Journal of Computational and Applied Mathematics 415 (2022) 114450

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Convergence and optimality of adaptive multigrid method for multiple eigenvalue problems[☆]

Fei Xu [a], Manting Xie [b], Qiumei Huang [a], Meiling Yue [c], Hongkun Ma [d,*]

[a] Faculty of Science, Beijing University of Technology, Beijing, 100124, China
[b] Center for Applied Mathematics, Tianjin University, Tianjin 300072, China
[c] School of Science, Beijing Technology and Business University, Beijing 100048, China
[d] Zhuhai Huafa Investment Holdings Group Co. Ltd., Hengqin 519000, China

ARTICLE INFO

Article history:
Received 18 July 2020
Received in revised form 2 April 2022

MSC:
65N30
35J61
65M55
65B99

Keywords:
Multiple eigenvalue problems
Adaptive finite element method
Multigrid method
Convergence and optimality complexity

ABSTRACT

A new type of adaptive multigrid method is presented for multiple eigenvalue problems based on multilevel correction scheme and adaptive multigrid method. Different from the classical adaptive finite element method which requires to solve eigenvalue problems on the adaptively refined triangulations, with our approach we just need to solve several linear boundary value problems in the current refined space and an eigenvalue problem in a very low dimensional space. Further, the involved boundary value problems are solved by an adaptive multigrid iteration. Since there is no eigenvalue problem to be solved on the refined triangulations, which is quite time-consuming, the proposed method can achieve the same efficiency as that of the adaptive multigrid method for the associated linear boundary value problems. Besides, the corresponding convergence and optimal complexity are verified theoretically and demonstrated numerically.

1. Introduction

Solving large-scale eigenvalue problem is one of the fundamental problems in modern science and engineering field. It is always a very difficult task to solve high dimensional eigenvalue problems especially for multiple eigenvalue problems, which come from practical physical and chemical sciences [1–3]. Different from the case of the boundary value problems, there are no many efficient numerical methods for solving eigenvalue problems with optimal complexity. The aim of this paper is to design a new type of adaptive multigrid method for multiple eigenvalue problems, by combining the multigrid method and adaptive finite element method.

Since the adaptive finite element method (AFEM) was proposed by Babuška and his collaborators in [4], it has been widely used to solve partial differential equations with singularities. The convergence and optimal complexity of AFEM have been much studied in recent years. For linear partial differential equations, especially, for the Poisson equation and its variants, the theory is well-developed. For instance, Dörfler [5] introduced Dörfler's marking and proved strict energy error reduction for the Laplace problem provided the initial mesh is fine enough. Following their work, Dörfler and Wilderotter [6], Morin, nochetto and Siebert [7], Binev, dehmen and Devore [8], Mekchay and Nochetto [9],

☆ This work is supported in part by the General projects of science and technology plan of Beijing Municipal Education Commission (Grant No. KM202110005011) and the National Science Foundation of China (Grant Nos. 11801021, 11971047, 72103210).

* Corresponding author.
E-mail addresses: xufei@lsec.cc.ac.cn (F. Xu), mtxie@tju.edu.cn (M. Xie), qmhuang@bjut.edu.cn (Q. Huang), yuemeiling@lsec.cc.ac.cn (M. Yue), mahongkun1212@163.com (H. Ma).

https://doi.org/10.1016/j.cam.2022.114450
0377-0427/

Stevenson [10] and Cascon et al. [11] have further studied the adaptive convergence of the standard finite element methods. Stevenson [10] and Cascon et al. [11] also analyzed the complexity of the adaptive method. For eigenvalue problem, Carstensen et al. [12,13] considered AFEM for simple eigenvalue problems and eigenvalue clusters. Moreover, the error reduction and optimal complexity analysis of AFEM can be found in [14] based on the connections between boundary value problem and eigenvalue problem. In [15,16], some methods are introduced to derive the a priori (without any computational effort) estimates of the eigenvalues of large matrices approximated by several Galerkin techniques based on the GLT theory and on the notion of symbol [17]. In [18,19], the techniques are provided for showing how close the eigenvalues of the finite dimensional approximations are with respect to the eigenvalues of the continuous problem. For more results about eigenvalue problems, please refer to [8–10,20–26] and the references cited therein.

In this paper, we will propose and analyze a new type of adaptive multigrid method to solve the multiple eigenvalue problems based on the adaptive mesh refinement, multigrid method and the recent work on the multilevel correction method [27–33]. Different from the classical approach which solves the large-scale eigenvalue problem in the new finite element space after each mesh refinement, with our approach we only need to solve several linear boundary value problems on the current refined mesh and then correct the approximate solution by solving a low dimensional eigenvalue problem in a specially designed correction space. During the adaptively refining process, the size of the low dimensional eigenvalue problems will be fixed. Further, the involved boundary value problems are solved by the adaptive multigrid method, which was initially proposed by Brandt in [34]. For more results about the adaptive multigrid method, please refer to [35–38] and the references cited therein. Since the main computation of the proposed algorithm is solving the linear boundary value problems on the adaptively refined partitions, the cost of the new adaptive multigrid method will not be more expensive than the adaptive multigrid method for the associated boundary value problems. In addition, we prove the convergence and optimal complexity of the new algorithm by adopting the techniques in [11,14].

In this study, we will research the following elliptic eigenvalue problem: Find (λ, u) such that

$$\begin{cases} -\nabla\cdot(\mathcal{A}\nabla u)+\phi u &= \lambda u, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\mathcal{A}$ is a symmetric and positive definite matrix with elements belong to $W^{1,\infty}$, $\phi \in L^\infty$ is a nonnegative function, $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ is a bounded domain with Lipschitz boundary $\partial\Omega$.

An outline of this paper is as follows. In Section 2, we will introduce some notations and recall some preliminaries of the standard AFEM for the boundary value problems. In Section 3, we construct the adaptive multigrid method for multiple eigenvalue problems. The corresponding convergence and complexity analysis are presented in Section 4. Finally, some numerical examples are presented in the last section to illustrate the efficiency of the proposed algorithm.

2. Preliminaries of standard AFEM for boundary value problem

In this section, we should review some basic results of AFEM [5,7,9,11] for linear boundary value problem, which will be a basis of the following analysis for multiple eigenvalue problems. In this study, we use the standard notation for Sobolev spaces $W^{s,p}(\Omega)$ and their associated norms $\|\cdot\|_{s,p,\Omega}$ and seminorms $|\cdot|_{s,p,\Omega}$ (see, e.g., [39]). For $p = 2$, we denote $H^s(\Omega) = W^{s,2}(\Omega)$ and $H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$. For simplicity, we use V to denote $H_0^1(\Omega)$ in the rest of the paper.

In this section, we consider the following elliptic boundary value problem:

$$\begin{cases} \mathcal{L}u_\ell = -\nabla\cdot(\mathcal{A}\nabla u_\ell)+\phi u_\ell &= f_\ell, \quad \text{in } \Omega, \quad \ell = i, \ldots, i+q-1, \\ u_\ell &= 0, \quad \text{on } \partial\Omega. \end{cases} \tag{2}$$

Remark 2.1. In fact, (2) is a linear system composed of q boundary value problems. The reason why we study (2) in this section is that we will encounter the equation as (2) in our analysis for multiple eigenvalue problem. Here, i and q denote positive integers which will be introduced in the next section.

The weak form of (2) is defined as follows: Find $u_\ell \in V$ such that

$$a(u_\ell, v_\ell) = b(f_\ell, v_\ell), \quad \forall v_\ell \in V, \quad \ell = i, \ldots, i+q-1, \tag{3}$$

where

$$a(u_\ell, v_\ell) = \int_\Omega \big(\mathcal{A}\nabla u_\ell\cdot\nabla v_\ell + \phi u_\ell v_\ell\big)\mathrm{d}\Omega, \quad b(f_\ell, v_\ell) = \int_\Omega f_\ell v_\ell \mathrm{d}\Omega.$$

From the properties of $\mathcal{A}$ and ϕ, the bilinear form $a(\cdot,\cdot)$ is bounded over V

$$|a(w, v)| \lesssim \|w\|_{a,\Omega}\|v\|_{a,\Omega}, \quad \forall w, v \in V,$$

and satisfies

$$c_a\|w\|_{1,\Omega} \le \|w\|_{a,\Omega} \le C_a\|w\|_{1,\Omega},$$

where the energy norm $\|\cdot\|_{a,\Omega}$ is defined by $\|w\|_{a,\Omega} = \sqrt{a(w, w)}$.

Now, we begin to define the finite element approximation of the boundary value problem (2). First we decompose the computing domain Ω to generate a conforming triangulation $\mathcal{T}_h$. The diameter of a cell $K \in \mathcal{T}_h$ is denoted by h_K and the mesh size h describes the maximum diameter of all cells $K \in \mathcal{T}_h$. Based on the partition $\mathcal{T}_h$, we can construct a finite element space denoted by $V_h \subset V$ which is composed of piecewise polynomials.

The standard finite element method for (3) is to solve the following discrete elliptic boundary value problems: Find $u_{h,\ell} \in V_h$ such that

$$a(u_{h,\ell}, v_{h,\ell}) = b(f_\ell, v_{h,\ell}), \quad \forall v_{h,\ell} \in V_h, \quad \ell = i, \ldots, i+q-1. \tag{4}$$

For the purpose of analysis, we define the Galerkin projection $P_h : V \to V_h$ by

$$a(u - P_h u, v_h) = 0, \qquad \forall v_h \in V_h. \tag{5}$$

Then we have

$$\|P_h u\|_{a,\Omega} \leq \|u\|_{a,\Omega}, \qquad \forall u \in V. \tag{6}$$

In this paper, we use $\mathcal{E}_h$ to denote the set of interior faces of $\mathcal{T}_h$. Based on the conclusions of AFEM for boundary value problems (see, e.g. [5,7,9,11]), we define the element residual $\widetilde{\mathcal{R}}_K(u_{h,\ell})$ and the jump residual $\widetilde{J}_e(u_{h,\ell})$ as follows:

$$\widetilde{\mathcal{R}}_K(u_{h,\ell}) := f_\ell - \mathcal{L}u_{h,\ell} = f_\ell + \nabla\cdot(\mathcal{A}\nabla u_{h,\ell}) - \phi u_{h,\ell}, \quad \text{in } K \in \mathcal{T}_h, \tag{7}$$

$$\widetilde{J}_e(u_{h,\ell}) := -\mathcal{A}\nabla u_{h,\ell}^+ \cdot \nu^+ - \mathcal{A}\nabla u_{h,\ell}^- \cdot \nu^- = [\mathcal{A}\nabla u_{h,\ell}]\cdot \nu_e, \ \text{ on } e \in \mathcal{E}_h, \tag{8}$$

where e is the common side of elements K^+ and K^- with the unit outward normals ν^+ and ν^-, respectively, and $\nu_e = \nu^-$. For $K \in \mathcal{T}_h$, we define the local error indicator $\widetilde{\eta}_h(u_{h,\ell}, K)$ and the oscillation $\widetilde{osc}_h^2(u_{h,\ell}, K)$ by

$$\widetilde{\eta}_h^2(u_{h,\ell}, K) := h_K^2\|\widetilde{\mathcal{R}}_K(u_{h,\ell})\|_{0,K}^2 + \sum_{e\in\mathcal{E}_h, e\subset\partial K} h_e\|\widetilde{J}_e(u_{h,\ell})\|_{0,e}^2,$$

$$\widetilde{osc}_h^2(u_{h,\ell}, K) := h_K^2\|(I - \mathbb{P}_K)\widetilde{\mathcal{R}}_K(u_{h,\ell})\|_{0,K}^2 + \sum_{e\in\mathcal{E}_h, e\subset\partial K} h_e\|(I - \mathbb{P}_e)\widetilde{J}_e(u_{h,\ell})\|_{0,e}^2,$$

where I denotes the identity operator, $\mathbb{P}_K$ and $\mathbb{P}_e$ denote the L^2-projection operators to polynomials of some degree on K and e, respectively.

Given a subset $\omega \subset \Omega$, we define the error estimator $\widetilde{\eta}_h(u_{h,\ell}, \omega)$ and the oscillation $\widetilde{osc}_h(u_{h,\ell}, \omega)$ by

$$\widetilde{\eta}_h^2(u_{h,\ell}, \omega) = \sum_{K\in\mathcal{T}_h, K\subset\omega} \widetilde{\eta}_h^2(u_{h,\ell}, K) \quad \text{and} \quad \widetilde{osc}_h^2(u_{h,\ell}, \omega) = \sum_{K\in\mathcal{T}_h, K\subset\omega} \widetilde{osc}_h^2(u_{h,\ell}, K).$$

Now we recall the reliability and efficiency of the abovementioned residual type a posteriori error estimator in the following lemma (see, e.g., [9,11,40]).

Lemma 2.1. *There exist some constants $\widetilde{C}_1$, $\widetilde{C}_2$ and $\widetilde{C}_3$, which depend on the shape-regularity of $\mathcal{T}_h$, such that the following reliability and efficiency hold*

$$\|u_\ell - u_{h,\ell}\|_{a,\Omega}^2 \leq \widetilde{C}_1\widetilde{\eta}_h^2(u_{h,\ell}; \mathcal{T}_h), \tag{9}$$

$$\widetilde{C}_2\widetilde{\eta}_h^2(u_{h,\ell}; \mathcal{T}_h) \leq \|u_\ell - u_{h,\ell}\|_{a,\Omega}^2 + \widetilde{C}_3\widetilde{osc}_h^2(u_{h,\ell}; \mathcal{T}_h). \tag{10}$$

The standard AFEM can be written as a loop of the following form

Solve $\to$ **Estimate** $\to$ **Mark** $\to$ **Refine**.

More precisely, to get $\mathcal{T}_{h_{k+1}}$ from $\mathcal{T}_{h_k}$, we first solve the discrete equation on $\mathcal{T}_{h_k}$ to get the approximate solution and then calculate the a posteriori error estimator on each mesh element. Next we mark elements to be subdivided according to the values of the a posteriori error estimator and refine these elements in such a way that the triangulation is still shape regular and conforming.

For simplicity, we use $\sum_\ell$ to represent $\sum_{\ell=i}^{i+q-1}$ in the rest of this paper, and the index ℓ may be different at different places. For any finite element function $U_h = (u_{h,i}, \ldots, u_{h,i+q-1}) \in (V_h)^q$, we denote

$$\widetilde{\eta}_h^2(U_h, K) = \sum_\ell \widetilde{\eta}_h^2(u_{h,\ell}, K) \quad \text{and} \quad \widetilde{osc}_h^2(U_h, K) = \sum_\ell \widetilde{osc}_h^2(u_{h,\ell}, K).$$

For any $U = (u_i, \ldots, u_{i+q-1}) \in (V)^q$, we denote

$$\|U\|_{a,\Omega} = (\sum_\ell \|u_\ell\|_{a,\Omega}^2)^{1/2}.$$

In order to simplify the description of the AFEM, we first introduce some modules for the boundary value problem:

- $\Phi = \text{BVP\_SOLVE}(\{f_\ell\}_{\ell=i}^{i+q-1}, V_h)$: Solve the boundary value problem (4) in the finite element space V_h and return the discrete solution $\Phi \in (V_h)^q$.
- $\Phi = \text{MGBVP\_SOLVE}(\{f_\ell\}_{\ell=i}^{i+q-1}, \Phi_0, V_h)$: Solve the boundary value problem (4) by the adaptive multigrid method with the initial value $\Phi_0 \in (V_h)^q$ in the finite element space V_h and return the iteration solution $\Phi \in (V_h)^q$.
- $\{\widetilde{\eta}_h(U_h; K)\}_{K\in\mathcal{T}_h} = \text{BVP\_ESTIMATE}(U_h, \mathcal{T}_h)$: Compute the local a posteriori error indicator $\widetilde{\eta}_h(U_h; K)$ on each mesh element $K \in \mathcal{T}_h$.
- $\mathcal{M}_h = \text{BVP\_MARK}(\theta, \{\widetilde{\eta}_h(U_h; K)\}_{K\in\mathcal{T}_h}, \mathcal{T}_h)$: Construct a subset $\mathcal{M}_h$ by Dörfler's marking strategy presented in [5], i.e., construct a minimal subset $\mathcal{M}_h$ from $\mathcal{T}_h$ by selecting some elements in $\mathcal{T}_h$ such that

  $$\widetilde{\eta}_h(U_h; \mathcal{M}_h) \geq \theta \widetilde{\eta}_h(U_h; \mathcal{T}_h)$$

  and mark all the elements in $\mathcal{M}_h$.
- $(\mathcal{T}_{h_{k+1}}, V_{h_{k+1}}) = \text{REFINE}(\mathcal{M}_{h_k}, \mathcal{T}_{h_k})$: Output a conforming refinement $\mathcal{T}_{h_{k+1}}$ according to $\mathcal{M}_{h_k}$ where all elements of $\mathcal{M}_{h_k}$ are refined and construct the finite element space $V_{h_{k+1}}$.

The basic loop of the classical AFEM for the elliptic boundary value problem (3) is presented in Algorithm 2.1.

Algorithm 2.1 (*Adaptive Finite Element Method*)**.**

1. Given a parameter $0 < \theta < 1$ and an initial mesh $\mathcal{T}_{h_1}$. Set $k := 1$.
2. $U_{h_k} = \text{BVP\_}\mathit{SOLVE}(\{f_\ell\}_{\ell=i}^{i+q-1}, V_{h_k})$;
3. $\{\widetilde{\eta}_{h_k}(U_{h_k}; K)\}_{K\in\mathcal{T}_{h_k}} = \text{BVP\_}\mathit{ESTIMATE}(U_{h_k}, \mathcal{T}_{h_k})$;
4. $\mathcal{M}_{h_k} = \text{BVP\_}\mathit{MARK}(\theta, \{\widetilde{\eta}_{h_k}(U_{h_k}; K)\}_{K\in\mathcal{T}_{h_k}}, \mathcal{T}_{h_k})$;
5. $(\mathcal{T}_{h_{k+1}}, V_{h_{k+1}}) = \text{REFINE}(\mathcal{M}_{h_k}, \mathcal{T}_{h_k})$;
6. Set $k := k + 1$ and go to step 2.

Now, we recall the well-known convergence result of the AFEM for the elliptic boundary value problem (see [11,14,40]). The following lemma is an extension of corresponding result for the case of $q = 1$ in [11] by some primary operations and it will be used in our analysis.

Lemma 2.2 (*[11, Theorem 4.1]*)**.** *Let $\{U_{h_k}\}_{k\in\mathbb{N}}$ be a sequence of finite element solutions produced by Algorithm 2.1. Then, there exist two constants $\widetilde{\gamma} > 0$ and $\xi \in (0, 1)$ depending only on the shape regularity of meshes and the marking parameter θ, such that any two consecutive iterations satisfy*

$$\|U - U_{h_{k+1}}\|_{a,\Omega}^2 + \widetilde{\gamma}\widetilde{\eta}_{h_{k+1}}^2(U_{h_{k+1}}, \mathcal{T}_{h_{k+1}}) \leq \xi^2(\|U - U_{h_k}\|_{a,\Omega}^2 + \widetilde{\gamma}\widetilde{\eta}_{h_k}^2(U_{h_k}, \mathcal{T}_{h_k})). \tag{11}$$

3. Adaptive multigrid method for multiple eigenvalue problems

In this section, we design a novel adaptive multigrid method for solving the multiple eigenvalue problems based on the multilevel correction scheme and adaptive multigrid method.

3.1. Finite element method for eigenvalue problems

First, we recall some basic theoretical results of the finite element method for the eigenvalue problems in this subsection.

The corresponding variational form for the eigenvalue problem (1) can be described as follows: Find $(\lambda, u) \in \mathbb{R} \times V$ such that $b(u, u) = 1$ and

$$a(u, v) = \lambda b(u, v), \quad \forall v \in V. \tag{12}$$

As we know, eigenvalue problem (12) has an eigenvalue sequence (see [41,42]):

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots, \ \lim_{k\to\infty} \lambda_k = \infty$$

and the corresponding eigenfunctions

$$u_1, u_2, \ldots, u_k, \ldots,$$

where $b(u_i, u_j) = \delta_{ij}$ and λ_j are repeated according to their geometric multiplicity in the sequence $\{\lambda_j\}$.

The following property about the eigenvalue and eigenfunction approximation is useful (see [41,43]).

Lemma 3.1. *Let (λ, u) be an eigenpair of (12). For any $w \in V \setminus \{0\}$, there holds the following expansion*

$$\frac{a(w, w)}{b(w, w)} - \lambda = \frac{a(w - u, w - u)}{b(w, w)} - \lambda \frac{b(w - u, w - u)}{b(w, w)}. \tag{13}$$

The standard finite element scheme for eigenvalue problem (12) is described as follows: Find $(\bar{\lambda}_h, \bar{u}_h) \in \mathbb{R} \times V_h$ such that $b(\bar{u}_h, \bar{u}_h) = 1$ and

$$a(\bar{u}_h, v_h) = \bar{\lambda}_h b(\bar{u}_h, v_h), \qquad \forall v_h \in V_h. \tag{14}$$

From [41–43], the discrete eigenvalue problem (14) has an eigenvalue sequence

$$0 < \bar{\lambda}_{h,1} \le \bar{\lambda}_{h,2} \le \cdots \le \bar{\lambda}_{h,k} \le \cdots \le \bar{\lambda}_{h,N_h}$$

and the corresponding eigenfunctions

$$\bar{u}_{h,1}, \bar{u}_{h,2}, \ldots, \bar{u}_{h,k}, \ldots, \bar{u}_{h,N_h},$$

where $b(\bar{u}_{h,i}, \bar{u}_{h,j}) = \delta_{i,j}$, $1 \le i, j \le N_h$ and N_h denotes the dimension of V_h.

Let $M(\lambda_i)$ denote the eigenfunction space corresponding to the eigenvalue λ_i which is defined by

$$M(\lambda_i) = \{w \in V : w \text{ is an eigenfunction of (12) corresponding to } \lambda_i \text{ and } b(w, w) = 1\}.$$

For generality, let q be the multiplicity of the desired eigenvalue. It means $\lambda_i = \cdots = \lambda_{i+q-1}$. We use $(\bar{\lambda}_{h,i}, \bar{u}_{h,i}), \ldots, (\bar{\lambda}_{h,i+q-1}, \bar{u}_{h,i+q-1})$ to denote the eigenpair approximations for the eigenvalues $\lambda_i = \cdots = \lambda_{i+q-1}$ and their corresponding eigenfunction space $M(\lambda_i)$. Let

$$M_h(\lambda_i) = \mathrm{span}\{\bar{u}_{h,i}, \ldots, \bar{u}_{h,i+q-1}\}. \tag{15}$$

For two subspaces X and Y of V, we denote

$$\hat{\Theta}(X, Y) = \sup_{w \in X, \|w\|_0 = 1} \inf_{v \in Y} \|w - v\|_{a,\Omega}, \quad \hat{\Phi}(X, Y) = \sup_{w \in X, \|w\|_0 = 1} \inf_{v \in Y} \|w - v\|_{0,\Omega}.$$

Then we define the gaps between $M(\lambda_i)$ and $M_h(\lambda_i)$ in $\|\cdot\|_{a,\Omega}$ as

$$\Theta(M(\lambda_i), M_h(\lambda_i)) = \max\{\hat{\Theta}(M(\lambda_i), M_h(\lambda_i)), \hat{\Theta}(M_h(\lambda_i), M(\lambda_i))\} \tag{16}$$

and in $\|\cdot\|_{0,\Omega}$ as

$$\Phi(M(\lambda_i), M_h(\lambda_i)) = \max\{\hat{\Phi}(M(\lambda_i), M_h(\lambda_i)), \hat{\Phi}(M_h(\lambda_i), M(\lambda_i))\}. \tag{17}$$

For $\hat{\Theta}(X, Y)$ defined above, we have (see, e.g., Theorem 6.1 of [41]) the following lemma.

Lemma 3.2. *If dim X = dim $Y < \infty$, then $\hat{\Theta}(X, Y) \le \hat{\Theta}(Y, X)[1 - \hat{\Theta}(Y, X)]^{-1}$.*

Let $T : L^2(\Omega) \to V$ be the operator defined by

$$a(Tw, v) = b(w, v), \quad \forall v \in V, \tag{18}$$

and $T_h : L^2(\Omega) \to V_h$ be the operator defined by

$$a(T_h w, v_h) = b(w, v_h), \quad \forall v_h \in V_h. \tag{19}$$

Let Γ be a circle in the complex plane centered at λ_i^{-1} and not enclosing any other eigenvalues of T. Define the spectral projection associated with T and λ_i as follows (see [41,43])

$$E = E(\lambda_i) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma (z - T)^{-1} dz. \tag{20}$$

For h sufficiently small, except $\bar{\lambda}_{h,i}^{-1}, \ldots, \bar{\lambda}_{h,i+q-1}^{-1}$, there is no other eigenvalue of T_h contained in Γ. So we can define the spectral projection associated with T_h and $\bar{\lambda}_{h,i}, \ldots, \bar{\lambda}_{h,i+q-1}$ as

$$E_h = E_h(\lambda_i) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma (z - T_h)^{-1} dz. \tag{21}$$

For the eigenpair approximation by the finite element method, the following two lemmas (see [41,43]) give key estimates.

Lemma 3.3. *Let $\lambda_i = \cdots = \lambda_{i+q-1}$ be any eigenvalues of (12) with multiplicity q and $\bar{u}_{h,\ell}$ with $\|\bar{u}_{h,\ell}\|_{0,\Omega} = 1$ be the eigenfunction corresponding to $\bar{\lambda}_{h,\ell}$ ($\ell = i, \ldots, i+q-1$). Then, there holds*

$$\|u - E_h u\|_{0,\Omega} \lesssim \eta_a(h)\|u - E_h u\|_{a,\Omega}, \quad \|u - E_h u\|_{a,\Omega} \lesssim \delta_h(\lambda_i), \quad \forall u \in M(\lambda_i),$$
$$\|\bar{u}_{h,\ell} - E\bar{u}_{h,\ell}\|_{0,\Omega} \lesssim \eta_a(h)\|\bar{u}_{h,\ell} - E\bar{u}_{h,\ell}\|_{a,\Omega}, \quad \|\bar{u}_{h,\ell} - E\bar{u}_{h,\ell}\|_{a,\Omega} \lesssim \delta_h(\lambda_i),$$
$$\bar{\lambda}_{h,\ell} - \lambda_i \lesssim \delta_h^2(\lambda_i),$$

where

$$\delta_h(\lambda_i) = \sup_{w \in M(\lambda_i)} \inf_{v_h \in V_h} \|w - v_h\|_{a,\Omega}, \quad \eta_a(h) = \sup_{f \in L^2(\Omega), \|f\|_{0,\Omega} = 1} \inf_{v_h \in V_h} \|Tf - v_h\|_{a,\Omega}.$$

Lemma 3.4. *For any $u \in M(\lambda_i)$, we have*

$$1 \leq \frac{\|u - E_h u\|_{a,\Omega}}{\|u - P_h u\|_{a,\Omega}} = 1 + \mathcal{O}(\nu(h)), \tag{22}$$

where $\nu(h)$ is defined as follows

$$\nu(h) = \sup_{f \in V, \|f\|_{a,\Omega}=1} \inf_{v_h \in V_h} \|Tf - v_h\|_{a,\Omega}$$

and $\nu(h) \to 0$ as $h \to 0$.

We will also use the following two lemmas in our analysis (see [14]).

Lemma 3.5 ([14, Corollary 2.10]). *For any $u \in M(\lambda_i)$, there holds*

$$1 - C\eta_a(h)\delta_h(\lambda_i) \leq \|E_h u\|_{0,\Omega}^2 \leq 1, \tag{23}$$

where C is a constant not depending on mesh size.

Lemma 3.6 ([14, Corollary 2.11]). *For any $u_j, u_\ell \in M(\lambda_i)$ with $b(u_j, u_\ell) = \delta_{j,\ell}$ ($j, \ell = i, \ldots, i+q-1$), we have*

$$b(E_h u_j, E_h u_\ell) = \delta_{j,\ell} + \mathcal{O}\big(\eta_a(h)\delta_h(\lambda_i)\big). \tag{24}$$

3.2. Adaptive multigrid method

In this subsection, we propose an adaptive multigrid method based on the combination of the multilevel correction method, multigrid method and adaptive mesh refinement.

According to the element residual $\widetilde{\mathcal{R}}_K(u_{h,\ell})$ and the jump residual $\widetilde{J}_e(u_{h,\ell})$ of the boundary value problem (2), we define the element residual and the jump residual of the eigenvalue problem (12) as follows:

$$\mathcal{R}_K(u_{h,\ell}) := \lambda_{h,\ell} u_{h,\ell} - \phi u_{h,\ell} + \nabla \cdot (\mathcal{A}\nabla u_{h,\ell}), \quad \text{in } K \in \mathcal{T}_h, \tag{25}$$

$$J_e(u_{h,\ell}) := -\mathcal{A}\nabla u_{h,\ell}^+ \cdot \nu^+ - \mathcal{A}\nabla u_{h,\ell}^- \cdot \nu^- = [\mathcal{A}\nabla u_{h,\ell}] \cdot \nu_e, \ \text{ on } e \in \mathcal{E}_h. \tag{26}$$

For $K \in \mathcal{T}_h$, we define the local error estimator $\eta_h(u_{h,\ell}, K)$ and the oscillation $osc_h(u_{h,\ell}, K)$ for the eigenvalue problem (12) by

$$\eta_h^2(u_{h,\ell}, K) := h_K^2 \|\mathcal{R}_K(u_{h,\ell})\|_{0,K}^2 + \sum_{e \in \mathcal{E}_h, e \subset \partial K} h_e \|J_e(u_{h,\ell})\|_{0,e}^2,$$

$$osc_h^2(u_{h,\ell}, K) := h_K^2 \|(I - \mathbb{P}_K)\mathcal{R}_K(u_{h,\ell})\|_{0,K}^2 + \sum_{e \in \mathcal{E}_h, e \subset \partial K} h_e \|(I - \mathbb{P}_e)J_e(u_{h,\ell})\|_{0,e}^2.$$

For any $U_h = (u_{h,i}, \ldots, u_{h,i+q-1}) \in (V_h)^q$, we set

$$\eta_h^2(U_h, K) = \sum_\ell \eta_h^2(u_{h,\ell}, K) \quad \text{and} \quad osc_h^2(U_h, K) = \sum_\ell osc_h^2(u_{h,\ell}, K).$$

Similarly, we also introduce some modules of our adaptive multigrid algorithm for multiple eigenvalue problem as follows:

- $(\Lambda, \Phi) = \text{EG\_SOLVE}(V_h)$: Solve the eigenvalue problem (14) in the finite element space V_h and return the desired q eigenpair approximations $(\Lambda, \Phi) \in \mathbb{R}^q \times (V_h)^q$.
- $\{\eta_h(U_h; K)\}_{K \in \mathcal{T}_h} = \text{EG\_ESTIMATE}(U_h, \mathcal{T}_h)$: Compute the local error indicators on each element.
- $\mathcal{M}_h = \text{EG\_MARK}(\theta, \{\eta_h(U_h; K)\}_{K \in \mathcal{T}_h}, \mathcal{T}_h)$: Construct a minimal subset $\mathcal{M}_h$ from $\mathcal{T}_h$ by selecting some elements in $\mathcal{T}_h$ such that

  $$\eta_h(U_h; \mathcal{M}_h) \geq \theta \eta_h(U_h; \mathcal{T}_h), \tag{27}$$

  and mark all elements in $\mathcal{M}_h$.

Then the adaptive multigrid method for multiple eigenvalue problem is defined in Algorithm 3.1. Instead of solving an eigenvalue problem in each adaptive finite element space, with our approach, we only need to solve several linear boundary value problems by adaptive multigrid method in the adaptively refined space and a small-scale eigenvalue problem in a low dimensional space. The idea is an extension of two-grid method for the eigenvalue problem [44] to multigrid case by adding a special correction step. The correction step can not only keep the H^1-norm accuracy of the approximate eigenfunction obtained from the adaptive space, but also gives a higher order L^2-norm accuracy. Since there is no eigenvalue problem solving in the refined triangulations directly, which needs more computation and memory than solving the associated boundary value problems, the proposed algorithm has a higher efficiency than the standard adaptive finite element method.

Algorithm 3.1 (*Adaptive Multigrid Method*)**.**

1. Given a parameter $0 < \theta < 1$. Generate a coarse mesh $\mathcal{T}_H$ on computing domain Ω and construct the corresponding finite element space V_H. Pick up an initial mesh $\mathcal{T}_{h_1}$ which is produced by refining $\mathcal{T}_H$ several times in the uniform way. Then build the initial finite element space V_{h_1} on $\mathcal{T}_{h_1}$. Let p denote the multigrid iteration times. Set $k := 1$ and do the following loops:
2. $(\Lambda_{h_k}, U_{h_k}) = \begin{cases} \text{EG\_}\mathit{SOLVE}(V_{h_1}), & \text{when } k = 1; \\ \text{EG\_}\mathit{SOLVE}(V_H \oplus \text{span}\{\breve{U}_{h_k}\}), & \text{when } k > 1; \end{cases}$
3. $\{\eta_{h_k}(U_{h_k}; K)\}_{K\in\mathcal{T}_{h_k}} = \text{EG\_}\mathit{ESTIMATE}(U_{h_k}, \mathcal{T}_{h_k})$;
4. $\mathcal{M}_{h_k} = \text{EG\_}\mathit{MARK}(\theta, \{\eta_{h_k}(U_{h_k}; K)\}_{K\in\mathcal{T}_{h_k}}, \mathcal{T}_{h_k})$;
5. $(\mathcal{T}_{h_{k+1}}, V_{h_{k+1}}) = \text{REFINE}(\mathcal{M}_{h_k}, \mathcal{T}_{h_k})$;
6. (a) set $U_{h_{k+1}}^{(0)} = U_{h_k}$;
   (b) For $\ell = 0, \cdots, p-1$, $U_{h_{k+1}}^{(\ell+1)} = \text{MGBVP\_}\mathit{SOLVE}(\Lambda_{h_k} U_{h_k}, U_{h_{k+1}}^{(\ell)}, V_{h_{k+1}})$;
   (c) Set $\breve{U}_{h_{k+1}} = U_{h_{k+1}}^{(p)}$;
7. Set $k := k + 1$ and go to step 2.

Remark 3.1. In Algorithm 3.1, we only need to solve a series of boundary value problems on adaptive spaces in step 6 and solve some low dimensional eigenvalue problems in step 2. The dimension of these eigenvalue problems (dim $(V_H) + q$) remains unchanged during the adaptive refinement, thus the overall efficiency of Algorithm 3.1 will not be significantly more expensive than the adaptive multigrid method for the corresponding elliptic boundary value problem.

In the 6-th step of Algorithm 3.1, the multigrid method is adopted for the linearized boundary value problems which includes pre-smoothing, coarse grid correction and post smoothing. Here, we choose some linear smoothers such as Richardson, Jacobi, Gauss–Seidel and symmetrized Gauss–Seidel iteration in the multigrid method.

Define $M_{H,h_k}(\lambda_i) = \text{span}\{u_{h_k,i}, \ldots, u_{h_k,i+q-1}\}$. In the following analysis, we need some crude a priori error estimates presented in the following lemma.

Lemma 3.7. *The approximate eigenfunction space $M_{H,h_k}(\lambda_i)$ obtained by Algorithm 3.1 has the following error estimates*

$$\hat{\Theta}(M(\lambda_i), M_{H,h_k}(\lambda_i)) \lesssim \delta_H(\lambda_i), \tag{28}$$

$$\hat{\Phi}(M(\lambda_i), M_{H,h_k}(\lambda_i)) \lesssim \eta_a(H)\delta_H(\lambda_i). \tag{29}$$

For each eigenvalue, we have

$$\lambda_{h_k,\ell} - \lambda_i \lesssim \delta_H^2(\lambda_i), \quad \textit{for } \ell = i, \ldots, i+q-1. \tag{30}$$

Proof. From Lemma 3.3, there holds

$$\begin{aligned} \hat{\Theta}(M(\lambda_i), M_{H,h_k}(\lambda_i)) &\lesssim \sup_{u\in M(\lambda_i)} \inf_{v_{h_k}\in V_H\oplus\text{span}\{\breve{U}_{h_k}\}} \|u - v_{h_k}\|_{a,\Omega} \\ &\lesssim \sup_{u\in M(\lambda_i)} \inf_{v_{h_k}\in V_H} \|u - v_{h_k}\|_{a,\Omega} := \delta_H(\lambda_i). \end{aligned}$$

Similarly, from Lemma 3.3, the following estimates hold

$$\hat{\Phi}(M(\lambda_i), M_{H,h_k}(\lambda_i)) \lesssim \eta_a(H)\delta_H(\lambda_i) \tag{31}$$

and

$$\lambda_{h_k,\ell} - \lambda_i \lesssim \delta_H^2(\lambda_i). \tag{32}$$

Then we complete the proof. □

3.3. The connections between eigenvalue problems and boundary value problems

In order to analyze the convergence and complexity property of Algorithm 3.1, we establish the connections between the solutions of the eigenvalue problem (12) and the associated boundary value problem (2) in this subsection.

From (18), eigenvalue problems (12) and (14) can be rewritten as

$$u = T\big(\lambda u\big) \quad \text{and} \quad \bar{u}_h = T_h\big(\bar{\lambda}_h \bar{u}_h\big). \tag{33}$$

For any $u \in M(\lambda_i)$, we define the spectral projection from V to $M_{H,h_k}(\lambda_i)$ by $\bar{E}_{h_k} : V \to M_{H,h_k}(\lambda_i)$. Then there exist q constants $\{\alpha_{h_k,\ell}\}_{\ell=i}^{i+q-1}$ such that

$$\bar{E}_{h_k} u = \sum_{\ell} \alpha_{h_k,\ell} u_{h_k,\ell}. \tag{34}$$

Further, from Lemma 3.5, we obtain that $\sum_\ell \alpha^2_{h_k,\ell} \leq 1$.

Define $\lambda^{h_k} = \frac{a(\bar{E}_{h_k}u, \bar{E}_{h_k}u)}{b(\bar{E}_{h_k}u, \bar{E}_{h_k}u)}$, we have

$$\lambda^{h_k} = \frac{1}{\sum_\ell \alpha^2_{h_k,\ell}} \sum_\ell \alpha^2_{h_k,\ell}\lambda_{h_k,\ell}, \tag{35}$$

which together with Lemma 3.1 yields

$$|\lambda_i - \lambda^{h_k}| \leq \frac{\|u - \bar{E}_{h_k}u\|^2_{a,\Omega}}{\|\bar{E}_{h_k}u\|^2_{0,\Omega}} = \frac{\|u - \bar{E}_{h_k}u\|^2_{a,\Omega}}{\sum_\ell \alpha^2_{h_k,\ell}}. \tag{36}$$

Define $w^{h_k} \in V$ by

$$w^{h_k} = \sum_\ell \alpha_{h_k,\ell}\lambda_{h_k,\ell} T u_{h_k,\ell}. \tag{37}$$

For the approximate solution $\breve{U}_{h_{k+1}} = (\breve{u}_{h_{k+1},i}, \ldots, \breve{u}_{h_{k+1},i+q-1})$ derived in the 6-th step of Algorithm 3.1, let us denote

$$\widetilde{w}^{h_{k+1}} = \sum_\ell \alpha_{h_k,\ell}\breve{u}_{h_{k+1},\ell}.$$

Then based on the structure of the multigrid iteration and the involved linear smoothers in Algorithm 3.1, $\widetilde{w}^{h_{k+1}}$ is the multigrid approximate solution for the finite element solution $P_{h_{k+1}}w^{h_k}$ with the initial value $\bar{E}_{h_k}u(= \sum_\ell \alpha_{h_k,\ell}u_{h_k,\ell})$.

Based on the above discussions, we can derive the following theorem.

Theorem 3.1. *Assume the convergence rate of the multigrid iteration used in Algorithm 3.1 is ν. Given $u \in M(\lambda_i)$, the following connections hold*

$$\begin{aligned} \|u - \bar{E}_{h_{k+1}}u\|_{a,\Omega} &= \|w^{h_{k+1}} - P_{h_{k+1}}w^{h_{k+1}}\|_{a,\Omega} \\ &\quad + \mathcal{O}(r(V_H,\nu))(\|u - \bar{E}_{h_{k+1}}u\|_{a,\Omega} + \|u - \bar{E}_{h_k}u\|_{a,\Omega}), \end{aligned} \tag{38}$$

$$\begin{aligned} \|u - \bar{E}_{h_{k+1}}u\|_{a,\Omega} &= \|w^{h_k} - P_{h_{k+1}}w^{h_k}\|_{a,\Omega} \\ &\quad + \mathcal{O}(r(V_H,\nu))(\|u - \bar{E}_{h_{k+1}}u\|_{a,\Omega} + \|u - \bar{E}_{h_k}u\|_{a,\Omega}), \end{aligned} \tag{39}$$

with $r(V_H,\nu) = \eta_a(H) + \nu^p$.

Proof. $u - \bar{E}_{h_{k+1}}u$ can be decomposed as follows

$$\begin{aligned} u - \bar{E}_{h_{k+1}}u &= (u - w^{h_{k+1}}) + (w^{h_{k+1}} - P_{h_{k+1}}w^{h_{k+1}}) + P_{h_{k+1}}(w^{h_{k+1}} - w^{h_k}) \\ &\quad + (P_{h_{k+1}}w^{h_k} - \bar{E}_{h_{k+1}}u). \end{aligned} \tag{40}$$

For the first part of (40), associating with (12), (33), (34) and (37), we have

$$\begin{aligned} &\|u - w^{h_{k+1}}\|^2_{a,\Omega} = a(u - w^{h_{k+1}}, u - w^{h_{k+1}}) \\ &= \Big(\lambda_i u - \sum_\ell \alpha_{h_{k+1},\ell}\lambda_{h_{k+1},\ell}u_{h_{k+1},\ell}, u - w^{h_{k+1}}\Big) \\ &= \Big(\lambda_i(u - \bar{E}_{h_{k+1}}u) + \sum_\ell \alpha_{h_{k+1},\ell}(\lambda_i - \lambda_{h_{k+1},\ell})u_{h_{k+1},\ell}, u - w^{h_{k+1}}\Big). \end{aligned} \tag{41}$$

The second term of (41) can be estimated as follows

$$\begin{aligned} &\Big(\sum_\ell \alpha_{h_{k+1},\ell}(\lambda_i - \lambda_{h_{k+1},\ell})u_{h_{k+1},\ell}, u - w^{h_{k+1}}\Big) \\ &\lesssim \sum_\ell |\alpha_{h_{k+1},\ell}(\lambda_i - \lambda_{h_{k+1},\ell})|\|u_{h_{k+1},\ell}\|_{0,\Omega}\|u - w^{h_{k+1}}\|_{0,\Omega} \\ &\lesssim \Big(\sum_\ell \alpha^2_{h_{k+1},\ell}(\lambda_{h_{k+1},\ell} - \lambda_i)\Big)^{1/2}\Big(\sum_\ell(\lambda_{h_{k+1},\ell} - \lambda_i)\Big)^{1/2}\|u - w^{h_{k+1}}\|_{a,\Omega} \\ &\lesssim \big(\sum_\ell \alpha^2_{h_{k+1},\ell}\big)^{1/2}(\lambda^{h_{k+1}} - \lambda_i)^{1/2}\Big(\sum_\ell(\lambda_{h_{k+1},\ell} - \lambda_i)\Big)^{1/2}\|u - w^{h_{k+1}}\|_{a,\Omega} \\ &\lesssim \eta_a(H)\|u - \bar{E}_{h_{k+1}}u\|_{a,\Omega}\|u - w^{h_{k+1}}\|_{a,\Omega}. \end{aligned} \tag{42}$$

Combining (41) and (42) leads to

$$\begin{aligned}\|u-w^{h_{k+1}}\|_{a,\Omega} &\lesssim \|u-\bar{E}_{h_{k+1}}u\|_{0,\Omega}+\eta_a(H)\|u-\bar{E}_{h_{k+1}}u\|_{a,\Omega}\\ &\lesssim \eta_a(H)\|u-\bar{E}_{h_{k+1}}u\|_{a,\Omega}.\end{aligned}\tag{43}$$

With regard to the third part of (40), referring to (6) and the proved result (43), we have the following estimates

$$\begin{aligned}\|P_{h_{k+1}}(w^{h_{k+1}}-w^{h_k})\|_{a,\Omega} &\le \|u-w^{h_{k+1}}\|_{a,\Omega}+\|u-w^{h_k}\|_{a,\Omega}\\ &\lesssim \eta_a(H)\big(\|u-\bar{E}_{h_{k+1}}u\|_{a,\Omega}+\|u-\bar{E}_{h_k}u\|_{a,\Omega}\big).\end{aligned}\tag{44}$$

For the last term of (40), since $\widetilde{w}^{h_{k+1}}-\bar{E}_{h_{k+1}}u\in V_H\oplus\mathrm{span}\{\breve{U}_{h_{k+1}}\}$, we use (34) and (37) to show the following inequalities

$$\begin{aligned}&\|P_{h_{k+1}}w^{h_k}-\bar{E}_{h_{k+1}}u\|_{a,\Omega}^2=a(P_{h_{k+1}}w^{h_k}-\bar{E}_{h_{k+1}}u,P_{h_{k+1}}w^{h_k}-\bar{E}_{h_{k+1}}u)\\ =\;&a(P_{h_{k+1}}w^{h_k}-\bar{E}_{h_{k+1}}u,P_{h_{k+1}}w^{h_k}-\widetilde{w}^{h_{k+1}}+\widetilde{w}^{h_{k+1}}-\bar{E}_{h_{k+1}}u)\\ =\;&a(P_{h_{k+1}}w^{h_k}-\bar{E}_{h_{k+1}}u,P_{h_{k+1}}w^{h_k}-\widetilde{w}^{h_{k+1}})\\ &+\Big(\sum_\ell\alpha_{h_k,\ell}(\lambda_{h_k,\ell}-\lambda_i)u_{h_k,\ell}+\lambda_i(\bar{E}_{h_k}u-\bar{E}_{h_{k+1}}u)\\ &+\sum_\ell\alpha_{h_{k+1},\ell}(\lambda_i-\lambda_{h_{k+1},\ell})u_{h_{k+1},\ell},\widetilde{w}^{h_{k+1}}-\bar{E}_{h_{k+1}}u\Big)\\ \lesssim\;&\|P_{h_{k+1}}w^{h_k}-\bar{E}_{h_{k+1}}u\|_{a,\Omega}\|P_{h_{k+1}}w^{h_k}-\widetilde{w}^{h_{k+1}}\|_{a,\Omega}\\ &+\eta_a(H)(\|u-\bar{E}_{h_k}u\|_{a,\Omega}+\|u-\bar{E}_{h_{k+1}}u\|_{a,\Omega})\|\widetilde{w}^{h_{k+1}}-\bar{E}_{h_{k+1}}u\|_{a,\Omega}\\ \lesssim\;&\|P_{h_{k+1}}w^{h_k}-\bar{E}_{h_{k+1}}u\|_{a,\Omega}\|P_{h_{k+1}}w^{h_k}-\widetilde{w}^{h_{k+1}}\|_{a,\Omega}+\eta_a(H)(\|u-\bar{E}_{h_k}u\|_{a,\Omega}\\ &+\|u-\bar{E}_{h_{k+1}}u\|_{a,\Omega})(\|\widetilde{w}^{h_{k+1}}-P_{h_{k+1}}w^{h_k}\|_{a,\Omega}+\|P_{h_{k+1}}w^{h_k}-\bar{E}_{h_{k+1}}u\|_{a,\Omega}).\end{aligned}\tag{45}$$

From the convergence rate ν of the multigrid method, we can derive

$$\begin{aligned}\|P_{h_{k+1}}w^{h_k}-\widetilde{w}^{h_{k+1}}\|_{a,\Omega} &\le \nu^p\|P_{h_{k+1}}w^{h_k}-\bar{E}_{h_k}u\|_{a,\Omega}\\ &\le \nu^p(\|P_{h_{k+1}}w^{h_k}-P_{h_{k+1}}u\|_{a,\Omega}+\|P_{h_{k+1}}u-u\|_{a,\Omega}+\|u-\bar{E}_{h_k}u\|_{a,\Omega})\\ &\le \nu^p(\|w^{h_k}-u\|_{a,\Omega}+\|\bar{E}_{h_{k+1}}u-u\|_{a,\Omega}+\|u-\bar{E}_{h_k}u\|_{a,\Omega})\\ &\le \nu^p(1+C\eta_a(H))(\|u-\bar{E}_{h_{k+1}}u\|_{a,\Omega}+\|u-\bar{E}_{h_k}u\|_{a,\Omega}).\end{aligned}\tag{46}$$

Combining (45) and (46) leads to

$$\begin{aligned}&\|P_{h_{k+1}}w^{h_k}-\bar{E}_{h_{k+1}}u\|_{a,\Omega}^2\\ \le\;&(\nu^p+\eta_a(H))(\|u-\bar{E}_{h_{k+1}}u\|_{a,\Omega}+\|u-\bar{E}_{h_k}u\|_{a,\Omega})\|P_{h_{k+1}}w^{h_k}-\bar{E}_{h_{k+1}}u\|_{a,\Omega}\\ &+\nu^p\eta_a(H)(\|u-\bar{E}_{h_{k+1}}u\|_{a,\Omega}+\|u-\bar{E}_{h_k}u\|_{a,\Omega})^2.\end{aligned}\tag{47}$$

Thus the following estimate holds

$$\|P_{h_{k+1}}w^{h_k}-\bar{E}_{h_{k+1}}u\|_{a,\Omega}\lesssim(\nu^p+\eta_a(H))\big(\|u-\bar{E}_{h_k}u\|_{a,\Omega}+\|u-\bar{E}_{h_{k+1}}u\|_{a,\Omega}\big).\tag{48}$$

Using (43), (44) and (48), we can easily prove (38).

The second identity (39) can be proved by the same technique using the decomposition of $u-\bar{E}_{h_{k+1}}u$ as follows

$$u-\bar{E}_{h_{k+1}}u=(u-w^{h_k})+(w^{h_k}-P_{h_{k+1}}w^{h_k})+(P_{h_{k+1}}w^{h_k}-\bar{E}_{h_{k+1}}u).$$

So we complete the proof. □

Theorem 3.1 has built the connections between the error estimates of the eigenvalue problem and the associated boundary value problem. Since the difference is a higher order term and the theoretical results of the boundary value problem have already been well analyzed, we can derive the theoretical results of adaptive multigrid method for the eigenvalue problem by following the procedure of adaptive finite element method for linear elliptic boundary value problem.

For the projection $\bar{E}_hu=\sum_\ell\alpha_{h,\ell}u_{h,\ell}$, we define the element residual and the jump residual as follows:

$$\mathcal{R}_K(\bar{E}_hu):=\sum_\ell\alpha_{h,\ell}\lambda_{h,\ell}u_{h,\ell}-\phi\bar{E}_hu+\nabla\cdot(\mathcal{A}\nabla\bar{E}_hu),\quad\text{in }K\in\mathcal{T}_h,\tag{49}$$

$$J_e(\bar{E}_hu):=-\mathcal{A}\nabla\bar{E}_hu^+\cdot\nu^+-\mathcal{A}\nabla\bar{E}_hu^-\cdot\nu^-:=[\mathcal{A}\nabla\bar{E}_hu]\cdot\nu_e,\ \text{ on }e\in\mathcal{E}_h.\tag{50}$$

Then we define the local error estimator and the oscillation by

$$\eta_h^2(\bar{E}_h u, K) := h_K^2 \|\mathcal{R}_K(\bar{E}_h u)\|_{0,K}^2 + \sum_{e\in\mathcal{E}_h\cap\partial K} h_e \|\mathcal{J}_e(\bar{E}_h u)\|_{0,e}^2,$$

$$osc_h^2(\bar{E}_h u, K) := h_K^2 \|(I-\mathbb{P}_K)\mathcal{R}_K(\bar{E}_h u)\|_{0,K}^2 + \sum_{e\in\mathcal{E}_h\cap\partial K} h_e \|(I-\mathbb{P}_e)\mathcal{J}_e(\bar{E}_h u)\|_{0,e}^2.$$

Similarly to Theorem 3.1, we have the following two theorems about the error estimator and oscillation by combining the definitions of the error estimator and oscillation, Sobolev trace theorem and the inverse inequality of the finite element method.

Theorem 3.2. *Given any $u \in M(\lambda_i)$, the following connections between the a posteriori error estimators hold*

$$\eta_{h_{k+1}}(\bar{E}_{h_{k+1}} u, \mathcal{T}_{h_{k+1}}) = \widetilde{\eta}_{h_{k+1}}(P_{h_{k+1}} w^{h_{k+1}}, \mathcal{T}_{h_{k+1}}) + \mathcal{O}(r(V_H,\nu))(\|u-\bar{E}_{h_{k+1}}u\|_{a,\Omega} + \|u-\bar{E}_{h_k}u\|_{a,\Omega}), \tag{51}$$

$$\eta_{h_{k+1}}(\bar{E}_{h_{k+1}} u, \mathcal{T}_{h_{k+1}}) = \widetilde{\eta}_{h_{k+1}}(P_{h_{k+1}} w^{h_k}, \mathcal{T}_{h_{k+1}}) + \mathcal{O}(r(V_H,\nu))(\|u-\bar{E}_{h_{k+1}}u\|_{a,\Omega} + \|u-\bar{E}_{h_k}u\|_{a,\Omega}). \tag{52}$$

Theorem 3.3. *Given any $u \in M(\lambda_i)$, the following connections between the oscillations hold*

$$osc_{h_{k+1}}(\bar{E}_{h_{k+1}} u, \mathcal{T}_{h_{k+1}}) = \widetilde{osc}_{h_{k+1}}(P_{h_{k+1}} w^{h_{k+1}}, \mathcal{T}_{h_{k+1}}) + \mathcal{O}(r(V_H,\nu))(\|u-\bar{E}_{h_{k+1}}u\|_{a,\Omega} + \|u-\bar{E}_{h_k}u\|_{a,\Omega}), \tag{53}$$

$$osc_{h_{k+1}}(\bar{E}_{h_{k+1}} u, \mathcal{T}_{h_{k+1}}) = \widetilde{osc}_{h_{k+1}}(P_{h_{k+1}} w^{h_k}, \mathcal{T}_{h_{k+1}}) + \mathcal{O}(r(V_H,\nu))(\|u-\bar{E}_{h_{k+1}}u\|_{a,\Omega} + \|u-\bar{E}_{h_k}u\|_{a,\Omega}). \tag{54}$$

3.4. The efficiency and reliability of the residual type a posteriori error estimator

Now we propose the efficiency and reliability of the residual type a posteriori error estimator for eigenvalue problems as Lemma 2.1. In the rest of this paper, we assume the mesh size H and ν^p are small enough such that

$$r(V_H,\nu)\|u-\bar{E}_{h_k}u\|_{a,\Omega}^2 \le \|u-\bar{E}_{h_{k+1}}u\|_{a,\Omega}^2, \qquad \text{for } k \ge 1. \tag{55}$$

Remark 3.2. It is worth mentioning that the assumption (55) indicates that the initial mesh should be small enough such that the error does not change too much after each refinement step. From another point of view, there exist both sharp upper bound and lower bound for $\|u-\bar{E}_{h_k}u\|_{a,\Omega}$ (see [45]) when the exact eigenfunction u does not belong to finite element space V_{h_k}. Thus such an assumption is reasonable generally.

Besides, in order to meet such a condition, we may need to execute several times multigrid iteration step ($p \ge 1$). But in our numerical experiments, one or two times iteration steps are enough to derive the optimal accuracy due to the efficiency of the multigrid method.

Based on Theorems 3.1–3.3 and (55), we can obtain the following reliability and efficiency of the a posteriori error estimator for eigenvalue problem by applying Lemma 2.1.

Lemma 3.8. *There exist some constants depending only on the shape regularity of $\mathcal{T}_{h_{k+1}}$ such that*

$$\|u-\bar{E}_{h_{k+1}}u\|_{a,\Omega}^2 \le C_1 \eta_{h_{k+1}}^2(\bar{E}_{h_{k+1}}u, \mathcal{T}_{h_{k+1}}), \tag{56}$$

$$C_2 \eta_{h_{k+1}}^2(\bar{E}_{h_{k+1}}u, \mathcal{T}_{h_{k+1}}) \le \|u-\bar{E}_{h_{k+1}}u\|_{a,\Omega}^2 + C_3 osc_{h_{k+1}}^2(\bar{E}_{h_{k+1}}u, \mathcal{T}_{h_{k+1}}). \tag{57}$$

Consequently, we have

$$|\lambda_i - \lambda^{h_{k+1}}| \lesssim \eta_{h_{k+1}}^2(\bar{E}_{h_{k+1}}u, \mathcal{T}_{h_{k+1}}). \tag{58}$$

Proof. Since w^{h_k} is the exact solution of the elliptic boundary value problem, from Lemma 2.1, we have

$$\|w^{h_k} - P_{h_{k+1}}w^{h_k}\|_{a,\Omega}^2 \le \widetilde{C}_1 \widetilde{\eta}_{h_{k+1}}^2(P_{h_{k+1}}w^{h_k}, \mathcal{T}_{h_{k+1}}), \tag{59}$$

$$\widetilde{C}_2 \widetilde{\eta}_{h_{k+1}}^2(P_{h_{k+1}}w^{h_k}, \mathcal{T}_{h_{k+1}}) \le \|w^{h_k} - P_{h_{k+1}}w^{h_k}\|_{a,\Omega}^2 + \widetilde{C}_3 \widetilde{osc}_{h_{k+1}}^2(P_{h_{k+1}}w^{h_k}, \mathcal{T}_{h_{k+1}}). \tag{60}$$

Then we can derive the desired results by applying Lemmas 2.1, 3.1, Theorems 3.1–3.3. □

In order to further built the efficiency and reliability for the residual type a posteriori error estimator $\eta_{h_k}(U_{h_k}, \mathcal{T}_{h_k})$ which was actually used in Algorithm 3.1, the following connections between $\eta_{h_k}(\bar{E}_{h_k}U, \mathcal{T}_{h_k})$ and $\eta_{h_k}(U_{h_k}, \mathcal{T}_{h_k})$ will play a crucial role in our analysis. For readability purpose of this paper, the detailed proof of Lemma 3.9 was proposed in Appendix.

Lemma 3.9. *For any orthogonal basis $\{u_\ell\}_{\ell=i}^{i+q-1}$ of $M(\lambda_i)$, we have the following estimates*

$$\frac{1}{q}\eta_{h_k}^2(\bar{E}_{h_k}U, K) \le \eta_{h_k}^2(U_{h_k}, K) \le \frac{2q}{1 - C\eta_a^2(H)}\big(\eta_{h_k}^2(\bar{E}_{h_k}U, K) + R(U_{h_k})\big), \tag{61}$$

where $\bar{E}_{h_k}U = (\bar{E}_{h_k}u_i, \dots, \bar{E}_{h_k}u_{i+q-1})$, $R(U_{h_k}) = \sum_\ell \sum_s h_K^2|\lambda_{h_k,\ell} - \lambda_{h_k,s}|^2\|u_{h_k,s}\|_{0,K}^2$. Consequently

$$\eta_{h_k}^2(\bar{E}_{h_k}U, \mathcal{T}_{h_k}) \cong \eta_{h_k}^2(U_{h_k}, \mathcal{T}_{h_k}) \text{ and } osc_{h_k}^2(\bar{E}_{h_k}U, \mathcal{T}_{h_k}) \cong osc_{h_k}^2(U_{h_k}, \mathcal{T}_{h_k}). \tag{62}$$

Theorem 3.4. *Let $\lambda_i \in \mathbb{R}$ be an eigenvalue of (12) with multiplicity q, we have the following estimates for the eigenpair approximations obtained from Algorithm 3.1*

$$\Theta^2(M(\lambda_i), M_{H,h_{k+1}}(\lambda_i)) \le C_1\eta_{h_{k+1}}^2(U_{h_{k+1}}, \mathcal{T}_{h_{k+1}}), \tag{63}$$

$$C_2\eta_{h_{k+1}}^2(U_{h_{k+1}}, \mathcal{T}_{h_{k+1}}) \le \Theta^2\big(M(\lambda_i), M_{H,h_{k+1}}(\lambda_i)\big) + C_3 osc_{h_{k+1}}^2(U_{h_{k+1}}, \mathcal{T}_{h_{k+1}}). \tag{64}$$

Proof. Let $\{u_i, \dots, u_{i+q-1}\}$ be an orthogonal basis of $M(\lambda_i)$. On the one hand

$$\begin{aligned}\sup_{u\in M(\lambda_i)} \inf_{v\in M_{H,h_{k+1}}(\lambda_i)} \|u - v\|_{a,\Omega} &\le \sup_{u\in M(\lambda_i)} \|u - \bar{E}_{h_{k+1}}u\|_{a,\Omega} \\ &\lesssim \max_{\ell=i,\dots,i+q-1} \|u_\ell - \bar{E}_{h_{k+1}}u_\ell\|_{a,\Omega} \\ &\lesssim \|U - \bar{E}_{h_{k+1}}U\|_{a,\Omega}.\end{aligned} \tag{65}$$

On the other hand

$$\begin{aligned}\sup_{u\in M(\lambda_i)} \inf_{v\in M_{H,h_{k+1}}(\lambda_i)} \|u - v\|_{a,\Omega} &\ge \sup_{u\in M(\lambda_i)} \|u - \bar{P}_{h_{k+1}}u\|_{a,\Omega} \\ &\gtrsim \max_{\ell=i,\dots,i+q-1} \|u_\ell - \bar{P}_{h_{k+1}}u_\ell\|_{a,\Omega} \\ &\gtrsim \|U - \bar{P}_{h_{k+1}}U\|_{a,\Omega},\end{aligned} \tag{66}$$

where $\bar{P}_{h_{k+1}}$ is the Galerkin projection from V to $V_H \oplus \text{span}\{\breve{U}_{h_{k+1}}\}$ defined by

$$a(u - \bar{P}_{h_{k+1}}u, v_{H,h_{k+1}}) = 0, \quad \forall v_{H,h_{k+1}} \in V_H \oplus \text{span}\{\breve{U}_{h_{k+1}}\}.$$

From Lemma 3.8, we have

$$\|U - \bar{E}_{h_{k+1}}U\|_{a,\Omega}^2 \le C_1\eta_{h_{k+1}}^2(\bar{E}_{h_{k+1}}U, \mathcal{T}_{h_{k+1}}), \tag{67}$$

$$C_2\eta_{h_{k+1}}^2(\bar{E}_{h_{k+1}}U, \mathcal{T}_{h_{k+1}}) \le \|U - \bar{E}_{h_{k+1}}U\|_{a,\Omega}^2 + C_3 osc_{h_{k+1}}^2(\bar{E}_{h_{k+1}}U, \mathcal{T}_{h_{k+1}}). \tag{68}$$

Hence, we obtained from Lemma 3.4 that

$$\|U - \bar{P}_{h_{k+1}}U\|_{a,\Omega}^2 \le C_1\eta_{h_{k+1}}^2(\bar{E}_{h_{k+1}}U, \mathcal{T}_{h_{k+1}}), \tag{69}$$

$$C_2\eta_{h_{k+1}}^2(\bar{E}_{h_{k+1}}U, \mathcal{T}_{h_{k+1}}) \le \|u - \bar{P}_{h_{k+1}}U\|_{a,\Omega}^2 + C_3 osc_{h_{k+1}}^2(\bar{E}_{h_{k+1}}U, \mathcal{T}_{h_{k+1}}) \tag{70}$$

when H is small enough.

Combining (65)–(70) and Lemma 3.9, we get

$$\sup_{u\in M(\lambda_i)} \inf_{v\in M_{H,h_{k+1}}(\lambda_i)} \|u - v\|_{a,\Omega}^2 \le C_1\eta_{h_{k+1}}^2(U_{h_{k+1}}, \mathcal{T}_{h_{k+1}}),$$

$$C_2\eta_{h_{k+1}}^2(U_{h_{k+1}}, \mathcal{T}_{h_{k+1}}) \le \sup_{u\in M(\lambda_i)} \inf_{v\in M_{H,h_{k+1}}(\lambda_i)} \|u - v\|_{a,\Omega}^2 + C_3 osc_{h_{k+1}}^2(U_{h_{k+1}}, \mathcal{T}_{h_{k+1}}).$$

Namely

$$\hat{\Theta}^2(M(\lambda_i), M_{H,h_{k+1}}(\lambda_i)) \le C_1\eta_{h_{k+1}}^2(U_{h_{k+1}}, \mathcal{T}_{h_{k+1}}), \tag{71}$$

$$C_2\eta_{h_{k+1}}^2(U_{h_{k+1}}, \mathcal{T}_{h_{k+1}}) \le \hat{\Theta}^2(M(\lambda_i), M_{H,h_{k+1}}(\lambda_i)) + C_3 osc_{h_{k+1}}^2(U_{h_{k+1}}, \mathcal{T}_{h_{k+1}}). \tag{72}$$

From (71), (72) and Lemma 3.2, we can derive the desired estimates. □

4. Convergence and optimal complexity of adaptive multigrid method

In this section, we will study the convergence property and complexity analysis of Algorithm 3.1.

4.1. Convergence property

This subsection is devoted to introducing the convergence property of Algorithm 3.1 based on the existing results for the elliptic boundary value problem and the connections between the elliptic boundary value problem and eigenvalue problem presented in Theorems 3.1–3.3.

Lemma 4.1. *Let $\theta \in (0, 1)$ be a given constant. If there holds*

$$\eta^2_{h_k}(U_{h_k}, \mathcal{M}_{h_k}) \geq \theta \eta^2_{h_k}(U_{h_k}, \mathcal{T}_{h_k}), \tag{73}$$

then for any orthogonal basis $\{u_\ell\}_{\ell=i}^{i+q-1}$ of $M(\lambda_i)$, there exists a constant $\bar{\theta} \in (0, 1)$ such that

$$\eta^2_{h_k}(\bar{E}_{h_k}U, \mathcal{M}_{h_k}) \geq \bar{\theta} \eta^2_{h_k}(\bar{E}_{h_k}U, \mathcal{T}_{h_k}), \tag{74}$$

where $\bar{E}_{h_k}U = (\bar{E}_{h_k}u_i, \ldots, \bar{E}_{h_k}u_{i+q-1})$.

Proof. From Lemma 3.9 and (73), we have

$$\begin{aligned} &\frac{2q}{1 - C\eta_a^2(H)}\Big(\eta^2_{h_k}(\bar{E}_{h_k}U, \mathcal{M}_{h_k}) + \sum_\ell \sum_s |\lambda_{h_k,\ell} - \lambda_{h_k,s}|^2 \eta_a^2(H) \|u_{h_k,s}\|^2_{0,\mathcal{M}_{h_k}}\Big) \\ &\geq \eta^2_{h_k}(U_{h_k}, \mathcal{M}_{h_k}) \ \geq \ \theta \eta^2_{h_k}(U_{h_k}, \mathcal{T}_{h_k}). \end{aligned} \tag{75}$$

By using the similar procedure as that, for deducing Lemma 3.9, we get

$$\sum_\ell \sum_s |\lambda_{h_k,\ell} - \lambda_{h_k,s}|^2 \eta_a^2(H) \|u_{h_k,s}\|^2_{0,\mathcal{M}_{h_k}} \lesssim \eta_a^4(H) \eta^2_{h_k}(U_{h_k}, \mathcal{T}_{h_k}). \tag{76}$$

Combining (75) and (76) leads to

$$\frac{2q}{1 - C\eta_a^2(H)} \eta^2_{h_k}(\bar{E}_{h_k}U, \mathcal{M}_{h_k}) + \frac{C\eta_a^4(H)}{1 - C\eta_a^2(H)} \eta^2_{h_k}(U_{h_k}, \mathcal{T}_{h_k}) \geq \theta \eta^2_{h_k}(U_{h_k}, \mathcal{T}_{h_k}).$$

Consequently

$$\begin{aligned} \frac{2q}{1 - C\eta_a^2(H)} \eta^2_{h_k}(\bar{E}_{h_k}U, \mathcal{M}_{h_k}) &\geq (\theta - \frac{C\eta_a^4(H)}{1 - C\eta_a^2(H)}) \eta^2_{h_k}(U_{h_k}, \mathcal{T}_{h_k}) \\ &\geq \frac{1}{q}(\theta - \frac{C\eta_a^4(H)}{1 - C\eta_a^2(H)}) \eta^2_{h_k}(\bar{E}_{h_k}U, \mathcal{T}_{h_k}). \end{aligned}$$

Hence, there exists a constant C such that

$$\begin{aligned} \eta^2_{h_k}(\bar{E}_{h_k}U, \mathcal{M}_{h_k}) &\geq \frac{1}{2q^2}\Big(\theta(1 - C\eta_a^2(H)) - C\eta_a^4(H)\Big) \eta^2_{h_k}(\bar{E}_{h_k}U, \mathcal{T}_{h_k}) \\ &\geq \frac{\theta}{2q^2}\big(1 - C\eta_a^2(H)\big) \eta^2_{h_k}(\bar{E}_{h_k}U, \mathcal{T}_{h_k}), \end{aligned}$$

provided H is small enough.

Taking $\bar{\theta} = \frac{\theta}{2q^2}\big(1 - C\eta_a^2(H)\big)$, we can derive the desired result. □

Now we prove the following convergence result which describes error reduction of Algorithm 3.1 by using the obtained conclusions.

Lemma 4.2. *Let $\theta \in (0, 1)$, $\lambda_i \in \mathbb{R}$ be some eigenvalue of (12) with multiplicity q and the corresponding eigenspace being $M(\lambda_i) = span\{u_i, \ldots, u_{i+q-1}\}$, $\{(\lambda_{h_k,\ell}, u_{h_k,\ell}), \ell = i, \ldots, i+q-1\}$ be a sequence of finite element solutions produced by Algorithm 3.1. Then there exist constants $\gamma > 0$ and $\widetilde{\alpha} \in (0, 1)$ depending only on the shape regularity of meshes and marking parameter θ such that*

$$\|U - \bar{E}_{h_{k+1}}U\|^2_{a,\Omega} + \gamma \eta^2_{h_{k+1}}(\bar{E}_{h_{k+1}}U, \mathcal{T}_{h_{k+1}}) \leq \widetilde{\alpha}^2\big(\|U - \bar{E}_{h_k}U\|^2_{a,\Omega} + \gamma \eta^2_{h_k}(\bar{E}_{h_k}U, \mathcal{T}_{h_k})\big). \tag{77}$$

Proof. Since the following marking strategy is used in Algorithm 3.1

$$\eta^2_{h_k}(U_{h_k}, \mathcal{M}_{h_k}) \geq \theta \eta^2_{h_k}(U_{h_k}, \mathcal{T}_{h_k}). \tag{78}$$

Then, from Lemma 4.1, there exists a constant $\bar{\theta} \in (0, 1)$ such that

$$\eta_{h_k}^2(\bar{E}_{h_k}U, \mathcal{M}_{h_k}) \geq \bar{\theta}\eta_{h_k}^2(\bar{E}_{h_k}U, \mathcal{T}_{h_k}). \tag{79}$$

For any $u_\ell \in M(\lambda_i)$, define $w^{h_k,\ell}$ according to (34) and (37). Set $W^{h_k} = (w^{h_k,\ell})_{\ell=i}^{i+q-1}$. From Theorem 3.2, (55) and (79), there exists a constant $\theta' \in (0, 1)$ such that

$$\widetilde{\eta}_{h_k}^2(P_{h_k}W^{h_k}, \mathcal{M}_{h_k}) \geq \theta'\widetilde{\eta}_{h_k}^2(P_{h_k}W^{h_k}, \mathcal{T}_{h_k}). \tag{80}$$

The inequality (80) means that we derive the Dörfler's marking strategy for elliptic boundary value problem, then we conclude from Lemma 2.2 that there exist constants $\widetilde{\gamma} > 0$ and $\xi \in (0, 1)$ satisfying

$$\begin{aligned} &\|W^{h_k} - P_{h_{k+1}}W^{h_k}\|_{a,\Omega} + \widetilde{\gamma}\widetilde{\eta}_{h_{k+1}}^2(P_{h_{k+1}}W^{h_k}, \mathcal{T}_{h_{k+1}}) \\ &\leq \xi^2\big(\|W^{h_k} - P_{h_k}W^{h_k}\|_{a,\Omega} + \widetilde{\gamma}\widetilde{\eta}_{h_k}^2(P_{h_k}W^{h_k}, \mathcal{T}_{h_k})\big). \end{aligned} \tag{81}$$

Combining (39), (52), (55) and (81) leads to

$$\begin{aligned} &\|U - \bar{E}_{h_{k+1}}U\|_{a,\Omega}^2 + \widetilde{\gamma}\eta_{h_{k+1}}^2(\bar{E}_{h_{k+1}}U, \mathcal{T}_{h_{k+1}}) \\ &\leq (1+\delta_1)\big(\|W^{h_k} - P_{h_{k+1}}W^{h_k}\|_{a,\Omega}^2 + \widetilde{\gamma}\widetilde{\eta}_{h_{k+1}}^2(P_{h_{k+1}}W^{h_k}, \mathcal{T}_{h_{k+1}})\big) \\ &\quad + C\delta_1^{-1}r^2(V_H, \nu)(\|U - \bar{E}_{h_{k+1}}U\|_{a,\Omega}^2 + \|U - \bar{E}_{h_k}U\|_{a,\Omega}^2) \\ &\leq (1+\delta_1)\big(\|W^{h_k} - P_{h_{k+1}}W^{h_k}\|_{a,\Omega}^2 + \widetilde{\gamma}\widetilde{\eta}_{h_{k+1}}^2(P_{h_{k+1}}W^{h_k}, \mathcal{T}_{h_{k+1}})\big) \\ &\quad + C\delta_1^{-1}r(V_H, \nu)(\|U - \bar{E}_{h_{k+1}}U\|_{a,\Omega}^2 + \widetilde{\gamma}\eta_{h_{k+1}}^2(\bar{E}_{h_{k+1}}U, \mathcal{T}_{h_{k+1}})). \end{aligned} \tag{82}$$

Simplifying the above formula leads to the following estimates when $r(V_H, \nu)$ is small enough

$$\begin{aligned} &\|U - \bar{E}_{h_{k+1}}U\|_{a,\Omega}^2 + \widetilde{\gamma}\eta_{h_{k+1}}^2(\bar{E}_{h_{k+1}}U, \mathcal{T}_{h_{k+1}}) \\ &\leq \frac{1+\delta_1}{1 - C\delta_1^{-1}r(V_H, \nu)}\big(\|W^{h_k} - P_{h_{k+1}}W^{h_k}\|_{a,\Omega}^2 + \widetilde{\gamma}\widetilde{\eta}_{h_{k+1}}^2(P_{h_{k+1}}W^{h_k}, \mathcal{T}_{h_{k+1}})\big) \\ &\leq \frac{(1+\delta_1)\xi^2}{1 - C\delta_1^{-1}r(V_H, \nu)}\big(\|W^{h_k} - P_{h_k}W^{h_k}\|_{a,\Omega}^2 + \widetilde{\gamma}\widetilde{\eta}_{h_k}^2(P_{h_k}W^{h_k}, \mathcal{T}_{h_k})\big). \end{aligned} \tag{83}$$

Using a similar argument on the righthand term of (83), we have

$$\begin{aligned} &\|W^{h_k} - P_{h_k}W^{h_k}\|_{a,\Omega}^2 + \widetilde{\gamma}\widetilde{\eta}_{h_k}^2(P_{h_k}W^{h_k}, \mathcal{T}_{h_k}) \\ &\leq (1 + \delta_2 + C\delta_2^{-1}r(V_H, \nu))(\|U - \bar{E}_{h_k}U\|_{a,\Omega}^2 + \widetilde{\gamma}\eta_{h_k}^2(\bar{E}_{h_k}U, \mathcal{T}_{h_k})). \end{aligned} \tag{84}$$

From (83) and (84), the following estimate holds

$$\begin{aligned} &\|U - \bar{E}_{h_{k+1}}U\|_{a,\Omega}^2 + \widetilde{\gamma}\eta_{h_{k+1}}^2(\bar{E}_{h_{k+1}}U, \mathcal{T}_{h_{k+1}}) \\ &\leq \frac{(1+\delta_1)(1+\delta_2 + C\delta_2^{-1}r(V_H, \nu))\xi^2}{1 - C\delta_1^{-1}r(V_H, \nu)}(\|U - \bar{E}_{h_k}U\|_{a,\Omega}^2 + \widetilde{\gamma}\eta_{h_k}^2(\bar{E}_{h_k}U, \mathcal{T}_{h_k})). \end{aligned} \tag{85}$$

Set

$$\widetilde{\alpha}^2 := \frac{(1+\delta_1)\big(1+\delta_2 + C\delta_2^{-1}r(V_H, \nu)\big)\xi^2}{1 - C\delta_1^{-1}r(V_H, \nu)}, \quad \gamma = \widetilde{\gamma},$$

we can derive

$$\|U - \bar{E}_{h_{k+1}}U\|_{a,\Omega}^2 + \gamma\eta_{h_{k+1}}^2(\bar{E}_{h_{k+1}}U, \mathcal{T}_{h_{k+1}}) \leq \widetilde{\alpha}^2\big(\|U - \bar{E}_{h_k}U\|_{a,\Omega}^2 + \gamma\eta_{h_k}^2(\bar{E}_{h_k}U, \mathcal{T}_{h_k})\big),$$

which is just the desired result (77). □

Similar to Theorem 3.4, we can also get the contraction property for the gap between $M(\lambda_i)$ and its finite element approximation $M_{H,h_n}(\lambda_i)$.

Theorem 4.1. *Let $\lambda_i \in \mathbb{R}$ be some eigenvalue of (12) with multiplicity q and the corresponding eigenspace being $M(\lambda_i) = \{u_i, \dots, u_{i+q-1}\}$, $\{(\lambda_{h_n,\ell}, u_{h_n,\ell}), \ell = i, \dots, i+q-1\}$ be a sequence of finite element solutions produced by Algorithm 3.1. Then, there exists a constant $\widetilde{\alpha} \in (0, 1)$, depending only on the shape regularity of meshes and the marking parameter θ such that*

$$\Theta(M(\lambda_i), M_{H,h_n}(\lambda_i)) \lesssim \widetilde{\alpha}^n. \tag{86}$$

Proof. For any $u \in M(\lambda_i)$, there exist q constants $\{\alpha_\ell\}$ such that $u = \sum_\ell \alpha_\ell u_\ell$. Thus, we have

$$\begin{aligned} \|u - \bar{E}_{h_n} u\|_{a,\Omega}^2 &= \|\sum_\ell \alpha_\ell (u_\ell - \bar{E}_{h_n} u_\ell)\|_{a,\Omega}^2 \\ &\le \sum_\ell |\alpha_\ell|^2 \sum_\ell \|u_\ell - \bar{E}_{h_n} u_\ell\|_{a,\Omega}^2 = \sum_\ell \|u_\ell - \bar{E}_{h_n} u_\ell\|_{a,\Omega}^2 \\ &\le \sum_\ell \big(\|u_\ell - \bar{E}_{h_n} u_\ell\|_{a,\Omega}^2 + \gamma \eta_{h_n}^2(\bar{E}_{h_n} u_\ell, \mathcal{T}_{h_n})\big) \lesssim \widetilde{\alpha}^{2n}. \end{aligned}$$

Therefore, from the definition of $\hat{\Theta}$, there holds

$$\hat{\Theta}(M(\lambda_i), M_{H,h_n}(\lambda_i)) \lesssim \widetilde{\alpha}^n.$$

Using Lemma 3.2, we then arrive at (86). □

4.2. Complexity analysis

In this subsection, we propose the complexity analysis of Algorithm 3.1. As in the normal analysis of AFEM for the boundary value problems, in order to state the result of the complexity estimate, we introduce a function approximation class as follows (cf. [11])

$$\mathcal{A}^s := \{v \in H_0^1(\Omega) : |v|_s < \infty\},$$

where $|\cdot|_s$ is defined as follows

$$|v|_s := \sup_{\varepsilon>0} \varepsilon \inf_{\{\mathcal{T}_{h_k} \subset \mathcal{T}_{h_1} : \inf\left(\|v - v_{h_k}\|_{a,\Omega}^2 + osc_{h_k}^2(v_{h_k}, \mathcal{T}_{h_k})\right)^{1/2} \le \varepsilon\}} (\#\mathcal{T}_{h_k} - \#\mathcal{T}_{h_1})^s$$

and $\mathcal{T}_{h_k} \subset \mathcal{T}_{h_1}$ means $\mathcal{T}_{h_k}$ is a refinement of $\mathcal{T}_{h_1}$. In this study, we use $\#\mathcal{T}$ to denote the number of elements in the mesh $\mathcal{T}$. So $\mathcal{A}^s$ is the class of functions that can be approximated within a given tolerance ε by continuous piecewise polynomial functions over a partition $\mathcal{T}_{h_k}$ satisfying $\#\mathcal{T}_{h_k} - \#\mathcal{T}_{h_1} \lesssim \varepsilon^{-1/s}|v|_s^{1/s}$.

Notice that the convergence result presented in Lemma 4.2 is the same as that in [11,14]. By using the same technique, we can prove that Algorithm 3.1 has the following optimal complexity. Please refer to papers [11,14] for the detailed proof.

Lemma 4.3. *Let $\lambda_i \in \mathbb{R}$ be an eigenvalue of (12) with multiplicity q and the corresponding eigenspace being $M(\lambda_i) = \{u_i, \ldots, u_{i+q-1}\}$, $\{(\lambda_{h_k,\ell}, u_{h_k,\ell}), \ell = i, \ldots, i+q-1\}$ be a sequence of finite element solutions produced by Algorithm 3.1. Then, the following optimality holds*

$$\|U - \bar{E}_{h_k} U\|_{a,\Omega}^2 + osc_{h_k}^2(\bar{E}_{h_k} U, \mathcal{T}_{h_k}) \lesssim (\#\mathcal{T}_{h_k} - \#\mathcal{T}_{h_0})^{-2s}, \tag{87}$$

$$|\lambda^{h_k} - \lambda_i| \lesssim (\#\mathcal{T}_{h_k} - \#\mathcal{T}_{h_0})^{-2s}, \tag{88}$$

provided H is small enough.

Similar to Theorem 4.1, we obtain the following conclusion from Lemma 4.3.

Theorem 4.2. *Let $\lambda_i \in \mathbb{R}$ be an eigenvalue of (12) with multiplicity q and the corresponding eigenspace being $M(\lambda_i) = \{u_i, \ldots, u_{i+q-1}\}$, $\{(\lambda_{h_n,\ell}, u_{h_n,\ell}), \ell = i, \ldots, i+q-1\}$ be a sequence of finite element solutions produced by Algorithm 3.1. Then, the kth iterate solution space $M_{H,h_k}(\lambda_i)$ satisfies*

$$\Theta(M(\lambda_i), M_{H,h_k}(\lambda_i)) \lesssim (\#\mathcal{T}_{h_k} - \#\mathcal{T}_{h_0})^{-s},$$

provided H is small enough.

Now we come to briefly estimate the computational work of Algorithm 3.1. Here we have to use additionally, that the sequence of unknowns belongs to a geometric progression (see e.g. [46]):

$$N_k < \sigma_0 N_k \le N_{k+1} < \sigma_1 N_k, \quad k = 1, 2, \ldots \tag{89}$$

Theorem 4.3. *Assume the multiple eigenvalue problem solving in the coarse spaces V_H and V_{h_1} need work M_H and M_1, respectively, and the work of the multigrid solver for the involved boundary value problems in V_{h_k} is $O(N_k)$ for $k = 2, 3, \ldots, n$. Then the total computational work of Algorithm 3.1 can be bounded by $O\big(M_1 + M_H \log(N_n) + N_n\big)$ and furthermore $O\big(N_n\big)$ provided M_H and M_1 are small enough.*

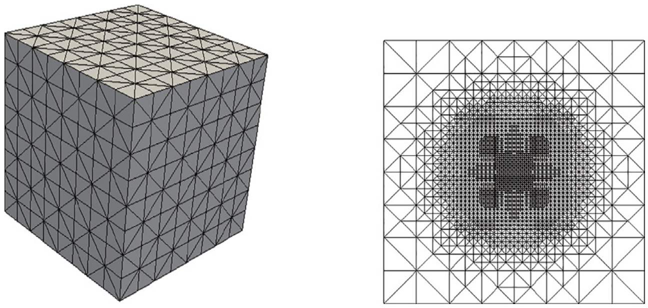

Fig. 1. The initial mesh and the triangulations after adaptive iterations for Example 1.

Proof. Let W denote the whole computational work of Algorithm 3.1, W_k denote the work on the kth level for $k = 1, \ldots, n$. From the definition of Algorithm 3.1 and (89), it follows that

$$\begin{aligned} W &= \sum_{k=1}^{n} W_k = \mathcal{O}\Big(M_1 + \sum_{k=2}^{n}(N_k + M_H)\Big) \\ &= \mathcal{O}\Big(M_1 + M_H(n-1) + N_n \sum_{k=2}^{n}\Big(\frac{1}{\sigma_0}\Big)^{(n-k)}\Big) \\ &= \mathcal{O}\Big(M_1 + M_H \log(N_n) + N_n\Big). \end{aligned}$$

Thus, the computational work W can be bounded by $O\big(M_1 + M_H \log(N_n) + N_n\big)$, and moreover, by $\mathcal{O}(N_n)$ if M_H and M_1 are small enough. □

5. Numerical experiments

In this section, we present two numerical examples for the second order elliptic eigenvalue problems by Algorithm 3.1. In these numerical examples, the well known implicitly restarted Lanczos method, which is included in the popular package ARPACK, is adopted to solve the small-scale eigenvalue problems. We set $p = 2$ in Algorithm 3.1, and each adaptive multigrid iteration step is executed with one multigrid V-cycle as the basic iteration using two times Gauss–Seidel iterations on those newly refined elements and their neighbors.

In Algorithm 3.1, we need to provide the multiplicity of the desired eigenvalue, which is usually unknown. In this case, we can first compute the approximate eigenpairs on the initial mesh and then deduce the multiplicity of the desired eigenvalue. Besides, in most case, people need to solve the smallest N eigenvalues, or the largest N eigenvalues, or the N eigenvalues closed to a special value. Then we can use our algorithm to solve such problems through using the desired N approximate solutions to construct the low dimensional correction space, which is involved in the second step of Algorithm 3.1.

Example 1. In the first example, we consider the following harmonic oscillator equation (see [47])

$$\left\{ \begin{array}{rcll} -\frac{1}{2}\Delta u + \frac{1}{2}|x|^2 u &=& \lambda u, & \text{in } \Omega, \\ u &=& 0, & \text{on } \partial\Omega, \end{array} \right. \tag{90}$$

where $\Omega = \mathbb{R}^3$ and $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. The eigenvalues of (90) are $\lambda_n = n + \frac{1}{2}$ with multiplicity $n(n+1)/2$ $(n = 1, 2, \ldots)$ and eigenfunction is $u_n = \kappa e^{-|x|^2/2} H_n(x)$ with any nonzero constant κ and $H_n(x) = (-1)^n e^{x^2}(d^n/dx^n)e^{-x^2}$. Since the solution of (90) exponentially decays, we set $\Omega = (-4, 4)^3$ in our computation.

We calculate the approximations of the first two smallest eigenvalues λ_1 and λ_2 with multiplicity 1 and multiplicity 3, respectively. The eigenvalue problem is solved by Algorithm 3.1 with the parameters $\theta = 0.4$. Figs. 1 shows the initial mesh and the triangulations after 15 times adaptive refinements. Fig. 2 shows the corresponding error estimates. It is shown in Fig. 2 that the eigenpair approximations by Algorithm 3.1 have the optimal convergence rate which coincide with the theoretical results.

In order to show the efficiency of Algorithm 3.1 more intuitively, we present the computational time of Algorithm 3.1 and direct AFEM in Fig. 3, and the eigenvalue problems involved in the direct AFEM are solved by the popular package ARPACK. Fig. 3 shows that Algorithm 3.1 has a better efficiency than the direct AFEM. Meanwhile, Algorithm 3.1 has the linear computational complexity.

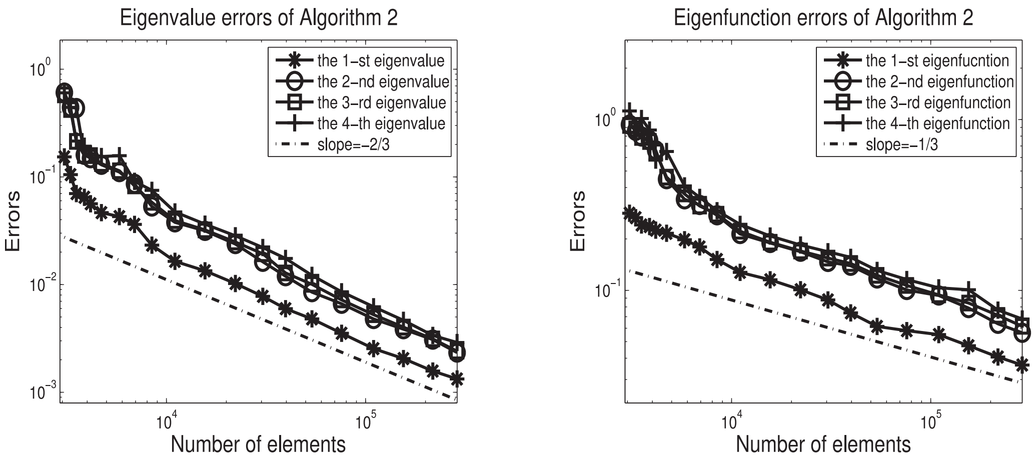

Fig. 2. The errors of the eigenpair approximations by Algorithm 3.1 for Example 1.

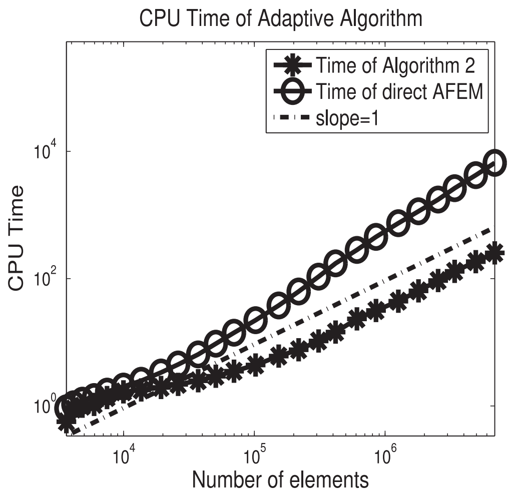

Fig. 3. The computational time (in second) by Algorithm 3.1 and direct AFEM for Example 1.

Example 2. In the second example, we consider the following second order elliptic eigenvalue problem

$$\left\{\begin{array}{rcll} -\nabla\cdot(\mathcal{A}\nabla u)+\varphi u &=& \lambda u, & \text{in } \Omega,\\ u &=& 0, & \text{on } \partial\Omega,\\ \|u\|_{0,\Omega} &=& 1, & \end{array}\right. \tag{91}$$

with

$$\mathcal{A}=\left(\begin{array}{ccc} 1+(x_1-\frac{1}{2})^2 & (x_1-\frac{1}{2})(x_2-\frac{1}{2}) & (x_1-\frac{1}{2})(x_3-\frac{1}{2})\\ (x_1-\frac{1}{2})(x_2-\frac{1}{2}) & 1+(x_2-\frac{1}{2})^2 & (x_2-\frac{1}{2})(x_3-\frac{1}{2})\\ (x_1-\frac{1}{2})(x_3-\frac{1}{2}) & (x_2-\frac{1}{2})(x_3-\frac{1}{2}) & 1+(x_3-\frac{1}{2})^2 \end{array}\right),$$

$\varphi = e^{(x_1-\frac{1}{2})(x_2-\frac{1}{2})(x_3-\frac{1}{2})}$ and $\Omega = (-1,1)^3\backslash[0,1)^3$. Hence, eigenfunctions with singularities are expected due to the nonconvex property.

Since the exact eigenvalues are not known, we choose adequately accurate approximations on finer finite element space as the exact eigenpairs for numerical tests. In this example, we give the numerical results for the first five eigenpair approximations of Algorithm 3.1 with the parameter $\theta = 0.4$. Fig. 4 shows the triangulation after 15 times adaptive iterations and the corresponding section along XY plane. Fig. 5 gives the numerical results of eigenpair approximations which show the optimal convergence rate of Algorithm 3.1.

Similarly, we also present the computational time of Algorithm 3.1 and direct AFEM in Fig. 6. The eigenvalue problems involved in the direct AFEM are also solved by the popular package ARPACK. Fig. 6 shows that Algorithm 3.1 has a better efficiency than the direct AFEM. Meanwhile, Algorithm 3.1 has the linear computational complexity.

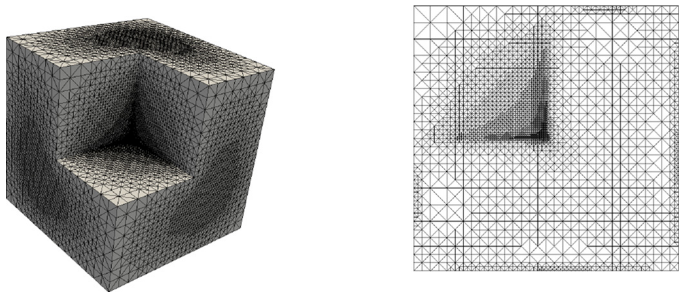

Fig. 4. The triangulation after adaptive iterations and the section along the X–Y plane for Example 2.

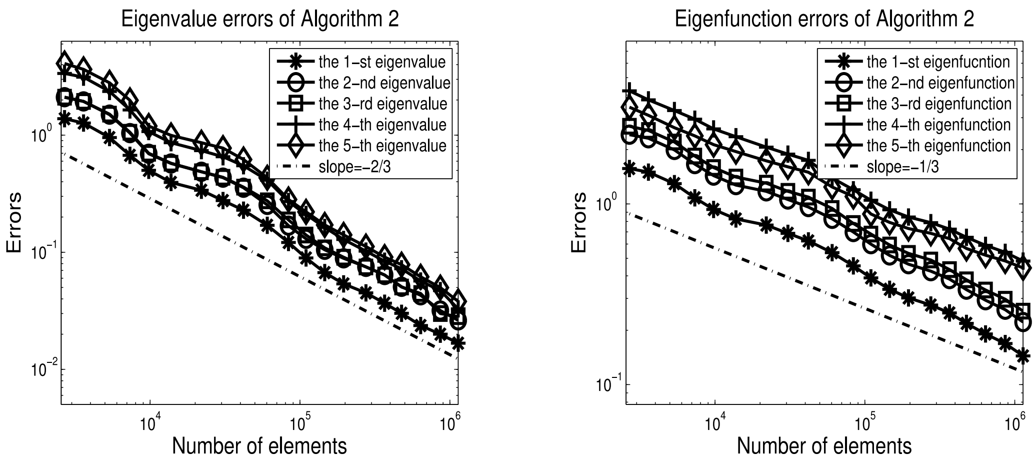

Fig. 5. The errors of the eigenpair approximations by Algorithm 3.1 for Example 2.

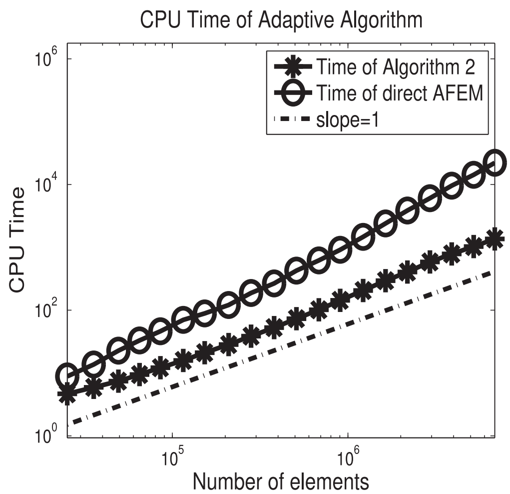

Fig. 6. The computational time (in second) by Algorithm 3.1 and direct AFEM for Example 2.

CRediT authorship contribution statement

Fei Xu: Methodology, Conceptualization, Writing – original draft. **Manting Xie:** Software. **Qiumei Huang:** Supervision. **Meiling Yue:** Funding acquisition. **Hongkun Ma:** Writing – review & editing.

Appendix. Proof of Lemma 3.9

Proof. For $\bar{E}_{h_k}u_\ell$, there exist q constants $\beta^\ell_{h_k,j}$, $j=i,\ldots,i+q-1$ such that

$$\bar{E}_{h_k}u_\ell = \sum_{j=i}^{i+q-1}\beta^\ell_{h_k,j}u_{h_k,j}, \quad \forall \ell = i,\ldots,i+q-1. \tag{92}$$

Thus, we have

$$\begin{aligned}
&\eta^2_{h_k}(\bar{E}_{h_k}u_\ell, K) = \eta^2_{h_k}(\sum_j \beta^\ell_{h_k,j}u_{h_k,j}, K)\\
&= h_K^2\|\sum_j \beta^\ell_{h_k,j}\big(\lambda_{h_k,j}u_{h_k,j} - \phi u_{h_k,j} + \nabla\cdot(\mathcal{A}\nabla u_{h_k,j})\big)\|^2_{0,K}\\
&\quad + \sum_{e\in\mathcal{E}_{h_k}, e\subset\partial K} h_e\big\|J_e(\sum_j \beta^\ell_{h_k,j}u_{h_k,j})\big\|^2_{0,e}\\
&\le \sum_j(\beta^\ell_{h_k,j})^2\sum_j\big(h_K^2\|\mathcal{R}_K(u_{h_k,j})\|^2_{0,K} + \sum_{e\in\mathcal{E}_{h_k}, e\subset\partial K} h_e\big\|J_e(u_{h_k,j})\big\|^2_{0,e}\big)\\
&= \sum_j(\beta^\ell_{h_k,j})^2\sum_j \eta^2_{h_k}(u_{h_k,j}, K) = \sum_j(\beta^\ell_{h_k,j})^2\eta^2_{h_k}(U_{h_k}, K).
\end{aligned}$$

Note that Lemma 3.5 indicates $\|\bar{E}_{h_k}u_\ell\|_{0,\Omega}\le 1$, namely $\sum_j(\beta^\ell_{h_k,j})^2\le 1$. Then we arrive at

$$\eta^2_{h_k}(\bar{E}_{h_k}U, K) = \sum_\ell \eta^2_{h_k}(\bar{E}_{h_k}u_\ell, K) \le q\eta^2_{h_k}(U_{h_k}, K). \tag{93}$$

Besides, since $\bar{E}_{h_k}: M(\lambda_i)\to M_{H,h_k}(\lambda_i)$ is one-to-one and onto when H is sufficiently small (see p.283 of [43]), we get that $\{\bar{E}_{h_k}u_\ell\}_{\ell=i}^{i+q-1}$ is a basis of $M_{H,h_k}(\lambda_i)$, namely $M_{H,h_k}(\lambda_i) = \mathrm{span}\{u_{h_k,i},\ldots,u_{h_k,i+q-1}\} = \mathrm{span}\{\bar{E}_{h_k}u_i,\ldots,\bar{E}_{h_k}u_{i+q-1}\}$. So, there exist q constants $\hat{\beta}^\ell_{h_k,j}$ $(j=i,\ldots,i+q-1)$ such that

$$u_{h_k,\ell} = \sum_{j=i}^{i+q-1}\hat{\beta}^\ell_{h_k,j}\bar{E}_{h_k}u_j, \quad \forall \ell = i,\ldots,i+q-1. \tag{94}$$

Similarly, from the definition of $\eta^2_{h_k}(u_{h_k,\ell}, K)$, (92) and (94), we get

$$\begin{aligned}
&\eta^2_{h_k}(u_{h_k,\ell}, K)\\
&= h_K^2\|\lambda_{h_k,\ell}u_{h_k,\ell} - \phi u_{h_k,\ell} + \nabla\cdot(\mathcal{A}\nabla u_{h_k,\ell})\|^2_{0,K} + \sum_{e\in\mathcal{E}_{h_k}, e\subset\partial K} h_e\big\|J_e(u_{h_k,\ell})\big\|^2_{0,e}\\
&= h_K^2\|\sum_j \lambda_{h_k,\ell}\hat{\beta}^\ell_{h_k,j}\bar{E}_{h_k}u_j - \phi\sum_j\hat{\beta}^\ell_{h_k,j}\bar{E}_{h_k}u_j + \nabla\cdot\big(\mathcal{A}\nabla(\sum_j\hat{\beta}^\ell_{h_k,j}\bar{E}_{h_k}u_j)\big)\|^2_{0,K}\\
&\quad + \sum_{e\in\mathcal{E}_{h_k}, e\subset\partial K} h_e\big\|J_e(\sum_j\hat{\beta}^\ell_{h_k,j}\bar{E}_{h_k}u_j)\big\|^2_{0,e}\\
&\le \sum_j(\hat{\beta}^\ell_{h_k,j})^2\sum_j\Big(h_K^2\big\|\lambda_{h_k,\ell}\bar{E}_{h_k}u_j - \phi\bar{E}_{h_k}u_j + \nabla\cdot(\mathcal{A}\nabla\bar{E}_{h_k}u_j)\big\|^2_{0,K}\\
&\quad + \sum_{e\in\mathcal{E}_{h_k}, e\subset\partial K} h_e\big\|J_e(\bar{E}_{h_k}u_j)\big\|^2_{0,e}\Big)\\
&= \sum_j(\hat{\beta}^\ell_{h_k,j})^2\sum_j\Big(h_K^2\big\|\lambda_{h_k,\ell}\sum_s\beta^j_{h_k,s}u_{h_k,s} - \phi\bar{E}_{h_k}u_j + \nabla\cdot(\mathcal{A}\nabla\bar{E}_{h_k}u_j)\big\|^2_{0,K}\\
&\quad + \sum_{e\in\mathcal{E}_{h_k}, e\subset\partial K} h_e\big\|J_e(\bar{E}_{h_k}u_j)\big\|^2_{0,e}\Big)\\
&\le \sum_j(\hat{\beta}^\ell_{h_k,j})^2\sum_j\Big(2h_K^2\big\|\sum_s\lambda_{h_k,s}\beta^j_{h_k,s}u_{h_k,s} - \phi\bar{E}_{h_k}u_j + \nabla\cdot(\mathcal{A}\nabla\bar{E}_{h_k}u_j)\big\|^2_{0,K}
\end{aligned}$$

$$+\sum_{e\in\mathcal{E}_{h_k},e\subset\partial K} h_e\left\|J_e(\bar{E}_{h_k}u_j)\right\|^2_{0,e}+2h_K^2\left\|\sum_s\beta^j_{h_k,s}(\lambda_{h_k,\ell}-\lambda_{h_k,s})u_{h_k,s}\right\|^2_{0,K}\Bigg)$$
$$\leq 2\sum_j(\hat{\beta}^\ell_{h_k,j})^2\Big(\sum_j\eta^2_{h_k}(\bar{E}_{h_k}u_j,K)+q\sum_s h_K^2|\lambda_{h_k,\ell}-\lambda_{h_k,s}|^2\left\|u_{h_k,s}\right\|^2_{0,K}\Big). \quad (95)$$

Note that

$$1=b(u_{h_k,\ell},u_{h_k,\ell})=b(\sum_j\hat{\beta}^\ell_{h_k,j}\bar{E}_{h_k}u_j,\sum_j\hat{\beta}^\ell_{h_k,j}\bar{E}_{h_k}u_j)$$
$$=\sum_j(\hat{\beta}^\ell_{h_k,j})^2 b(\bar{E}_{h_k}u_j,\bar{E}_{h_k}u_j)+\sum_{s\neq j}\hat{\beta}^\ell_{h_k,s}\hat{\beta}^\ell_{h_k,j}b(\bar{E}_{h_k}u_s,\bar{E}_{h_k}u_j). \quad (96)$$

Further, from Lemma 3.6, we can derive the following estimate for the right-hand terms of (96)

$$(1-C\eta_a^2(H))\sum_j(\hat{\beta}^\ell_{h_k,j})^2\leq\sum_j(\hat{\beta}^\ell_{h_k,j})^2 b(\bar{E}_{h_k}u_j,\bar{E}_{h_k}u_j)\leq(1+C\eta_a^2(H))\sum_j(\hat{\beta}^\ell_{h_k,j})^2 \quad (97)$$

and

$$\Big|\sum_{s\neq j}\hat{\beta}^\ell_{h_k,s}\hat{\beta}^\ell_{h_k,j}b(\bar{E}_{h_k}u_s,\bar{E}_{h_k}u_j)\Big|\lesssim\sum_j(\hat{\beta}^\ell_{h_k,j})^2\eta_a^2(H). \quad (98)$$

By (96), (97) and (98), we can derive

$$\frac{1}{1+C\eta_a^2(H)}\leq\sum_j(\hat{\beta}^\ell_{h_k,j})^2\leq\frac{1}{1-C\eta_a^2(H)}. \quad (99)$$

Combining (95) and (99) leads to

$$\begin{aligned}&\eta^2_{h_k}(U_{h_k},K)\\ \leq&\frac{2}{1-C\eta_a^2(H)}\sum_\ell\sum_j\eta^2_{h_k}(\bar{E}_{h_k}u_j,K)\\ &+\frac{2q}{1-C\eta_a^2(H)}\sum_\ell\sum_s h_K^2|\lambda_{h_k,\ell}-\lambda_{h_k,s}|^2\|u_{h_k,s}\|^2_{0,K}\\ =&\frac{2q}{1-C\eta_a^2(H)}\Big(\eta^2_{h_k}(\bar{E}_{h_k}U,K)+\sum_\ell\sum_s h_K^2|\lambda_{h_k,\ell}-\lambda_{h_k,s}|^2\|u_{h_k,s}\|^2_{0,K}\Big),\end{aligned} \quad (100)$$

which is just the desired estimate (61).

Note that (58) implies

$$|\lambda_i-\lambda_{h_k,\ell}|\lesssim\eta^2_{h_k}(u_{h_k,\ell},\mathcal{T}_{h_k}),\quad\forall\ell=i,\ldots,i+q-1.$$

By the fact that $\|u_{h_k,\ell}\|_{0,\Omega}=1$ and the abovementioned inequality, we have

$$\begin{aligned}&\sum_\ell\sum_s|\lambda_{h_k,\ell}-\lambda_{h_k,s}|^2\sum_{K\in\mathcal{T}_{h_k}}h_K^2\|u_{h_k,s}\|^2_{0,K}\\ \lesssim&\ \eta_a^4(H)\sum_\ell\sum_s|\lambda_{h_k,\ell}-\lambda_{h_k,s}|\\ \lesssim&\ q\eta_a^4(H)\sum_\ell|\lambda_i-\lambda_{h_k,\ell}|\ \lesssim\ \eta_a^4(H)\eta^2_{h_k}(U_{h_k},\mathcal{T}_{h_k}).\end{aligned}$$

Therefore, we arrive at

$$\eta^2_{h_k}(U_{h_k},\mathcal{T}_{h_k})\leq\frac{2q}{1-C\eta_a^2(H)}\Big(\eta^2_{h_k}(\bar{E}_{h_k}U,\mathcal{T}_{h_k})+C\eta_a^4(H)\eta^2_{h_k}(U_{h_k},\mathcal{T}_{h_k})\Big),$$

when H is small enough, that is

$$\eta^2_{h_k}(U_{h_k},\mathcal{T}_{h_k})\leq\frac{2q}{1-C\eta_a^2(H)}\eta^2_{h_k}(\bar{E}_{h_k}U,\mathcal{T}_{h_k}). \quad (101)$$

Then we derive the first equality of (62) from (93) and (101), and the second equality can be proved similarly. □

References

[1] X. Dai, X. Gong, Z. Yang, D. Zhang, A. Zhou, Finite volume discretizations for eigenvalue problems with applications to electronic structure calculations, Multiscale Model. Simul. 9 (2011) 208–240.
[2] R.M. Martin, Electronic Structure: Basic Theory and Practical Method, Cambridge University Press, Cambridge, 2004.
[3] L. Shen, A. Zhou, A defect correction scheme for finite element eigenvalues with applications to quantum chemistry, SIAM J. Sci. Comput. 28 (2006) 321–338.
[4] I. Babuška, W.C. Rheinboldt, Error estimates for adaptive finite element computations, SIAM J. Numer. Anal. 15 (1978) 736–754.
[5] W. Döfler, A convergent adaptive algorithm for Poisson's equation, SIAM J. Numer. Anal. 33 (1996) 1106–1124.
[6] W. Döfler, O. Wilderotter, An adaptive finite element method for a linear elliptic equation with variable coecients, ZAMM Z. Angew. Math. Mech. 80 (2000) 481–491.
[7] P. Morin, R.H. Nochetto, K. Siebert, Convergence of adaptive finite element methods, SIAM Rev. 44 (2002) 631–658.
[8] P. Binev, W. Dahmen, R. DeVore, Adaptive finite element methods with convergence rates, Numer. Math. 97 (2004) 219–268.
[9] K. Mekchay, R.H. Nochetto, Convergence of adaptive finite element methods for general second order linear ellipic PDEs, SIAM J. Numer. Anal. 43 (2005) 1803–1827.
[10] R. Stevension, Optimality of a standard adaptive finite element method, Found. Comput. Math. 7 (2) (2007) 245–269.
[11] J.M. Cascon, C. Kreuzer, R.H. Nochetto, K.G. Siebert, Quasi-optimal convergence rate for an adaptive finite element method, SIAM J. Numer. Anal. 46 (5) (2008) 2524–2550.
[12] C. Carstensen, D. Gallistl, M. Schedensack, Adaptive nonconforming Crouzeix–Raviart FEM for eigenvalue problems, Math. Comp. 84 (293) (2014) 1061–1087.
[13] C. Carstensen, J. Gedicke, An oscillation-free adaptive FEM for symmetric eigenvalue problems, Numer. Math. 118 (3) (2011) 401–427.
[14] X. Dai, L. He, A. Zhou, Convergence and quasi-optimal complexity of adaptive finite element computations for multiple eigenvalues, IMA J. Numer. Anal. 35 (2015) 1934–1977.
[15] S.-E. Ekström, I. Furci, C. Garoni, C. Manni, S. Serra-Capizzano, H. Speleers, Numer. Linear Algebra Appl. (2018) e2198, http://dx.doi.org/10.1002/nla.2198.
[16] S.-E. Ekström, C. Garoni, T.J.R. Hughes, A. Reali, S. Serra-Capizzano, H. Speleers, Symbol-based analysis of finite element and isogeometric B-spline discretizations of eigenvalue problems: Exposition and review, Arch. Comput. Methods Eng. 26 (5) (2019) 1639–1690, http://dx.doi.org/10.1007/s11831-018-9295-y.
[17] C. Garoni, S. Serra-Capizzano, The Theory of Multilevel Generalized Locally Toeplitz Sequences: Theory and Applications, Vol. II, in: Springer-Springer Monographs in Mathematics, 2018.
[18] D. Bianchi, Analysis of the spectral symbol associated to discretization schemes of linear selfadjoint differential operators, 2020, arXiv:2004.10058.
[19] D. Bianchi, S. Serra-Capizzano, Spectral Analysis of Finite Dimensional Approximations of 1-D Waves in Non-Uniform Grids, Vol. 55, Calcolo, 2018, p. 47.
[20] D. Arnold, A. Mukherjee, L. Pouly, Locally adapted tetrahedral meshes using bisection, SIAM J. Sci. Comput. 22 (2000) 431–448.
[21] R. Becker, R. Rannacher, An optimal control approach to a posteriori error estimation in finite element methods, Acta Numer. 10 (2001) 1–102.
[22] S. Brenner, L. Scott, The Mathematical Theory of Finite Element Methods, Springer-Verlag, New York, 1994.
[23] R. Durán, C. Padra, R. Rodrǵuez, A posteriori error estimates for the finite element approximation of eigenvalue problems, Math. Models Methods Appl. Sci. 13 (2003) 1219–1229.
[24] E. Garau, P. Morin, C. Zuppa, Convergence of adaptive finite element methods for eigenvalue problems, Math. Models Methods Appl. Sci. 19 (05) (2009) 721–747.
[25] V. Heuveline, R. Rannacher, A posteriori error control for finite element approximations of ellipic eigenvalue problems, Adv. Comput. Math. 15 (2001) 107–138.
[26] M. Larson, A posteriori and a priori error analysis for finite element approximations of self-adjoint elliptic eigenvalue problems, SIAM J. Numer. Anal. 38 (2001) 608–625.
[27] H. Chen, H. Xie, F. Xu, A full multigrid method for eigenvalue problems, J. Comput. Phys. 322 (2017) 747–759.
[28] Q. Hong, H. Xie, F. Xu, A multilevel correction type of adaptive finite element method for eigenvalue problems, SIAM J. Sci. Comput. 40 (6) (2018) A4208–A4235.
[29] Q. Lin, H. Xie, A multi-level correction scheme for eigenvalue problems, Math. Comp. 84 (2015) 71–88.
[30] Q. Lin, H. Xie, An observation on aubin-nitsche lemma and its applications, Math. Pract. Theory. 41 (17) (2011) 247–258.
[31] Q. Lin, H. Xie, F. Xu, Multilevel correction adaptive finite element method for semilinear elliptic equation, Appl. Math. 60 (5) (2015) 527–550.
[32] H. Xie, A multigrid method for eigenvalue problem, J. Comput. Phys. 274 (2014) 550–561.
[33] H. Xie, A type of multilevel method for the steklov eigenvalue problem, IMA J. Numer. Anal. 34 (2) (2014) 592–608.
[34] A. Brandt, Multi-level adaptive solutions to boundary-value problems, Math. Comp. 31 (1977) 333–390.
[35] L. Chen, R. Nochetto, J. Xu, Optimal multilevel methods for graded bisection grids, Numer. Math. 120 (2012) 1–34.
[36] S. McCormick, Fast adaptive composite grid (FAC) methods, in: K. Böhmer, H.J. Stetter (Eds.), Defect Correction Methods: Theory and Applications, in: Computing Supplementum, vol. 5, Sringer Verlag, Wien, 1984, pp. 115–121.
[37] U. Trottenberg, A. Schuller, Multigrid, Academic Press, 2015.
[38] H. Wu, Z. Chen, Uniform convergence of multigrid V-cycle on adaptively refined finite element meshes for second order elliptic problems, Sci. China Ser. A 49 (2006) 1405–1429.
[39] R.A. Adams, Sobolev Spaces, Academic Press, New York, 1975.
[40] R. Verfürth, A Review of a Posteriori Error Estimates and Adaptive Mesh-Refinement Techniques, Wiley-Teubner, New York, 1996.
[41] I. Babuška, J. Osborn, Eigenvalue problems, in: P.G. Lions, P.G. Ciarlet (Eds.), Handbook of Numerical Analysis, II, in: Finite Element Methods (Part 1), North-Holland, Amsterdam, 1991, pp. 641–787.
[42] F. Chatelin, Spectral Approximation of Linear Operators, Academic Press Inc, New York, 1983.
[43] I. Babuška, J. Osborn, Finite element-Galerkin approximation of the eigenvalues and eigenvectors of selfadjoint problems, Math. Comp. 52 (1989) 275–297.
[44] J. Xu, A. Zhou, A two-grid discretization scheme for eigenvalue problems, Math. Comp. 70 (233) (2001) 17–25.
[45] Q. Lin, H. Xie, J. Xu, Lower bounds of the discretization for piecewise polynomials, Math. Comp. 83 (2014) 1–13.
[46] F. Bornemann, P. Deuflhard, The cascadic multigrid method for elliptic problems, Numer. Math. 75 (1996) 135–152.
[47] W. Greiner, Quantum Mechanics: An Introduction, third ed., Springer, Berlin, 1994.