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A posteriori and superconvergence error analysis for finite element approximation of the Steklov eigenvalue problem

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ABSTRACT

In the current paper, we introduce an error analysis method and a new procedure to accelerate the convergence of finite element (FE) approximation of the Steklov eigenvalue problem. The error analysis consists of three steps. First, we introduce an optimal residual type the a posteriori error estimator, and prove its efficiency and reliability. Next, we present a residual type the a priori estimate in terms of derivatives of the eigenfunctions. Finally, we prove accurate the a priori error estimates by combining the a priori residual estimate and the a posteriori error estimates. The new procedure for accelerating the convergence comes from a postprocessing technique, in which we solve an auxiliary source problem on argument spaces. The argument space can be obtained similarly as in the two-space method by increasing the order of polynomials by one. We end the paper by reporting the results of a couple of numerical tests, which allow us to assess the performance of the new error analysis and the postprocessing method.

1. Introduction

The Steklov eigenvalue problems are the eigenvalue problems that the eigenvalue is on the boundary condition. In this paper, we are concerned with the second-order type, which goes as follows,

$$\begin{cases} -\Delta u = 0, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = \lambda u, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here, Ω is a bounded domain with a Lipschitz boundary and n is the unit outward normal on the boundary. Such a problem has an increasing sequence of eigenvalues (see [3]):

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots .$$

Steklov first proposed this problem and studied the bounded domains in the plane in [23]. Since then, the problem was also found in many other physics fields, for instance, in the study of the surface wave [4], in the study of the vibration modes of a structure in contact with an incompressible fluid [5], mechanical oscillators in a viscous fluid [10, 20], in the antiplane shearing on a system of collinear faults [7], etc.

Researchers have applied different numerical methods to deal with this problem. Bramble and Osborn [6], Andreev and Todorov [2] studied the conforming finite element methods for the problem. Yang [28] applied the nonconforming finite element method to the problem and gave the lower bounds for the eigenvalue. Han and Guan [11], Han, Guan and He [12], Huang and Lü [14] and Tang, Guan and Han [24] studied the boundary element method for the problem. Xie [26] and Han, Li, and Xie [13] proposed a multilevel correction method for the problem and largely increased the computation efficiency. Weng, Zhai, and Feng [25] introduced the two-grid method for the problem. In the paper, the standard finite element method is applied to the Steklov eigenvalue problem, and a new error analysis method is developed with the help of the method in [15].

For the main content of the paper, we would like to introduce some works on the topics of the a posterior error analysis of the Steklov eigenvalue problem and the post-processing method as well. Armentano and Padra [1] analyzed the residual type of the a posteriori error estimators for the linear FE approximations and proved the efficiency and reliability. Yang and Bi [27] provided the new local a posteriori error estimates and the local a priori error estimates in $(\|\cdot\|_{1,\Omega_0})$ norm for conforming elements eigenfunction. Russo and Alonso [22] provided

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a posteriori error estimates of the non-conforming Crouzeix–Raviart FE approximations of the Steklov eigenvalue problem. For the post-processing method, Lin and Lü [19], Lin and Lin [18], Lin, Huang and Li [17] applied the Richardson extrapolation for the elliptic eigenvalue problems. Chen and Lin [8] extended the Richardson extrapolation to the Stokes eigenvalue problems. Especially, for the Steklov eigenvalue problem, Li, Lin, and Zhang [16] used the Richardson extrapolation to improve the accuracy of the approximation interpolation, the Rayleigh quotient accelerating techniques, and an interpolation postprocessing method to get the superconvergence results of the bilinear finite element.

In this paper, we mainly focus on two things: new error analysis and a new method for accelerating the convergence of FE approximations of Steklov eigenvalue problems. We provide the new error analysis in three steps. First, we introduce an optimal residual type a posteriori error estimator, and prove its efficiency and reliability. Next, we present a residual type a priori estimate in terms of derivatives of the eigenfunctions. Finally, we prove accurate a priori error estimates by combining the a priori residual estimate and the a posteriori error estimate. The new procedure for accelerating the convergence comes from a postprocessing technique, in which we solve an auxiliary source problem on an argument space. The argument space can be obtained similarly as in the two-space method by increasing the order of the polynomial by one.

The rest of the paper is organized as follows. In Section 2, we introduce the abstractly formulated eigenvalue problem along with the main theorem, which provides the lower eigenvalue bounds. Some results from the previous section are applied to the Steklov eigenvalue problem to obtain lower eigenvalue bounds, taking care to give explicit error estimates for the projection operator in Section 3. Section 4 presents some computation results to demonstrate the efficiency of our proposed method for bounding eigenvalues. Finally, in Section 5, we summarize the results of this paper and discuss issues with the current algorithm.

2. Preliminaries and main results

2.1. The finite element methods

In this subsection, we introduce the FE methods for the problem (1.1).

Mesh. Assume that Ω_h is a family of shape-regular partitions of the domain Ω which is the union of disjoint open element domains K such that the nonempty intersection of a distinct pair of elements is a single node or edge. h_K denotes the diameter of K. As usual, $h=\max_{K\in\Omega_h}$ and $h_K=\mathrm{diam}(K)$.

Edges. The set of edges (or faces) of the partition Ω_h is denoted by $\partial\Omega_h$. Γ_{int} denotes the union of all the interior faces of Ω_h, and the set of faces that are not located in the boundary $\partial\Omega$, i.e.,

$$\partial\Omega_h=\{\partial K : K\in\Omega_h\},\qquad \Gamma_{int}=\bigcup_{K\in\Omega_h}\bigcup_{F\in\partial K\setminus\partial\Omega}F.$$

Jumps. For each element $K\in\Omega_h$ and a function $v\in H^s(\Omega_h)$, we denote the interior (exterior) trace on ∂K by $v_K^+(v_K^-)$. Furthermore, the inner trace and outer trace of the boundary $\partial\Omega$ are defined as follows: $v^+=v(x)$ and $v^-=0$. So the jump $[v]$ and average value $\{v\}$ are naturally introduced

$$[v]|_{F_{ij}}=v|_{K_i}-v|_{K_j},\quad \{v\}_{F_{ij}}=\frac{1}{2}(v|_{K_i}+v|_{K_j}),$$

here $F_{ij}=\partial K_i\cap\partial K_j$ is the common edge.

Spaces, norms and inner-products. Let $(\cdot,\cdot)_K$ and $\langle\cdot,\cdot\rangle_F$ denote the usual scalar products in $L^2(K)$ and $L^2(F)$, and $\|\cdot\|_{L^2(K)}$ and $\|\cdot\|_{L^2(F)}$ the corresponding norms. We also use the following notations:

$$(u,w)_\Omega:=\int_\Omega u(\mathbf{x})w(\mathbf{x})\mathrm{d}\mathbf{x},$$

$$(u,w)_{\Omega_h}:=\sum_{K\in\Omega_h}(u,w)_K=\sum_{K\in\Omega_h}\int_K u(\mathbf{x})w(\mathbf{x})\mathrm{d}\mathbf{x},$$

$$\langle u,w\rangle_{\partial\Omega}:=\sum_{F\in\partial\Omega}\langle u,w\rangle_F=\sum_{F\in\partial\Omega}\int_F u(\mathbf{x})w(\mathbf{x})\mathrm{d}s,$$

$$\langle u,\mathbf{v}\cdot\mathbf{n}\rangle_{\partial\Omega_h}:=\sum_{K\in\Omega_h}\int_{\partial K}u(\mathbf{x})\big(\mathbf{v}(\mathbf{x})\cdot\mathbf{n}\big)\mathrm{d}s,$$

$$\langle u,\mathbf{v}\cdot\mathbf{n}\rangle_{\Gamma_{int}}:=\sum_{F\in\Gamma_{int}}\int_F u(\mathbf{x})\big(\mathbf{v}(\mathbf{x})\cdot\mathbf{n}\big)\mathrm{d}s,$$

where $\mathbf{v}\cdot\mathbf{n}$ is the vector inner product. In addition, $H^s(K)$ is the standard Sobolev space. The associated norm and seminorm are defined, respectively, by

$$\|v\|_{H^s(\Omega_h)}=\left(\sum_{K\in\Omega_h}\|v\|^2_{H^s(K)}\right)^{\frac{1}{2}},$$

and

$$\|v\|_{S,\Omega_h}=\left(\sum_{K\in\Omega_h}\|v\|^2_{S,K}\right)^{\frac{1}{2}},$$

where $\|\cdot\|_{H^s(K)}$ is the Sobolev norm on K, and $\|v\|_{S,K}:=\|\nabla v\|^2_{L^2(K)}$. For simplicity, we denote $\|\cdot\|_S:=\|\cdot\|_{S,\Omega_h}$ and $V:=H^1(\Omega)$.

The finite element space is defined as follows

$$V_h=\{v\in C(\Omega_h) : v|_K\in\mathscr{P}^r(K),\ \ \forall K\in\Omega_h\},$$

where $\mathscr{P}^r(K)$ denotes the set of polynomials of total degree r on the element K.

For $v\in H^s(\Omega_h)$, we define the following mesh-dependent norm

$$\|v\|^2_{\partial\Omega_h}=\sum_{F\in\partial\Omega_h}\int_F[v]^2\mathrm{d}s.$$

FE approximation. We describe the FE approximation of the Steklov model problem (1.1). First, we introduce the classical weak formulation of (1.1): Find the eigenpair $(\lambda,u)\in\mathbb{R}\times V$ satisfying

$$(\nabla u,\nabla v)_\Omega=\lambda\langle u,v\rangle_{\partial\Omega},\quad \forall v\in V. \tag{2.1}$$

Now we define the finite element approximation scheme corresponding with (2.1): Find the eigenpair $(\lambda_h,u_h)\in\mathbb{R}\times V_h$ satisfying

$$(\nabla u_h,\nabla v_h)_{\Omega_h}=\lambda_h\langle u_h,v_h\rangle_{\partial\Omega},\quad \forall v_h\in V_h. \tag{2.2}$$

Projections and Interpolation. In our subsequent error analysis, some appropriate interpolant and projection operators play important roles. Here we recall their properties. First, we list the standard approximation results of [1]: For any $v\in H^s(\Omega)$, there exists an interpolant operator Π such that $\Pi v\in V_h$ and

$$\|v-\Pi v\|_{L^2(K)}\le Ch\|\nabla v\|_{L^2(\hat{K})}, \tag{2.3}$$

$$\|v-\Pi v\|^2_{L^2(F)}\le Ch\|\nabla v\|^2_{L^2(\hat{F})}, \tag{2.4}$$

for any element $K\in\Omega_h$, where $\hat{K}$ is the union of all the elements sharing a vertex with K and $\hat{F}$ is the union of all the elements sharing a vertex with F.

For an arbitrary function $v_h\in V_h$, and an arbitrary element $K\in\Omega_h$, the image $P_\Lambda v_h$ to K is a element of the invariant eigen-subspace $\mathcal{E}(\Lambda)$ associated with the eigenvalue subset Λ that satisfies

$$(P_\Lambda v_h-v_h,u)_K=0,\ \forall u\in\mathcal{E}(\Lambda). \tag{2.5}$$

For any face $F\in\Gamma$ and given a function $\xi\in L^2(\Gamma)$, the image $P_\partial\xi$ to a face F of K is a element of $\mathscr{P}_r(F)$ that satisfies

$$\langle P_\partial\xi-\xi,\mu\rangle_F=0,\ \forall\mu\in\mathscr{P}_r(F). \tag{2.6}$$

Adjoint equations. To prove our error estimates in the approximation of the eigenvalue λ and eigenfunction u, we need to introduce the adjoint equations

$$\begin{cases} -\Delta\phi = \theta, & \text{in } \Omega, \\ \dfrac{\partial \phi}{\partial \mathbf{n}} - \lambda\phi = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.7}$$

with a regularity assumption

$$\|\phi\|_{H^2(\Omega)} \le C\|\theta\|_{L^2(\Omega)}.$$

2.2. Main results

Let $\lambda_i\,(i = 1,2,...,\infty)$ and Λ denote the i-th eigenvalue and a subset of the spectrum of the Steklov eigenvalue problem, respectively. $\mathcal{E}(\Lambda)$ denotes the invariant space associated with the eigenvalues in Λ. For example, $\mathcal{E}(\lambda_i)$ is the eigen-subspace of the individual eigenvalue λ, i.e., $\Lambda = \{\lambda_i\}$.

Let (λ_h, u_h) be an approximation eigenpair with the normalization $\|u_h\|_{L^2(\partial\Omega)} = 1$. We define the error of eigenfunction in u_h with respect to $\mathcal{E}(\Lambda)$ by

$$Q_\Lambda u_h := u_h - P_\Lambda u_h = (I - P_\Lambda)u_h.$$

For example, if Λ is the set of individual eigenvalue λ, then Q_Λ describes the error of the eigen-subspace $\mathcal{E}(\Lambda)$. In order to continue to discuss our error analysis, we need the following assumption: There exists a sufficiently small constant $\delta \in [0,1)$ such that

$$\max_{\lambda_i \notin \Lambda}\left|\frac{\lambda_h - \lambda}{\lambda_i - \lambda}\right| \le \delta, \qquad \|Q_\Lambda u_h\|_{L^2(\Omega)} \le \delta, \qquad \|u_h - P_\partial u_h\|_{L^2(\partial\Omega)} \le \delta. \tag{2.8}$$

Theorem 2.1 *(Reliability). Let $\{\lambda_{j,h}, u_{j,h}\}$ with $\|u_h\|_{L^2(\partial\Omega)} = 1$ be an eigenpair solution of (2.2) and $\{u, \lambda\}$ with $\|u\|_{L^2(\partial\Omega)} = 1$ be a solution of (1.1). Assume that (2.8) holds. Then we have the following the a posterior error estimates: for the error of eigenfunctions,*

$$\|D^m Q_\Lambda u_{j,h}\|_{L^2(\Omega)} \le Ch^{2-m}\sum_{K\in\Omega_h}\mathcal{R}_{K,F}(u_{j,h},\lambda_{j,h}) \qquad m = 0,1.$$

and for the error of the eigenvalues,

$$\lambda_{j,h} - \lambda_j \le Ch^2\sum_{K\in\Omega_h}\mathcal{R}_{K,F}(u_{j,h},\lambda_{j,h}).$$

where the estimator is defined by

$$\mathcal{R}_{K,F}(u_{j,h},\lambda_{j,h}) = CR_K(u_{j,h}) + Ch^{-\frac{1}{2}}R_F(u_{j,h},\lambda_{j,h})$$

where $R_K(u_{j,h}) = \|\Delta u_{j,h}\|_{L^2(K)}$, and

$$R_F(u_{j,h},\lambda_{j,h}) = \begin{cases} \frac{1}{2}\|[\frac{\partial u_{j,h}}{\partial \mathbf{n}}]\|_{L^2(F)}, & F \in \Gamma_{int} \\ \|\frac{\partial u_{j,h}}{\partial \mathbf{n}} - \lambda_{j,h}u_{j,h}\|_{L^2(F)}, & F \in \partial\Omega. \end{cases}$$

According to the results of Theorem 2.1, we provide a natural result for discrete invariant subspaces in the following corollary. These bounds of corollary are very useful when we approximate the eigen-subspace of the multiple eigenvalue or eigenfunctions associated with a small cluster of closed eigenvalues. $\mathcal{E}_h(\Lambda_h)$ denotes the discrete invariant subspace of the discrete eigenvalue subset of Λ_h. We define a natural measure of the distance between the space $\mathcal{E} := \mathcal{E}(\Lambda)$ and the space $\mathcal{E}_h := \mathcal{E}_h(\Lambda_h)$ as follows:

$$\text{dist}(\mathcal{E},\mathcal{E}_h)_{L^2(\Omega)} = \sup_{\substack{\xi\in\mathcal{E}_h \\ \|\xi\|_{L^2(\Omega)}=1}} \|(I - P_\Lambda)\xi\|_{L^2(\Omega)},$$
$$\text{dist}(\mathcal{E},\mathcal{E}_h)_S = \sup_{\substack{\xi\in\mathcal{E}_h \\ \|\xi\|_{L^2(\Omega)}=1}} \|(I - P_\Lambda)\xi\|_S,$$

in the sense of the different norms.

Corollary 2.1. *Let $\mathcal{E}_h$ be an approximation of $\mathcal{E}$ and the conditions of Theorem 2.1 hold. Then for the error between $\mathcal{E}_h$ and $\mathcal{E}$, we have the error estimates:*

$$\text{dist}(\mathcal{E},\mathcal{E}_h)_{L^2(\Omega)} \le Ch^2\sum_{K\in\Omega_h}\mathcal{R}_{K,F}(\mathcal{E}_h,\Lambda_h),$$
$$\text{dist}(\mathcal{E},\mathcal{E}_h)_S \le Ch\sum_{K\in\Omega_h}\mathcal{R}_{K,F}(\mathcal{E}_h,\Lambda_h),$$

where $\mathcal{R}_{K,F}(\mathcal{E}_h,\Lambda_h)$ is the vector function whose j-th components are

$$\sum_{K\in\Omega_h}\mathcal{R}_{K,F}(u_{j,h},\lambda_{j,h}), \quad j = 1,2,\ldots,m,$$

and m is the multiplicity of the eigenvalue.

Theorem 2.2 *(Efficiency). Let the eigenpair $\{u_h,\lambda_h\}$ with $\|u_h\|_{L^2(\partial\Omega)} = 1$ be the solution of (2.2) and the eigenpair $\{u,\lambda\}$ with $\|u\|_{L^2(\partial\Omega)} = 1$ be the solution of (1.1). Assume that (2.8) holds. Then, for the error of the eigenfunction, we have the following lower bound estimate:*

$$\sum_{K\in\Omega_h} h_K\mathcal{R}_{K,F} \le C\|Q_\Lambda u_h\|_S.$$

Now we turn to describe the a priori error estimates.

Theorem 2.3 *(The a priori error estimate). Assume that the conditions of Theorem 2.1 hold. Then, for the eigenfunction error, we have the following a priori estimates:*

$$\|Q_\Lambda u_h\|_{L^2(\Omega)} \le Ch^{2\min\{r+1,s\}}\|P_\Lambda u_h\|_{H^s(\Omega)},$$
$$\|Q_\Lambda u_h\|_S \le Ch^{\min\{r+1,s\}-1}\|P_\Lambda u_h\|_{H^s(\Omega)}.$$

For the eigenvalue error, we have

$$\lambda_h - \lambda \le Ch^{2\min\{r+1,s\}-2}\|P_\Lambda u_h\|_{H^s(\Omega)}.$$

We end Section 2 by illustrating how to exploit the superconvergence property to post-process λ_h and u_h to get a better approximation solution to the eigenvalue λ defined as follows.

We introduce a better approximation $(\hat{\lambda}_h, \hat{u}_h)$ of (λ, u), as the element of $\mathbb{R}\times\widetilde{V}_h$ $(V_h \subset \widetilde{V}_h \subset V)$ presented by the following algorithm.

Algorithm 2.1 *(H). Superconvergence algorithm*

1. *Find the eigenvalue problem (2.2) for $(\lambda_h, u_h) \in \mathbb{R}\times V_h$.*
2. *Find the solution of the following source problem with Neumann boundary condition $\frac{\partial \widetilde{u}_h}{\partial \mathbf{n}} = u_h$: Find $\widetilde{u}_h \in \widetilde{V}_h$ such that*

$$(\nabla\widetilde{u}_h, \nabla v_h)_{\Omega_h} = \langle u_h, v_h\rangle_{\partial\Omega}, \quad \forall v_h \in \widetilde{V}_h. \tag{2.9}$$

3. *Compute $\hat{\lambda}_h = \frac{1}{\langle u_h,\widetilde{u}_h\rangle_{\partial\Omega}}$.*
4. *Evaluate $\hat{u}_h = \hat{\lambda}_h\widetilde{u}_h$.*

The fact that $\hat{\lambda}_h$ provides a better approximation to the eigenvalue λ than λ_h is discussed in the following result.

Theorem 2.4. *Assume the conditions of Theorem 2.1 hold and $\hat{\lambda}_h$ comes from Algorithm 2.1. Then, for the eigenvalue error, we have the following superconvergence*

$$\hat{\lambda}_h - \lambda \le Ch^{2\min\{r+1,s\}-1}\|P_\Lambda u_h\|_{H^s(\Omega)}.$$

3. Proofs of main theorems

3.1. The a posteriori error estimates

The main purpose of this subsection is to present the proof of the reliability (Theorem 2.1) for eigenfunctions and the eigenvalues and efficiency (Theorem 2.2) for the residual error estimator introduced in Section 2. Our idea is to convert the a posteriori error representation into an a posteriori error estimator which is used in adaptive algorithms. In the following, we begin with the first step of the proof that states the following intermediate result for the errors of the eigenvalue and eigenfunction approximations.

Lemma 3.1. *Assume that u_h is the solution of (2.2) and $\theta \in L^2(\Omega)$. Then we have the following identity:*

$$(u_h,\theta)_{\Omega_h} = (-\Delta u_h, \phi-\Pi\phi)_{\Omega_h} + \left\langle \frac{\partial u_h}{\partial \mathbf{n}} - \lambda_h u_h, \phi - \Pi\phi \right\rangle_{\partial\Omega} + \left\langle \left[\frac{\partial u_h}{\partial \mathbf{n}}\right], \phi-\Pi\phi \right\rangle_{\Gamma_{int}} + \sum_{\lambda_i\notin\Lambda} \frac{\lambda-\lambda_h}{\lambda-\lambda_i}(\theta,\varphi_i)_{\Omega_h}\langle u_h,\varphi_i\rangle_{\partial\Omega} \quad (3.1)$$

where $\{\varphi_i\}_{i=1}^{\infty}$ are an orthogonal basis of eigenfunctions in $L^2(\Omega)$ associated with eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$.

Proof. Using the dual problem (2.7) and the integration by parts, we have

$$\begin{aligned}(u_h,\theta)_{\Omega_h} &= (u_h,-\Delta\phi)_{\Omega_h} = -\left\langle u_h, \frac{\partial\phi}{\partial\mathbf{n}}\right\rangle_{\partial\Omega_h} + (\nabla u_h,\nabla\phi)_{\Omega_h}\\
&= -\langle u_h,\lambda\phi\rangle_{\partial\Omega} + \left\langle \frac{\partial u_h}{\partial\mathbf{n}},\phi\right\rangle_{\partial\Omega_h} + (-\Delta u_h,\phi)_{\Omega_h}\\
&= -\lambda\langle u_h,\phi\rangle_{\partial\Omega} + \left\langle \frac{\partial u_h}{\partial\mathbf{n}},\phi\right\rangle_{\partial\Omega_h} + (-\Delta u_h,\phi-\Pi\phi)_{\Omega_h}\\
&\quad + (-\Delta u_h,\Pi\phi)_{\Omega_h}\\
&= -\lambda\langle u_h,\phi\rangle_{\partial\Omega} + \left\langle \frac{\partial u_h}{\partial\mathbf{n}},\phi\right\rangle_{\partial\Omega_h} + (-\Delta u_h,\phi-\Pi\phi)_{\Omega_h}\\
&\quad - \left\langle \frac{\partial u_h}{\partial\mathbf{n}},\Pi\phi\right\rangle_{\partial\Omega_h} + (\nabla u_h,\nabla\Pi\phi)_{\Omega_h}\\
&= -\lambda\langle u_h,\phi\rangle_{\partial\Omega} + \left\langle \frac{\partial u_h}{\partial\mathbf{n}},\phi\right\rangle_{\partial\Omega_h} + (-\Delta u_h,\phi-\Pi\phi)_{\Omega_h}\\
&\quad - \left\langle \frac{\partial u_h}{\partial\mathbf{n}},\Pi\phi\right\rangle_{\partial\Omega_h} + \lambda_h\langle u_h,\Pi\phi\rangle_{\partial\Omega}\end{aligned}$$

by the integration of parts and the discrete problem (2.2). Next, we consider the combination of the integrations on the boundary, i.e.,

$$\left\langle \frac{\partial u_h}{\partial\mathbf{n}},\phi\right\rangle_{\partial\Omega_h} - \left\langle \frac{\partial u_h}{\partial\mathbf{n}},\Pi\phi\right\rangle_{\partial\Omega_h} = \left\langle \left[\frac{\partial u_h}{\partial\mathbf{n}}\right],\phi-\Pi\phi\right\rangle_{\Gamma_{int}} + \left\langle \frac{\partial u_h}{\partial\mathbf{n}},\phi-\Pi\phi\right\rangle_{\partial\Omega}$$

and

$$\langle\lambda_h u_h,\Pi\phi\rangle_{\partial\Omega} - \langle\lambda u_h,\phi\rangle_{\partial\Omega} = \langle\lambda_h u_h,\Pi\phi-\phi\rangle_{\partial\Omega} + (\lambda_h-\lambda)\langle u_h,\phi\rangle_{\partial\Omega}.$$

Inserting the above two identities into the first equation induces that

$$(u_h,\theta)_{\Omega_h} = (-\Delta u_h,\phi-\Pi\phi)_{\Omega_h} + \left\langle \frac{\partial u_h}{\partial\mathbf{n}} - \lambda_h u_h,\phi-\Pi\phi\right\rangle_{\partial\Omega} + \left\langle \left[\frac{\partial u_h}{\partial\mathbf{n}}\right],\phi-\Pi\phi\right\rangle_{\Gamma_{int}} + (\lambda_h-\lambda)\langle u_h,\phi\rangle_{\partial\Omega}.$$

For the above representation, we need to express the last term, i.e. the solution of dual problem (2.7) can be expressed by θ and the eigenfunctions ψ_i.

Assume that $\check{\lambda}\notin\Lambda$, $\lambda\in\Lambda$ and (λ,u) and $(\check{\lambda},\phi)$ are the solutions of the problem (1.1) and the dual problem (2.7), respectively. By using the problem (1.1), the integration by parts, and the dual problem (2.7), we have

$$(-\Delta\phi,u)_\Omega = -\check{\lambda}\langle\phi,u\rangle_{\partial\Omega} + \lambda\langle\phi,u\rangle_{\partial\Omega} + (\phi,-\Delta u)_\Omega = (\lambda-\check{\lambda})\langle\phi,u\rangle_{\partial\Omega}.$$

On the other hand, using the dual problem (2.7) with $\theta = u_h - P_\Lambda u_h = Q_\Lambda u_h$, we have

$$(-\Delta\phi,u)_\Omega = (\theta,u)_\Omega = (u_h - P_\Lambda u_h, u)_\Omega = 0,$$

by the definition of the projection P_Λ. From the above two equations, $\check{\lambda}\notin\Lambda$ and $\lambda\in\Lambda$, we have

$$\langle\phi,u\rangle_{\partial\Omega} = 0.$$

i.e. the dual solution ϕ is orthogonal to an arbitrary eigenfunction of subspace $\mathcal{E}(\Lambda)$ in the sense of inner product $\langle\cdot,\cdot\rangle_{\partial\Omega}$. Since $\{\varphi_i\}_{i=1}^{\infty}$ are an orthogonal basis of eigenfunctions in L^2 associated with eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$, ϕ can be expressed by

$$\phi = \sum_{\lambda_i\notin\Lambda} x_i\varphi_i,$$

where x_i is the unknown coefficient. To obtain them, we need to discuss the property of eigenfunction φ_i on the boundary. Since φ_i and φ_j are the solutions to the problem (1.1) with the differential eigenvalues λ_i and λ_j, we have that

$$\langle\varphi_i,\varphi_j\rangle_{\partial\Omega} = \frac{1}{\lambda_i}\int_{\partial\Omega}\frac{\partial\varphi_i}{\partial\mathbf{n}}\varphi_j = \frac{1}{\lambda_i}\int_\Omega \nabla\varphi_i\nabla\varphi_j = \frac{\lambda_j}{\lambda_i}\langle\varphi_i,\varphi_j\rangle_{\partial\Omega},$$

by Gauss formula, which implies that

$$\langle\varphi_i,\varphi_j\rangle_{\partial\Omega} = 0.$$

It follows from $\phi = \sum_{\lambda_i\notin\Lambda} x_i\varphi_i$ that

$$\begin{aligned}x_i &= \langle\phi,\varphi_i\rangle_{\partial\Omega} = \langle\phi,\frac{1}{\lambda_i}\frac{\partial\varphi_i}{\partial\mathbf{n}}\rangle_{\partial\Omega}\\
&= \frac{1}{\lambda_i}\int_\Omega(\Delta\varphi_i)\phi + \frac{1}{\lambda_i}\int_\Omega\nabla\varphi_i\cdot\nabla\phi \quad \text{(by Gauss formula)}\\
&= \frac{1}{\lambda_i}\int_{\partial\Omega}\frac{\partial\phi}{\partial\mathbf{n}}\varphi_i - \frac{1}{\lambda_i}\int_\Omega\varphi_i(\Delta\phi) \quad \text{(by } \Delta\varphi_i = 0 \text{ and integration by parts)}\\
&= \frac{\lambda}{\lambda_i}\langle\phi,\varphi_i\rangle_{\partial\Omega} - \frac{1}{\lambda_i}(\theta,\varphi_i)_\Omega. \quad \text{(by the dual problem)}\end{aligned}$$

So we have

$$x_i = \frac{(\varphi_i,\theta)_\Omega}{\lambda-\lambda_i}, \text{ i.e., } \phi = \sum_{i\notin\Lambda}\frac{(\theta,\varphi_i)_\Omega}{\lambda_i-\lambda}\varphi_i.$$

Inserting the above expression into $(\lambda_h-\lambda)\langle u_h,\phi\rangle_{\partial\Omega}$ completes the proof. □

From Lemma 3.1, it is easy to see the first three terms are easy to estimate by the approximation property and the stability of the dual problem, while the last term will be tackled in the next lemma.

Lemma 3.2. *Assume that u_h and ϕ are the solutions of (2.2) and (2.7) and the projections P_Λ and P_∂ are defined in Section 2. Then*

$$|\langle u_h,\phi\rangle_{\partial\Omega}| \le Ch^2 R_K(u_h,\lambda_h)_{\Omega_h} + \max_{\lambda_i\notin\Lambda}\left|\frac{\lambda-\lambda_h}{\lambda-\lambda_i}\right|^2 \|D^m(u_h-P_\Lambda u_h)\|^2_{L^2(\Omega)}. \quad (3.2)$$

Proof. It follows from the fact that φ_i is orthogonal to the subspace $\mathcal{E}(\Lambda)$, $P_\partial u_h$ and $P_\Lambda u_h \in \mathcal{E}(\Lambda)$ that

$$\langle u_h,\phi\rangle_{\partial\Omega}=\sum_{\lambda_i\notin\Lambda}\frac{(\theta,\varphi_i)_\Omega}{\lambda-\lambda_i}\langle u_h,\varphi_i\rangle_{\partial\Omega}=\sum_{\lambda_i\notin\Lambda}\frac{(\theta,\varphi_i)_\Omega}{\lambda-\lambda_i}\langle u_h-P_\partial u_h,\varphi_i\rangle_{\partial\Omega}$$
$$\overset{\theta=(-\Delta)^m Q_\Lambda u_h}{=\!=\!=}\sum_{\lambda_i\notin\Lambda}\frac{\lambda_i^m\langle Q_\Lambda u_h,\varphi_i\rangle}{\lambda-\lambda_i}\langle u_h-P_\partial u_h,\varphi_i\rangle_{\partial\Omega}$$
$$\le\max_{\lambda_i\notin\Lambda}\frac{1}{|\lambda-\lambda_i|}\left(\sum_{\lambda_i\notin\Lambda}\lambda_i^{2m}\langle Q_\Lambda u_h,\varphi_i\rangle_{\Omega_h}^2\right)^{\frac12}$$
$$\times\left(\sum_{\lambda_i\notin\Lambda}\langle u_h-P_\partial u_h,\varphi_i\rangle_{\partial\Omega}^2\right)^{\frac12}$$
$$\le C\max_{\lambda_i\notin\Lambda}\frac{1}{|\lambda-\lambda_i|}\|D^mQ_\Lambda u_h\|_{L^2(\Omega)}\|u_h-P_\partial u_h\|_{L^2(\partial\Omega)},\qquad(3.3)$$

where we use the facts that $(\nabla Q_\Lambda u_h,\nabla\varphi_i)_\Omega=\lambda_i\langle Q_\Lambda u_h,\varphi_i\rangle_{\partial\Omega}$ and

$$\sum_{\lambda_i\notin\Lambda}\lambda_i^{2m}\langle Q_\Lambda u_h,\varphi_i\rangle_{\partial\Omega}^2=\sum_{\lambda_i\notin\Lambda}(\nabla Q_\Lambda u_h,\nabla\varphi_i)_\Omega^2$$
$$\le\|DQ_\Lambda u_h\|_{L^2(\Omega)}^2\sum_{\lambda_i\notin\Lambda}\|\nabla\varphi_i\|_{L^2(\Omega)}^2$$
$$=C\|DQ_\Lambda u_h\|_{L^2(\Omega)}^2.$$

We are now ready to bound $\|u_h-P_\partial u_h\|_{L^2(\partial\Omega)}$ in (3.3). Consider the following auxiliary problem

$$\begin{cases}-\Delta\phi_1&=0,\quad\text{in }\Omega,\\ \frac{\partial\phi_1}{\partial\mathbf{n}}-\lambda\phi_1&=\theta_1,\quad\text{on }\partial\Omega.\end{cases}\qquad(3.4)$$

The following identity plays an important role in bounding $\|u_h-P_\partial u\|_{L^2(\partial\Omega)}$. Using similar techniques as in Lemma 3.1, we have

$$\langle u_h,\theta_1\rangle_{\partial\Omega}=\langle u_h,\frac{\partial\phi_1}{\partial\mathbf{n}}\rangle_{\partial\Omega}-\langle u_h,\lambda\phi_1\rangle_{\partial\Omega}$$
$$=(u_h,\Delta\phi_1)_{\Omega_h}+(\nabla u_h,\nabla\phi_1)_{\Omega_h}-\langle u_h,\lambda\phi_1\rangle_{\partial\Omega}$$
$$=(-\Delta u_h,\phi_1-\Pi\phi_1)_{\Omega_h}-\left\langle\frac{\partial u_h}{\partial\mathbf{n}},\phi_1-\Pi\phi_1\right\rangle_{\Gamma_{int}}$$
$$-\left\langle\frac{\partial u_h}{\partial\mathbf{n}}-\lambda_hu_h,\phi_1-\Pi\phi_1\right\rangle_{\partial\Omega}$$
$$-(\lambda_h-\lambda)\langle u_h,\phi_1\rangle_{\partial\Omega},$$

by using the Gauss formula in the second and the third lines, and fact $\Delta\phi_1=0$, and the discrete problem (2.2) in the third line. It follows from the problem (3.4), integration by parts, the orthogonality of the projection P_∂ and $\theta_1|_{\partial\Omega}=(u_h-P_\partial u_h)|_{\partial\Omega}$ that

$$\langle\varphi_i,\phi_1\rangle_{\partial\Omega}=\frac1\lambda\left\langle\varphi_i,\frac{\partial\phi_1}{\partial\mathbf{n}}\right\rangle_{\partial\Omega}-\frac1\lambda\langle\varphi_i,\theta_1\rangle_{\partial\Omega}=\frac{\lambda_i}{\lambda}\langle\varphi_i,\phi_1\rangle_{\partial\Omega}.$$

By $\lambda\notin\Lambda$ and $\lambda_i\in\Lambda$, we have, for $u\in\mathcal{E}(\Lambda)$, $\langle u,\phi_1\rangle_{\partial\Omega}=0$. For $\langle u_h,\phi_1\rangle$, we again use the same techniques as in Lemma 3.1 in order to conclude that

$$\langle u_h,\phi_1\rangle_{\partial\Omega}\le\max_{\lambda_i\notin\Lambda}\frac{1}{|\lambda-\lambda_i|}\|D^m(u_h-P_\Lambda u_h)\|_{L^2(\Omega)}\|u_h-P_\partial u_h\|_{L^2(\Omega)}.$$

At the same time, taking $\theta_1|_{\partial\Omega}=(u_h-P_\partial u_h)|_{\partial\Omega}$ and using the definition of P_∂, we have

$$\langle u_h,\theta_1\rangle_{\partial\Omega}=\langle u_h,u_h-P_\partial u_h\rangle_{\partial\Omega}=\|u_h-P_\partial u_h\|_{L^2(\partial\Omega)}^2.$$

Combining all the intermediate steps implies that

$$\|u_h-P_\partial u_h\|_{L^2(\partial\Omega)}^2\le|(-\Delta u_h,\phi_1-\Pi\phi_1)_{\Omega_h}|+\left|\left\langle\frac{\partial u_h}{\partial\mathbf{n}},\Pi\phi_1-\phi_1\right\rangle_{\Gamma_{int}}\right|$$
$$+\left|\left\langle\frac{\partial u_h}{\partial\mathbf{n}}-\lambda_hu_h,\phi_1-\Pi\phi_1\right\rangle_{\partial\Omega}\right|$$
$$+\max_{\lambda_i\notin\Lambda}\left|\frac{\lambda-\lambda_h}{\lambda-\lambda_i}\right|\|D^m(u_h-P_\Lambda u_h)\|_{L^2(\Omega)}\|u_h-P_\partial u_h\|_{L^2(\partial\Omega)}.$$

We estimate the first term of the above inequality by the Cauchy-Schwartz inequality and the approximation property (2.3) to get

$$|(-\Delta u_h,\phi_1-\Pi\phi_1)_{\Omega_h}|\le\|\Delta u_h\|_{L^2(\Omega)}\cdot Ch^2\|\phi_1\|_{H^2(\Omega)}.$$

Next, we turn to bound the second and the third terms. It follows from the approximation property (2.4) and again the Cauchy-Schwartz inequality that we have

$$\left|\left\langle\left[\left[\frac{\partial u_h}{\partial\mathbf{n}}\right]\right],\phi_1-\Pi\phi_1\right\rangle_{\Gamma_{int}}\right|+\left|\left\langle\frac{\partial u_h}{\partial\mathbf{n}}-\lambda_hu_h,\phi_1-\Pi\phi_1\right\rangle_{\partial\Omega}\right|$$
$$\le C\left\|\left[\left[\frac{\partial u_h}{\partial\mathbf{n}}\right]\right]\right\|_{\Gamma_{int}}h^{\frac32}\|\phi_1\|_{H^2}+C\left\|\frac{\partial u_h}{\partial\mathbf{n}}-\lambda_hu_h\right\|_{L^2(\partial\Omega)}h^{\frac32}\|\phi_1\|_{H^2(\Omega)}.$$

By using the simple and miscellaneous calculation for the above threes inequalities, we have

$$\|u_h-P_\partial u_h\|_{L^2(\partial\Omega)}^2\le Ch^2R_K(u_h,\lambda_h)+\max_{\lambda_i\notin\Lambda}\left|\frac{\lambda-\lambda_h}{\lambda-\lambda_i}\right|^2\|D^m(u_h-P_\Lambda u_h)\|_{L^2(\Omega)}^2.$$

Substituting the above inequality into (3.3) completes the proof. □

Now we first present the residual of the approximation solution according to the identity in Lemma 3.1. Then we bound the right-hand sides in the error expression formulas in terms of the residuals and the regularity assumption of the dual problem.

The residuals are the combination of the following three parts: the residual of equation $\|\Delta u_h\|_{L^2(\Omega)}$ arising from the element domain, the jump $\left\|\left[\left[\frac{\partial u_h}{\partial\mathbf{n}}\right]\right]\right\|_{L^2(\Gamma_{int})}$ describing the magnitude of the discontinuous of the normal derivatives across the interior edges of the element and the residual $\left\|\frac{\partial u_h}{\partial\mathbf{n}}-\lambda_hu_h\right\|_{L^2(\partial\Omega)}$ of the boundary condition arising from the boundary of domain Ω.

The proof of Theorem 2.1. For the residual in (3.1), we obtain similar results by similar techniques in Lemma 3.2. Then we aim to bound $\|D^m(u_h-P_\Lambda u_h)\|_{L^2(\Omega)}=\|D^mQ_\Lambda u_h\|_{L^2(\Omega)}$.

Taking $\theta=(-\Delta)^mQ_\Lambda u_h$ and using the definition of P_Λ, we have

$$(u_h,\theta)_{\Omega_h}=\left(u_h,(-\Delta)^mQ_\Lambda u_h\right)_{\Omega_h}=\|D^mQ_\Lambda u_h\|_{L^2(\Omega)},\quad m=0,1.$$

Inserting the above equation and (3.2) into (3.1) and using the approximation properties (2.3)–(2.4) and Cauchy-Schwartz inequality imply the desired results for the eigenfunction.

Finally, we turn to the error estimate of the eigenvalue. The main idea of error estimate is to express the eigenvalue error representation formula similar to the eigenfunction.

To obtain a representation formula of eigenvalue error, we choose $\theta=0$ in the dual problem, which implies that (3.1) can be written as follows:

$$0=(-\Delta u_h,\phi-\Pi\phi)_{\Omega_h}+\left\langle\frac{\partial u_h}{\partial\mathbf{n}}-\lambda_hu_h,\phi-\Pi\phi\right\rangle_{\partial\Omega}$$
$$+\left\langle\left[\frac{\partial u_h}{\partial\mathbf{n}}\right],\phi-\Pi\phi\right\rangle_{\Gamma_{int}}+(\lambda_h-\lambda)\langle u_h,\phi\rangle_{\partial\Omega}.$$

On the other hand, we observe the fact that the Steklov eigenvalue problem (1.1) is the same as the dual problem with $\theta=0$, i.e., $P_\partial u_h=\phi$ is the solution of the dual problem. Further, using the assumption (2.8), we have

$$\langle u_h,\phi\rangle_{\partial\Omega}=\langle u_h,P_\partial u_h\rangle_{\partial\Omega}=\langle u_h-P_\partial u_h,P_\partial u_h\rangle_{\partial\Omega}+\langle P_\partial u_h,P_\partial u_h\rangle_{\partial\Omega}$$
$$=\|P_\partial u_h\|_{L^2(\partial\Omega)}^2=1-\|u_h-P_\partial u_h\|_{L^2(\partial\Omega)}^2\ge1-\delta.$$

Inserting this inequality into the identity above, we obtain the eigenvalue error representation formula

$$(1-\delta)(\lambda_h-\lambda)\le -(-\Delta u_h,\phi-\Pi\phi)_{\Omega_h}-\left\langle\left[\frac{\partial u_h}{\partial \mathbf{n}}\right],\phi-\Pi\phi\right\rangle_{\Gamma_{int}} - \left\langle\frac{\partial u_h}{\partial \mathbf{n}}-\lambda_h u_h,\phi-\Pi\phi\right\rangle_{\partial\Omega}.$$

For the above inequality, using the approximation property (2.3)–(2.4) and Cauchy–Schwartz inequality completes the proof of Theorem 2.1. □

The remainder of this subsection is devoted to the proof of efficiency for the error estimators $R_K(u_{j,h})$ and $R_F(\lambda_{j,h},u_{j,h})$. The efficiency ensures that the true error will be an upper bound for the resulting estimators up to a generic constant and some high-order terms.

The proof relies on the bubble functions which are developed in [29]. In general, they are positive, smooth, real-valued, local compact supports and bounded by 1 in the sense of the L^∞-norm. In order to describe some properties of the bubble function, we introduce the denotations as follows.

Let b_K be the standard polynomial bubble function with support in the element K which vanished on the edge of K. Similarly, for any interior edge F, the polynomial bubble function is denoted by b_F which vanishes outside the closure of $K_F^+\cup K_F^-$, where K_F^+ and K_F^- are the two adjacent elements sharing the common edge or face F. Some properties of the bubble function are collected in the following lemma which is seen in Lemma 3.3 from [29].

Property 3.1. *For all polynomial functions $v\in\mathscr{P}^r(K)$ and the bubble functions b_K and b_F, we have*

$$\|b_K v\|_{L^2(K)}\le C\|v\|_{L^2(K)}, \tag{3.5}$$

$$\|v\|_{L^2(K)}\le C\|b_K^{\frac12}v\|_{L^2(K)}, \tag{3.6}$$

$$\|b_K v\|_{S,K}\le h_K^{-1}\|v\|_{L^2(K)}. \tag{3.7}$$

For all polynomial functions $\omega\in\mathscr{P}^r(F)$, we have

$$\|b_F\omega\|_{L^2(F)}\le C\|\omega\|_{L^2(F)}, \tag{3.8}$$

$$\|\omega\|_{L^2(F)}\le C\|b_F^{\frac12}\omega\|_{L^2(F)}, \tag{3.9}$$

$$\|b_F\omega\|_{L^2(K_F^+\cup K_F^-)}\le Ch_F^{\frac12}\|\omega\|_{L^2(F)}, \tag{3.10}$$

$$\|b_F\omega\|_{S,K_F^+\cup K_F^-}\le Ch_F^{-\frac12}\|\omega\|_{L^2(F)}. \tag{3.11}$$

Lemma 3.3. *Assume that $(\lambda_{j,h},u_{j,h})$ is an eigenpair solution of (2.2) which converges to eigenvalue λ_j. Then we have the following local bound:*

$$\left(\sum_{K\in\Omega_h}h_K^2\|\Delta u_h\|_{L^2(K)}^2\right)^{\frac12}\le C\mathrm{dist}(u_{j,h},E(\lambda_j))_{S,\Omega_h},$$

where u_j be the minimizer of $\mathrm{dist}(v,V)_{E,\Omega_h}=\min_{\omega\in V}\|v-\omega\|_{S,\Omega_h}$.

Proof. For arbitrary element K, let $v_h|_K=h_K^2\Delta u_{j,h}\cdot b_K$, then

$$h_K^2R_K^2(u_{j,h})=h_K^2\|\Delta u_{j,h}\|_{L^2(K)}^2\le C\int_K\Delta u_{j,h}\cdot v_h dx$$

by using (3.6). Since $\Delta u_j=0$ is satisfied in a distributed sense, we then have

$$\begin{aligned}R_K^2(u_{j,h})&\le C\int_K\big(\Delta(u_{j,h}-u_j)\big)v_h dx\\&=-C\int_K\nabla(u_{j,h}-u_j)\cdot\nabla v_h dx\end{aligned}$$

by the integration of parts and the fact that $v_h|_{\partial K}=0$. Using Cauchy-Schwartz inequality, inverse inequality, and (3.5), we have

$$\begin{aligned}R_K^2(u_{j,h})&\le C\|\nabla(u_{j,h}-u_j)\|_{L^2(K)}\cdot\|\nabla v_h\|_{L^2(K)}\\&\le C\|\nabla(u_{j,h}-u_j)\|_{L^2(K)}\cdot Ch_K^{-1}\|v_h\|_{L^2(K)}\\&\le Ch_K^{-1}\|\nabla(u_{j,h}-u_j)\|_{L^2(K)}\cdot h_K^2\|\Delta u_{j,h}\|_{L^2(K)}.\end{aligned}$$

Using the fact $R_K(u_{j,h})=\|\Delta u_{j,k}\|_{L^2(K)}$, we end up with

$$R_K(u_h)\le Ch_K\|\nabla(u_{j,h}-u_j)\|_{L^2(K)}.\quad\square$$

Lemma 3.4. *Let $(\lambda_{j,h},u_{j,h})$ be an eigenpair solution of the problem (2.2) which converges to λ_j of multiplicity $R\ge1$. Then we have*

$$\begin{aligned}R_F(u_{j,h},\lambda_{j,h})=&\left(\sum_{F\in\Gamma_{int}}h_K\left\|\frac{\partial u_{j,h}}{\partial\mathbf{n}}\right\|_{L^2(F)}^2\right)^{\frac12}\\&+\left(\sum_{F\in\partial\Omega}h_K\left\|\frac{\partial u_{j,h}}{\partial\mathbf{n}}-\lambda u_{j,h}\right\|_{L^2(F)}^2\right)^{\frac12}\\&\le\mathrm{dist}(u_{j,h},E(\lambda_j))_{S,\Omega_h}+h\|\lambda_ju_j-\lambda_{j,h}u_{j,h}\|_{L^2(\partial\Omega)}.\end{aligned}$$

Proof. For the interior faces and domain faces, taking $v_h|_{F\in\Gamma_{int}}=h_F\left[\frac{\partial u_{j,h}}{\partial\mathbf{n}}\right]b_F$ and $v_h|_{F\in\partial\Omega}=h_F\left(\frac{\partial u_h}{\partial\mathbf{n}}-\lambda_hu_h\right)b_F$ respectively, and using (3.9), we have

$$\left(\sum_{F\in\Gamma_{int}}h_F\left\|\frac{\partial u_{j,h}}{\partial\mathbf{n}}\right\|_{L^2(F)}^2\right)^{\frac12}\le C\sum_{F\in\Gamma_{int}}\int_F\left[\frac{\partial u_{j,h}}{\partial\mathbf{n}}\right]v_h\mathrm{d}s,$$

and

$$\left(\sum_{F\in\partial\Omega}h_F\left\|\frac{\partial u_{j,h}}{\partial\mathbf{n}}-\lambda u_{j,h}\right\|_{L^2(F)}^2\right)^{\frac12}\le C\sum_{F\in\partial\Omega}\int_F\left(\frac{\partial u_{j,h}}{\partial\mathbf{n}}-\lambda_{j,h}u_{j,h}\right)v_h\mathrm{d}s.$$

Using the fact that $\left[\frac{\partial u}{\partial\mathbf{n}}\right]\Big|_F=0$ for any face F and $\frac{\partial u_j}{\partial\mathbf{n}}|_{F\in\partial\Omega}=\lambda_ju_j$, we have

$$\begin{aligned}R_F^2(u_{j,h},\lambda_{j,h})\le&\,C\sum_{F\in\Gamma_{int}}\int_F\left[\frac{\partial u_{j,h}}{\partial\mathbf{n}}-\frac{\partial u_j}{\partial\mathbf{n}}\right]\cdot v_h\mathrm{d}s\\&+C\sum_{F\in\partial\Omega}\int_F\left(\frac{\partial u_{j,h}}{\partial\mathbf{n}}-\frac{\partial u_j}{\partial\mathbf{n}}+\lambda_ju_j-\lambda_{j,h}u_{j,h}\right)v_h\mathrm{d}s\\=&\,C\sum_{F\in\Gamma_{int}}\int_{K_F^+\cup K_F^-}\Big((\Delta u_{j,h})v_h+\nabla(u_{j,h}-u_j)\cdot\nabla v_h\Big)\mathrm{d}\mathbf{x}\\&+C\sum_{F\in\partial\Omega}\int_F(\lambda_ju_j-\lambda_{j,h}u_{j,h})v_h\mathrm{d}s,\end{aligned}$$

by using the Gauss formula and $\Delta u_j=0$ in the third line. We bound the right-side hand of the term by term in the above equation. For the first term, it follows from the above Lemma 3.3, Cauchy-Schwartz inequality, and (3.10) that

$$\begin{aligned}&\sum_{F\in\Gamma_{int}}\int_{K_F^+\cup K_F^-}\Delta u_{j,h}\cdot v_h dx\\&\le C\sum_{F\in\Gamma_{int}}h_K\|\Delta u_{j,h}\|_{L^2(K_F^+\cup K_F^-)}\cdot h_K^{-1}\|v_h\|_{L^2(K_F^+\cup K_F^-)}\\&\le C\sum_{F\in\Gamma_{int}}\|u_{j,h}-u_j\|_{S,K_F^+\cup K_F^-}\cdot h_K^{-1}\|v_h\|_{L^2(K_F^+\cup K_F^-)}\\&\le C\left(\sum_{K\in\Omega_h}\|u_{j,h}-u_j\|_{S,K}^2\right)^{\frac12}\cdot\left(\sum_{F\in\Gamma_{int}}h_K^{-2}\|v_h\|_{L^2(K_F^+\cup K_F^-)}^2\right)^{\frac12}\\&\le\|u_{j,h}-u_j\|_S\cdot R_F(u_{j,h},\lambda_{j,h}).\end{aligned}$$

For the second term, using (3.11) and again Cauchy-Schwartz inequality, we have

$$\sum_{F\in\Gamma_{int}}\int_{K_F^+\cup K_F^-}\nabla(u_{j,h}-u_j)\cdot\nabla v_h\mathrm{d}\mathbf{x}$$
$$\leq \|u_{j,h}-u_j\|_S\|v_h\|_S\leq C\|u_{j,h}-u_j\|_S\cdot R_F(u_{j,h},\lambda_{j,h}).$$

For the last term, using (3.8), we have

$$\sum_{F\in\partial\Omega}\int_F(\lambda_j u_j-\lambda_{j,h}u_{j,h})v_h\mathrm{d}s$$
$$\leq C\sum_{F\in\partial\Omega}\|\lambda_j u_j-\lambda_{j,h}u_{j,h}\|_{L^2(F)}\left\|\frac{\partial u_{j,h}}{\partial\mathbf{n}}-\lambda_{j,h}u_{j,h}\right\|_{L^2(F)}$$
$$\leq Ch\|\lambda_j u_j-\lambda_{j,h}u_{j,h}\|_{L^2(\partial\Omega)}\cdot\left\|\frac{\partial u_{j,h}}{\partial\mathbf{n}}-\lambda_{j,h}u_{j,h}\right\|_{L^2(\partial\Omega)}.$$

Combining the all above intermediate steps completes the proof. □

The proof of Theorem 2.2

Proof. The efficiency in Theorem 2.2 follows immediately from Lemma 3.3 and Lemma 3.4. □

3.2. A priori error estimate

In this subsection, we present the proof of the a posteriori-priori error results for the standard finite element method applied to eigenvalue problems. For the FE approximation of the problem (1.1) by the discrete problem (2.2), our priori error analysis depends on the following Ritz projection $R_h : H^1(\Omega)\to V_h$, defined by

$$(\nabla u,\nabla v)_\Omega=(\nabla R_h u,\nabla v)_\Omega,\quad \forall v\in V_h. \tag{3.12}$$

The main implementation in the a priori error analysis of Theorem 2.3 is the following approximation proposition: the weighted a priori error estimate of the Ritz projection.

Property 3.2 *([15]). Assume that the partition is uniform. If $v\in H^s, 1\leq s\leq p+1$. Then we have*

$$\|h^{\alpha-1}D(I-R_h)v\|_{L^2(\Omega)}\leq C\|h^{\alpha+s-2}D^{\alpha+s-1}v\|_{L^2(\Omega)},\quad 1\leq s\leq p+1,$$

where $\alpha=1,2,3$.

The proof of Theorem 2.3:

Proof. To prove the result, we use $\widehat{V}$ to denote $H^2\cap\ker\Delta$. Let $T: L^2(\Omega)\to\widehat{V}$ with $\frac{\partial Tv}{\partial\mathbf{n}}=\lambda_h v$ for any $v\in L^2(\Omega)$. Set $u=Tu_h$ in Ritz projection (3.12), we have

$$\begin{aligned}(\nabla R_hTu_h,\nabla v_h)_{\Omega_h}&=(\nabla Tu_h,\nabla v_h)_{\Omega_h}\\&=\int_{\partial\Omega}\frac{\partial Tu_h}{\partial\mathbf{n}}v_h\mathrm{d}s-\int_\Omega\Delta Tu_h\cdot v_h\mathrm{d}\mathbf{x}\\&=\lambda_h\langle u_h,v_h\rangle_{\partial\Omega},\end{aligned}$$

by the integration of parts and the definition of the operator T. Using the approximation problem and the above equation, we induce that

$$u_h=R_hTu_h.$$

So we can express the interior part of the residual as follows

$$\Delta u_h=\Delta R_hTu_h-\Delta Tu_h=\Delta(R_h-I)Tu_h.$$

The rest of the proof is similar to the proof of [15, Theorem 4.2], and it is omitted for brevity. □

3.3. Postprocessing

In this subsection, employing the idea developed in [21], we extend the methods to the FE approximation of the Steklov eigenvalue problem (1.1). We describe and prove a postprocessing algorithm that presents a better approximation with a superconvergence for the post-processed eigenvalues. The essence of our new method consists of the following two steps: First, solve the finite element approximation of the eigenvalue problem for a given finite element space V_h. Next, solve the additional source problem with the Neumann condition $\frac{\partial\widetilde{u}_h}{\partial\mathbf{n}}=u_h$ on an argument space. We introduce an additional FE space $\widetilde{V}_h$ such that

$$V_h\subset\widetilde{V}_h=\{v: v\in H^1(\Omega), v|_K\in\mathscr{P}^{r+1}(K),\forall K\in\Omega_h\}\subset V.$$

Considering the following finite element approximation of elliptic problem (source problem): Find $\widetilde{u}_h\in\widetilde{V}_h$ such that

$$(\nabla\widetilde{u}_h,\nabla v_h)_{\Omega_h}=\langle u_h,v_h\rangle_{\partial\Omega},\quad\forall v_h\in\widetilde{V}_h. \tag{3.13}$$

After doing the necessary work, now we describe a postprocessing Algorithm 2.1, which will present a much better approximation $\widehat{\lambda}_h$ of the eigenvalues. In order to prove the better approximation property of $\widehat{\lambda}_h$, we introduce an auxiliary problem defined as follows: Find the solution $\widetilde{u}\in V$ such that

$$(\nabla\widetilde{u},\nabla v)_{\Omega_h}=\langle u_h,v\rangle_{\partial\Omega},\quad\forall v\in V \tag{3.14}$$

Then we can evaluate the real number

$$\widetilde{\lambda}=\frac{1}{\langle u_h,\widetilde{u}\rangle_{\partial\Omega}}. \tag{3.15}$$

Lemma 3.5. *(λ,u) and (λ_h,u_h) are two eigenpairs of problem (1.1) and discrete problem (2.2), respectively. Assume that $\|u\|_{L^2(\partial\Omega)}=\|u_h\|_{L^2(\partial\Omega)}=1$. Assume that $\widetilde{u}$ be the solution of the source problem (3.14) with Neumann boundary condition $\frac{\partial\widetilde{u}}{\partial\mathbf{n}}|_{\partial\Omega}=u_h$. $\widetilde{\lambda}$ is computed by (3.15). Then*

$$|\lambda-\widetilde{\lambda}|\leq C\|u-u_h\|^2_{L^2(\partial\Omega)}.$$

Proof. Consider the Neumann problem with Neumann boundary condition $g\in L^2(\partial\Omega)$: Find $\xi\in\widehat{V}$ such that

$$(\nabla\xi,\nabla v)_\Omega=\langle g,v\rangle_{\partial\Omega},\quad\forall v\in\widehat{V}. \tag{3.16}$$

The solution ξ of the above problem defines the operator $\mathcal{S}: L^2(\partial\Omega)\to\widehat{V}$,

$$\mathcal{S}g=\xi.$$

So the solution to the Steklov eigenvalue problem (1.1) can be expressed through the operator $\mathcal{S}$: $u=\lambda\mathcal{S}u$. Indeed, consider the problem (3.16) with $g=\lambda u$. Therefore, the solutions of (3.16) and (3.14) are $u=\lambda\mathcal{S}u$ and $\widetilde{u}=\mathcal{S}u_h$, respectively. By the definition (3.15) of $\widetilde{\lambda}$, $u=\lambda\mathcal{S}u$ and $\|u\|_{L^2(\partial\Omega)}=\|u_h\|_{L^2(\partial\Omega)}=1$, we have

$$\begin{aligned}\lambda^{-1}-\widetilde{\lambda}^{-1}&=\langle u,\mathcal{S}u\rangle_{\partial\Omega}-\langle u_h,\mathcal{S}u_h\rangle_{\partial\Omega}\\&=2\langle u,\mathcal{S}u\rangle_{\partial\Omega}-2\langle u_h,\mathcal{S}u\rangle_{\partial\Omega}-\langle u-u_h,\mathcal{S}(u-u_h)\rangle_{\partial\Omega}\\&=\frac{1}{\lambda}\langle u-u_h,u-u_h\rangle_{\partial\Omega}-\langle u-u_h,\mathcal{S}(u-u_h)\rangle_{\partial\Omega},\end{aligned}$$

by the symmetry of the operator $\mathcal{S}$ in $\langle\cdot,\cdot\rangle_{\partial\Omega}$. We obtain that

$$|\lambda-\widetilde{\lambda}|\leq(\widetilde{\lambda}+\lambda\widetilde{\lambda}\|\mathcal{S}\|_{L^2(\partial\Omega)})\|u-u_h\|^2_{L^2(\partial\Omega)}.\quad\square$$

Next, we estimate $\|u-u_h\|_{L^2(\partial\Omega)}$. By the Ritz projection and its approximation property, trace theorem and inverse inequality, we have

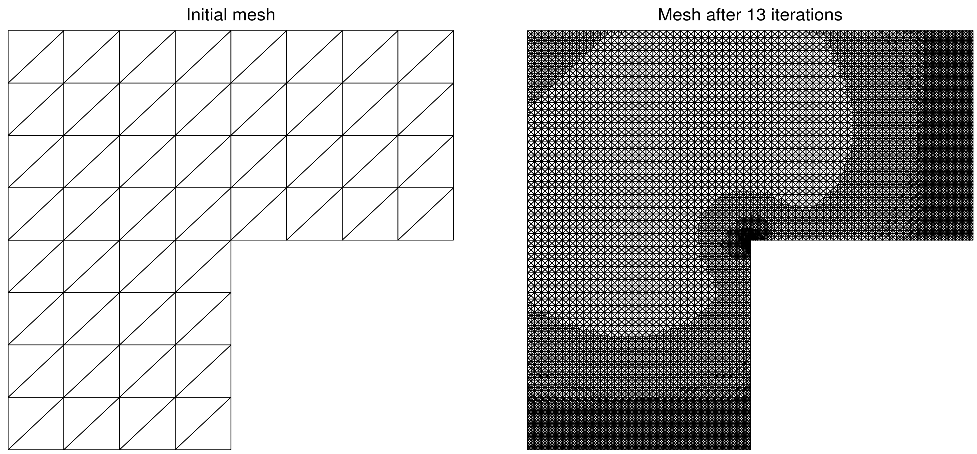

Fig. 1. The initial mesh (left) and the one after 9 adaptive refinements for the first five eigenvalues (right) for the L–shaped domain.

$$
\begin{aligned}
\|u-u_h\|_{L^2(\partial\Omega)} &\le \|u-R_h u\|_{L^2(\partial\Omega)} + \|R_h u - u_h\|_{L^2(\partial\Omega)} \\
&\le C\|u-R_h u\|^{\frac{1}{2}}_{L^2(\Omega)}\|u-R_h u\|^{\frac{1}{2}}_{H^1(\Omega)} + Ch^{-\frac{1}{2}}\|R_h u - u_h\|_{L^2(\Omega)} \\
&\le Ch^{min\{r+\frac{1}{2},s\}}\|u\|_{H^s(\Omega)} + Ch^{\frac{1}{2}}(\|R_h u - u\| + \|u-u_h\|_{0,\Omega}) \\
&\le Ch^{min\{r+\frac{1}{2},s\}}\|u\|_{H^s(\Omega)}.
\end{aligned}
$$

The proof of Theorem 2.4

Proof. By the triangle inequality, we have

$$|\lambda-\hat{\lambda}_h| \le |\lambda-\tilde{\lambda}| + |\tilde{\lambda}-\hat{\lambda}_h|.$$

Lemma 3.3 has shown that the first term is bounded. So the only second term needs to be estimated. The techniques are similar to those in the estimate of the first term $\lambda-\tilde{\lambda}$. Using the definition of $\tilde{\lambda}$ and $\hat{\lambda}_h$, and equations (3.13) and (3.14), we have

$$
\begin{aligned}
\tilde{\lambda}^{-1}-\hat{\lambda}_h^{-1} &= \langle u_h,\tilde{u}\rangle_{\partial\Omega} - \langle u_h,\tilde{u}_h\rangle_{\partial\Omega} = (\nabla\tilde{u},\nabla\tilde{u})_\Omega - (\nabla\tilde{u}_h,\nabla\tilde{u}_h)_{\Omega_h} \\
&= (\nabla(\tilde{u}-\tilde{u}_h),\nabla(\tilde{u}-\tilde{u}_h))_{\Omega_h},
\end{aligned}
$$

by the orthogonal property $(\nabla(\tilde{u}-\tilde{u}_h),\nabla\tilde{u}_h)_{\Omega_h}=0$. Indeed, let $v=v_h\in\tilde{V}$ in (3.13) and subtract (3.13) from (3.14), we obtain the orthogonality. On the other hand, $\tilde{u}_h\in\tilde{V}_h$ is the finite element approximation of the problem (3.14). Using the standard finite element error estimate [9], we have

$$\|\nabla(\tilde{u}-\tilde{u}_h)\|_{L^2(\Omega)} \le Ch^{\min\{r+1,s\}}\|\tilde{u}\|_{H^s(\Omega)}.$$

The above equation and inequality then lead to

$$|\tilde{\lambda}^{-1}-\hat{\lambda}_h^{-1}| \le Ch^{2\min\{r+1,s\}}\|\tilde{u}\|_{H^s(\Omega)},$$

i.e.,

$$|\tilde{\lambda}-\lambda_h| \le C\tilde{\lambda}\tilde{\lambda}_h h^{2\min\{r+1,s\}}\|\tilde{u}\|_{H^s(\Omega)},$$

which together with Lemma 3.3 completes the superconvergence proof. □

4. Numerical experiment

This section presents two numerical examples that would allow us to assess the theoretical results proved above. First, we illustrate the behavior of an adaptive algorithm that is driven by the error estimator in the L-shaped domain. Then the superconvergence of the postprocessing algorithm is investigated on model problems. With those aims, we implement in MATLAB code a first-order finite element on triangular meshes.

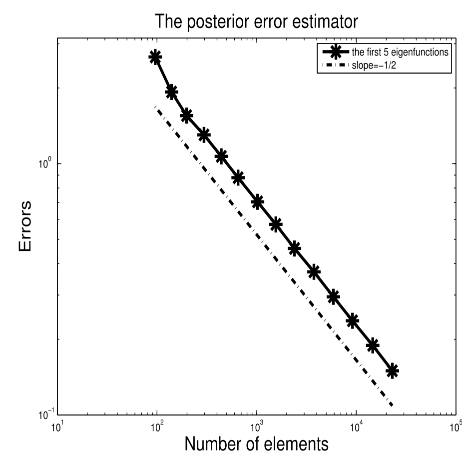

Fig. 2. Mesh efficiency of the a posteriori error estimator of the first five eigenfunctions for adaptive refinements with FEM ("Adaptive FEM") on the L–shaped domain.

4.1. Experiment 1: adaptive algorithm

The first example is the Steklov eigenvalue problem (1.1) on the L-shaped domain $\Omega=[-1,1]^2/([0,1]\times[-1,0])$. This problem is very popular in the numerical experiment because the regularity is broken on the original point (reentrant corner). The initial mesh is a uniform structured mesh of 96 elements. In Figs. 1–3, we show the efficiency, that is the observed accuracy of TOL versus the number of elements, of the meshes obtained by using estimator, and global (uniform) refinement. We see that estimator mesh refinement yields more economical meshes than simple uniform refinement.

Fig. 1 shows the initial grid and the adaptively refined grids obtained with adaptive procedures on the L–shaped domain by using the estimator for the first five eigenvalues.

We show the efficiency of the a posteriori error estimator with that achieved by the first five eigenfunctions in Fig. 2.

Fig. 3 illustrates the error curves for the obtained first five eigenvalues on the adaptively refined meshes with FEM schemes. We also see that the estimated error reflects the predicted behavior with a line of slope −1, which corresponds with the optimal convergence order.

4.2. Experiment 2: superconvergence of eigenvalue

The efficiency of the postprocessing techniques is illustrated in problem (1.1). Assume that eigenfunctions are known and the eigen-

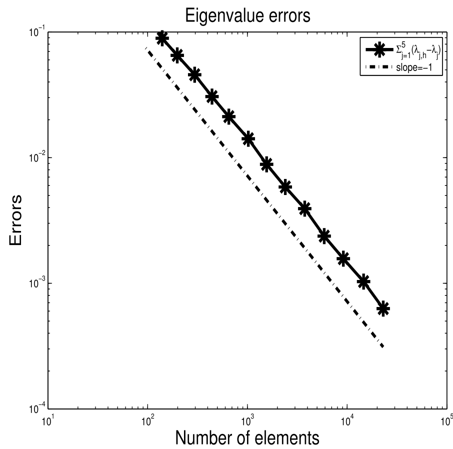

Fig. 3. The errors for the first five eigenvalues using adaptive FEM on the L-shaped domain.

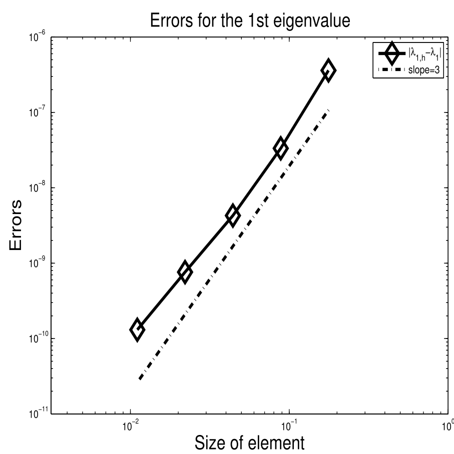

Fig. 4. The convergence rate of the first eigenvalue for the postprocessing method on the unit square domain.

functions of the example are enough smooth. Therefore, there are no restrictions concerning regularity.

We consider the Steklov eigenvalue problem

$$\begin{cases} -\Delta u = 0, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} - \lambda u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega = [0,1]^2$. The exact eigenpair (λ, u) of this problem is unknown, we use the accurate enough approximation $\lambda = 0.2400790830800452$ given by extrapolation method in [18]. In the example, the order of polynomials is 1. In Fig. 4 we plot the true error for the first eigenvalue against the size of the mesh. We also see that the estimated error reflects the predicted behavior with a line of slope 3 which corresponds to the superconvergence rate.

Fig. 5 presents the superconvergence behavior of the first four eigenvalues using the FE method's postprocessing procedure. Because of the postprocessing techniques, it is ready to see that the convergence rate has been considerably accelerated. A line of the slope is 3, which corresponds with the superconvergence.

5. Conclusion

In this paper, a new error analysis technique is presented for the FE approximation of the Steklov eigenvalue problem. The error estimates of the eigenvalue are reliable and efficient as well as the energy error estimates of the eigenfunctions. Our numerical experiment 1 has illustrated the efficiency of the resulting error estimators which generate the optimal grids. In addition, we propose a post-processing technique that provides superconvergence for the eigenvalues. For other classes of problems, such as the fourth-order Steklov eigenvalue problem, the spectral superconvergence of finite element methods is under investigation.

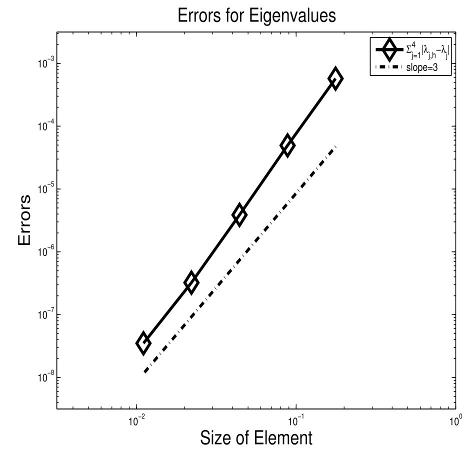

Fig. 5. The convergence rate of the first four eigenvalues for the postprocessing method on the unit square domain.

Data availability

Data will be made available on request.

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References

[1] M. Armentano, C. Padra, A posteriori error estimates for the Steklov eigenvalue problem, Appl. Numer. Math. 58 (2008) 593–601.
[2] A. Andreev, T. Todorov, Isoparametric finite element approximation of a Steklov eigenvalue problem, IMA J. Numer. Anal. 24 (2004) 309–322.
[3] I. Babuska, J. Osborn, Finite element Galerkin approximation of the eigenvalues and eigenfunctions of selfadjoint problems, Math. Comput. 52 (1989) 275–297.
[4] S. Bergmann, M. Schiffer, Kernel Functions and Elliptic Differential Equations in Mathematical Physics, Academic Press, New York, 1953.
[5] A. Bermùdez, R. Rodrìguez, D. Santamarina, A finite element solution of an added mass formulation for coupled fluid-solid vibrations, Numer. Math. 87 (2000) 201–227.
[6] J. Bramble, J. Osborn, Approximation of Steklov eigenvalues of non-selfadjoint second-order elliptic operators, in: Math. Found. Finite Element Method Applications PDE, Academic Press, New York, 1972, pp. 387–408.
[7] D. Bucur, I. Ionescu, Asymptotic analysis and scaling of friction parameters, Z. Angew. Math. Phys. 57 (2006) 1–15.
[8] W. Chen, Q. Lin, Approximation of an eigenvalue problem associated with the Stokes problem by the stream function-vorticity-pressure method, Appl. Math. 51 (2006) 73–88.
[9] P. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam, 1978.
[10] C. Conca, J. Planchard, M. Vanninathan, Fluid and Periodic Structures, John Wiley & Sons, New York, 1995.
[11] H. Han, Z. Guan, An analysis of the boundary element approximation of Steklov eigenvalue problems, in: Numerical Methods for Partial Differential Equations, World Scientific, Singapore, 1994, pp. 35–51.

[12] H. Han, Z. Guan, B. He, Boundary element approximation of Steklov eigenvalue problem, J. Chin. Univ. Appl. Math. Ser. A 9 (1994) 231–238.
[13] X. Han, Y. Li, H. Xie, A multilevel correction method for Steklov eigenvalue problem by nonconforming finite element methods, Numer. Math., Theory Methods Appl. 8 (03) (2015) 383–405.
[14] J. Huang, T. Lü, The mechanical quadrature methods and their extrapolation for solving BIE of Steklov eigenvalue problems, J. Comput. Math. 22 (5) (2004) 719–726.
[15] M. Larson, A posteriori and a priori error analysis for finite element approximations of self-adjoint eigenvalue problems, SIAM J. Numer. Anal. 38 (2000) 562–580.
[16] M. Li, Q. Lin and, S. Zhang, Extrapolation and superconvergence of the Steklov eigenvalue problem, Adv. Comput. Math. 33 (1) (2010) 25–44.
[17] Q. Lin, H. Huang, Z. Li, New expansion of numerical eigenvalue for $-\Delta u = \lambda u$ by nonconforming elements, Math. Comput. 77 (2008) 2061–2084.
[18] Q. Lin, J. Lin, Finite Element Methods: Accuracy and Improvement, China Sci. Tech. Press, Beijing, 2005.
[19] Q. Lin, T. Lü, Asymptotic expansions for finite element eigenvalues and finite element solution, Bonner Math. Schr. 158 (1984) 1–10.
[20] J. Planchard, J. Thomas, On the dynamic stability of cylinders placed in cross-flow, J. Fluids Struct. 7 (1993) 321–339.
[21] M. Racheva, A. Andreev, Superconvergence postprocessing for eigenvalues, Comput. Methods Appl. Math. 2 (2) (2002) 171–185.
[22] A. Russo, A. Alonso, A posteriori error estimates for nonconforming approximations of Steklov eigenvalue problems, Comput. Math. Appl. 62 (11) (2011) 4100–4117.
[23] M. Stekloff, Sur les problèmes fondamentaux de la physique mathématique, Ann. Sci. Éc. Norm. Supér. 19 (1902) 455–490 (in French).
[24] W. Tang, Z. Guan, H. Han, Boundary element approximation of Steklov eigenvalue problem for Helmholtz equation, J. Comput. Math. 2 (1998) 165–178.
[25] Z. Weng, S. Zhai, X. Feng, An improved two-grid finite element method for the Steklov eigenvalue problem, Appl. Math. Model. 39 (2015) 2962–2972.
[26] H. Xie, A type of multilevel method for the Steklov eigenvalue problem, IMA J. Numer. Anal. 34 (2) (2014) 592–608.
[27] Y. Yang, B. Hai, Local a priori/a posteriori error estimates of conforming finite elements approximation for Steklov eigenvalue problems, Sci. China Math. 57 (6) (2014) 1319–1329.
[28] Y. Yang, Q. Li, S. Li, Nonconforming finite element approximations of the Steklov eigenvalue problem, Appl. Numer. Math. 59 (10) (2009) 2388–2401.
[29] R. Verfurth, A Review of a Posteriori Estimation and Adaptive Mesh-Refinement Techniques, Wiley-Teubner Series Advances in Numerical Mathematics, John Wiley, Chichester, 1996.