^{\frac{3}{2}}} \int_{\mathbb{R}} \mathcal{I}m(x v \bar{v}_x) \psi_x \left(\frac{x}{\sqrt{t+a}} \right) dx.
\end{aligned}$$

We distribute the last term between the quadratic terms. Using Young's inequality, we have

$$\left|\frac{1}{4(t+a)^{\frac{3}{2}}}\int_{\mathbb{R}}\mathcal{Im}(xv\bar{v}_x)\psi_x\left(\frac{x}{\sqrt{t+a}}\right)dx\right|\leqslant$$
$$\frac{1}{8\sqrt{t+a}}\int_{\mathbb{R}}|v_x|^2\psi_x\left(\frac{x}{\sqrt{t+a}}\right)dx+\frac{1}{8(t+a)^{\frac{3}{2}}}\int_{\mathbb{R}}|v|^2\left(\frac{x}{\sqrt{t+a}}\right)^2\psi_x\left(\frac{x}{\sqrt{t+a}}\right)dx.$$

In addition, since ψ_x is supported on $[-1,1]$, we have

$$\frac{1}{8(t+a)^{\frac{3}{2}}}\int_{\mathbb{R}}|v|^2\left(\frac{x}{\sqrt{t+a}}\right)^2\psi_x\left(\frac{x}{\sqrt{t+a}}\right)dx\leqslant\frac{1}{8(t+a)^{\frac{3}{2}}}\int_{\mathbb{R}}|v|^2\psi_x\left(\frac{x}{\sqrt{t+a}}\right)dx.$$

We also apply Young's inequality to the derivative part of the nonlinear term:

$$\left|\frac{1}{\sqrt{t+a}}\int_{\mathbb{R}}\frac{1}{2}\mathcal{Im}\left(|v|^2\bar{v}v_x\right)\psi_x\left(\frac{x}{\sqrt{t+a}}\right)dx\right|\leqslant$$
$$\frac{1}{8\sqrt{t+a}}\int_{\mathbb{R}}|v_x|^2\psi_x\left(\frac{x}{\sqrt{t+a}}\right)dx+\frac{1}{2\sqrt{t+a}}\int_{\mathbb{R}}|v|^6\psi_x\left(\frac{x}{\sqrt{t+a}}\right)dx.$$

Summarizing, we have obtained that

$$\frac{\partial}{\partial t}\mathcal{I}_j(u)\leqslant-\frac{1}{\sqrt{t+a}}\int_{\mathbb{R}}\left(\frac{3}{4}|v_x|^2-\frac{1}{2}|v|^6+\frac{\tilde{\sigma}_j}{8}|v|^4\right)\psi_x\left(\frac{x}{\sqrt{t+a}}\right)dx$$
$$+\frac{1}{4(t+a)^{\frac{3}{2}}}\int_{\mathbb{R}}|v|^2\left(\frac{1}{2}\psi_x+\psi_{xxx}\right)\left(\frac{x}{\sqrt{t+a}}\right)dx. \quad (33)$$

By assumption (4) we have $\sigma_j>0$, thus we also have $\tilde{\sigma}_j>0$ (for δ and $1/L$ small enough). Therefore, to obtain the (quasi)-monotonicity of $\mathcal{I}_j$, it is sufficient to bound the L^2-term and the nonlinear term with power 6. This is allowed by the following claims.

Claim 5.4. For δ and $1/L$ small enough and for any $t\in[0,t^\star]$, we have

$$\int_{|x|<\sqrt{t+a}}|v|^2dx\leqslant 2\|\varepsilon\|_{L^2}^2+O\left(e^{-\frac{1}{16}\omega_\star(c_\star^\sigma t+L)}\right).$$

Claim 5.5. For δ and $1/L$ small enough and for any $t\in[0,t^\star]$, we have

$$\int_{\mathbb{R}}|v|^6\psi_x\left(\frac{x}{\sqrt{t+a}}\right)dx$$
$$\leqslant\frac{1}{4}\int_{\mathbb{R}}|v_x|^2\psi_x\left(\frac{x}{\sqrt{t+a}}\right)dx+\frac{1}{t+a}\|\varepsilon\|_{L^2}^2+O\left(e^{-\frac{1}{16}\omega_\star(c_\star^\sigma t+L)}\right).$$

Proof of Claim 5.4. By definition of v as a Galilean transform of u, we have

$$|v(t,x)|^2=|u(t,x+x_j^\sigma+\tilde{\sigma}_jt)|^2$$
$$\leqslant 2\sum_{k=1}^{N}\left|\tilde{R}_k(t,x+x_j^\sigma+\tilde{\sigma}_jt)\right|^2+2\left|\varepsilon(t,x+x_j^\sigma+\tilde{\sigma}_jt)\right|^2. \quad (34)$$

By exponential decay of the solitons profiles and the control (21) on the modulation parameters $(\tilde{\omega}_k)$ and $(\tilde{c}_k)$, we have

$$\sum_{k=1}^{N}\left|\tilde{R}_k(t,x+x_j^\sigma+\tilde{\sigma}_j t)\right|^2 \lesssim \sum_{k=1}^{N} e^{-\omega_\star|x-\tilde{x}_k(t)+x_j^\sigma+\tilde{\sigma}_j t|}.$$

Assume that $|x|<\sqrt{t-a}$. We have

$$|x-\tilde{x}_k(t)+x_j^\sigma+\tilde{\sigma}_j t| \geqslant |\tilde{x}_k(t)-x_j^\sigma-\tilde{\sigma}_j t|-|x|,$$

which for $|x|<\sqrt{t+a}<\sqrt{t}+\sqrt{a}=\sqrt{t}+\frac{L}{8}$ gives

$$|x-\tilde{x}_k(t)+x_j^\sigma+\tilde{\sigma}_j t| \geqslant |\tilde{x}_k(t)-x_j^\sigma-\tilde{\sigma}_j t|-\sqrt{t}-\frac{L}{8}.$$

If $k\geqslant j$, then using the dynamical system (23) verified by the modulation parameters we get

$$\partial_t\tilde{x}_k \geqslant \partial_t\tilde{x}_j \geqslant c_j-\frac{c_\star^\sigma}{8}.$$

Since in addition $\tilde{x}_k(0)\geqslant\tilde{x}_j(0)$, we have

$$\tilde{x}_k(t)-x_j^\sigma-\tilde{\sigma}_j t \geqslant \left(c_j-\frac{c_\star^\sigma}{8}\right)t+\tilde{x}_j(0)-x_j^\sigma-\tilde{\sigma}_j t \geqslant \frac{c_\star^\sigma}{8}t+\frac{\tilde{x}_j(0)-\tilde{x}_{j-1}(0)}{2},$$

where for the last inequality we have used $x_j^\sigma=(\tilde{x}_j(0)+\tilde{x}_{j-1}(0))/2$ and $c_j-\sigma_j\geqslant\frac{c_\star^\sigma}{4}$. Therefore, using $\tilde{x}_j(0)>\tilde{x}_{j-1}(0)+L/2$, we get

$$|x-\tilde{x}_k(t)+x_j^\sigma+\tilde{\sigma}_j t| \geqslant \frac{c_\star^\sigma}{8}t-\sqrt{t}+\frac{L}{8}. \tag{35}$$

Choose now L large enough so that $\min_{t\geqslant0}\left(\frac{c_\star^\sigma}{16}t-\sqrt{t}+\frac{L}{16}\right)>0$. Then

$$\frac{c_\star^\sigma}{8}t-\sqrt{t}+\frac{L}{8}=\frac{c_\star^\sigma}{16}t+\frac{L}{16}+\left(\frac{c_\star^\sigma}{16}t-\sqrt{t}+\frac{L}{16}\right)\geqslant\frac{c_\star^\sigma}{16}t+\frac{L}{16},$$

and we infer from (35) that

$$|x-\tilde{x}_k(t)+x_j^\sigma+\tilde{\sigma}_j t| \geqslant \frac{c_\star^\sigma}{16}t+\frac{L}{16}.$$

Arguing in a similar fashion for $k\leqslant j-1$ allows us to obtain, for L large enough,

$$\sum_{k=1}^{N}\left|\tilde{R}_k(t,x+x_j^\sigma+\tilde{\sigma}_j t)\right|^2 \lesssim \sum_{k=1}^{N} e^{-\frac{1}{16}\omega_\star(c_\star^\sigma t+L)}. \tag{36}$$

Combining (34) and (36) gives the desired conclusion. □

For further reference, we state here a Lemma which can be obtained using similar arguments as in Claim 5.4.

Lemma 5.6 (Interaction Estimates Two)**.** *There exists a function* $g\in L_t^\infty L_x^1(\mathbb{R},\mathbb{R}^d)\cap L_t^\infty L_x^\infty(\mathbb{R},\mathbb{R}^d)$ *such that for all* $j,k=1,\dots,N$ *such that* $j\neq k$, *we have*

$$e^{-\omega_\star|x-\tilde{x}_j|}\chi_k(t,x)\leqslant e^{-\frac{1}{32}\omega_\star(c_\star^\sigma t+L)}g(t,x),$$

where

$$c_\star^\sigma=\min\left\{|\tilde{\sigma}_j-c_k|;\ j,k=1,\dots,N,\ j\neq 1\right\}. \tag{37}$$

Let us recall without proof the following technical lemma from [37, 38] that is used for the proof of Claim 5.5.

Lemma 5.7. *Let $w \in H^1(\mathbb{R})$ and let $h \geqslant 0$ be a $\mathcal{C}^2$ bounded function such that $\sqrt{h}$ is $\mathcal{C}^1$ and $(h_x)^2 \lesssim h$. Then*

$$\int_{\mathbb{R}} |w|^6 h dx \leqslant 8 \left(\int_{\mathrm{supp}(h)} |w|^2 dx \right)^2 \left(\int_{\mathbb{R}} |w_x|^2 h dx + \int_{\mathbb{R}} |w|^2 \frac{(h_x)^2}{h} dx \right),$$

where $\mathrm{supp}(h)$ *denotes the support of* h.

Proof of Claim 5.5. From the technical result Lemma 5.7, we infer that

$$\int_{\mathbb{R}} |v|^6 \psi_x \left(\frac{x}{\sqrt{t+a}} \right) dx \leqslant 8 \left(\int_{|x|<\sqrt{t+a}} |v|^2 dx \right)^2 \times \\ \times \left(\int_{\mathbb{R}} |v_x|^2 \psi_x \left(\frac{x}{\sqrt{t+a}} \right) dx + \frac{1}{t+a} \int_{\mathbb{R}} |v|^2 \frac{(\psi_{xx})^2}{\psi_x} \left(\frac{x}{\sqrt{t+a}} \right) dx \right).$$

Using Claim 5.4 and the fact that by construction $\frac{(\psi_{xx})^2}{\psi_x} \lesssim 1$, we get for δ and $1/L$ small enough that

$$\begin{aligned} \int_{\mathbb{R}} |v|^6 \psi_x \left(\frac{x}{\sqrt{t+a}} \right) dx & \\ \leqslant \frac{1}{4} \int_{\mathbb{R}} |v_x|^2 \psi_x \left(\frac{x}{\sqrt{t+a}} \right) dx &+ \frac{1}{2(t+a)} \int_{|x|<\sqrt{t+a}} |v|^2 dx, \\ \leqslant \frac{1}{4} \int_{\mathbb{R}} |v_x|^2 \psi_x \left(\frac{x}{\sqrt{t+a}} \right) dx &+ \frac{1}{t+a} \|\varepsilon\|_{L^2}^2 + O(e^{-\frac{1}{16}\omega_\star (c_\star^\sigma t + L)}). \end{aligned}$$

This finishes the proof of Claim 5.5. □

Let us now conclude the proof of Proposition 5.3. Coming back to (33) and using $\tilde{\sigma}_j > 0$ and Claims 5.4 and 5.5, we get

$$\partial_t \mathcal{I}_j(t) \lesssim \frac{1}{(t+a)^{\frac{3}{2}}} \|\varepsilon(t)\|_{L^2}^2 + e^{-\frac{1}{16}\omega_\star (c_\star^\sigma t + L)}.$$

Integrating between 0 and t, we obtain (using in particular $a = L^2/64$)

$$\mathcal{I}_j(t) - \mathcal{I}_j(0) \lesssim \frac{1}{L} \sup_{s \in [0,t^\star]} \|\varepsilon(s)\|_{L^2}^2 + e^{-\frac{1}{16}\omega_\star (c_\star^\sigma t + L)},$$

and this finishes the proof. □

For convenience, we also introduce here functionals similar to $\mathcal{I}_j$ but with a different parameter σ. They will be useful when we will control the modulation parameters. For $j = 2, \dots, N$, let τ_j be such that

$$\tilde{c}_{j-1}(0) < \tau_j < \tilde{c}_j(0).$$

For any $j = 2, \dots, N$, we define

$$\psi_{j,\tau_j}(t,x) = \psi \left(\frac{x - x_j^\sigma - \tau_j t}{\sqrt{t+a}} \right), \quad \mathcal{I}_{j,\tau_j}(t) = \frac{1}{2} \int_{\mathbb{R}} \left(\frac{\tau_j}{2} |u|^2 + \mathcal{Im}(u \bar{u}_x) \right) \psi_{j,\tau_j} dx.$$

Then following the same proof as for Proposition 5.3 we get the following result.

Proposition 5.8 (Monotonicity Three). *If δ and $1/L$ are small enough, then for any $j = 2, \dots, N$ and $t \in [0, t^\star]$ we have*

$$\mathcal{I}_{j,\tau_j}(t) - \mathcal{I}_{j,\tau_j}(0) \lesssim \frac{1}{L} \sup_{s\in[0,t^\star]} \|\varepsilon(s)\|_{L^2}^2 + e^{-\frac{1}{16}\omega_\star (c_\star^\tau t+L)},$$

where

$$c_\star^\tau = \min\{|\tau_j - c_k|;\ j,k = 1,\dots,N,\ j \neq 1\}.$$

6. Linearized Action Functional and Coercivity for N Solitons

For $j = 1, \dots, N$, we define an action functional related to the j-th soliton by

$$S_j(v) = E(v) + \tilde\omega_j(0) M(v) + \tilde c_j(0) P(v).$$

For the sum of N solitons we define an action-like functional $\mathcal{S}$ which will correspond to S_j locally around the j-th soliton. The functional $\mathcal{S}$ is given by

$$\mathcal{S}(t) = E(u(t)) + \mathcal{I}(t),$$

where $\mathcal{I}$ is the functional composed of localized masses and momenta defined in (32).

It is classical when working with solitons and related solutions of nonlinear dispersive equations to introduce functionals related to the second variation of the action. In our context, we will work with the functional $\mathcal{H}$, obtained as follows.

We set

$$\bar c_\star = \min\{c_\star, c_\star^\sigma\},$$

where $c_\star$ and $c_\star^\sigma$ are given by (18) and (37).

Lemma 6.1 (Expansion of the action). *For $t \in [0, t^\star]$, we have*

$$\mathcal{S}(t) = \sum_{j=1}^N S_j(\phi_{\tilde\omega_j(0),\tilde c_j(0)}) + \frac{1}{2}\mathcal{H}(t) + \sum_{j=1}^N O\left(|\tilde\omega_j(t) - \tilde\omega_j(0)|^2\right) + o(\|\varepsilon\|_{H^1}^2) + O(e^{-\frac{1}{32}\omega_\star(\bar c_\star t + L)}), \quad (38)$$

where

$$\mathcal{H}(t) = \|\varepsilon_x\|_{L^2}^2 + \sum_{j=1}^N \mathcal{I}m \int_{\mathbb{R}} \left(|\tilde R_j|^2 \bar\varepsilon\varepsilon_x + \tilde R_j \partial_x \tilde R_j (\bar\varepsilon)^2 + \overline{\tilde R_j}\partial_x \tilde R_j |\varepsilon|^2\right) dx + \sum_{j=1}^N \left(\tilde\omega_j(t)\int_{\mathbb{R}} |\varepsilon|^2 \chi_j dx + \tilde c_j(t)\mathcal{I}m \int_{\mathbb{R}} \varepsilon\bar\varepsilon_x \chi_j dx\right).$$

Proof. Writing $u = \sum_{j=1}^N \tilde R_j + \varepsilon$, we expand in the components of $\mathcal{S}$. For the energy, we have

$$E(u) = E\left(\sum_{j=1}^N \tilde R_j\right) + E'\left(\sum_{j=1}^N \tilde R_j\right)\varepsilon + \frac{1}{2}\left\langle E''\left(\sum_{j=1}^N \tilde R_j\right)\varepsilon, \varepsilon\right\rangle + o(\|\varepsilon\|_{H^1}^2).$$

From Lemma 4.3 (Interaction Estimates One), we have

$$E\Big(\sum_{j=1}^N \tilde R_j\Big) = \sum_{j=1}^N E\left(\tilde R_j\right) + O(e^{-\frac14\omega_\star(c_\star t+L)}),$$

$$E'\Big(\sum_{j=1}^N \tilde R_j\Big)\varepsilon = \sum_{j=1}^N E'\left(\tilde R_j\right)\varepsilon + O(e^{-\frac14\omega_\star(c_\star t+L)}),$$

$$\Big\langle E''\Big(\sum_{j=1}^N \tilde R_j\Big)\varepsilon,\varepsilon\Big\rangle = \sum_{j=1}^N \left\langle E''\left(\tilde R_j\right)\varepsilon,\varepsilon\right\rangle + O(e^{-\frac14\omega_\star(c_\star t+L)}).$$

From Lemma 5.6 (Interaction Estimates Two), we also have

$$\frac12\int_{\mathbb R}|u|^2\chi_j dx = \sum_{j=1}^N M(\tilde R_j) + 2\sum_{j=1}^N M'(\tilde R_j)\varepsilon + \frac12\int_{\mathbb R}|\varepsilon|^2\chi_j dx + O(e^{-\frac{1}{32}\omega_\star(c_\star^\sigma t+L)}),$$

$$\frac12\mathcal{I}m\int_{\mathbb R}u\bar u_x\chi_j dx = \sum_{j=1}^N P(\tilde R_j) + 2\sum_{j=1}^N P'(\tilde R_j)\varepsilon + \frac12\mathcal{I}m\int_{\mathbb R}\varepsilon\bar\varepsilon_x\chi_j dx + O(e^{-\frac{1}{32}\omega_\star(c_\star^\sigma t+L)}).$$

Recall that, for $j=1,\dots,N$, the function $\tilde R_j$ verifies

$$E'(\tilde R_j) + \tilde\omega_j(t)M'(\tilde R_j) + \tilde c_j(t)P'(\tilde R_j) = 0.$$

Therefore, we have

$$\begin{aligned}\mathcal{S}(t) = \sum_{j=1}^N S_j(\tilde R_j) &+ \sum_{j=1}^N(\tilde\omega_j(0)-\tilde\omega_j(t))M'(\tilde R_j)\varepsilon + \sum_{j=1}^N(\tilde c_j(0)-\tilde c_j(t))P'(\tilde R_j)\varepsilon \\ &+ \frac12\sum_{j=1}^N\left\langle E''\left(\tilde R_j\right)\varepsilon,\varepsilon\right\rangle + \sum_{j=1}^N\tilde\omega_j(0)\frac12\int_{\mathbb R}|\varepsilon|^2\chi_j dx + \sum_{j=1}^N\tilde c_j(0)\frac12\int_{\mathbb R}\varepsilon\bar\varepsilon_x\chi_j dx \\ &+ O(e^{-\frac{1}{32}\omega_\star(barc_\star t+L)}) + o(\|\varepsilon\|_{H^1}^2). \end{aligned}\tag{39}$$

A Taylor expansion in $\tilde\omega_j$ gives (using that $\tilde R_j(0)$ is a critical point of S_j)

$$S_j(\tilde R_j(t)) = S_j(\tilde R_j(0)) + O(|\tilde\omega_j(t)-\tilde\omega_j(0)|^2). \tag{40}$$

Moreover, using phase and translation invariance, we have

$$S_j(\tilde R_j(0)) = S_j(\phi_{\tilde\omega_j(0),\tilde c_j(0)}). \tag{41}$$

Using the relation (20) between $\tilde\omega$ and $\tilde c$ and the last orthogonality condition on ε in (19), for all $j=1,\dots,N$ we obtain

$$\begin{aligned}(\tilde\omega_j(0)-\tilde\omega_j(t))&M'(\tilde R_j)\varepsilon + (\tilde c_j(0)-\tilde c_j(t))P'(\tilde R_j)\varepsilon \\ &= (\tilde\omega_j(0)-\tilde\omega_j(t))\left(M'(\tilde R_j)+\mu_j P'(\tilde R_j)\right)\varepsilon \\ &= (\tilde\omega_j(0)-\tilde\omega_j(t))\left(\tilde R_j + i\mu_j\partial_x\tilde R_j,\varepsilon\right)_2 = 0.\end{aligned}$$

From Young's inequality and using again the relation (20) between $\tilde\omega$ and $\tilde c$, for all $j=1,\dots,N$ we have

$$\left|(\tilde\omega_j(0)-\tilde\omega(t))\frac12\int_{\mathbb R}|\varepsilon|^2\chi_j dx + (\tilde c_j(0)-c_j(t))\frac12\int_{\mathbb R}\varepsilon\bar\varepsilon_x\chi_j dx\right| \lesssim |\tilde\omega_j(0)-\tilde\omega(t)|^2 + \|\varepsilon\|_{H^1}^4.$$

Therefore, the term in the second line of (39) becomes

$$\frac{1}{2}\sum_{j=1}^{N}\left\langle E''\left(\tilde{R}_j\right)\varepsilon,\varepsilon\right\rangle+\sum_{j=1}^{N}\tilde{\omega}_j(0)\frac{1}{2}\int_{\mathbb{R}}|\varepsilon|^2\chi_j dx+\sum_{j=1}^{N}\tilde{c}_j(0)\frac{1}{2}\int_{\mathbb{R}}\varepsilon\bar{\varepsilon}_x\chi_j dx$$
$$=\frac{1}{2}\mathcal{H}(t)+\sum_{j=1}^{N}O(|\tilde{\omega}_j(t)-\tilde{\omega}_j(0)|^2)+o(\|\varepsilon\|_{H^1}^2).$$

Together with (39), (40) and (41), this gives the desired result (38). □

As a consequence of the coercivity of each linearized action S_j'' given by Proposition 2.1, we have global coercivity for $\mathcal{H}$.

Lemma 6.2 (Coercivity)**.** *There exists $\kappa>0$ such that for all $t\in[0,t^\star]$ we have*

$$\mathcal{H}(t)\geqslant\kappa\|\varepsilon\|_{H^1}^2.$$

Before doing the proof, we explain how to control $\|\varepsilon\|_{H^1}$ using the coercivity property Lemma 6.2 and the first monotonicity result Proposition 5.1.

Lemma 6.3. *For all $t\in[0,t^\star]$, we have*

$$\|\varepsilon(t)\|_{H^1}^2\lesssim\frac{1}{L}\sup_{s\in[0,t]}\|\varepsilon(s)\|_{L^2}^2+\|\varepsilon(0)\|_{H^1}^2+\sum_{j=1}^{N}|\tilde{\omega}_j(t)-\tilde{\omega}_j(0)|^2+e^{-\frac{1}{32}\omega_\star L}.$$

Proof. From the Taylor-like expansion of $\mathcal{S}$ in Lemma 6.1 at 0 and $t\in[0,t^\star]$, we get

$$\mathcal{S}(t)-\mathcal{S}(0)=\frac{1}{2}\left(\mathcal{H}(t)-\mathcal{H}(0)\right)$$
$$+\sum_{j=1}^{N}O\left(|\tilde{\omega}_j(t)-\tilde{\omega}_j(0)|^2\right)+o(\|\varepsilon(t)\|_{H^1}^2)+o(\|\varepsilon(0)\|_{H^1}^2)+O(e^{-\frac{1}{32}\omega_\star L}).$$

By definition of $\mathcal{S}$ and conservation of the energy, we have

$$\mathcal{S}(t)-\mathcal{S}(0)=\mathcal{I}(t)-\mathcal{I}(0).$$

In addition, $\mathcal{H}$ is quadratic in ε, hence

$$\mathcal{H}(0)\lesssim\|\varepsilon(0)\|_{H^1}^2.$$

Taking into account the coercivity of $\mathcal{H}$ given by Lemma 6.2, we obtain

$$\|\varepsilon(t)\|_{H^1}^2\lesssim\mathcal{H}(t)\lesssim\mathcal{I}(t)-\mathcal{I}(0)+\sum_{j=1}^{N}|\tilde{\omega}_j(t)-\tilde{\omega}_j(0)|^2+\|\varepsilon(0)\|_{H^1}^2+e^{-\frac{1}{32}\omega_\star L}. \quad (42)$$

The conclusion then follows from the first monotonicity result Proposition 5.1. □

Proof of Lemma 6.2. We write $\mathcal{H}(t)$ as

$$\begin{aligned}
\mathcal{H}(t) =& \sum_{j=1}^{N} \int \big(|\varepsilon_x|^2 + \tilde{\omega}_j(t)|\varepsilon|^2 + \tilde{c}_j(t)\mathcal{Im}(\varepsilon\bar{\varepsilon}_x)\big)\chi_j(t)\,dx \\
&+ \sum_{j=1}^{N} \mathcal{Im}\int_{\mathbb{R}} \Big(|\tilde{R}_j|^2\bar{\varepsilon}\varepsilon_x + \tilde{R}_j\partial_x\tilde{R}_j(\bar{\varepsilon})^2 + \overline{\tilde{R}_j}\partial_x\tilde{R}_j|\varepsilon|^2\Big)\,dx \\
=& \sum_{j=1}^{N} \int \Big(|\varepsilon_x|^2 + \tilde{\omega}_j(t)|\varepsilon|^2 + \tilde{c}_j(t)\mathcal{Im}(\varepsilon\bar{\varepsilon}_x) \\
&+ \mathcal{Im}\Big(|\tilde{R}_j|^2\bar{\varepsilon}\varepsilon_x + \tilde{R}_j\partial_x\tilde{R}_j(\bar{\varepsilon})^2 + \overline{\tilde{R}_j}\partial_x\tilde{R}_j|\varepsilon|^2\Big)\Big)\chi_j(t)\,dx + O\big(e^{-\frac{1}{16}\omega_\star L}\big)\|\varepsilon\|_{H^1}^2 \\
\triangleq& \sum_{j=1}^{N} \mathcal{H}_j(t) + O\big(e^{-\frac{1}{16}\omega_\star L}\big)\|\varepsilon\|_{H^1}^2.
\end{aligned}$$

Now we need another cut-off function defined as in [37]. Let Λ be a smooth positive function satisfying

$$\Lambda = 1 \text{ on } [-1,1], \quad \Lambda \sim e^{-|x|} \text{ on } \mathbb{R}, \quad |\Lambda'| \lesssim \Lambda.$$

Moreover, we fix $B : 1 \ll B \ll L$, and denote $\Lambda_j(x) = \Lambda(\frac{x-\tilde{x}_j(t)}{B})$. Then for some small $c > 0$,

$$\chi_j \geq \Lambda_j - e^{-cL/B}, \quad \operatorname{supp}(\chi_j - \Lambda_j) \subset \{x : |x - \tilde{x}_j(t)| \geq B\}, \quad |\Lambda_j'| \lesssim \frac{1}{B}\Lambda_j.$$

Let $z_j = \varepsilon\sqrt{\Lambda_j}$. Then

$$|\partial_x\varepsilon|^2\,\Lambda_j = |\partial_x z_j|^2 + \frac{1}{4}|z_j|^2\left(\frac{\Lambda_j'}{\Lambda_j}\right)^2 - \mathcal{Re}\left(z_j\partial_x\bar{z}_j\frac{\Lambda_j'}{\Lambda_j}\right).$$

This implies

$$\begin{aligned}
\int_{\mathbb{R}} |\partial_x z_j|^2 dx - \frac{C}{B}\int_R (|\partial_x z_j|^2 + |z_j|^2)dx& \\
\leqslant \int_{\mathbb{R}} |\partial_x\varepsilon|^2\,\Lambda_j dx \leqslant& \\
\int_{\mathbb{R}} |\partial_x z_j|^2 dx + \frac{C}{B}\int_R (|\partial_x z_j|^2 + |z_j|^2)dx.& \quad (43)
\end{aligned}$$

Moreover, we have

$$\varepsilon\partial_x\bar{\varepsilon} = \frac{z_j\partial_x\bar{z}_j}{\Lambda_j} - |z_j|^2\frac{\Lambda_j'}{2\Lambda_j^2},$$

and therefore,

$$\mathcal{Im}(\varepsilon\partial_x\bar{\varepsilon})\Lambda_j = \mathcal{Im}(z\partial_x\bar{z}_j). \quad (44)$$

Now, using the localization of $\tilde{R}_j$, we rewrite $\mathcal{H}_j$ as

$$\begin{aligned}\mathcal{H}_j(t) = \int \Big[& |\varepsilon_x|^2 + \tilde{\omega}_j(t)|\varepsilon|^2 + \tilde{c}_j(t)\mathcal{Im}(\varepsilon\bar{\varepsilon}_x) \\ & + \mathcal{Im}\Big(|\tilde{R}_j|^2\bar{\varepsilon}\varepsilon_x + \tilde{R}_j\partial_x\tilde{R}_j(\bar{\varepsilon})^2 + \overline{\tilde{R}_j}\partial_x\tilde{R}_j|\varepsilon|^2\Big)\Big]\Lambda_j(t)\,dx \\ & + \int \Big(|\varepsilon_x|^2 + \tilde{\omega}_j(t)|\varepsilon|^2 + \tilde{c}_j(t)\mathcal{Im}\big(\varepsilon\bar{\varepsilon}_x\big)\Big)(\chi_j - \Lambda_j)\,dx \\ & + O(e^{-\frac{1}{16}\omega_\star B})\|\varepsilon\|_{H^1}^2. \end{aligned} \tag{45}$$

Using (43) and (44) we have

$$\begin{aligned} & \int \Big[|\varepsilon_x|^2 + \tilde{\omega}_j(t)|\varepsilon|^2 + \tilde{c}_j(t)\mathcal{Im}(\varepsilon\bar{\varepsilon}_x) \\ & \quad + \mathcal{Im}\Big(|\tilde{R}_j|^2\bar{\varepsilon}\varepsilon_x + \tilde{R}_j\partial_x\tilde{R}_j(\bar{\varepsilon})^2 + \overline{\tilde{R}_j}\partial_x\tilde{R}_j|\varepsilon|^2\Big)\Big]\Lambda_j(t)\,dx \\ \geq & \int \Big[|\partial_x z_j|^2 + \tilde{\omega}_j(t)|z_j|^2 + \tilde{c}_j(t)\mathcal{Im}\big(z_j\partial_x\bar{z}_j\big) \\ & + \mathcal{Im}\Big(|\tilde{R}_j|^2\overline{z_j}\partial_x z_j + \tilde{R}_j\partial_x\tilde{R}_j(\overline{z_j})^2 + \overline{\tilde{R}_j}\partial_x\tilde{R}_j|z_j|^2\Big)\Big]\,dx - \frac{C}{B}\int |z_j|^2\,dx \\ = & H_{\tilde{\omega}(t),\tilde{c}_j(t)}(z_j) - \frac{C}{B}\int |z_j|^2\,dx. \end{aligned} \tag{46}$$

From the the orthogonality conditions (19), we have for $j = 1, \ldots, N$,

$$\Big|\Big(z_j, i\tilde{R}_j\Big)_2\Big| = \Big|\Big(z_j, \partial_x\tilde{R}_j\Big)_2\Big| = \Big|\Big(z_j, M'(\tilde{R}_j) + \mu_j P'(\tilde{R}_j)\Big)_2\Big| \lesssim e^{-\frac{1}{16}\omega_\star B}\|z_j\|_{L^2},$$

after suitable perturbation (for example, let $\tilde{z}_j = z_j + a_1 i\tilde{R}_j + a_2\partial_x\tilde{R}_j + a_3(M'(\tilde{R}_j) + \mu_j P'(\tilde{R}_j))$ for $|a_j| \lesssim e^{-\frac{1}{16}\omega_\star B}\|z\|_{L^2}, j = 1, 2, 3$), we obtain from Proposition 2.1 and (43), that for some small constant $\kappa' > 0$,

$$H_{\tilde{\omega}(t),\tilde{c}_j(t)}(z_j) \geq \kappa' \int (|\partial_x z_j|^2 + |z_j|^2)\,dx \geq \left(\kappa' - \frac{C}{B}\right)\int (|\partial_x\varepsilon|^2 + |\varepsilon|^2)\Lambda_j\,dx.$$

Hence, from (46), we obtain that

$$\begin{aligned} \Big| \int \Big[& |\varepsilon_x|^2 + \tilde{\omega}_j(t)|\varepsilon|^2 + \tilde{c}_j(t)\mathcal{Im}(\varepsilon\bar{\varepsilon}_x) \\ & + \mathcal{Im}\Big(|\tilde{R}_j|^2\bar{\varepsilon}\varepsilon_x + \tilde{R}_j\partial_x\tilde{R}_j(\bar{\varepsilon})^2 + \overline{\tilde{R}_j}\partial_x\tilde{R}_j|\varepsilon|^2\Big)\Big]\Lambda_j(t)\,dx\Big| \\ & \geq \kappa'' \int (|\partial_x\varepsilon|^2 + |\varepsilon|^2)\Lambda_j\,dx, \end{aligned} \tag{47}$$

where $\kappa'' = \kappa' - \frac{2C}{B}$. Furthermore, since $\tilde{c}_j^2 < 4\tilde{\omega}_j$, we have

$$\tilde{\nu}(|\partial_x\varepsilon|^2 + |\varepsilon|^2) \geq |\partial_x\varepsilon|^2 + \tilde{\omega}_j(t)|\varepsilon|^2 + \tilde{c}_j(t)\mathcal{Im}\big(\bar{\varepsilon}\partial_x\varepsilon\big) \geq \nu(|\partial_x\varepsilon|^2 + |\varepsilon|^2),$$

for some small $\nu > 0$ and $\tilde{\nu} = 3(1 + \omega_j^2)$. Hence, using $\chi_j \geq \Lambda_j - e^{-cL/B}$, we find

$$\begin{aligned} & \int \Big[|\partial_x\varepsilon|^2 + \tilde{\omega}_j(t)|\varepsilon|^2 + \tilde{c}_j(t)\mathcal{Im}\big(\bar{\varepsilon}\partial_x\varepsilon\big)\Big](\chi_j - \Lambda_j)\,dx \\ & \qquad \geq \nu' \int \big(|\partial_x\varepsilon|^2 + |\varepsilon|^2\big)(\chi_j - \Lambda_j)\,dx, \end{aligned} \tag{48}$$

where $\nu' = \nu - \widetilde{\nu} e^{-cL/B}$. Choosing L large enough, we have $\nu' > 0$. Inserting (47) and (48) into (45), we obtain that

$$\mathcal{H}_j(t) \geq 2\kappa \int \left(|\partial_x \varepsilon|^2 + |\varepsilon|^2\right)\chi_j \, dx,$$

where $\kappa = \frac{1}{2}\min\{\nu', \kappa''\}$. Since $\sum_{j=1}^{N} \chi_j = 1$, this proves the lemma. □

7. Control of the modulation parameters

With Lemma 6.3 in hands, the only thing left to control are the modulation parameters $\tilde{\omega}_j$. We prove the following result.

Lemma 7.1. *For all $t \in [0, t^\star]$, we have*

$$\sum_{j=1}^{N} |\tilde{\omega}_j(t) - \tilde{\omega}_j(0)| \lesssim \sup_{s\in[0,t]} \|\varepsilon(s)\|_{H^1}^2 + e^{-\frac{1}{32}\omega_\star L}.$$

Getting control over the modulation parameters is not an easy task for Schrödinger like equations. Indeed, a useful tool for that aim are monotonicity properties of localized conservation laws. For the Korteweg-de Vries equation, the localized mass satisfies this monotonicity property and can be used directly to control the modulation parameters (see [36]). For (dNLS) and (NLS), the monotonicity is verified only for the momentum and nothing similar is available for the mass (or the energy). This is the reason why one has to use several cut-off functions, in order to transfer the information from the momentum to the Q_j-quantity defined below. This was one of the main ideas introduced in [37]. The argument here is however more involved due to our choice of orthogonality conditions.

We first make the following claim.

Claim 7.2. For all $t \in [0, t^\star]$, we have

$$\sum_{j=2}^{N} |\mathcal{I}_j(t) - \mathcal{I}_j(0)| \lesssim \frac{1}{L} \sup_{s\in[0,t]} \|\varepsilon(s)\|_{H^1}^2 + \|\varepsilon(0)\|_{H^1}^2 + \sum_{j=1}^{N} |\tilde{\omega}_j(t) - \tilde{\omega}_j(0)|^2 + e^{-\frac{1}{32}\omega_\star L}.$$

Proof. From (42) in the proof of Lemma 6.3, the decomposition of $\mathcal{I}$ from Lemma 5.2, and conservation of mass and momentum, we get

$$\|\varepsilon(t)\|_{H^1}^2 \lesssim \sum_{j=2}^{N} (\tilde{c}_j(0) - \tilde{c}_{j-1}(0))\,(\mathcal{I}_j(t) - \mathcal{I}_j(0)) + \sum_{j=1}^{N} |\tilde{\omega}_j(t) - \tilde{\omega}_j(0)|^2 + \|\varepsilon(0)\|_{H^1}^2 + e^{-\frac{1}{32}\omega_\star L}. \quad (49)$$

On one hand, for all $j = 1, \ldots, N$ such that $\mathcal{I}_j(t) - \mathcal{I}_j(0) \geqslant 0$, by Proposition 5.3 we have

$$|\mathcal{I}_j(t) - \mathcal{I}_j(0)| \lesssim \frac{1}{L} \sup_{s\in[0,t]} \|\varepsilon(s)\|_{L^2}^2 + e^{-\frac{1}{16}\omega_\star L}. \quad (50)$$

On the other hand, since by assumption $c_j > c_{j-1}$, we have for $\delta, 1/L$ small enough that $\tilde{c}_j(0) - \tilde{c}_{j-1}(0) > 0$ for all $j = 1, \ldots, N$. Thus for all $j = 1, \ldots, N$ such that

$\mathcal{I}_j(t) - \mathcal{I}_j(0) < 0$, (49)-(50) imply

$$|\mathcal{I}_j(t) - \mathcal{I}_j(0)| \lesssim \frac{1}{L} \sup_{s\in[0,t]} \|\varepsilon(s)\|_{H^1}^2 + \sum_{j=1}^{N} |\tilde{\omega}_j(t) - \tilde{\omega}_j(0)|^2 + \|\varepsilon(0)\|_{H^1}^2 + e^{-\frac{1}{32}\omega_\star L}.$$

Combining these two facts gives the desired conclusion. □

As announced, we introduce the conserved quantity Q_j combining the mass and momentum:

$$Q_j(u) = M(u) + \mu_j P(u).$$

Remark that, due to the choice of μ_j after (16), and using the explicit calculations (70) and (71), we have

$$\frac{d}{d\tilde{\omega}_j} Q_j(\phi_{\tilde{\omega}_j,\tilde{c}_j})\Big|_{\tilde{\omega}_j=\omega_j} = \left(\frac{c_j}{\omega_j} - 4\mu_j + c_j\mu_j^2\right) \frac{1}{\sqrt{4\omega_j - c_j^2}} < 0. \tag{51}$$

Moreover, due to the orthogonality condition (19), we have

$$Q_j(\tilde{R}_j(t) + \varepsilon) = Q_j(\tilde{R}_j(t)) + O(\|\varepsilon\|_{H^1}^2).$$

Using $(Q_j)_{j=1,\ldots,N}$, we will be able to control the parameter $(\omega_j)_{j=1,\ldots,N}$. To do this, we first need the following claim.

Claim 7.3. For any $j = 1, \ldots, N$ we have

$$\Big| Q_j\left(\tilde{R}_j(t)\right) - Q_j\left(\tilde{R}_j(0)\right) \Big| \lesssim \sup_{s\in[0,t]} \|\varepsilon(s)\|_{H^1}^2 + \sum_{k=1}^{N} |\tilde{\omega}_k(t) - \tilde{\omega}_k(0)|^2 + e^{-\frac{1}{32}\omega_\star L}.$$

Proof. Let $j = 1, \ldots, N$. We set the notations,

$$m(u) = \frac{1}{2}|u|^2; \quad p(u) = \frac{1}{2}\mathcal{Im}(u\bar{u}_x); \quad q_j(u) = m(u) + \mu_j p(u).$$

Then we can rewrite $\mathcal{I}_j$ as

$$\begin{aligned} \mathcal{I}_j(t) &= \int_{\mathbb{R}} \left(\frac{1}{2}\tilde{\sigma}_j m(u) + p(u)\right) \psi_j \, dx \\ &= \frac{1}{2}\tilde{\sigma}_j \left(2\tilde{\sigma}_j^{-1} - \mu_j\right) \int_{\mathbb{R}} \left(\left(2\tilde{\sigma}_j^{-1} - \mu_j\right)^{-1} q_j(u) + p(u)\right) \psi_j \, dx. \end{aligned} \tag{52}$$

We assume that $2\tilde{\sigma}_j^{-1} - \mu_j \neq 0$ (otherwise, the formulas (54)–(57) can be obtained more simply). We introduce the constants

$$\lambda_j = \left(\frac{1}{2}\tilde{\sigma}_j \left(2\tilde{\sigma}_j^{-1} - \mu_j\right)\right)^{-1}, \qquad a_j = \left(2\tilde{\sigma}_j^{-1} - \mu_j\right)^{-1}.$$

Then (52) becomes

$$\lambda_j \mathcal{I}_j(t) = \int_{\mathbb{R}} (a_j q_j(u) + p(u)) \psi_j \, dx. \tag{53}$$

Similarly, we rewrite $\mathcal{I}_{j,\tau_j}$ as

$$\lambda_{j,\tau_j} \mathcal{I}_{j,\tau_j}(t) = \int_{\mathbb{R}} \left(a_{j,\tau_j} q_j(u) + p(u)\right) \psi_{j,\tau_j} \, dx,$$

where we have set

$$\lambda_{j,\tau_j} = \left(\frac{1}{2}\tau_j\left(2\tau_j^{-1} - \mu_j\right)\right)^{-1}, \qquad a_{j,\tau_j} = \left(2\tau_j^{-1} - \mu_j\right)^{-1}.$$

In addition to (53), we also need another formula of $\mathcal{I}_j(t)$ based on q_{j-1} when $j \geqslant 2$.

$$\mathcal{I}_j(t) = \frac{1}{2}\tilde{\sigma}_j\left(2\tilde{\sigma}_j^{-1} - \mu_{j-1}\right)\int_{\mathbb{R}}\left(\left(2\tilde{\sigma}_j^{-1} - \mu_{j-1}\right)^{-1} q_{j-1}(u) + p(u)\right)\psi_j\,dx.$$

Here, we also assume that $2\tilde{\sigma}_j^{-1} - \mu_{j-1} \neq 0$. Let

$$\gamma_j = \left(\frac{1}{2}\tilde{\sigma}_j\left(2\tilde{\sigma}_j^{-1} - \mu_{j-1}\right)\right)^{-1}, \qquad b_j = \left(2\tilde{\sigma}_j^{-1} - \mu_{j-1}\right)^{-1},$$

then

$$\gamma_j\mathcal{I}_j(t) = \int_{\mathbb{R}}\left(b_j q_{j-1}(u) + p(u)\right)\psi_j\,dx.$$

Similarly, we also rewrite $\mathcal{I}_{j,\tau_j}$ as

$$\gamma_{j,\tau_j}\mathcal{I}_{j,\tau_j}(t) = \int_{\mathbb{R}}\left(b_{j,\tau_j} q_{j-1}(u) + p(u)\right)\psi_{j,\tau_j}\,dx,$$

where we have set

$$\gamma_{j,\tau_j} = \left(\frac{1}{2}\tau_j\left(2\tau_j^{-1} - \mu_{j-1}\right)\right)^{-1}, \qquad b_{j,\tau_j} = \left(2\tau_j^{-1} - \mu_{j-1}\right)^{-1}.$$

Take a small constant $\epsilon_0 > 0$, and set the constants $\tau_j^1, \dots, \tau_j^4$ such that

$$a_{j,\tau_j^1} = a_j + \epsilon_0, \qquad a_{j,\tau_j^2} = a_j - \epsilon_0,$$
$$b_{j+1,\tau_j^3} = b_{j+1} + \epsilon_0, \qquad b_{j+1,\tau_j^4} = b_{j+1} - \epsilon_0.$$

Then for $j = 1, \dots, N$ we have the following identity

$$\lambda_{j,\tau_j^1}\mathcal{I}_{j,\tau_j^1}(t) - \lambda_j\mathcal{I}_j(t) = \epsilon_0\int_{\mathbb{R}} q_j(u)\psi_j\,dx + \int_{\mathbb{R}}(a_{j,\tau_j^1}q_j(u) + p(u))(\psi_{j,\tau_j} - \psi_j)dx.$$

The function $(\psi_j - \psi_{j,\tau_j})$ is zero for $x < \tilde{x}_{j-1} + \frac{(c_{j-1}+\min(\tau_j,\tilde{\sigma}_j))}{2}t$ and for $x > \tilde{x}_j - \frac{(\max(\tau_j,\tilde{\sigma}_j)+c_j)}{2}t$. Hence, due to the exponential localization of the solitons around each $\tilde{x}_k$ and using the lower bound $L/2$ on the distance between $\tilde{x}_j - \tilde{x}_k$ given by (22) we have

$$\left|\int_{\mathbb{R}}\left(a_{j,\tau_j^1}q_j\left(\sum_{k=1}^N \tilde{R}_k\right) + p\left(\sum_{k=1}^N \tilde{R}_k\right)\right)(\psi_{j,\tau_j} - \psi_j)dx\right| \lesssim e^{-\frac{1}{32}\omega_\star L}.$$

Similar estimates can be obtained replacing τ_j^1 by τ_j^l, $l = 2, 3, 4$ and j by $j+1$. As a consequence, for $j = 1, \dots, N$ we have the following identities,

$$\lambda_{j,\tau_j^1}\mathcal{I}_{j,\tau_j^1}(t) - \lambda_j\mathcal{I}_j(t) = \epsilon_0\int_{\mathbb{R}} q_j(u)\psi_j\,dx + O(e^{-\frac{1}{32}\omega_\star L}), \tag{54}$$

$$\lambda_{j,\tau_j^2}\mathcal{I}_{j,\tau_j^2}(t) - \lambda_j\mathcal{I}_j(t) = -\epsilon_0\int_{\mathbb{R}} q_j(u)\psi_j\,dx + O(e^{-\frac{1}{32}\omega_\star L}), \tag{55}$$

and for $j = 1, \dots, N-1$ we have

$$\gamma_{j+1,\tau_j^3}\mathcal{I}_{j+1,\tau_j^3}(t) - \gamma_{j+1}\mathcal{I}_{j+1}(t) = \epsilon_0 \int_{\mathbb{R}} q_j(u)\psi_{j+1}\,dx + O(e^{-\frac{1}{32}\omega_\star L}), \tag{56}$$

$$\gamma_{j+1,\tau_j^4}\mathcal{I}_{j+1,\tau_j^4}(t) - \gamma_{j+1}\mathcal{I}_{j+1}(t) = -\epsilon_0 \int_{\mathbb{R}} q_j(u)\psi_{j+1}\,dx + O(e^{-\frac{1}{32}\omega_\star L}). \tag{57}$$

We assume that λ_j, γ_{j+1} are both positive, the other cases being treated similarly. Choosing ϵ_0 small enough, we can assume $\lambda_{j,\tau_j^1}, \lambda_{j,\tau_j^2}, \gamma_{j+1,\tau_j^3}, \gamma_{j+1,\tau_j^4}$ are also positive. Then from (54)–(57) for $j = 1, \dots, N-1$, we have

$$\begin{aligned}
&\epsilon_0 \int_{\mathbb{R}} q_j(u)\chi_j(t,x)\,dx \\
&\quad= \Big(\lambda_{j,\tau_j^1}\mathcal{I}_{j,\tau_j^1}(t) - \lambda_j\mathcal{I}_j(t)\Big) + \Big(\gamma_{j+1,\tau_j^4}\mathcal{I}_{j+1,\tau_j^4}(t) - \gamma_{j+1}\mathcal{I}_{j+1}(t)\Big) + O(e^{-\frac{1}{32}\omega_\star L}),\\
&-\epsilon_0 \int_{\mathbb{R}} q_j(u)\chi_j(t,x)\,dx \\
&\quad= \Big(\lambda_{j,\tau_j^2}\mathcal{I}_{j,\tau_j^2}(t) - \lambda_j\mathcal{I}_j(t)\Big) + \Big(\gamma_{j+1,\tau_j^3}\mathcal{I}_{j+1,\tau_j^3}(t) - \gamma_{j+1}\mathcal{I}_{j+1}(t)\Big) + O(e^{-\frac{1}{32}\omega_\star L}).
\end{aligned}$$

Moreover, since $u = \sum_{k=1}^N \tilde{R}_k(t) + \varepsilon$, and the support of χ_j is far away from the center of the soliton $\tilde{R}_k(t)$ when $k \neq j$, we have

$$\int_{\mathbb{R}} q_j(u)\chi_j\,dx = Q_j(\tilde{R}_j(t)) + O(\|\varepsilon\|_{H^1}^2) + O(e^{-\frac{1}{32}\omega_\star L}),$$

where we have used the orthogonality conditions (19) to cancel the first order term. Therefore,

$$\begin{aligned}
\epsilon_0 \left(Q_j\left(\tilde{R}_j(t)\right) - Q_j\left(\tilde{R}_j(0)\right)\right) &= \epsilon_0 \int_{\mathbb{R}} q_j(u(t))\chi_j(t)\,dx - \epsilon_0 \int_{\mathbb{R}} q_j(u(0))\chi_j(0)\,dx \\
&\qquad + O(\|\varepsilon(t)\|_{H^1}^2 + \|\varepsilon(0)\|_{H^1}^2 + e^{-\frac{1}{32}\omega_\star L}) \\
&= \Big(\lambda_{j,\tau_j^1}\mathcal{I}_{j,\tau_j^1}(t) - \lambda_{j,\tau_j^1}\mathcal{I}_{j,\tau_j^1}(0)\Big) + \Big(\gamma_{j+1,\tau_j^4}\mathcal{I}_{j+1,\tau_j^4}(t) - \gamma_{j+1,\tau_j^4}\mathcal{I}_{j+1,\tau_j^4}(0)\Big) \\
&\qquad - (\lambda_j\mathcal{I}_j(t) - \lambda_j\mathcal{I}_j(0)) - (\gamma_{j+1}\mathcal{I}_{j+1}(t) - \gamma_{j+1}\mathcal{I}_{j+1}(0)) \\
&\qquad + O\Big(\sup_{s\in[0,t]}\|\varepsilon(s)\|_{H^1}^2 + e^{-\frac{1}{32}\omega_\star L}\Big).
\end{aligned}$$

Now we use Proposition 5.8 to control the first and second terms, and we use Claim 7.2 to control the third and fourth terms. For $j = 2, \dots, N-1$, we obtain

$$Q_j\left(\tilde{R}_j(t)\right) - Q_j\left(\tilde{R}_j(0)\right) \lesssim \sup_{s\in[0,t]}\|\varepsilon(s)\|_{H^1}^2 + \sum_{j=1}^N |\tilde{\omega}_j(t) - \tilde{\omega}_j(0)|^2 + e^{-\frac{1}{32}\omega_\star L}. \tag{58}$$

Similarly,

$$\begin{aligned}-\epsilon_0\left(Q_j\left(\tilde R_j(t)\right)-Q_j\left(\tilde R_j(0)\right)\right)&=-\epsilon_0\int_{\mathbb R}q_j(u(t))\chi_j(t)\,dx+\epsilon_0\int_{\mathbb R}q_j(u(0))\chi_j(0)\,dx\\&\quad+O(\|\varepsilon(t)\|_{H^1}^2+\|\varepsilon(0)\|_{H^1}^2+e^{-\frac{1}{32}\omega_\star L})\\&=\left(\lambda_{j,\tau_j^2}\mathcal I_{j,\tau_j^2}(t)-\lambda_{j,\tau_j^2}\mathcal I_{j,\tau_j^2}(0)\right)+\left(\gamma_{j+1,\tau_j^3}\mathcal I_{j+1,\tau_j^3}(t)-\gamma_{j+1,\tau_j^3}\mathcal I_{j+1,\tau_j^3}(0)\right)\\&\quad-(\lambda_j\mathcal I_j(t)-\lambda_j\mathcal I_j(0))-(\gamma_{j+1}\mathcal I_{j+1}(t)-\gamma_{j+1}\mathcal I_{j+1}(0))\\&\quad+O(\sup_{s\in[0,t]}\|\varepsilon(s)\|_{H^1}^2+e^{-\frac{1}{32}\omega_\star L}).\end{aligned}$$

Hence, arguing as before, for $j=2,\dots,N-1$, we obtain

$$-\left(Q_j\left(\tilde R_j(t)\right)-Q_j\left(\tilde R_j(0)\right)\right)\lesssim\sup_{s\in[0,t]}\|\varepsilon(s)\|_{H^1}^2+\sum_{j=1}^N|\tilde\omega_j(t)-\tilde\omega_j(0)|^2+e^{-\frac{1}{32}\omega_\star L}.\tag{59}$$

Now combining (58) and (59), we get for $j=2,\dots,N-1$,

$$\left|Q_j\left(\tilde R_j(t)\right)-Q_j\left(\tilde R_j(0)\right)\right|\lesssim\sup_{s\in[0,t]}\|\varepsilon(s)\|_{H^1}^2+\sum_{j=1}^N|\tilde\omega_j(t)-\tilde\omega_j(0)|^2+e^{-\frac{1}{32}\omega_\star L}.\tag{60}$$

Now we remain to treat the case $j=1$ and $j=N$. For $j=N$, since

$$\int_{\mathbb R}q_N(u)\psi_N\,dx=Q_N(\tilde R_N(t))+O\Big(\sup_{s\in[0,t]}\|\varepsilon(s)\|_{H^1}^2+e^{-\frac{1}{32}\omega_\star L}\Big),$$

from (54), we have

$$\begin{aligned}&\epsilon_0\left(Q_N\left(\tilde R_N(t)\right)-Q_N\left(\tilde R_N(0)\right)\right)\\&=\epsilon_0\int_{\mathbb R}q_N(u(t))\psi_N(t)\,dx-\epsilon_0\int_{\mathbb R}q_N(u(0))\psi_N(0)\,dx+O\Big(\sup_{s\in[0,t]}\|\varepsilon(s)\|_{H^1}^2+e^{-\frac{1}{32}\omega_\star L}\Big)\\&=\left(\lambda_{N,\tau_N^1}\mathcal I_{N,\tau_N^1}(t)-\lambda_{N,\tau_N^1}\mathcal I_{N,\tau_N^1}(0)\right)-(\lambda_N\mathcal I_N(t)-\lambda_N\mathcal I_N(0))\\&\quad+O\Big(\sup_{s\in[0,t]}\|\varepsilon(s)\|_{H^1}^2+e^{-\frac{1}{32}\omega_\star L}\Big).\end{aligned}$$

Again, we use Proposition 5.8 to control the first term, and use Claim 7.2 to control the second term, we obtain

$$Q_N\left(\tilde R_N(t)\right)-Q_N\left(\tilde R_N(t)\right)\lesssim\sup_{s\in[0,t]}\|\varepsilon(s)\|_{H^1}^2+\sum_{j=1}^N|\tilde\omega_j(t)-\tilde\omega_j(t)|^2+e^{-\frac{1}{32}\omega_\star L}.$$

Using (55) instead, we obtain from similar arguments that

$$-Q_N\left(\tilde R_N(0)\right)-Q_N\left(\tilde R_N(t)\right)\lesssim\sup_{s\in[0,t]}\|\varepsilon(s)\|_{H^1}^2+\sum_{j=1}^N|\tilde\omega_j(t)-\tilde\omega_j(0)|^2+e^{-\frac{1}{32}\omega_\star L}.$$

Therefore, we get (60) when $j=N$.

At last, we consider the case $j = 1$. We use (56) and (57) to get

$$-\epsilon_0 \int_{\mathbb{R}} q_1(u)\psi_1\, dx = -\epsilon_0 Q_1(u) + \left(\gamma_{2,\tau_1^3}\mathcal{I}_{2,\tau_1^3}(t) - \gamma_2\mathcal{I}_2(t)\right),$$
$$\epsilon_0 \int_{\mathbb{R}} q_1(u)\psi_1\, dx = \epsilon_0 Q_1(u) + \left(\gamma_{2,\tau_1^4}\mathcal{I}_{2,\tau_1^4}(t) - \gamma_2\mathcal{I}_2(t)\right).$$

Then by mass and momentum conservation laws, and similar arguments as before, we also obtain (60) when $j = 1$. □

With these preliminaries out of the way, let us prove Lemma 7.1.

Proof of Lemma 7.1. From (51) and (24), we know that for all $\tilde{\omega}$ and $\tilde{c}$ with the relationship (20), we have

$$\frac{d}{d\tilde{\omega}_j} Q_j(\phi_{\tilde{\omega}_j,\tilde{c}_j}) \neq 0.$$

Consequently, for any $j = 1, \ldots, N$ we have

$$|\tilde{\omega}_j(t) - \tilde{\omega}_j(0)| \lesssim \left| Q_j\left(\tilde{R}_j(t)\right) - Q_j\left(\tilde{R}_j(0)\right) \right|$$

Hence, from Claim 7.3, for $j = 1, \ldots, N$, we obtain

$$|\tilde{\omega}_j(t) - \tilde{\omega}_j(0)| \lesssim \sup_{s\in[0,t]} \|\varepsilon(s)\|_{H^1}^2 + \sum_{k=1}^N |\tilde{\omega}_k(t) - \tilde{\omega}_k(0)|^2 + e^{-\frac{1}{32}\omega_\star L}.$$

This allows us to infer that for $j = 1, \ldots, N$ we have

$$|\tilde{\omega}_j(t) - \tilde{\omega}_j(0)| \lesssim \sup_{s\in[0,t]} \|\varepsilon(s)\|_{H^1}^2 + e^{-\frac{1}{32}\omega_\star L},$$

and this concludes the proof of Lemma 7.1. □

8. Proof of the Bootstrap Result

Proof of Proposition 3.1. With Lemmas 6.3 and 7.1 in hands, we can now conclude the proof of Proposition 3.1. For δ and $1/L$ small enough, we have

$$\|\varepsilon(t)\|_{H^1}^2 \leqslant \frac{1}{2} \sup_{s\in[0,t]} \|\varepsilon(s)\|_{H^1}^2 + C\|\varepsilon(0)\|_{H^1}^2 + Ce^{-\frac{1}{32}\omega_\star L}.$$

Therefore, for all $t \in [0, t^\star]$, we have

$$\frac{1}{2} \sup_{s\in[0,t]} \|\varepsilon(s)\|_{H^1}^2 \leqslant C\|\varepsilon(0)\|_{H^1}^2 + Ce^{-\frac{1}{32}\omega_\star L}. \tag{61}$$

Plugging that back in the control on the modulation parameters Lemma 7.1 we obtain

$$\sum_{j=1}^N |\tilde{\omega}_j(t) - \tilde{\omega}_j(0)| \leqslant C\|\varepsilon(0)\|_{H^1}^2 + Ce^{-\frac{1}{32}\omega_\star L}.$$

From the modulation result Proposition 4.1 and the bootstrap assumption we also have at time $t = 0$ the following estimate,

$$\sum_{j=1}^N |\tilde{\omega}_j(0) - \omega_j| \leqslant \tilde{C}\delta.$$

Recall also that

$$\|\varepsilon(0)\|_{H^1} \leqslant \delta. \tag{62}$$

We combine now (61)-(62) to conclude the proof:

$$\begin{aligned}
&\left\|u-\sum_{j=1}^{N}e^{\tilde\theta_j}\phi_j(\cdot-\tilde x_j)\right\|_{H^1}\\
&\qquad\leqslant\left\|u-\sum_{j=1}^{N}\tilde R_j\right\|_{H^1}+\left\|\sum_{j=1}^{N}\tilde R_j-\sum_{j=1}^{N}e^{\tilde\theta_j}\phi_j(\cdot-\tilde x_j)\right\|_{H^1}\\
&\qquad\leqslant\|\varepsilon\|_{H^1}+C\sum_{j=1}^{N}|\tilde\omega_j(t)-\omega_j|\\
&\leqslant\|\varepsilon\|_{H^1}+C\sum_{j=1}^{N}|\tilde\omega_j(t)-\tilde\omega_j(0)|+C\sum_{j=1}^{N}|\tilde\omega_j(0)-\omega_j|\leqslant C_0\left(\delta+e^{-\frac{1}{32}\omega_\star L}\right).
\end{aligned}$$

Note that this last constant C_0 is *independent* of A_0. Hence we may chose $A_0=2C_0$ and this concludes the proof. □

Appendix A. Some Explicit Formulas

In this section, we give explicit formulas for quantities evaluated on $\phi_{\omega,c}$. Since they are obtained by elementary calculations using the explicit formula (5)-(6) for $\phi_{\omega,c}$, we omit the details. We start with remarkable L^p-norms.

$$\|\phi_{\omega,c}\|_{L^2}^2=8\arctan\sqrt{\frac{2\sqrt{\omega}+c}{2\sqrt{\omega}-c}},\tag{63}$$

$$\|\phi_{\omega,c}\|_{L^4}^4=16c\arctan\sqrt{\frac{2\sqrt{\omega}+c}{2\sqrt{\omega}-c}}+8\sqrt{4\omega-c^2},\tag{64}$$

$$\|\phi_{\omega,c}\|_{L^6}^6=32(c^2+2\omega)\arctan\sqrt{\frac{2\sqrt{\omega}+c}{2\sqrt{\omega}-c}}+24c\sqrt{4\omega-c^2}.\tag{65}$$

We also have

$$\|\partial_x\phi_{\omega,c}\|_{L^2}^2=8\omega\arctan\sqrt{\frac{2\sqrt{\omega}+c}{2\sqrt{\omega}-c}}.\tag{66}$$

The mass, momentum and energy are given by

$$M(\phi_{\omega,c})=4\arctan\sqrt{\frac{2\sqrt{\omega}+c}{2\sqrt{\omega}-c}},\tag{67}$$

$$P(\phi_{\omega,c})=\sqrt{4\omega-c^2},\tag{68}$$

$$E(\phi_{\omega,c})=-\frac{c}{2}\sqrt{4\omega-c^2}.\tag{69}$$

Moreover, we have

$$\partial_\omega M(\phi_{\omega,c}) = -\frac{c}{\omega\sqrt{4\omega - c^2}}, \tag{70}$$

$$\partial_c M(\phi_{\omega,c}) = \partial_\omega P(\phi_{\omega,c}) = \frac{2}{\sqrt{4\omega - c^2}}, \tag{71}$$

$$\partial_c P(\phi_{\omega,c}) = -\frac{c}{\sqrt{4\omega - c^2}}. \tag{72}$$

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