Numerical Algorithms
https://doi.org/10.1007/s11075-023-01643-5

ORIGINAL PAPER

Local and parallel multigrid method for semilinear Neumann problem with nonlinear boundary condition

Fei Xu[1] · Bingyi Wang[1] · Manting Xie[2]

Received: 19 April 2022 / Accepted: 31 July 2023

Abstract

A novel local and parallel multigrid method is proposed in this study for solving the semilinear Neumann problem with nonlinear boundary condition. Instead of solving the semilinear Neumann problem directly in the fine finite element space, we transform it into a linear boundary value problem defined in each level of a multigrid sequence and a small-scale semilinear Neumann problem defined in a low-dimensional correction subspace. Furthermore, the linear boundary value problem can be efficiently solved using local and parallel methods. The proposed process derives an optimal error estimate with linear computational complexity. Additionally, compared with existing multigrid methods for semilinear Neumann problems that require bounded second order derivatives of nonlinear terms, ours only needs bounded first order derivatives. A rigorous theoretical analysis is proposed in this paper, which differs from the maturely developed theories for equations with Dirichlet boundary conditions.

Keywords Semilinear neumann problem · Nonlinear boundary condition · Local and parallel · Multigrid method

Mathematics Subject Classification (2010) 65F15 · 65H17 · 65N25 · 65N30 · 65Y05

This work was supported by the Beijing Municipal Natural Science Foundation (Grant No. 1232001) General projects of science and technology plan of Beijing Municipal Education Commission (Grant No. KM202110005011) and the National Natural Science Foundation of China (Grant Nos. 11801021).

✉ Manting Xie
mtxie@tju.edu.cn

Fei Xu
xufei@lsec.cc.ac.cn

Bingyi Wang
boycecore@outlook.com

1 Beijing Institute for Scientific and Engineering Computing, Faculty of Science, Beijing University of Technology, Beijing 100124, People's Republic of China

2 Center for Applied Mathematics, Tianjin University, Tianjin 300072, China

Published online: 18 August 2023

1 Introduction

In [32], Xu and Zhou first proposed the local and parallel method to solve linear elliptic boundary value problems. Through this method, large-scale equations can be decomposed into some small-scale subproblems. Thus, the simulation efficiency, especially for large-scale partial differential equations in practical applications, can greatly benefit from such a technique. As we know, the low-frequency components describe the global appearances of a solution, whereas the high-frequency components describe the local appearances. So generally, local and parallel method first uses a coarse mesh to approximate the low-frequency components and then use a fine mesh to correct the resulting residual via some local and parallel procedures. Meanwhile, this method is naturally suitable for parallel computing based on the domain decomposition. Owing to these advantages, local and parallel method has been widely used to solve various mathematical models, for instance [2, 3, 9–12, 14, 15, 17, 20–22, 24–26, 29, 31–34, 37–40]. However, to date, local and parallel methods have been used to solve various equations with Dirichlet boundary condition. Meanwhile, the local and parallel two-grid method has a strict restriction on the mesh size ratio between the coarse mesh and the fine mesh.

To overcome these problems, we propose a local and parallel method to solve the semilinear Neumann problem with nonlinear boundary condition based on multigrid discretization. First, we will provide rigorous error estimates for the approximate solution of the semilinear Neumann problem based on some weak assumptions for the nonlinear terms; then, we propose a new type of local and parallel method based on the local and parallel technique alongside the multilevel correction technique [7, 16, 18, 28]. Instead of solving the semilinear Neumann problem directly in the fine finite element space, we transform the problem into a linear boundary value problem defined in each level of a multigrid sequence and a small-scale semilinear Neumann problem defined in a low-dimensional correction subspace. The linear boundary value problem can then be solved efficiently by the local and parallel method. Furthermore, the dimension of the correction subspace is small and fixed, which is independent of the fine space. This is the main difference between our algorithm and the local and parallel two-grid methods. Thus, the computing time for the small-scale semilinear Neumann problem can be negligible with mesh refinement. We can derive the optimal error estimates with linear computational complexity for the proposed local and parallel multigrid method. Additionally, compared with the existing multigrid methods for semilinear Neumann problems, which require bounded second order derivatives of nonlinear terms [10, 14, 23, 33], our method only needs bounded first order derivatives. Rigorous theoretical analysis is also proposed, which differs from the maturely developed theories for equations having Dirichlet boundary conditions.

The remainder of this paper is organized as follows. In Section 2, some basic finite element error estimates for the linear elliptic boundary value problem are presented. In Section 3, we introduce the semilinear Neumann problem with a nonlinear boundary condition to be solved and provide rigourous finite element error estimates. The novel algorithm for the semilinear Neumann problem with a nonlinear boundary condition and the theoretical analysis are discussed in Section 4. In Section 5, we present some numerical experiments to support our theory and illustrate the efficiency of our algorithm. Finally, some concluding remarks are given in the last section.

2 Finite element method for linear elliptic boundary value problem

In this paper, Ω denotes a bounded domain in $\mathbb{R}^d$ $(d \geq 1)$. We use the standard symbols $H^s(\Omega)$ and $H^s(\partial\Omega)$ to denote the Sobolev spaces defined in Ω and the boundary $\partial\Omega$. The corresponding norms are denoted by $\|\cdot\|_{s,\Omega}$ and $\|\cdot\|_{s,\partial\Omega}$, respectively. In the rest of this paper, the letter C denotes a mesh-independent constant. For the three nested domains $G \subset D \subset \Omega$, we use $G \subset\subset D$ to denote $dist(\partial D \backslash \partial\Omega, \partial G \backslash \partial\Omega) > 0$, as depicted in Fig. 1 (see also [32]).

In this section, for the following analysis, we study the linear elliptic boundary value problem:

$$\begin{cases} Lu := -\nabla \cdot (\mathcal{A}\nabla u) + \phi u = b, & \text{in } \Omega, \\ \qquad\qquad\quad (\mathcal{A}\nabla u) \cdot n = g, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where the coefficient $\mathcal{A}$ is a symmetric positive definite matrix with suitable regularity and ϕ is a nonnegative function bounded from below and above by positive constants.

The variational form of (1) can be described as follows: Find $u \in H^1(\Omega)$ such that

$$a(u, v) = (b, v) + \langle g, v \rangle, \quad \forall v \in H^1(\Omega),$$

where

$$a(u, v) = \int_\Omega (\mathcal{A}\nabla u \cdot \nabla v + \phi uv) d\Omega, \quad (b, v) = \int_\Omega bv d\Omega, \quad \langle g, v \rangle = \int_{\partial\Omega} gv ds.$$

Next, we introduce the finite element method to solve the linear elliptic boundary value problem (1). Let us decompose the computing domain Ω into shape-regular elements, denoted by the symbol $\mathcal{T}_h(\Omega)$. For a mesh element $K \in \mathcal{T}_h(\Omega)$, the associated diameter is denoted as h_K. Besides, given any point $x \in \Omega$, we denote $h(x) := h_K$ where $x \in K$, $h(x) = \max_{\{K:x\in K\}}\{h_K\}$ when x is defined on skeleton, and we also denote $h_\Omega := \max_{x\in\Omega} h(x)$. Based on $\mathcal{T}_h(\Omega)$, the finite element space $V^h(\Omega) \subset H^1(\Omega)$ is defined as

$$V^h(\Omega) = \left\{ v \in C(\bar{\Omega}) : v|_K \in \mathcal{P}_k, \forall K \in \mathcal{T}_h(\Omega) \right\}. \tag{2}$$

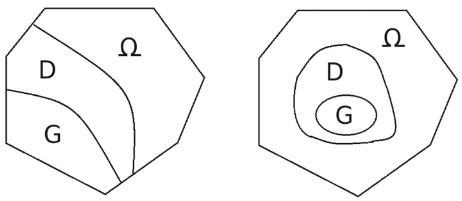

Fig. 1 $G \subset\subset D \subset \Omega$

For $G \subset \Omega$, $V^h(G)$ and $\mathcal{T}_h(G)$ denote the restriction of $V^h(\Omega)$ and $\mathcal{T}_h(\Omega)$ to G. For any $G \subset \Omega$ mentioned in this paper, we assume that it aligns with the triangulation $\mathcal{T}_h(\Omega)$. Next, we also need to define the following two spaces:

$$H_\Gamma^1(G) = \left\{ v \in H^1(G) : \text{supp } v \subset\subset G \right\} \tag{3}$$

and

$$V_\Gamma^h(G) = \left\{ v \in V^h(\Omega) : \text{supp } v \subset\subset G \right\}. \tag{4}$$

Thus, for any $v \in H_\Gamma^1(G)$, v equals zero on the boundary $\partial G \backslash \partial \Omega$, and v may not equal zero on the boundary $\partial G \cap \partial \Omega$.

From [6, 8, 32, 34], the fractional norm property can be derived:

Lemma 1 *For any subset $G \subset \Omega$, the following fractional norm property holds true*

$$\inf_{v \in V_\Gamma^h(G)} \|w - v\|_{1,G} \lesssim \|w\|_{1/2,\partial G \backslash \partial \Omega}, \quad \forall w \in V^h(G). \tag{5}$$

Proof Define $V_0^h(G) = V^h(G) \cap H_0^1(G)$. For any $w \in V^h(G)$, let v be the solution of the following equation: Find $v \in V_\Gamma^h(G)$ such that $v|_{\partial G \cap \partial \Omega} = w$ and

$$(\nabla v, \nabla \phi) = (\nabla w, \nabla \phi), \quad \forall \phi \in V_0^h(G).$$

Then, $v - w$ is discrete harmonic. The desired result then follows from the following estimate for discrete harmonic functions [35]:

$$\|w - v\|_{1,G} \lesssim \|w - v\|_{1/2,\partial G} = \|w\|_{1/2,\partial G \backslash \partial \Omega}.$$

□

Let us define a projection operator $P_h : H^1(\Omega) \to V^h(\Omega)$ in the following way:

$$a(u - P_h u, v_h) = 0, \quad \forall v_h \in V^h(\Omega). \tag{6}$$

Then, we can derive

$$\|u - P_h u\|_{1,\Omega} \lesssim \inf_{v_h \in V^h(\Omega)} \|u - v_h\|_{1,\Omega},$$

and the following estimates also hold true:

Lemma 2 *The following estimates for the projection operator hold true*

$$\begin{aligned}\|(I-P_h)Tg\|_{1,\Omega} &\lesssim \rho_\Omega(h)\|g\|_{0,\partial\Omega}, \quad \forall g \in L^2(\partial\Omega),\\ \|(I-P_h)T'b\|_{1,\Omega} &\lesssim r_\Omega(h)\|b\|_{0,\Omega}, \quad \forall b \in L^2(\Omega),\\ \|u-P_hu\|_{0,\partial\Omega} &\lesssim \rho_\Omega(h)\|u-P_hu\|_{1,\Omega}, \quad \forall u \in H^1(\Omega),\\ \|u-P_hu\|_{0,\Omega} &\lesssim r_\Omega(h)\|u-P_hu\|_{1,\Omega}, \quad \forall u \in H^1(\Omega),\end{aligned}$$

where

$$\rho_\Omega(h) = \sup_{g\in L^2(\partial\Omega),\|g\|_{0,\partial\Omega}=1} \inf_{v_h\in V^h(\Omega)} \|Tg - v_h\|_{1,\Omega} \tag{7}$$

with the operator $T : L^2(\partial\Omega) \to H^1(\Omega)$ *being defined by*

$$a(Tg, v) = \langle g, v\rangle, \quad \forall v \in H^1(\Omega), \tag{8}$$

and

$$r_\Omega(h) = \sup_{b\in L^2(\Omega),\|b\|_{0,\Omega}=1} \inf_{v_h\in V^h(\Omega)} \left\|T'b - v_h\right\|_{1,\Omega} \tag{9}$$

with the operator $T' : L^2(\Omega) \to H^1(\Omega)$ *being defined by*

$$a(T'b, v) = (b, v), \quad \forall v \in H^1(\Omega). \tag{10}$$

In this study, we assume that the finite element space satisfies the following conditions (see [32, 34]):
A.1. There exists $\gamma > 1$ such that

$$h_\Omega^\gamma \lesssim h(x), \quad \forall x \in \Omega. \tag{11}$$

A.2. For $G \subset \Omega$ and any $v \in V^h(G)$, the following inverse estimate holds true

$$\|v\|_{1,G} \lesssim \|h^{-1}v\|_{0,G}. \tag{12}$$

A.3. For $G \subset \Omega$ and $\omega \in C^\infty(\bar{\Omega})$ satisfying $(\mathrm{supp}\omega\backslash(\partial G \cap \partial\Omega)) \subset\subset G$, then for any $w \in V^h(G)$, there exists $v \in V_\Gamma^h(G)$ such that

$$\|h_G^{-1}(\omega w - v)\|_{1,G} \lesssim \|w\|_{1,G}. \tag{13}$$

Define

$$a_0(u, v) = \int_\Omega (\mathcal{A}\nabla u \cdot \nabla v)d\Omega. \tag{14}$$

Then, we can derive the following lemmas:

Lemma 3 ([32, 34]) *Let* $D \subset\subset \Omega_0 \subset \Omega$ *and* $\omega \in C^\infty(\bar{\Omega})$ *satisfy* $(\mathrm{supp}\omega \backslash (\partial\Omega_0 \cap \partial\Omega)) \subset\subset \Omega_0$. *We can derive the following inequality:*

$$a_0(\omega w, \omega w) \leq 2a(w, \omega^2 w) + C \|w\|^2_{0,\Omega_0}, \quad \forall w \in H^1(\Omega). \tag{15}$$

Lemma 4 *For* $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$ *and* $D \subset\subset \Omega_0 \subset \Omega$, *if* $w \in V^h(\Omega_0)$ *is the solution of the following equation*

$$a(w, v) = (f, v) + \langle g, v \rangle, \quad \forall v \in V^h_\Gamma(\Omega_0), \tag{16}$$

then there holds

$$\|w\|_{1,D} \lesssim \|w\|_{0,\Omega_0} + \|f\|_{0,\Omega_0} + \|g\|_{0,\partial\Omega\cap\partial\Omega_0}. \tag{17}$$

Proof Let $p \geq 2\gamma - 1$ be an integer and Ω_j be a series of nested domains satisfying

$$D \subset\subset \Omega_p \subset\subset \Omega_{p-1} \subset\subset \cdots \subset\subset \Omega_1 \subset\subset \Omega_0.$$

Let us choose a domain D_1 such that $D \subset\subset D_1 \subset\subset \Omega_p$, and choose $\omega \in C^\infty(\bar{\Omega})$ satisfying $\omega \in [0, 1]$, $\omega = 1$ on $\bar{D}_1$ and $(\mathrm{supp}\omega \setminus (\partial\Omega_p \cap \partial\Omega)) \subset\subset \Omega_p$. Based on A.3., there exists a function $v \in V^h_\Gamma(\Omega_p)$ such that

$$\|\omega^2 w - v\|_{1,\Omega_p} \lesssim h_{\Omega_0}\|w\|_{1,\Omega_p}. \tag{18}$$

Based on (18) and trace inequality, we can further derive

$$\begin{aligned} |(f, v)| &= |\int_\Omega f v d\Omega| \\ &\lesssim \|f\|_{0,\Omega_p}\|v\|_{0,\Omega_p} \\ &\lesssim \|f\|_{0,\Omega_p}\|v\|_{1,\Omega_p} \\ &\lesssim \|f\|_{0,\Omega_0}(h_{\Omega_0}\|w\|_{1,\Omega_p} + \|\omega^2 w\|_{1,\Omega}) \\ &\lesssim \|f\|_{0,\Omega_0}(h_{\Omega_0}\|w\|_{1,\Omega_p} + \|\omega w\|_{1,\Omega}) \end{aligned} \tag{19}$$

and

$$\begin{aligned} |\langle g, v \rangle| &= |\int_{\partial\Omega} g v ds| = |\int_{\partial\Omega_p\cap\partial\Omega} g v ds| \\ &\lesssim \|g\|_{0,\partial\Omega_p\cap\partial\Omega}\|v\|_{0,\partial\Omega_p\cap\partial\Omega} \\ &\lesssim \|g\|_{0,\partial\Omega_0\cap\partial\Omega}\|v\|_{1,\Omega_p} \\ &\lesssim \|g\|_{0,\partial\Omega_0\cap\partial\Omega}(h_{\Omega_0}\|w\|_{1,\Omega_p} + \|\omega w\|_{1,\Omega}). \end{aligned} \tag{20}$$

From (15), (16), (18), (19), and (20), there holds

$$\|\omega w\|^2_{1,\Omega} \lesssim a(w, \omega^2 w) + \|w\|^2_{0,\Omega_0}$$

$$
\begin{aligned}
&= a(w, \omega^2 w - v) + \|w\|_{0,\Omega_0}^2 + (f, v) + \langle g, v\rangle \\
&\le Ch_{\Omega_0}\|w\|_{1,\Omega_p}^2 + \|w\|_{0,\Omega_0}^2 + C(\|f\|_{0,\Omega_0} + \|g\|_{0,\partial\Omega_0\cap\partial\Omega})(h_{\Omega_0}\|w\|_{1,\Omega_p} + \|\omega w\|_{1,\Omega}) \\
&\le Ch_{\Omega_0}\|w\|_{1,\Omega_p}^2 + \|w\|_{0,\Omega_0}^2 + \tfrac{1}{2}C^2(\|f\|_{0,\Omega_0} + \|g\|_{0,\partial\Omega_0\cap\partial\Omega})^2 + \tfrac{1}{2}h_{\Omega_0}^2\|w\|_{1,\Omega_p}^2 \\
&\quad + \tfrac{1}{2}C^2(\|f\|_{0,\Omega_0} + \|g\|_{0,\partial\Omega_0\cap\partial\Omega})^2 + \tfrac{1}{2}\|\omega w\|_{1,\Omega}^2.
\end{aligned}
$$

Thus,

$$
\tfrac{1}{2}\|\omega w\|_{1,\Omega}^2 \le Ch_{\Omega_0}\|w\|_{1,\Omega_p}^2 + \|w\|_{0,\Omega_0}^2 + C^2(\|f\|_{0,\Omega_0} + \|g\|_{0,\partial\Omega_0\cap\partial\Omega})^2 + \tfrac{1}{2}h_{\Omega_0}^2\|w\|_{1,\Omega_p}^2,
$$

which implies

$$
\|w\|_{1,D} = \|\omega w\|_{1,D} \le \|\omega w\|_{1,\Omega} \lesssim h_{\Omega_0}^{\frac{1}{2}}\|w\|_{1,\Omega_p} + \|w\|_{0,\Omega_0} + \|f\|_{0,\Omega_0} + \|g\|_{0,\partial\Omega\cap\partial\Omega_0}. \tag{21}
$$

Similarly, the following estimates can be proved in the same way as (21):

$$
\|w\|_{1,\Omega_j} \lesssim h_{\Omega_0}^{\frac{1}{2}}\|w\|_{1,\Omega_{j-1}} + \|w\|_{0,\Omega_0} + \|f\|_{0,\Omega_0} + \|g\|_{0,\partial\Omega\cap\partial\Omega_0}, \quad j = 1, 2, \cdots, p. \tag{22}
$$

Using (21), (22), A.1, and A.2, we can obtain

$$
\begin{aligned}
\|w\|_{1,D} &\lesssim h_{\Omega_0}^{\frac{p+1}{2}}\|w\|_{1,\Omega_0} + \|w\|_{0,\Omega_0} + \|f\|_{0,\Omega_0} + \|g\|_{0,\partial\Omega\cap\partial\Omega_0} \\
&\lesssim h_{\Omega_0}^{\frac{p+1}{2}}\|h^{-1}w\|_{0,\Omega_0} + \|w\|_{0,\Omega_0} + \|f\|_{0,\Omega_0} + \|g\|_{0,\partial\Omega\cap\partial\Omega_0} \\
&\lesssim \|w\|_{0,\Omega_0} + \|f\|_{0,\Omega_0} + \|g\|_{0,\partial\Omega\cap\partial\Omega_0}.
\end{aligned}
$$

Then, we complete the proof. □

3 Finite element method for semilinear Neumann problem with nonlinear boundary condition

In this paper, we study the semilinear Neumann problem with nonlinear boundary condition:

$$
\begin{cases}
-\nabla \cdot (\mathcal{A}\nabla u) + \phi u + f(x, u) = b, & \text{in } \Omega, \\
(\mathcal{A}\nabla u)\cdot n + g(x, u) = 0, & \text{on } \partial\Omega,
\end{cases} \tag{23}
$$

where $f(x, u) \in C(\Omega \times \mathbb{R})$ denotes a nonlinear term with respect to u in the domain Ω, and $g(x, u) \in C(\partial\Omega \times \mathbb{R})$ denotes a nonlinear term with respect to u in the boundary $\partial\Omega$.

The variational form of (23) can be described as follows: Find $u \in H^1(\Omega)$ such that

$$
a(u, v) + (f(x, u), v) + \langle g(x, u), v\rangle = (b, v), \quad \forall v \in H^1(\Omega). \tag{24}
$$

The discrete form of (24) can be described as follows: Find $\bar{u}_h \in V^h(\Omega)$ such that

$$
a(\bar{u}_h, v_h) + (f(x, \bar{u}_h), v_h) + \langle g(x, \bar{u}_h), v_h\rangle = (b, v_h), \quad \forall v_h \in V^h(\Omega). \tag{25}
$$

To guarantee the well-posedness of the semilinear Neumann problems (24) and (25), we should make some assumptions on the nonlinear terms $f(x,u)$ and $g(x,u)$: There exist two mesh-independent constants C_f and C_g, such that

$$0 \le \frac{\partial f}{\partial v}(x,v) \le C_f \text{ and } 0 \le \frac{\partial g}{\partial v}(x,v) \le C_g, \quad \forall x \in \partial\Omega,\ \forall v \in \mathbb{R}. \tag{26}$$

Theorem 1 *If the condition (26) holds true, then (24) and (25) are uniquely solvable. Besides, there exist the following error estimates:*

$$\|u - \bar{u}_h\|_{1,\Omega} \lesssim \delta_h(u), \tag{27}$$

$$\|u - \bar{u}_h\|_{0,\Omega} \lesssim (r_\Omega(h) + \rho_\Omega(h))\|u - \bar{u}_h\|_{1,\Omega}, \tag{28}$$

$$\|u - \bar{u}_h\|_{0,\partial\Omega} \lesssim (r_\Omega(h) + \rho_\Omega(h))\|u - \bar{u}_h\|_{1,\Omega}, \tag{29}$$

where

$$\delta_h(u) = \inf_{v_h \in V^h(\Omega)} \|u - v_h\|_{1,\Omega}. \tag{30}$$

Proof Using Theorem 6.1 in [23], the semilinear Neumann problems (24) and (25) are uniquely solvable under the condition (26). Next, we prove the error estimates.

Using Lemma 2, (24), (25), and (26), there holds

$$\begin{aligned}
&\|\bar{u}_h - P_h u\|_{1,\Omega}^2 \\
&\lesssim a(\bar{u}_h - P_h u, \bar{u}_h - P_h u) \\
&\le a(\bar{u}_h - P_h u, \bar{u}_h - P_h u) + (f(x,\bar{u}_h) - f(x,P_h u), \bar{u}_h - P_h u) \\
&\quad + \langle g(x,\bar{u}_h) - g(x,P_h u), \bar{u}_h - P_h u\rangle \\
&= (b, \bar{u}_h - P_h u) - a(P_h u, \bar{u}_h - P_h u) - (f(x,P_h u), \bar{u}_h - P_h u) \\
&\quad - \langle g(x,P_h u), \bar{u}_h - P_h u\rangle \\
&= a(u - P_h u, \bar{u}_h - P_h u) + (f(x,u) - f(x,P_h u), \bar{u}_h - P_h u) \\
&\quad + \langle g(x,u) - g(x,P_h u), \bar{u}_h - P_h u\rangle \\
&= (f(x,u) - f(x,P_h u), \bar{u}_h - P_h u) + \langle g(x,u) - g(x,P_h u), \bar{u}_h - P_h u\rangle \\
&\le C_f\|u - P_h u\|_{0,\Omega}\|\bar{u}_h - P_h u\|_{0,\Omega} + C_g\|u - P_h u\|_{0,\partial\Omega}\|\bar{u}_h - P_h u\|_{0,\partial\Omega} \\
&\lesssim (\|u - P_h u\|_{0,\Omega} + \|u - P_h u\|_{0,\partial\Omega})\|\bar{u}_h - P_h u\|_{1,\Omega},
\end{aligned}$$

which yields

$$\begin{aligned}
\|\bar{u}_h - P_h u\|_{1,\Omega} &\lesssim \|u - P_h u\|_{0,\Omega} + \|u - P_h u\|_{0,\partial\Omega} \\
&\lesssim (r_\Omega(h) + \rho_\Omega(h))\|u - P_h u\|_{1,\Omega}.
\end{aligned} \tag{31}$$

Based on (31) and triangle inequality, we can obtain the following estimates:

$$\|u - \bar{u}_h\|_{1,\Omega} \le \|u - P_h u\|_{1,\Omega} + \|\bar{u}_h - P_h u\|_{1,\Omega}$$

$$\lesssim \delta_h(u) + (r_\Omega(h) + \rho_\Omega(h))\|u - P_h u\|_{1,\Omega}$$
$$\lesssim (1 + Cr_\Omega(h) + C\rho_\Omega(h))\delta_h(u), \tag{32}$$

which is just the first estimate (27).

Using Lemma 2, (31), and triangle inequality, we can obtain

$$\begin{aligned}\|u - \bar{u}_h\|_{0,\Omega} &\leq \|u - P_h u\|_{0,\Omega} + \|\bar{u}_h - P_h u\|_{0,\Omega}\\ &\leq \|u - P_h u\|_{0,\Omega} + C\|\bar{u}_h - P_h u\|_{1,\Omega}\\ &\lesssim (r_\Omega(h) + \rho_\Omega(h))\|u - P_h u\|_{1,\Omega}\\ &\lesssim (r_\Omega(h) + \rho_\Omega(h))\|u - \bar{u}_h\|_{1,\Omega},\end{aligned}$$

and

$$\begin{aligned}\|u - \bar{u}_h\|_{0,\partial\Omega} &\leq \|u - P_h u\|_{0,\partial\Omega} + \|\bar{u}_h - P_h u\|_{0,\partial\Omega}\\ &\leq \|u - P_h u\|_{0,\partial\Omega} + C\|\bar{u}_h - P_h u\|_{1,\Omega}\\ &\lesssim (r_\Omega(h) + \rho_\Omega(h))\|u - P_h u\|_{1,\Omega}\\ &\lesssim (r_\Omega(h) + \rho_\Omega(h))\|u - \bar{u}_h\|_{1,\Omega},\end{aligned}$$

which are the desired estimates (28) and (29). Then, the proof is completed. □

4 Local and parallel multigrid method for semilinear Neumann problem with nonlinear boundary condition

This section is devoted to introducing our novel local and parallel method based on multigrid discretization for solving the semilinear Neumann problem with nonlinear boundary condition. To design the algorithm, we need to construct a multilevel mesh sequence. First, let us generate a coarse mesh $\mathcal{T}_H$. Then, we construct an initial mesh $\mathcal{T}_{h_1}$ satisfying $\mathcal{T}_H \subset \mathcal{T}_{h_1}$. Next, each mesh $\mathcal{T}_{h_k}$ $(k \geq 2)$ is obtained from $\mathcal{T}_{h_{k-1}}$ through one-time uniform refinement. Finally, we can obtain the following mesh sequence:

$$\mathcal{T}_H(\Omega) \subset \mathcal{T}_{h_1}(\Omega) \subset \cdots \subset \mathcal{T}_{h_k}(\Omega) \subset \cdots \subset \mathcal{T}_{h_n}(\Omega). \tag{33}$$

Based on (33), we can construct a nested finite element space sequence such that

$$V^H(\Omega) \subset V^{h_1}(\Omega) \subset \cdots \subset V^{h_k}(\Omega) \subset \cdots \subset V^{h_n}(\Omega). \tag{34}$$

4.1 Local and parallel method for semilinear Neumann problem

In this subsection, we explain how to perform the novel local and parallel method in one level of the finite element space sequence, which is the basic component of the local and parallel multigrid method. Assume that we have obtained an approximate solution $u_{h_k} \in V^{h_k}(\Omega)$, then we design an algorithm to get a more accurate approximate solution $u_{h_{k+1}} \in V^{h_{k+1}}(\Omega)$.

Let us divide Ω into disjoint domains $D_1, \cdots, D_m$ satisfying $\bigcup_{j=1}^m \bar{D}_j = \bar{\Omega},\ D_i \cap D_j = \emptyset$, then let us enlarge and reduce D_j to obtain Ω_j and G_j, respectively, and all subdomains align with $\mathcal{T}_H(\Omega)$. Then, we can obtain some subdomains satisfying $G_j \subset\subset D_j \subset \Omega_j \subset \Omega$ for $j = 1, \cdots, m$. Finally, let us set $G_{m+1} = \Omega \setminus (\bigcup_{j=1}^m \bar{G}_j)$ (see Fig. 2).

In addition, we require that the decompositions satisfy

$$\sum_{j=1}^{m+1} \|v\|_{\ell,\Omega_j} \lesssim \|v\|_{\ell,\Omega} \text{ and } \sum_{j=1}^{m+1} \|v\|_{\ell,\partial\Omega_j\cap\partial\Omega} \lesssim \|v\|_{\ell,\partial\Omega}, \quad \forall v \in H^\ell(\Omega),\ \ell = 0, 1. \tag{35}$$

Then, in Algorithm 1, we propose the novel local and parallel method in the finite element space $V^{h_{k+1}}(\Omega)$ $(k \geq 1)$.

Algorithm 1 Local and parallel method for semilinear Neumann problem.

---

1. Solve the following linear boundary value problem in each subdomain $\Omega_j,\ j = 1, 2, \cdots, m$: Find $e^j_{h_{k+1}} \in V^{h_{k+1}}_\Gamma(\Omega_j)$, such that for any $v_{h_{k+1}} \in V^{h_{k+1}}_\Gamma(\Omega_j)$, there holds

$$a(e^j_{h_{k+1}}, v_{h_{k+1}}) = (b, v_{h_{k+1}}) - \langle g(x, u_{h_k}), v_{h_{k+1}}\rangle - (f(x, u_{h_k}), v_{h_{k+1}}) - a(u_{h_k}, v_{h_{k+1}}). \tag{36}$$

Set $\tilde{u}^j_{h_{k+1}} = u_{h_k} + e^j_{h_{k+1}} \in V^{h_{k+1}}(\Omega_j)$.

2. Solve the following linear boundary value problem in G_{m+1}: Find $\tilde{u}^{m+1}_{h_{k+1}} \in V^{h_{k+1}}(G_{m+1})$ such that $\tilde{u}^{m+1}_{h_{k+1}}|_{\partial G_j\cap\partial G_{m+1}} = \tilde{u}^j_{h_{k+1}},\ j = 1, \cdots, m$ and for any $v_{h_{k+1}} \in V^{h_{k+1}}_\Gamma(G_{m+1})$, there holds

$$a(\tilde{u}^{m+1}_{h_{k+1}}, v_{h_{k+1}}) = (b, v_{h_{k+1}}) - \langle g(x, u_{h_k}), v_{h_{k+1}}\rangle - (f(x, u_{h_k}), v_{h_{k+1}}). \tag{37}$$

3. Construct $\tilde{u}_{h_{k+1}} \in V^{h_{k+1}}(\Omega)$ such that

$$\tilde{u}_{h_{k+1}} = \tilde{u}^j_{h_{k+1}} \quad \text{in } G_j, \quad j = 1, \cdots, m+1. \tag{38}$$

4. Define a correction subspace $V^{H,h_{k+1}}(\Omega) = V^H(\Omega) + \text{span}\{\tilde{u}_{h_{k+1}}\}$ and solve the following small-scale semilinear Neumann problem: Find $u_{h_{k+1}} \in V^{H,h_{k+1}}(\Omega)$ such that for any $v_{H,h_{k+1}} \in V^{H,h_{k+1}}(\Omega)$, there holds

$$a(u_{h_{k+1}}, v_{H,h_{k+1}}) + \langle g(x, u_{h_{k+1}}), v_{H,h_{k+1}}\rangle + (f(x, u_{h_{k+1}}), v_{H,h_{k+1}}) = (b, v_{H,h_{k+1}}). \tag{39}$$

Summarize the above four steps into

$$u_{h_{k+1}} = Correction(V^H(\Omega), u_{h_k}, V^{h_{k+1}}(\Omega)).$$

Next, Theorem 2 rigorously proves that $u_{h_{k+1}} \in V^{h_{k+1}}(\Omega)$ derived using Algorithm 1 has a better accuracy than $u_{h_k} \in V^{h_k}(\Omega)$.

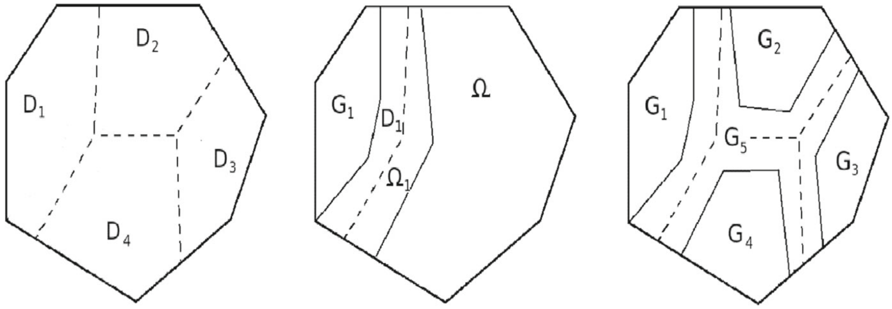

Fig. 2 $G_j \subset\subset D_j \subset \Omega$

Theorem 2 *Assume that* $u_{h_k} \in V^{h_k}(\Omega)$ *satisfies the following estimates:*

$$\|u - u_{h_k}\|_{1,\Omega} \lesssim \varepsilon_{h_k}(u), \tag{40}$$
$$\|u - u_{h_k}\|_{0,\Omega} \lesssim (r_\Omega(H) + \rho_\Omega(H))\varepsilon_{h_k}(u), \tag{41}$$
$$\|u - u_{h_k}\|_{0,\partial\Omega} \lesssim (r_\Omega(H) + \rho_\Omega(H))\varepsilon_{h_k}(u). \tag{42}$$

Then, the new approximation $u_{h_{k+1}} \in V^{h_{k+1}}(\Omega)$ *obtained by Algorithm 1 satisfies*

$$\|u - u_{h_{k+1}}\|_{1,\Omega} \lesssim \varepsilon_{h_{k+1}}(u), \tag{43}$$
$$\|u - u_{h_{k+1}}\|_{0,\Omega} \lesssim (r_\Omega(H) + \rho_\Omega(H))\varepsilon_{h_{k+1}}(u), \tag{44}$$
$$\|u - u_{h_{k+1}}\|_{0,\partial\Omega} \lesssim (r_\Omega(H) + \rho_\Omega(H))\varepsilon_{h_{k+1}}(u), \tag{45}$$

where $\varepsilon_{h_{k+1}}(u) := (r_\Omega(H) + \rho_\Omega(H))\varepsilon_{h_k}(u) + \delta_{h_{k+1}}(u)$.

Proof Based on Theorem 1 and (38), we can obtain

$$\begin{aligned}
&\|u - u_{h_{k+1}}\|_{1,\Omega} \\
&\lesssim \inf_{v_{H,h_{k+1}} \in V^{H,h_{k+1}}(\Omega)} \|u - v_{H,h_{k+1}}\|_{1,\Omega} \\
&\le \|u - \tilde{u}_{h_{k+1}}\|_{1,\Omega} \\
&\le \|u - P_{h_{k+1}}u\|_{1,\Omega} + \|\tilde{u}_{h_{k+1}} - P_{h_{k+1}}u\|_{1,\Omega} \\
&\lesssim \|u - P_{h_{k+1}}u\|_{1,\Omega} + \sum_{j=1}^{m} \|\tilde{u}^j_{h_{k+1}} - P_{h_{k+1}}u\|^2_{1,G_j} + \|\tilde{u}^{m+1}_{h_{k+1}} - P_{h_{k+1}}u\|^2_{1,G_{m+1}}.
\end{aligned} \tag{46}$$

To describe the proof more clearly, next the procedure is divided into three parts. The first part is used to estimate $\sum_{j=1}^{m} \|\tilde{u}^j_{h_{k+1}} - P_{h_{k+1}}u\|_{1,G_j}$. The second part is used to estimate $\|\tilde{u}^{m+1}_{h_{k+1}} - P_{h_{k+1}}u\|_{1,G_{m+1}}$. Then, based on (46), the final conclusion is proved in the third part.

Part 1: Based on (6), (24), and (36), the following equation holds true for $j = 1, 2, \cdots m$:

$$\begin{aligned} & a(\tilde{u}^j_{h_{k+1}} - P_{h_{k+1}} u, v) \\ = & (f(x,u) - f(x,u_{h_k}), v) + \langle g(x,u) - g(x,u_{h_k}), v\rangle, \quad \forall v \in V_\Gamma^{h_{k+1}}(\Omega_j). \end{aligned} \tag{47}$$

Using Lemma 4 for (47), there holds

$$\begin{aligned} & \|\tilde{u}^j_{h_{k+1}} - P_{h_{k+1}} u\|_{1,G_j} \\ \lesssim & \|\tilde{u}^j_{h_{k+1}} - P_{h_{k+1}} u\|_{0,\Omega_j} + \|f(x,u) - f(x,u_{h_k})\|_{0,\Omega_j} \\ & + \|g(x,u) - g(x,u_{h_k})\|_{0,\partial\Omega\cap\partial\Omega_j} \\ \lesssim & \|\tilde{u}^j_{h_{k+1}} - u_{h_k}\|_{0,\Omega_j} + \|u_{h_k} - P_{h_{k+1}} u\|_{0,\Omega_j} + \|f(x,u) - f(x,u_{h_k})\|_{0,\Omega_j} \\ & + \|g(x,u) - g(x,u_{h_k})\|_{0,\partial\Omega\cap\partial\Omega_j} \\ \lesssim & \|e^j_{h_{k+1}}\|_{0,\Omega_j} + \|u_{h_k} - P_{h_{k+1}} u\|_{0,\Omega_j} + \|f(x,u) - f(x,u_{h_k})\|_{0,\Omega_j} \\ & + \|g(x,u) - g(x,u_{h_k})\|_{0,\partial\Omega\cap\partial\Omega_j} \\ \lesssim & \|e^j_{h_{k+1}}\|_{0,\Omega_j} + \|u_{h_k} - P_{h_{k+1}} u\|_{0,\Omega_j} + \|u - u_{h_k}\|_{0,\Omega_j} \\ & + \|u - u_{h_k}\|_{0,\partial\Omega\cap\partial\Omega_j}. \end{aligned} \tag{48}$$

Next, we adopt Aubin-Nitsche technique to estimate $\|e^j_{h_{k+1}}\|_{0,\Omega_j}$ involved in (48): For any $\psi \in L^2(\Omega_j)$, there exists $w \in H^1_\Gamma(\Omega_j)$ such that

$$a(v, w) = (v, \psi), \qquad \forall v \in H^1_\Gamma(\Omega_j). \tag{49}$$

Based on the finite element method, there exists $w^j_{h_{k+1}} \in V_\Gamma^{h_{k+1}}(\Omega_j),\ w^j_H \in V_\Gamma^H(\Omega_j)$ such that

$$a(v_{h_{k+1}}, w^j_{h_{k+1}}) = (v_{h_{k+1}}, \psi), \quad \forall v_{h_{k+1}} \in V_\Gamma^{h_{k+1}}(\Omega_j), \tag{50}$$

$$a(v_H, w^j_H) = (v_H, \psi), \qquad \forall v_H \in V_\Gamma^H(\Omega_j). \tag{51}$$

The following two estimates are the standard finite element error estimates:

$$\|w - w^j_{h_{k+1}}\|_{1,\Omega_j} \lesssim r_{\Omega_j}(h_{k+1})\|\psi\|_{0,\Omega_j}, \tag{52}$$

$$\|w - w^j_H\|_{1,\Omega_j} \lesssim r_{\Omega_j}(H)\|\psi\|_{0,\Omega_j}. \tag{53}$$

Taking $v_{h_{k+1}} = e^j_{h_{k+1}}$ in (50), we can obtain

$$\begin{aligned} & (e^j_{h_{k+1}}, \psi) \\ = & a(e^j_{h_{k+1}}, w^j_{h_{k+1}}) \end{aligned}$$

$$
\begin{aligned}
&= (b, w_{h_{k+1}}^j) - \langle g(x,u_{h_k}), w_{h_{k+1}}^j\rangle - (f(x,u_{h_k}), w_{h_{k+1}}^j) - a(u_{h_k}, w_{h_{k+1}}^j)\\
&= \langle g(x,u) - g(x,u_{h_k}), w_{h_{k+1}}^j\rangle + (f(x,u) - f(x,u_{h_k}), w_{h_{k+1}}^j) + a(P_{h_{k+1}}u - u_{h_k}, w_{h_{k+1}}^j)\\
&= \langle g(x,u) - g(x,u_{h_k}), w_{h_{k+1}}^j - w_H^j\rangle + (f(x,u) - f(x,u_{h_k}), w_{h_{k+1}}^j - w_H^j)\\
&\quad + \langle g(x,u) - g(x,u_{h_k}), w_H^j\rangle + (f(x,u) - f(x,u_{h_k}), w_H^j) + a(P_{h_{k+1}}u - u_{h_k}, w_{h_{k+1}}^j)\\
&= \langle g(x,u) - g(x,u_{h_k}), w_{h_{k+1}}^j - w_H^j\rangle + (f(x,u) - f(x,u_{h_k}), w_{h_{k+1}}^j - w_H^j)\\
&\quad + a(P_{h_{k+1}}u - u_{h_k}, w_{h_{k+1}}^j - w_H^j). \qquad (54)
\end{aligned}
$$

Set $\psi = e_{h_{k+1}}^j$ in (54), there holds

$$
\begin{aligned}
&\|e_{h_{k+1}}^j\|_{0,\Omega_j}^2\\
&= \langle g(x,u) - g(x,u_{h_k}), w_{h_{k+1}}^j - w_H^j\rangle + (f(x,u) - f(x,u_{h_k}), w_{h_{k+1}}^j - w_H^j)\\
&\quad + a(P_{h_{k+1}}u - u_{h_k}, w_{h_{k+1}}^j - w_H^j)\\
&\lesssim \|u - u_{h_k}\|_{0,\partial\Omega\cap\partial\Omega_j}\|w_{h_{k+1}}^j - w_H^j\|_{0,\partial\Omega\cap\partial\Omega_j} + \|u - u_{h_k}\|_{0,\Omega_j}\|w_{h_{k+1}}^j - w_H^j\|_{0,\Omega_j}\\
&\quad + \|P_{h_{k+1}}u - u_{h_k}\|_{1,\Omega_j}\|w_{h_{k+1}}^j - w_H^j\|_{1,\Omega_j}\\
&\leq \left(\|w - w_{h_{k+1}}^j\|_{1,\Omega_j} + \|w - w_H^j\|_{1,\Omega_j}\right)\\
&\quad \left(\|P_{h_{k+1}}u - u_{h_k}\|_{1,\Omega_j} + \|u - u_{h_k}\|_{0,\partial\Omega\cap\partial\Omega_j} + \|u - u_{h_k}\|_{0,\Omega_j}\right)\\
&\lesssim r_{\Omega_j}(H)\|e_{h_{k+1}}^j\|_{0,\Omega_j}\left(\|P_{h_{k+1}}u - u_{h_k}\|_{1,\Omega_j} + \|u - u_{h_k}\|_{0,\partial\Omega\cap\partial\Omega_j} + \|u - u_{h_k}\|_{0,\Omega_j}\right),
\end{aligned}
$$

which yields

$$
\|e_{h_{k+1}}^j\|_{0,\Omega_j} \lesssim r_{\Omega_j}(H)\big(\|P_{h_{k+1}}u - u_{h_k}\|_{1,\Omega_j} + \|u - u_{h_k}\|_{0,\partial\Omega\cap\partial\Omega_j} + \|u - u_{h_k}\|_{0,\Omega_j}\big). \qquad (55)
$$

Combining (48) and (55), we can derive the following estimate:

$$
\begin{aligned}
\|\tilde{u}_{h_{k+1}}^j - P_{h_{k+1}}u\|_{1,G_j} \lesssim\ & r_{\Omega_j}(H)\|P_{h_{k+1}}u - u_{h_k}\|_{1,\Omega_j} + \|P_{h_{k+1}}u - u_{h_k}\|_{0,\Omega_j}\\
& + \|u - u_{h_k}\|_{0,\partial\Omega\cap\partial\Omega_j} + \|u - u_{h_k}\|_{0,\Omega_j}. \qquad (56)
\end{aligned}
$$

Part 2: Using (6), (24), and (37), for any $v_{h_{k+1}} \in V_\Gamma^{h_{k+1}}(G_{m+1})$, we can obtain

$$
\begin{aligned}
& a(\tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}}u, v_{h_{k+1}})\\
&= \langle g(x,u) - g(x,u_{h_k}), v_{h_{k+1}}\rangle + (f(x,u) - f(x,u_{h_k}), v_{h_{k+1}}). \qquad (57)
\end{aligned}
$$

Let us use $a_{G_{m+1}}(\cdot,\cdot)$ to denote the restriction of $a(\cdot,\cdot)$ on G_{m+1}. Then, for any $v \in V_\Gamma^{h_{k+1}}(G_{m+1})$, we can derive

$$
\|\tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}}u\|_{1,G_{m+1}}^2
$$

$$
\begin{aligned}
&\lesssim a_{G_{m+1}}(\tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}}u, \tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}}u) \\
&\lesssim a_{G_{m+1}}(\tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}}u, \tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}}u - v) \\
&\quad + \langle g(x,u) - g(x,u_{h_k}), v\rangle + (f(x,u) - f(x,u_{h_k}), v) \\
&\lesssim \|\tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}}u\|_{1,G_{m+1}} \inf_{\psi \in V_\Gamma^{h_{k+1}}(G_{m+1})} \|\tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}}u - \psi\|_{1,G_{m+1}} \\
&\quad + \|u - u_{h_k}\|_{0,\partial\Omega\cap\partial G_{m+1}} \Big(\|\tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}}u\|_{1,G_{m+1}} \\
&\quad + \inf_{\psi \in V_\Gamma^{h_{k+1}}(G_{m+1})} \|\tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}}u - \psi\|_{1,G_{m+1}} \Big) \\
&\quad + \|u - u_{h_k}\|_{0,G_{m+1}} \Big(\|\tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}}u\|_{0,G_{m+1}} \\
&\quad + \inf_{\psi \in V_\Gamma^{h_{k+1}}(G_{m+1})} \|\tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}}u - \psi\|_{0,G_{m+1}} \Big).
\end{aligned} \tag{58}
$$

Next, using Lemma 1 and trace theorem, (58) can be written as

$$
\begin{aligned}
&\|\tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}}u\|_{1,G_{m+1}}^2 \\
\lesssim\ &\|\tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}}u\|_{1,G_{m+1}} \|\tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}}u\|_{1/2,\partial G_{m+1}\setminus\partial\Omega} \\
&+ \big(\|u - u_{h_k}\|_{0,\partial\Omega\cap\partial G_{m+1}} + \|u - u_{h_k}\|_{0,G_{m+1}}\big)\big(\|\tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}}u\|_{1,G_{m+1}} \\
&+ \|\tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}}u\|_{1/2,\partial G_{m+1}\setminus\partial\Omega}\big) \\
\lesssim\ &\|\tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}}u\|_{1,G_{m+1}} \big(\|\tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}}u\|_{1/2,\partial G_{m+1}\setminus\partial\Omega} \\
&+ \|u - u_{h_k}\|_{0,\partial\Omega\cap\partial G_{m+1}} + \|u - u_{h_k}\|_{0,G_{m+1}}\big) \\
&+ \|u - u_{h_k}\|_{0,\partial\Omega\cap\partial G_{m+1}}^2 + \|u - u_{h_k}\|_{0,G_{m+1}}^2 + \|\tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}}u\|_{1/2,\partial G_{m+1}\setminus\partial\Omega}^2.
\end{aligned} \tag{59}
$$

Set

$$x := \|\tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}}u\|_{1,G_{m+1}},$$

$$m = \|\tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}}u\|_{1/2,\partial G_{m+1}\setminus\partial\Omega} + \|u - u_{h_k}\|_{0,\partial\Omega\cap\partial G_{m+1}} + \|u - u_{h_k}\|_{0,G_{m+1}},$$

$$n = \|u - u_{h_k}\|_{0,\partial\Omega\cap\partial G_{m+1}}^2 + \|u - u_{h_k}\|_{0,G_{m+1}}^2 + \|\tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}}u\|_{1/2,\partial G_{m+1}\setminus\partial\Omega}^2.$$

Thus, (59) means

$$x^2 \le Cmx + cn,$$

which indicates

$$x \le \frac{Cm + \sqrt{C^2m^2 + 4cn}}{2} \lesssim m + \sqrt{n}. \tag{60}$$

Since

$$\begin{aligned}\|\tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}}u\|^2_{\frac{1}{2},\partial G_{m+1}\backslash\partial\Omega} &\lesssim \sum_{j=1}^{m} \|\tilde{u}_{h_{k+1}}^{j} - P_{h_{k+1}}u\|^2_{\frac{1}{2},\partial G_j} \\ &\lesssim \sum_{j=1}^{m} \|\tilde{u}_{h_{k+1}}^{j} - P_{h_{k+1}}u\|^2_{1,G_j}, \end{aligned} \tag{61}$$

combining (60) and (61) leads to

$$\begin{aligned}&\|\tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}}u\|^2_{1,G_{m+1}} \\ \lesssim &\sum_{j=1}^{m} \|\tilde{u}_{h_{k+1}}^{j} - P_{h_{k+1}}u\|^2_{1,G_j} + \|u-u_{h_k}\|^2_{0,\partial\Omega\cap\partial G_{m+1}} + \|u-u_{h_k}\|^2_{0,G_{m+1}}. \end{aligned} \tag{62}$$

Part 3: From (35), (56), and (62), we obtain

$$\begin{aligned}&\|\tilde{u}_{h_{k+1}} - P_{h_{k+1}}u\|^2_{1,\Omega} \\ \lesssim &\sum_{j=1}^{m} \big(r^2_{\Omega_j}(H)\|P_{h_{k+1}}u - u_{h_k}\|^2_{1,\Omega_j} + \|P_{h_{k+1}}u - u_{h_k}\|^2_{0,\Omega_j} + \|u-u_{h_k}\|^2_{0,\partial\Omega\cap\partial\Omega_j} \\ &+\|u-u_{h_k}\|^2_{0,\Omega_j}\big) + \|u-u_{h_k}\|^2_{0,\partial\Omega\cap\partial G_{m+1}} + \|u-u_{h_k}\|^2_{0,G_{m+1}} \\ \lesssim &\, r^2_{\Omega}(H)\|P_{h_{k+1}}u - u_{h_k}\|^2_{1,\Omega} + \|P_{h_{k+1}}u - u_{h_k}\|^2_{0,\Omega} + \|u-u_{h_k}\|^2_{0,\partial\Omega} + \|u-u_{h_k}\|^2_{0,\Omega} \\ \lesssim &\, r^2_{\Omega}(H)\|u - u_{h_k}\|^2_{1,\Omega} + r^2_{\Omega}(H)\|u - P_{h_{k+1}}u\|^2_{1,\Omega} + \|u - P_{h_{k+1}}u\|^2_{0,\Omega} \\ &+\|u-u_{h_k}\|^2_{0,\partial\Omega} + \|u-u_{h_k}\|^2_{0,\Omega}, \end{aligned} \tag{63}$$

which indicates

$$\begin{aligned}\|\tilde{u}_{h_{k+1}} - P_{h_{k+1}}u\|_{1,\Omega} \lesssim &\,(r_\Omega(H) + \rho_\Omega(H))\|u-u_{h_k}\|_{1,\Omega} + r_\Omega(H)\|u - P_{h_{k+1}}u\|_{1,\Omega} \\ &+r_\Omega(h_{k+1})\delta_{h_{k+1}}(u). \end{aligned} \tag{64}$$

Using (46) and (64), we can derive

$$\begin{aligned}&\|u-u_{h_{k+1}}\|_{1,\Omega} \\ \lesssim &\,\|u-\tilde{u}_{h_{k+1}}\|_{1,\Omega} \\ \lesssim &\,\|u-P_{h_{k+1}}u\|_{1,\Omega} + (r_\Omega(H)+\rho_\Omega(H))\|u-u_{h_k}\|_{1,\Omega} + r_\Omega(h_{k+1})\delta_{h_{k+1}}(u) \\ \lesssim &\,(r_\Omega(H)+\rho_\Omega(H))\varepsilon_{h_k}(u) + \delta_{h_{k+1}}(u) \\ \lesssim &\,\varepsilon_{h_{k+1}}(u), \end{aligned}$$

where $\varepsilon_{h_{k+1}}(u) := (r_\Omega(H) + \rho_\Omega(H))\varepsilon_{h_k}(u) + \delta_{h_{k+1}}(u)$. Then, we derive the first desired result (43).

Let us define

$$\tilde{r}_\Omega(H) = \sup_{f\in L^2(\Omega), \|f\|_{0,\Omega}=1} \inf_{v_{H,h_{k+1}}\in V^{H,h_{k+1}}(\Omega)} \left\| Tf - v_{H,h_{k+1}} \right\|_{1,\Omega},$$

and

$$\tilde{\rho}_\Omega(H) = \sup_{g\in L^2(\partial\Omega), \|g\|_{0,\partial\Omega}=1} \inf_{v_{H,h_{k+1}}\in V^{H,h_{k+1}}(\Omega)} \left\| T'g - v_{H,h_{k+1}} \right\|_{1,\Omega}.$$

Using Theorem 1 again, there holds

$$\begin{aligned}\|u - u_{h_{k+1}}\|_{0,\Omega} &\lesssim (\tilde{r}_\Omega(H) + \tilde{\rho}_\Omega(H))\|u - u_{h_{k+1}}\|_{1,\Omega} \\ &\leq (r_\Omega(H) + \rho_\Omega(H))\|u - u_{h_{k+1}}\|_{1,\Omega},\end{aligned}$$

and

$$\begin{aligned}\|u - u_{h_{k+1}}\|_{0,\partial\Omega} &\lesssim (\tilde{r}_\Omega(H) + \tilde{\rho}_\Omega(H))\|u - u_{h_{k+1}}\|_{1,\Omega} \\ &\leq (r_\Omega(H) + \rho_\Omega(H))\|u - u_{h_{k+1}}\|_{1,\Omega}.\end{aligned}$$

Then, we derive the desired results (44) and (45). The proof is completed. □

4.2 Local and parallel multigrid method for semilinear Neumann problem

In this subsection, a new type of local and parallel multigrid for the semilinear Neumann problem (24) is designed based on Algorithm 1 and the multilevel mesh sequence (33). For the multilevel mesh sequence (33), any two consecutive meshes $\mathcal{T}_{h_k}(\Omega)$ and $\mathcal{T}_{h_{k-1}}(\Omega)$ are generated through a one-time unform refinement, such that the mesh sizes satisfy $h_k = \frac{1}{q}h_{k-1},\ k \geq 2$. Meanwhile, there holds

$$\delta_{h_k}(u) \approx \frac{1}{q}\delta_{h_{k-1}}(u), \quad q > 1. \tag{65}$$

Based on Algorithm 1, the local and parallel multigrid method for (24) is designed in Algorithm 2.

Theorem 3 *After implementing Algorithm 2, the final approximate solution u_{h_n} satisfies*

$$\|u - u_{h_n}\|_{1,\Omega} \lesssim \delta_{h_n}(u), \tag{66}$$

$$\|u - u_{h_n}\|_{0,\Omega} \lesssim (r_\Omega(H) + \rho_\Omega(H))\delta_{h_n}(u), \tag{67}$$

$$\|u - u_{h_n}\|_{0,\partial\Omega} \lesssim (r_\Omega(H) + \rho_\Omega(H))\delta_{h_n}(u). \tag{68}$$

Algorithm 2 Local and parallel multigrid method for semilinear Neumann problem.

1. Solve the semilinear Neumann problem (24) in the initial space: Find $u_{h_1} \in V^{h_1}(\Omega)$ such that

$$a(u_{h_1}, v_{h_1}) + (f(x, u_{h_1}), v_{h_1}) + \langle g(x, u_{h_1}), v_{h_1}\rangle = (b, v_{h_1}), \quad \forall v_{h_1} \in V^{h_1}(\Omega).$$

2. For $k = 1, \cdots, n-1$, do the following loop:

(i) Solve the following linear boundary value problem in each subdomain Ω_j, $j = 1, 2, \cdots, m$: Find $e^j_{h_{k+1}} \in V^{h_{k+1}}_{\Gamma}(\Omega_j)$, such that for any $v_{h_{k+1}} \in V^{h_{k+1}}_{\Gamma}(\Omega_j)$, there holds

$$a(e^j_{h_{k+1}}, v_{h_{k+1}}) = (b, v_{h_{k+1}}) - \langle g(x, u_{h_k}), v_{h_{k+1}}\rangle - (f(x, u_{h_k}), v_{h_{k+1}}) - a(u_{h_k}, v_{h_{k+1}}).$$

Set $\tilde{u}^j_{h_{k+1}} = u_{h_k} + e^j_{h_{k+1}} \in V^{h_{k+1}}(\Omega_j)$.

(ii) Solve the following linear boundary value problem in G_{m+1}: Find $\tilde{u}^{m+1}_{h_{k+1}} \in V^{h_{k+1}}(G_{m+1})$ such that $\tilde{u}^{m+1}_{h_{k+1}}|_{\partial G_j \cap \partial G_{m+1}} = \tilde{u}^j_{h_{k+1}}$, $j = 1, \cdots, m$ and for any $v_{h_{k+1}} \in V^{h_{k+1}}_{\Gamma}(G_{m+1})$, there holds

$$a(\tilde{u}^{m+1}_{h_{k+1}}, v_{h_{k+1}}) = (b, v_{h_{k+1}}) - \langle g(x, u_{h_k}), v_{h_{k+1}}\rangle - (f(x, u_{h_k}), v_{h_{k+1}}).$$

(iii) Construct $\tilde{u}_{h_{k+1}} \in V^{h_{k+1}}(\Omega)$ such that

$$\tilde{u}_{h_{k+1}} = \tilde{u}^j_{h_{k+1}} \quad \text{in } G_j, \quad j = 1, \cdots, m+1.$$

(iv) Define a correction subspace $V^{H,h_{k+1}}(\Omega) = V^H(\Omega) + \text{span}\{\tilde{u}_{h_{k+1}}\}$ and solve the following small-scale semilinear Neumann problem: Find $u_{h_{k+1}} \in V^{H,h_{k+1}}(\Omega)$ such that for any $v_{H,h_{k+1}} \in V^{H,h_{k+1}}(\Omega)$, there holds

$$a(u_{h_{k+1}}, v_{H,h_{k+1}}) + \langle g(x, u_{h_{k+1}}), v_{H,h_{k+1}}\rangle + (f(x, u_{h_{k+1}}), v_{H,h_{k+1}}) = (b, v_{H,h_{k+1}}).$$

End For.

The final approximation u_{h_n} is obtained in the finest space $V^{h_n}(\Omega)$.

under the condition $Cq(r_\Omega(H) + \rho_\Omega(H)) < 1$ for some constant C.

Proof Based on Theorem 1, the initial approximate solution u_{h_1} satisfies

$$\begin{aligned}
\left\|u - u_{h_1}\right\|_{1,\Omega} &\lesssim \delta_{h_1}(u), \\
\left\|u - u_{h_1}\right\|_{0,\Omega} &\lesssim (r_\Omega(h_1) + \rho_\Omega(h_1))\delta_{h_1}(u), \\
\left\|u - u_{h_1}\right\|_{0,\partial\Omega} &\lesssim (r_\Omega(h_1) + \rho_\Omega(h_1))\delta_{h_1}(u).
\end{aligned}$$

which means the initial condition of Theorem 2 can be met if we set $\varepsilon_{h_1}(u) := \delta_{h_1}(u)$.

Thus, using Theorem 2, we can obtain

$$\varepsilon_{h_{k+1}}(u) \lesssim (r_\Omega(H) + \rho_\Omega(H))\varepsilon_{h_k}(u) + \delta_{h_{k+1}}(u), \quad 1 \le k \le n-1. \tag{69}$$

Based on (65) and (69), we can derive

$$\begin{aligned}
\varepsilon_{h_n}(u) &\lesssim (r_\Omega(H)+\rho_\Omega(H))\varepsilon_{h_{n-1}}(u)+\delta_{h_n}(u)\\
&\lesssim (r_\Omega(H)+\rho_\Omega(H))^2\varepsilon_{h_{n-2}}(u)+(r_\Omega(H)+\rho_\Omega(H))\delta_{h_{n-1}}(u)+\delta_{h_n}(u)\\
&\lesssim \sum_{k=1}^{n}\big(r_\Omega(H)+\rho_\Omega(H)\big)^{n-k}\delta_{h_k}(u)\\
&\lesssim \sum_{k=1}^{n}\big(q(r_\Omega(H)+\rho_\Omega(H))\big)^{n-k}\delta_{h_n}(u)\\
&\lesssim \frac{\delta_{h_n}(u)}{1-q(\rho_\Omega(H)+r_\Omega(H))}\\
&\lesssim \delta_{h_n}(u).
\end{aligned} \tag{70}$$

Using Theorem 2 and (70) leads to

$$\|u-u_{h_n}\|_{1,\Omega}\lesssim \varepsilon_{h_n}(u)\lesssim \delta_{h_n}(u),$$

which is just the desired result (66).

Further using Theorem 2, we can obtain

$$\|u-u_{h_n}\|_{0,\Omega}\lesssim (r_\Omega(H)+\rho_\Omega(H))\|u-u_{h_n}\|_{1,\Omega}\lesssim (r_\Omega(H)+\rho_\Omega(H))\delta_{h_n}(u)$$

and

$$\|u-u_{h_n}\|_{0,\partial\Omega}\lesssim (r_\Omega(H)+\rho_\Omega(H))\|u-u_{h_n}\|_{1,\Omega}\lesssim (r_\Omega(H)+\rho_\Omega(H))\delta_{h_n}(u),$$

which are the desired results (67) and (68). Then, the proof is completed. □

4.3 Computational work of Algorithm 2

In this subsection, we estimate the computational work of Algorithm 2, finding that requires nearly the same computational work as that of solving the corresponding linear boundary value problem. Let us define

$$N_k^j=\dim V_\Gamma^{h_k}(\Omega_j)\quad\text{and}\quad N_k=\dim V^{h_k}(\Omega)\ \text{ for } k=1,\cdots n,\ j=1,\cdots,m+1.$$

Then, there holds

$$N_k^j\approx(\frac{1}{q})^{d(n-k)}N_n^j\quad\text{and}\quad N_k^j\approx(\frac{N_k}{m})\ \text{ for } k=1,\cdots,n,\ j=1,\cdots,m+1. \tag{71}$$

Theorem 4 *Assume solving the semilinear Neumann problem in the coarse spaces $V^H(\Omega)$ and $V^{h_1}(\Omega)$ requires work $O(M_H)$ and $O(M_{h_1})$, and solving the linear boundary value problem in $V_\Gamma^{h_k}(\Omega_j)$ requires work $O(N_k^j)$, where $k=2,\cdots,n,\ j=$*

$1, \cdots, m+1$. *Then, the computational work of each computing node involved in Algorithm 2 requires* $O(M_{h_1} + N_n/m + M_H \log N_n)$. *Furthermore, Algorithm 2 requires* $O(N_n/m)$ *when* $M_{h_1} \leq N_n/m$, $M_H \ll N_n/m$.

Proof Based on (71), the computational work of Algorithm 2 can be estimated by

$$\begin{aligned} \text{Total work} &= O(M_{h_1} + \sum_{k=2}^{n} (N_k/m + M_H)) \\ &= O(M_{h_1} + \sum_{k=2}^{n} N_k/m + (n-1)M_H) \\ &= O(M_{h_1} + \sum_{k=2}^{n} (\frac{1}{q})^{d(n-k)} N_n/m + (n-1)M_H) \\ &= O(M_{h_1} + N_n/m + M_H \log N_n). \end{aligned} \tag{72}$$

Furthermore, if $M_{h_1} \leq N_n/m$, $M_H \ll N_n/m$, (72) can be controlled by $O(N_n/m)$. Then, we complete the proof. □

5 Numerical examples

In this section, some numerical examples are presented to support our theoretical conclusions and illustrate the solving efficiency of Algorithm 2. All the linear equations involved in the numerical experiments are solved by conjugate gradient method.

5.1 Example 1

In the first example, we solve the following semilinear Neumann problem: Find $u \in H^1(\Omega)$ such that

$$\begin{cases} -\Delta u + u + f(x, u) = 1, & \text{in } \Omega, \\ \nabla u \cdot n + g(x, u) = 0, & \text{on } \partial\Omega, \end{cases} \tag{73}$$

where $\Omega = (0, 1)^2$, $f(x, u) = u^3$ and $g(x, u) = u^3$.

In order to use Algorithm 2, Ω is divided into four disjoint subdomains: $D_1 = (0.5, 1.0) \times (0.5, 1.0)$, $D_2 = (0.0, 0.5) \times (0.5, 1.0)$, $D_3 = (0.5, 1.0) \times (0.0, 0.5)$, $D_4 = (0.0, 0.5) \times (0.0, 0.5)$. Next, we construct G_j and Ω_j such that $G_j \subset\subset D_j \subset \Omega_j \subset \Omega$: $\Omega_1 = (0.375, 1.0) \times (0.375, 1.0)$, $\Omega_2 = (0.0, 0.625) \times (0.375, 1.0)$, $\Omega_3 = (0.375, 1.0) \times (0.0, 0.625)$, $\Omega_4 = (0.0, 0.625) \times (0.0, 0.625)$, $G_1 = (0.625, 1.0) \times (0.625, 1.0)$, $G_2 = (0.0, 0.375) \times (0.625, 1.0)$, $G_3 = (0.625, 1.0) \times (0.0, 0.375)$, $G_4 = (0.0, 0.375) \times (0.0, 0.375)$, and $G_5 = \Omega \backslash (\cup_{j=1}^{4} \bar{G}_j)$.

We use linear finite element space in this example, and the multilevel mesh sequence is produced through one-time uniform refinement with the refinement index of $q = 2$. The coarse $\mathcal{T}_H$ is the same as $\mathcal{T}_{h_1}$ with mesh sizes $H = h_1 = 1/8$ (see Fig. 3).

The numerical results derived by Algorithm 2 are presented in Fig. 3. Besides, we also use the direct finite element method to solve (75). The direct finite element method

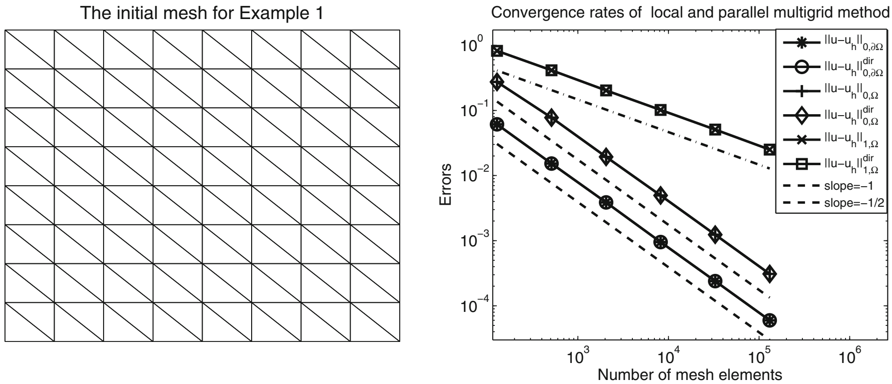

Fig. 3 The initial mesh (left) and error estimates (right) of Algorithm 2 for Example 1

means we solve semilinear Neumann problem (75) directly in the finest finite element space. That is, we use the fixed point iteration for the semilinear Neumann problem in the finest finite element space and the linear equation in each fixed point iteration step are solved by the classical finite element method. The results are also presented in Fig. 3. From Fig. 3, we can find that Algorithm 2 can produce an optimal approximate solution as the direct finite element method.

5.2 Example 2

In the second example, we solve the following semilinear Neumann problem by Algorithm 2: Find $u \in H^1(\Omega)$ such that

$$\begin{cases} -\nabla \cdot (\mathcal{A}\nabla u) + \phi u + f(x,u) = 0, & \text{in } \Omega, \\ (\mathcal{A}\nabla u) \cdot n + g(x,u) = 0, & \text{on } \partial\Omega, \end{cases} \tag{74}$$

where $\Omega = (0,1)^3, \phi = e^{(x_1-\frac{1}{2})(x_2-\frac{1}{2})(x_3-\frac{1}{2})}$, $f(x,u) = arctan(u), g(x,u) = |u|^{3/2}$ and

$$\mathcal{A} = \begin{pmatrix} 1+(x_1-\frac{1}{2})^2 & (x_1-\frac{1}{2})(x_2-\frac{1}{2}) & (x_1-\frac{1}{2})(x_3-\frac{1}{2}) \\ (x_1-\frac{1}{2})(x_2-\frac{1}{2}) & 1+(x_2-\frac{1}{2})^2 & (x_2-\frac{1}{2})(x_3-\frac{1}{2}) \\ (x_1-\frac{1}{2})(x_3-\frac{1}{2}) & (x_2-\frac{1}{2})(x_3-\frac{1}{2}) & 1+(x_3-\frac{1}{2})^2 \end{pmatrix}.$$

Similar to the first example, Ω is also divided into several disjoint subdomains $D_1, \cdots, D_8$: $D_1 = (0.5, 1.0) \times (0.5, 1.0) \times (0.0, 0.5)$, $D_2 = (0.0, 0.5) \times (0.5, 1.0) \times (0.0, 0.5)$, $D_3 = (0.5, 1.0) \times (0.0, 0.5) \times (0.0, 0.5)$, $D_4 = (0.0, 0.5) \times (0.0, 0.5) \times (0.0, 0.5)$, $D_5 = (0.5, 1.0) \times (0.5, 1.0) \times (0.5, 1.0)$, $D_6 = (0.0, 0.5) \times (0.5, 1.0) \times (0.5, 1.0)$, $D_7 = (0.5, 1.0) \times (0.0, 0.5) \times (0.5, 1.0)$, $D_8 = (0.0, 0.5) \times (0.0, 0.5) \times (0.5, 1.0)$, For the enlarged and reduced subdomains $G_j \subset\subset D_j \subset \Omega_j \subset \Omega$: $\Omega_1 = (0.375, 1.0) \times (0.375, 1.0) \times (0.0, 0.625)$, $\Omega_2 = (0.0, 0.625) \times (0.375, 1.0) \times (0.0, 0.625)$, $\Omega_3 = (0.375, 1.0) \times (0.0, 0.625) \times (0.0, 0.625)$, $\Omega_4 = (0.0, 0.625) \times (0.0, 0.625) \times (0.0, 0.625)$, $\Omega_5 = (0.375, 1.0) \times (0.375, 1.0) \times (0.375, 1.0)$, $\Omega_6 =$

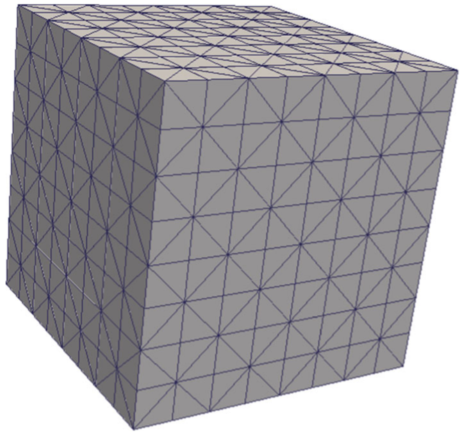

Fig. 4 The initial mesh of Algorithm 2 for Example 2

$(0.0, 0.625) \times (0.375, 1.0) \times (0.375, 1.0)$, $\Omega_7 = (0.375, 1.0) \times (0.0, 0.625) \times (0.375, 1.0)$, $\Omega_8 = (0.0, 0.625) \times (0.0, 0.625) \times (0.375, 1.0)$, $G_1 = (0.625, 1.0) \times (0.625, 1.0) \times (0.0, 0.375)$, $G_2 = (0.0, 0.375) \times (0.625, 1.0) \times (0.0, 0.375)$, $G_3 = (0.625, 1.0) \times (0.0, 0.375) \times (0.0, 0.375)$, $G_4 = (0.0, 0.375) \times (0.0, 0.375) \times (0.0, 0.375)$, $G_5 = (0.625, 1.0) \times (0.625, 1.0) \times (0.625, 1.0)$ $G_6 = (0.0, 0.375) \times (0.625, 1.0) \times (0.625, 1.0)$, $G_7 = (0.625, 1.0) \times (0.0, 0.375) \times (0.625, 1.0)$, $G_8 = (0.0, 0.375) \times (0.0, 0.375) \times (0.625, 1.0)$, and $G_9 = \Omega \backslash (\cup_{j=1}^{8} \bar{G}_j)$.

We also use the linear finite element space in this example, and the multilevel mesh sequence is produced through one-time uniform refinement with the refinement index of $q = 2$. The coarse $\mathcal{T}_H$ is the same as $\mathcal{T}_{h_1}$ with mesh sizes $H = h_1 = 1/8$ (see Fig. 4).

The numerical results of Algorithm 2 are presented in Fig. 5. Besides, we also use the direct finite element method to solve (74), and the results are also presented in Fig. 5. From Fig. 5, we also can find that Algorithm 2 can produce an optimal approximate solution as the direct finite element method.

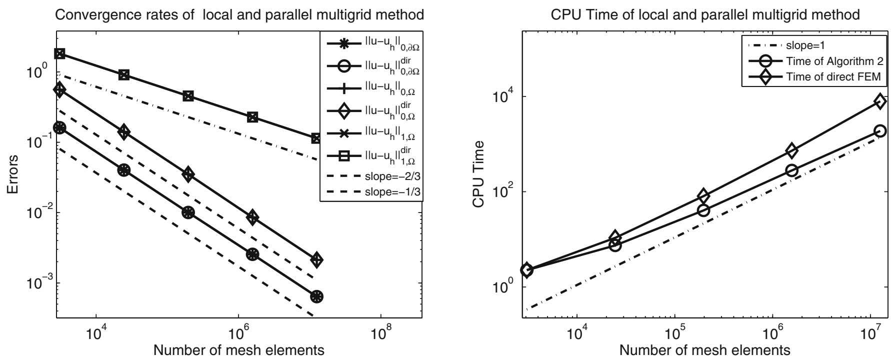

Fig. 5 Errors (left) and computational time (right) of Algorithm 2 for Example 2

In addition, to illustrate the efficiency of Algorithm 2 intuitively, we also present the computational time of Algorithm 2 and the direct finite element method in Fig. 5. Figure 5 shows that Algorithm 2 has a linear computational complexity. Meanwhile, Algorithm 2 has a higher solving efficiency than the direct finite element method.

5.3 Example 3

In the third example, we solve the following semilinear Neumann problem on the L-shape domain: Find $u \in H^1(\Omega)$ such that

$$\begin{cases} -\Delta u + u + f(x,u) = 1, & \text{in } \Omega, \\ \nabla u \cdot n + g(x,u) = 0, & \text{on } \partial\Omega, \end{cases} \tag{75}$$

where $\Omega = (0,1)^2/[1/2,1)^2$, $f(x,u) = u^3$ and $g(x,u) = u^3$.

In order to use Algorithm 2, Ω is divided into three disjoint subdomains: $D_1 = (0.0, 0.5) \times (0.5, 1.0)$, $D_2 = (0.5, 1.0) \times (0.0, 0.5)$, $D_3 = (0.0, 0.5) \times (0.0, 0.5)$. Next, we construct G_j and Ω_j such that $G_j \subset\subset D_j \subset \Omega_j \subset \Omega$: $\Omega_1 = (0.0, 0.625) \times (0.375, 1.0)$, $\Omega_2 = (0.375, 1.0) \times (0.0, 0.625)$, $\Omega_3 = (0.0, 0.625) \times (0.0, 0.625)$, $G_1 = (0.0, 0.375) \times (0.625, 1.0)$, $G_2 = (0.625, 1.0) \times (0.0, 0.375)$, $G_3 = (0.0, 0.375) \times (0.0, 0.375)$, and $G_4 = \Omega \backslash (\cup_{j=1}^{3} \bar{G}_j)$.

We use linear finite element space in this example, and the multilevel mesh sequence is produced through one-time uniform refinement with the refinement index of $q = 2$. The coarse $\mathcal{T}_H$ is the same as $\mathcal{T}_{h_1}$ with mesh sizes $H = h_1 = 1/8$ (see Fig. 6).

The numerical results derived by Algorithm 2 are presented in Fig. 6. Besides, we also use the direct finite element method to solve (75). That is, we solve semilinear Neumann problem (75) directly in the final finite element space. The results are also presented in Fig. 6. From Fig. 6, we can find that Algorithm 2 can produce the same accuracy as the direct finite element method.

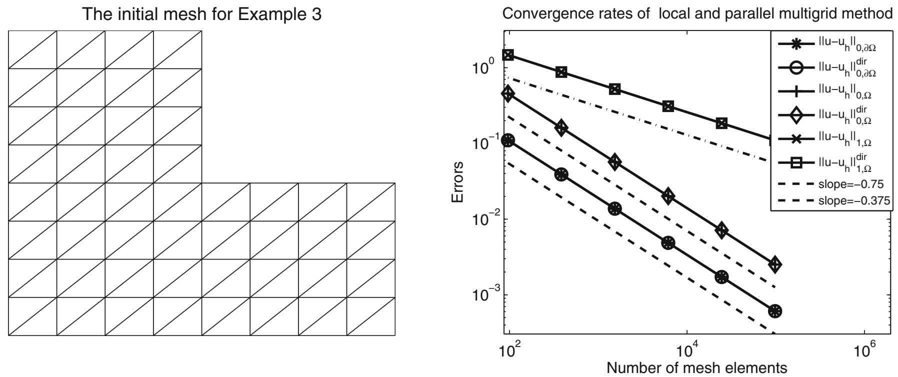

Fig. 6 The initial mesh (left) and error estimates (right) of Algorithm 2 for Example 3

5.4 Example 4

In the last example, we solve the following semilinear Neumann problem by Algorithm 2: Find $u \in H^1(\Omega)$ such that

$$\begin{cases} -\Delta u + u + f(x, u) = 0, & \text{in } \Omega, \\ \nabla u \cdot n + g(x, u) = 0, & \text{on } \partial\Omega, \end{cases} \tag{76}$$

where $\Omega = (0, 1)^3$, $f(x, u) = g(x, u) = \begin{cases} u^{3/2}, \text{ if } u \geq 0, \\ -|u|^{3/2}, \text{ if } u < 0. \end{cases}$ The nonlinear terms of the semilinear elliptic equation (76) have bounded first order derivative but unbounded second order derivative.

Similar to the first example, Ω is also divided into several disjoint subdomains $D_1, \cdots, D_8$: $D_1 = (0.5, 1.0) \times (0.5, 1.0) \times (0.0, 0.5)$, $D_2 = (0.0, 0.5) \times (0.5, 1.0) \times (0.0, 0.5)$, $D_3 = (0.5, 1.0) \times (0.0, 0.5) \times (0.0, 0.5)$, $D_4 = (0.0, 0.5) \times (0.0, 0.5) \times (0.0, 0.5)$, $D_5 = (0.5, 1.0) \times (0.5, 1.0) \times (0.5, 1.0)$, $D_6 = (0.0, 0.5) \times (0.5, 1.0) \times (0.5, 1.0)$, $D_7 = (0.5, 1.0) \times (0.0, 0.5) \times (0.5, 1.0)$, $D_8 = (0.0, 0.5) \times (0.0, 0.5) \times (0.5, 1.0)$, For the enlarged and reduced subdomains $G_j \subset\subset D_j \subset \Omega_j \subset \Omega$: $\Omega_1 = (0.375, 1.0) \times (0.375, 1.0) \times (0.0, 0.625)$, $\Omega_2 = (0.0, 0.625) \times (0.375, 1.0) \times (0.0, 0.625)$, $\Omega_3 = (0.375, 1.0) \times (0.0, 0.625) \times (0.0, 0.625)$, $\Omega_4 = (0.0, 0.625) \times (0.0, 0.625) \times (0.0, 0.625)$, $\Omega_5 = (0.375, 1.0) \times (0.375, 1.0) \times (0.375, 1.0)$, $\Omega_6 = (0.0, 0.625) \times (0.375, 1.0) \times (0.375, 1.0)$, $\Omega_7 = (0.375, 1.0) \times (0.0, 0.625) \times (0.375, 1.0)$, $\Omega_8 = (0.0, 0.625) \times (0.0, 0.625) \times (0.375, 1.0)$, $G_1 = (0.625, 1.0) \times (0.625, 1.0) \times (0.0, 0.375)$, $G_2 = (0.0, 0.375) \times (0.625, 1.0) \times (0.0, 0.375)$, $G_3 = (0.625, 1.0) \times (0.0, 0.375) \times (0.0, 0.375)$, $G_4 = (0.0, 0.375) \times (0.0, 0.375) \times (0.0, 0.375)$, $G_5 = (0.625, 1.0) \times (0.625, 1.0) \times (0.625, 1.0)$ $G_6 = (0.0, 0.375) \times (0.625, 1.0) \times (0.625, 1.0)$, $G_7 = (0.625, 1.0) \times (0.0, 0.375) \times (0.625, 1.0)$, $G_8 = (0.0, 0.375) \times (0.0, 0.375) \times (0.625, 1.0)$, and $G_9 = \Omega \backslash (\cup_{j=1}^{8} \bar{G}_j)$.

We also use the linear finite element space in this example, and the multilevel mesh sequence is produced through one-time uniform refinement with the refinement index of $q = 2$. We use the same coarse mesh $\mathcal{T}_H$ and $\mathcal{T}_{h_1}$ as the first example.

The numerical results of Algorithm 2 are presented in Fig. 7. Besides, we also use the direct finite element method to solve (76), and the results are also presented in Fig. 7. From Fig. 7, we also can find that Algorithm 2 can produce an optimal approximate solution as the direct finite element method.

In addition, to illustrate the efficiency of Algorithm 2 intuitively, we also present the computational time of Algorithm 2 and the direct finite element method in Fig. 7. Figure 7 shows that Algorithm 2 has a linear computational complexity. Meanwhile, Algorithm 2 has a higher solving efficiency than the direct finite element method.

6 Concluding remark

A new type of local and parallel method was designed to solve the semilinear Neumann problem with nonlinear boundary condition based on the multigrid discretization.

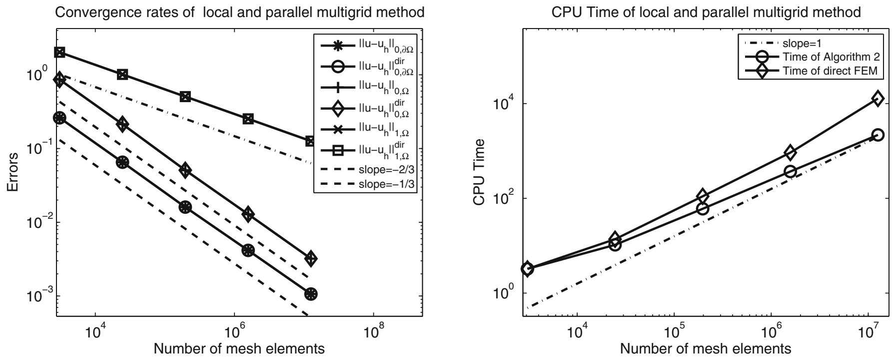

Fig. 7 Errors (left) and computational time (right) of Algorithm 2 for Example 4

Instead of solving the semilinear Neumann problem directly in the finest space, we transformed it into some linear boundary value problems in a multilevel mesh sequence and some small-scale semilinear Neumann problems in a low-dimensional correction subspace. The linear boundary value problems were efficiently solved through local and parallel technique. Meanwhile, the computational time for the small-scale semilinear Neumann problems can be negligible because the dimension of the correction subspace is small and remains fixed. Additionally, compared with the existing multigrid method for semilinear Neumann problems that require the second order derivatives of the nonlinear terms, our algorithm only requires the first order derivatives of the nonlinear terms. Rigorous theoretical analysis and some numerical experiments are presented to show the efficiency of the proposed algorithm.

Data Availability All data generated or analyzed during this study are included in this published article.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Adams, R.A.: Sobolev Spaces. Academic Press, New York (1975)
2. Bi, H., Li, Z., Yang, Y.: Local and parallel finite element algorithms for the Steklov eigenvalue problem. Numer. Methods Partial Differ. Equ. **32**(2), 399–417 (2016)
3. Bi, H., Yang, Y., Li, H.: Local and parallel finite element discretizations for eigenvalue problems. SIAM J. Sci. Comput. **15**(6), A2575–A2597 (2013)
4. Bramble, J.H., Pasciak, J.E.: New convergence estimates for multigrid algorithms. Math. Comp. **49**, 311–329 (1987)
5. Bramble, J.H., Zhang, X.: The analysis of multigrid methods, Handbook of Numerical Analysis. 173–415 (2000)
6. Brenner, S., Scott, L.: The Mathematical Theory of Finite Element Methods. Springer-Verlag, New York (1994)
7. Chen, H., Xie, H., Xu, F.: A full multigrid method for eigenvalue problems. J. Comput. Phys. **322**, 747–759 (2016)

8. Ciarlet, P.G.: The Finite Element Method for Elliptic Problems. North-Holland, Amsterdam (1978)
9. Dai, X., Zhou, A.: Three-scale finite element discretizations for quantum eigenvalue problems. SIAM J. Numer. Anal. **46**(1), 295–324 (2008)
10. Dong, X., He, Y., Wei, H., Zhang, Y.: Local and parallel finite element algorithm based on the partition of unity method for the incompressible MHD flow. Adv. Comput. Math. **44**(4), 1295–1319 (2018)
11. Du, G., Zuo, L.: Local and parallel finite element post-processing scheme for the Stokes problem. Comput. Math. Appl. **73**, 129–140 (2017)
12. Du, G., Hou, Y., Zuo, L.: A modified local and parallel finite element method for the mixed Stokes-Darcy model. J. Math. Anal. Appl. **435**(2), 1129–1145 (2016)
13. Grisvard, P.: Elliptic Problems in Nonsmooth Domains. Pitman, Boston, MA (1985)
14. He, Y., Mei, L., Shang, Y., Cui, J.: Newton iterative parallel finite element algorithm for the steady Navier-Stokes equations. J. Sci. Comput. **44**, 92–106 (2010)
15. He, Y., Xu, J., Zhou, A.: Local and parallel finite element algorithms for the Navier-Stokes problem. J. Comput. Math. **24**(3), 227–238 (2006)
16. Jia, S., Xie, H., Xie, M., Xu, F.: A full multigrid method for nonlinear eigenvalue problems. Sci. China Math. **59**, 2037–2048 (2016)
17. Li, Y., Han, X., Xie, H., You, C.: Local and parallel finite element algorithm based on multilevel discretization for eigenvalue problem. Int. J. Numer. Anal. Model. **13**(1), 73–89 (2016)
18. Lin, Q., Xie, H.: A multi-level correction scheme for eigenvalue problems. Math. Comp. **84**(291), 71–88 (2015)
19. Lin, Q., Xie, H., Xu, F.: Multilevel correction adaptive finite element method for semilinear elliptic equation. Appl. Math. **60**(5), 527–550 (2015)
20. Liu, Q., Hou, Y.: Local and parallel finite element algorithms for time-dependent convection-diffusion equations. Appl. Math. Mech. Engl. Ed. **30**, 787–794 (2009)
21. Ma, F., Ma, Y., Wo, W.: Local and parallel finite element algorithms based on two-grid discretization for steady Navier-Stokes equations. Appl. Math. Mech. **28**(1), 27–35 (2007)
22. Ma, Y., Zhang, Z., Ren, C.: Local and parallel finite element algorithms based on two-grid discretization for the stream function form of Navier-Stokes equations. Appl. Math. Comput. **175**, 786–813 (2006)
23. Shaidurov, V.: Multigrid Methods For Finite Elements. Springer (1995)
24. Shang, Y., Wang, K.: Local and parallel finite element algorithms based on two-grid discretizations for the transient Stokes equations. Numer. Algorithms **54**, 195–218 (2010)
25. Shang, Y., He, Y., Luo, Z.: A comparison of three kinds of local and parallel finite element algorithms based on two-grid discretizations for the stationary Navier-Stokes equations. Comput. Fluids **40**, 249–257 (2011)
26. Tang, Q., Huang, Y.: Local and parallel finite element algorithm based on Oseen-type iteration for the stationary incompressible MHD flow. J. Sci. Comput. **70**, 149–174 (2017)
27. Watson, E.B., Evans, D.V.: Resonant frequencies of a fluid in containers with internal bodies. J. Engrg. Math. **25**(2), 115–135 (1991)
28. Xie, H.: A multigrid method for eigenvalue problem. J. Comput. Phys. **274**, 550–561 (2014)
29. Xu, F., Huang, Q., Ma, H.: Local and parallel multigrid method for semilinear elliptic equations. Appl. Numer. Math. **162**, 20–34 (2021)
30. Xu, J.: Iterative methods by space decomposition and subspace correction. SIAM Rev. **34**(4), 581–613 (1992)
31. Xu, J., Zhou, A.: A two-grid discretization scheme for eigenvalue problems. Math. Comput. **70**(233), 17–25 (2001)
32. Xu, J., Zhou, A.: Local and parallel finite element algorithms based on two-grid discretizations. Math. Comput. **69**(231), 881–909 (1999)
33. Xu, J., Zhou, A.: Local and parallel finite element algorithms based on two-grid discretizations for nonlinear problems. Adv. Comput. Math. **14**, 293–327 (2001)
34. Xu, J., Zhou, A.: Local and parallel finite element algorithms for eigenvalue problems. Acta. Math. Appl. Sin. Engl. Ser. **18**, 185–200 (2002)
35. Xu, J., Zou, J.: Some non-overlapping domain decomposition methods. SIAM Rev. **40**(4), 857–914 (1998)
36. Yao, C., Li, F., Zhao, Y.: Superconvergence analysis of two-grid FEM for Maxwell's equations with a thermal effect. Comput. Math. Appl. **79**(12), 3378–3393 (2020)
37. Yu, J., Shi, F., Zheng, H.: Local and parallel finite element algorithms based on the partition of unity for the Stokes problem. SIAM J. Sci. Comput. **36**(5), C547–C567 (2014)

38. Zhao, R., Yang, Y., Bi, H.: Local and parallel finite element method for solving the biharmonic eigenvalue problem of plate vibration. Numer. Methods Partial Differ. Equ. **35**(2), 851–869 (2019)
39. Zheng, H., Yu, J., Shi, F.: Local and parallel finite element algorithm based on the partition of unity for incompressible flows. J. Sci. Comput. **65**(2), 512–532 (2015)
40. Zheng, H., Shi, F., Hou, Y., Zhao, J., Cao, Y., Zhao, R.: New local and parallel finite element algorithm based on the partition of unity. J. Math. Anal. Appl. **435**(1), 1–19 (2016)