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An efficient adaptive multigrid method for the elasticity eigenvalue problem

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Abstract
This paper aims to introduce a novel adaptive multigrid method for the elasticity eigenvalue problem. Different from the developing adaptive algorithms for the elasticity eigenvalue problem, the proposed approach transforms the elasticity eigenvalue problem into a series of boundary value problems in the adaptive spaces and some small-scale elasticity eigenvalue problems in a low-dimensional space. As our algorithm avoids solving large-scale elasticity eigenvalue problems, which is time-consuming, and the boundary value problem can be solved efficiently by the adaptive multigrid method, our algorithm can evidently improve the overall solving efficiency for the elasticity eigenvalue problem. Meanwhile, we present a rigorous theoretical analysis of the convergence and optimal complexity. Finally, some numerical experiments are presented to validate the theoretical conclusions and verify the numerical efficiency of our approach.

Keywords Elasticity eigenvalue problem · Finite element method · Adaptive multigrid method · Convergence and optimal complexity

Mathematics Subject Classification 65N30 · 65N25 · 65L15 · 65B99

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1 Introduction

The large-scale elasticity eigenvalue problem is a fundamental problem in the study of vibrations in elastic structures. However, studies of efficient algorithms for solving this problem are scarce compared to those for solving other types of eigenvalue problems. The corresponding conclusions can be found in [16, 22, 25, 33, 39, 47, 50] and the references therein. Therefore, we investigate an efficient numerical algorithm for the following elasticity eigenvalue problem:

$$\begin{cases} -\mathrm{div}\sigma(\mathbf{u}) = \lambda\mathbf{u}, & \text{in } \Omega, \\ \mathbf{u} = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here, $\mathbf{u} = (u_1, \ldots, u_d)^T$ is the desired vector function of displacement, $\sigma(\mathbf{u})$ is the symmetric Cauchy stress tensor satisfying

$$\sigma(\mathbf{u}) = 2\underline{\mu}\varepsilon(\mathbf{u}) + \underline{\lambda}(\nabla \cdot \mathbf{u})\mathbf{I},$$

where $\underline{\mu}$ and $\underline{\lambda}$ denote Lamé constants, $\varepsilon(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ denotes the linear strain tensor, and $\mathbf{I}$ denotes the identity matrix.

The multigrid method is a well-known optimal algorithm that can derive the optimal error estimates with linear computational complexity. For elliptic boundary value problems, the theoretical results are well developed. In [54], Xu presented a uniform framework for analyzing the multigrid method, domain decomposition method, and other iteration methods. Thus far, studies on the multigrid method have been mainly focused on boundary value problems. For eigenvalue problems, there exists no corresponding multigrid method that can obtain optimal solutions. In [55], a two-grid method was proposed for solving eigenvalue problems by combining two meshes of different scales. The optimal error estimates could be obtained by adjusting the mesh sizes appropriately. In [29, 56], a multigrid method was designed on the basis of the shift-inverse technique. The optimal approximations could be obtained through solving a nearly singular boundary value problem in each multigrid space. A recently designed multilevel correction method [12, 27, 30, 34, 35, 52] extended the two-grid method to multigrid method, which needs to solve only some boundary value problems in the multigrid spaces and some small-scale eigenvalue problems in a low-dimensional space.

For the elasticity eigenvalue problem, we often derive a singular eigenfunction when the computing domain is non-convex or the equation has a jump coefficient. In such cases, we need to adopt the adaptive finite element method (AFEM) which has been widely used to solve singular partial differential equations. The AFEM was first proposed by Babuška in [2, 3]. Since then, a mature theoretical system has been developed. In [46], Nochetto, Siebert, and Veeser reviewed existing findings on adaptive algorithms. Cascon [9] proposed the most widely used version of the adaptive algorithm. For further details on the AFEM for elliptic boundary value problems, readers may refer to [17, 24, 26, 43–45, 48] and the references therein. The AFEM is also an efficient technique for solving eigenvalue problems. Dai et al. [14] presented the convergence and optimal complexity analysis by establishing connections between the boundary value problems and the eigenvalue problems. More details about AFEM for eigenvalue problems can be found in [8, 18, 20, 21, 28, 32, 38, 42] and the references therein.

It is worth mentioning that the multigrid method and the AFEM have close connections, as the adaptive mesh refinement technique has been confirmed to be fully compatible with the multilevel mesh structure. On the basis of such connections, Brandt [4, 7] designed a type of multilevel adaptive technique (MLAT). Subsequently, McCormick [40] investigated the fast adaptive composite (FAC) grid method. For further details on the adaptive multigrid method, readers may refer to [13, 23, 41, 49, 51] and the references therein. Many adaptive multigrid algorithms have been developed for solving linear elasticity problems [5, 15, 31, 37]. However, studies on the elasticity eigenvalue problem are relatively scarce.

In this study, a new type of multilevel correction adaptive multigrid method is designed for solving the elasticity eigenvalue problem on the basis of our recent advances in the multilevel correction method [27, 30, 35, 53, 57] and adaptive multigrid method. Different from the developing AFEMs for eigenvalue problems, we do not need to solve eigenvalue problem directly in adaptive spaces in the new adaptive method, which is the key to improving efficiency. The main strategy is to transform the elasticity eigenvalue problem into some elasticity boundary value problems in the adaptive finite element spaces and some small-scale elasticity eigenvalue problems in a low-dimensional space. Though the AFEM can generate optimal mesh, we still need to solve the elasticity boundary value problem directly in each adaptive space. Since there exist many repeated mesh elements between two adjacent mesh levels, the actual computational work is still very large. To improve efficiency, we adopt the adaptive multigrid method for the associated elasticity boundary value problems. Further, the dimension of the small-scale elasticity eigenvalue problem is fixed and small in solving process; thus, the computation time is negligible if the size of the mesh becomes increasingly smaller after some refinement steps. As our method avoids solving large-scale elasticity eigenvalue problems, it improves the overall solving efficiency of the elasticity eigenvalue problem. Finally, we also present the rigorous theoretical analysis of its convergence and optimal complexity.

The overall structure of this paper is as follows. In Sect. 2, we review the classical AFEM for the elasticity boundary value problem. Section 3 introduces our novel adaptive multigrid method for solving the elasticity eigenvalue problem. Section 4 presents the corresponding convergence analysis. Section 5 describes some numerical experiments conducted to verify the solving efficiency and validate the theoretical analysis of our approach. Finally, Sect. 6 concludes the paper.

2 Preliminaries of classical AFEM for the elasticity boundary value problem

In this section, we review the classical AFEM for solving the elasticity boundary value problem. The presented conclusions will be used in our analysis of the elasticity eigenvalue problem. The standard notation for Sobolev space will be used in this paper.

Let $\Omega \subset \mathbb{R}^d (d = 2, 3)$ be a bounded domain with Lipschitz continuous boundary. The symbol $x \lesssim y$ means that $x \leq Cy$.

In this study, we first consider the elasticity boundary value problem:

$$\begin{cases} -\mathrm{div}\sigma(\mathbf{u}) = \mathbf{f}, & \text{in } \Omega, \\ \mathbf{u} = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Here, $\mathbf{f} = (f_1, \ldots, f_d)^T$ is the vector function of mass forces. For linear plane strain, the Lamé constants satisfy

$$\underline{\lambda} = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \underline{\mu} = \frac{E}{2(1+\nu)}, \tag{2.2}$$

where ν and E denote the Poisson coefficient and the elasticity modulus, respectively.

For simplicity, we use the symbol $(\cdot, \cdot)$ to denote the L^2-inner product in $L^2(\Omega)$, $[L^2(\Omega)]^d$, or $[L^2(\Omega)]^{d\times d}$, as required, and we use $\|\cdot\|_{0,\Omega}$ to denote the norm induced by $(\cdot, \cdot)$ hereafter.

The weak form of the linear elasticity boundary value problem (2.1) is defined as follows: Find $\mathbf{u} \in V := (H_0^1(\Omega))^d$ such that

$$a(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in V, \tag{2.3}$$

where

$$a(\mathbf{u}, \mathbf{v}) = 2\underline{\mu}(\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v})) + \underline{\lambda}(\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v}). \tag{2.4}$$

In [19], it has been proven that $a(\cdot, \cdot)$ satisfies

$$a(\mathbf{v}, \mathbf{v}) \geq c_a \|\mathbf{v}\|_{1,\Omega}^2, \quad a(\mathbf{u}, \mathbf{v}) \leq C_a \|\mathbf{u}\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega}, \quad \forall \mathbf{u}, \mathbf{v} \in V, \tag{2.5}$$

implying that we can define the energy norm as follows:

$$\|\mathbf{u}\|_{a,\Omega} = \sqrt{a(\mathbf{u}, \mathbf{u})}, \quad \forall \mathbf{u} \in V. \tag{2.6}$$

Now, we introduce the classical AFEM for solving the elasticity boundary value problem (2.3). First, let $\mathscr{T}_k$ be a regular mesh which is a decomposition of the computing domain Ω [14, 20]. Then we use $S_k \subset H_0^1(\Omega)$ to denote the corresponding finite element space, and denote $V_k = (S_k)^d$.

Then, the classical finite element scheme is to solve the following discrete elasticity boundary value problem: Find $\mathbf{u}_k \in V_k$ such that

$$a(\mathbf{u}_k, \mathbf{v}_k) = (\mathbf{f}, \mathbf{v}_k), \quad \forall \mathbf{v}_k \in V_k. \tag{2.7}$$

For the bilinear form $a(\cdot, \cdot)$, let us define a projection operator $P_k : V \to V_k$ by

$$a(\mathbf{u} - P_k\mathbf{u}, \mathbf{v}_k) = 0, \quad \forall \mathbf{v}_k \in V_k. \tag{2.8}$$

Then, we can obtain $\mathbf{u}_k = P_k\mathbf{u}$ and

$$\|P_k\mathbf{u}\|_{a,\Omega} \leq \|\mathbf{u}\|_{a,\Omega}, \qquad \forall \mathbf{u} \in V. \tag{2.9}$$

To derive a new adaptive mesh after solving (2.7), we need to use the a posteriori error estimator. Following the procedure of the classical AFEM (see, e.g. [9, 43, 44]), we first define the element residual $\widehat{\mathcal{R}}_T(\mathbf{u}_k)$ and the jump $\widehat{\mathcal{J}_E}(\mathbf{u}_k)$ as follows

$$\begin{aligned}
\widehat{\mathcal{R}}_T(\mathbf{u}_k) &:= \mathbf{f} + \mathrm{div}\sigma(\mathbf{u}), \quad \text{for } T \in \mathcal{T}_k,\\
\widehat{\mathcal{J}_E}(\mathbf{u}_k) &:= -\sigma(\mathbf{u}_k^+)\cdot\nu^+ - \sigma(\mathbf{u}_k^-)\cdot\nu^- := [\sigma(\mathbf{u}_k)\cdot\nu_E], \quad \text{for } E \in \mathcal{E}_k,
\end{aligned}$$

where $\mathcal{E}_k$ denote the set of interior faces for $d = 3$ (edges for $d = 2$) of $\mathcal{T}_k$, $\nu_E = \nu^-$, E denotes the common side of elements T^+ and T^- with outward normals ν^+ and ν^-, respectively.

On the basis of the two definitions, the local error estimators on each mesh element $T \in \mathcal{T}_k$ are defined by:

$$\begin{aligned}
\widehat{\eta}_k^2(\mathbf{u}_k; T) &:= h_T^2\|\widehat{\mathcal{R}}_T(\mathbf{u}_k)\|_{0,T}^2 + \sum_{E\in\mathcal{E}_k, E\subset\partial T} h_E\|\widehat{\mathcal{J}_E}(\mathbf{u}_k)\|_{0,E}^2,\\
\widehat{\mathrm{osc}}_k^2(\mathbf{u}_k; T) &:= h_T^2\|(I - P_T)\widehat{\mathcal{R}}_T(\mathbf{u}_k)\|_{0,T}^2\\
&\quad + \sum_{E\in\mathcal{E}_k, E\subset\partial T} h_E\|(I - P_E)\widehat{\mathcal{J}_E}(\mathbf{u}_k)\|_{0,E}^2,
\end{aligned}$$

where P_T and P_E denote the L^2-projections to polynomials of some degree on T and E.

Next, for a submesh $\mathcal{T}' \subset \mathcal{T}_k$, the global error estimators are defined by:

$$\widehat{\eta}_k^2(\mathbf{u}_k; \mathcal{T}') := \sum_{T\in\mathcal{T}'} \widehat{\eta}_k^2(\mathbf{u}_k; T) \quad \text{and} \quad \widehat{\mathrm{osc}}_k^2(\mathbf{u}_k; \mathcal{T}') := \sum_{T\in\mathcal{T}'} \widehat{\mathrm{osc}}_k^2(\mathbf{u}_k; T).$$

The procedure of the adaptive finite element method can be described as follows:

$$\textbf{Solve} \rightarrow \textbf{Estimate} \rightarrow \textbf{Mark} \rightarrow \textbf{Refine}.$$

Specifically, to obtain a new mesh $\mathcal{T}_{k+1}$ from $\mathcal{T}_k$, we first need to deal with the elasticity boundary value problem (2.7) on $\mathcal{T}_k$ to derive an approximation. Then, we calculate the local error estimators for all mesh elements. Next, we mark some mesh elements on the basis of the values of local error estimators. We use the bisection of elements for the marked mesh elements in this paper. Finally, we refine the marked elements such that the new mesh is still shape-regular and conforming.

To simplify the description of the classical AFEM to solve elasticity boundary value problem, we introduce the following notations:

- $\mathbf{u}_k = \mathrm{EBVP\_SOLVE}(\mathbf{f}, V_k)$: Solve the elasticity boundary value problem (2.3) in V_k and return the finite element solution $\mathbf{u}_k \in V_k$.

- $\mathbf{u}_k = \text{MGEBVP\_SOLVE}(\mathbf{f}, \mathbf{u}_0, V_k)$: Solve the elasticity boundary value problem (2.3) by the multigrid method with initial value $\mathbf{u}_0 \in V_k$ in V_k and return the multigrid approximation $\mathbf{u}_k \in V_k$.
- $\{\widehat{\eta}_k(\mathbf{u}_k; T)\}_{T\in\mathcal{T}_k} = \text{EBVP\_ESTIMATE}(\mathbf{u}_k, \mathcal{T}_k)$: Compute $\widehat{\eta}_k(\mathbf{u}_k; T)$ on each mesh element $T \in \mathcal{T}_k$.
- $\mathcal{M}_k = \text{EBVP\_MARK}(\theta, \widehat{\eta}_k(\mathbf{u}_k; T), \mathcal{T}_k)$: Select a subset $\mathcal{M}_k$ using Dörfler's marking strategy defined in [17]; in other words, choose a minimal subset $\mathcal{M}_k$ from $\mathcal{T}_k$ satisfying

$$\widehat{\eta}_k(\mathbf{u}_k; \mathcal{M}_k) \geq \theta \widehat{\eta}_k(\mathbf{u}_k; \mathcal{T}_k).$$

- $(\mathcal{T}_{k+1}, V_{k+1}) = \text{REFINE}(\mathcal{T}_k, \mathcal{M}_k)$: Generate a new mesh $\mathcal{T}_{k+1}$ and finite element space V_{k+1} according to $\mathcal{M}_k$ where at least all element of $\mathcal{M}_k$ are refined.

The classical AFEM for the elasticity boundary value problem (2.3) is summarized in Algorithm 2.1.

Algorithm 2.1 Adaptive Finite Element Method.
Given an initial mesh $\mathcal{T}_1$ and a refinement parameter $\theta \in (0, 1)$. Set $k := 1$ and execute the following loops:

1. $\mathbf{u}_k = \text{EBVP\_SOLVE}(\mathbf{f}, V_k)$;
2. $\{\widehat{\eta}_k(\mathbf{u}_k; T)\}_{T\in\mathcal{T}_k} = \text{EBVP\_ESTIMATE}(\mathbf{u}_k, \mathcal{T}_k)$;
3. $\mathcal{M}_k = \text{EBVP\_MARK}(\theta, \widehat{\eta}_k(\mathbf{u}_k; T), \mathcal{T}_k)$;
4. $(\mathcal{T}_{k+1}, V_{k+1}) = \text{REFINE}(\mathcal{T}_k, \mathcal{M}_k)$;
5. Set $k := k + 1$ and proceed to step 1.

The theoretical conclusions for the elasticity boundary value problem follow the classical adaptive finite element theory. Finally, we recall some conclusions on the AFEM for the elasticity boundary value problem in the following lemmas. The detailed proofs of the following lemmas are simple extensions of the corresponding results in subsection 2.1 of [14] by some simple operations.

Lemma 2.1 *The following reliability and efficiency of the a posteriori error estimator hold:*

$$\|\mathbf{u} - \mathbf{u}_k\|_{a,\Omega}^2 \leq \hat{C}_{\mathrm{u}} \widehat{\eta}_k^2(\mathbf{u}_k; \mathcal{T}_k) \tag{2.10}$$

and

$$\hat{C}_\ell \widehat{\eta}_k^2(\mathbf{u}_k; \mathcal{T}_k) \leq \|\mathbf{u} - \mathbf{u}_k\|_{a,\Omega}^2 + \widehat{\mathrm{osc}}_k^2(\mathbf{u}_k; \mathcal{T}_k), \tag{2.11}$$

where the coefficients depend only on the shape-regularity of $\mathcal{T}_k$.

Lemma 2.2 *The following estimate about the projection and the oscillation holds*

$$\|\mathbf{u} - P_k\mathbf{u}\|_{a,\Omega}^2 + \widehat{\mathrm{osc}}_k^2(P_k\mathbf{u}; \mathcal{T}_k) \leq \hat{C} \inf_{\mathbf{v}_k\in V_k} \left(\|\mathbf{u} - \mathbf{v}_k\|_{a,\Omega}^2 + \widehat{\mathrm{osc}}_k^2(\mathbf{v}_k; \mathcal{T}_k)\right),$$

where $\hat{C}$ depends only on the shape regularity of the initial mesh $\mathcal{T}_1$.

Lemma 2.3 *The finite element approximate solutions derived from Algorithm 2.1 satisfy the following convergence*

$$\|\mathbf{u}-\mathbf{u}_{k+1}\|_{a,\Omega}^2+\hat{\gamma}\widehat{\eta}_{k+1}^2(\mathbf{u}_{k+1};\mathcal{T}_{k+1})\leq\hat{\xi}^2\big(\|\mathbf{u}-\mathbf{u}_k\|_{a,\Omega}^2+\hat{\gamma}\widehat{\eta}_k^2(\mathbf{u}_k;\mathcal{T}_k)\big), \tag{2.12}$$

where $\hat{\gamma}>0$ *and* $\hat{\xi}\in(0,1)$ *are two constants that depend only on the shape regularity of meshes and marking parameter* θ.

In this paper, we assume that the marking parameter θ satisfies $\theta\in(0,\theta_*)$ with θ_* being defined in Assumption 5.8 of [9].

Lemma 2.4 *Suppose that* $\mathcal{T}_{k,*}$ *is derived by refining* $\mathcal{T}_k$*, and the two projections* $P_{k,*}\mathbf{u}$ *and* $P_k\mathbf{u}$ *satisfy*

$$\|\mathbf{u}-P_{k,*}\mathbf{u}\|_{a,\Omega}^2+\widehat{\operatorname{osc}}_{k,*}^2(P_{k,*}\mathbf{u};\mathcal{T}_{k,*})\leq\tilde{\xi}_0^2\big(\|\mathbf{u}-P_k\mathbf{u}\|_{a,\Omega}^2+\widehat{\operatorname{osc}}_k^2(P_k\mathbf{u};\mathcal{T}_k)\big)$$

with $\tilde{\xi}_0^2\in(0,\frac{1}{2})$. *Denote* $\tilde{\theta}=\theta_*(1-2\tilde{\xi}_0^2)^{\frac{1}{2}}$, *Then, we can derive the following estimate*

$$\widehat{\eta}_k(P_k\mathbf{u};\mathcal{T}_k\backslash(\mathcal{T}_{k,*}\cap\mathcal{T}_k))\geq\tilde{\theta}\widehat{\eta}_k(P_k\mathbf{u};\mathcal{T}_k).$$

3 Multilevel correction adaptive multigrid method for the elasticity eigenvalue problem

This section is devoted to proposing a novel multilevel correction adaptive multigrid method for the elasticity eigenvalue problem (1.1).

The weak form of the elasticity eigenvalue problem (1.1) can be written as follows: Find $(\lambda,\mathbf{u})\in\mathbb{R}\times V$ such that

$$a(\mathbf{u},\mathbf{v})=\lambda(\mathbf{u},\mathbf{v}),\quad\forall\mathbf{v}\in V. \tag{3.1}$$

From [1, 10], the elasticity eigenvalue problem (3.1) has eigenvalues:

$$0<\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_i\leq\cdots,\ \lim_{i\to\infty}\lambda_i=\infty$$

and the corresponding eigenfunctions:

$$\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_i,\ldots,$$

where $(\mathbf{u}_i,\mathbf{u}_j)=\delta_{ij}$.

The standard finite element method for (3.1) is to solve the following discrete elasticity eigenvalue problem: Find $(\bar{\lambda}_k,\bar{\mathbf{u}}_k)\in\mathbb{R}\times V_k$ such that

$$a(\bar{\mathbf{u}}_k,\mathbf{v}_k)=\bar{\lambda}_k(\bar{\mathbf{u}}_k,\mathbf{v}_k),\quad\forall\mathbf{v}_k\in V_k. \tag{3.2}$$

From [1, 10], the discrete elasticity eigenvalue problem (3.2) has eigenvalues:

$$0 < \bar{\lambda}_{k,1} \le \bar{\lambda}_{k,2} \le \cdots \le \bar{\lambda}_{k,N_k}$$

and the corresponding eigenfunctions:

$$\bar{\mathbf{u}}_{k,1}, \bar{\mathbf{u}}_{k,2}, \ldots, \bar{\mathbf{u}}_{k,N_k},$$

where $(\bar{\mathbf{u}}_{k,i}, \bar{\mathbf{u}}_{k,j}) = \delta_{i,j}$, $1 \le i, j \le N_k$ (N_k is the dimension of the finite element space V_k).

Denote $M(\lambda)$ as the eigenspace corresponding to λ as follows:

$$M(\lambda) = \{\mathbf{w} \in V : \mathbf{w} \text{ is an eigenfunction of (3.1) corresponding to } \lambda,\ \|\mathbf{w}\|_{0,\Omega} = 1\}.$$

Let us define

$$\delta_k(\lambda) = \sup_{\mathbf{w} \in M(\lambda)} \inf_{\mathbf{v}_k \in V_k} \|\mathbf{w} - \mathbf{v}_k\|_{a,\Omega}$$

and

$$\eta_a(V_k) = \sup_{\mathbf{f} \in (L^2(\Omega))^d, \|\mathbf{f}\|_{0,\Omega} = 1} \inf_{\mathbf{v}_k \in V_k} \|T\mathbf{f} - \mathbf{v}_k\|_{a,\Omega},$$

where $T : (L^2(\Omega))^d \to V$ is defined as

$$a(T\mathbf{f}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{f} \in (L^2(\Omega))^d \text{ and } \forall \mathbf{v} \in V. \tag{3.3}$$

For the standard finite element approximate eigenvalue and approximate eigenfunction, we can obtain (see [1, 10]):

Lemma 3.1 *An exact eigenpair* $(\lambda, \mathbf{u})$ *of (3.1) exists such that each approximation* $(\bar{\lambda}_k, \bar{\mathbf{u}}_k)$ *has the following estimates*

$$\|\mathbf{u} - \bar{\mathbf{u}}_k\|_{a,\Omega} \lesssim \delta_k(\lambda), \tag{3.4}$$

$$\|\mathbf{u} - \bar{\mathbf{u}}_k\|_{0,\Omega} \lesssim \eta_a(V_k)\|\bar{\mathbf{u}} - \bar{\mathbf{u}}_k\|_{a,\Omega}, \tag{3.5}$$

$$|\lambda - \bar{\lambda}_k| \lesssim \|\mathbf{u} - \bar{\mathbf{u}}_k\|^2_{a,\Omega}, \tag{3.6}$$

where the hidden coefficient depends on the desired eigenvalue but is independent of mesh size.

3.1 Multilevel correction adaptive multigrid method

A novel adaptive multigrid method is designed in this subsection for the elasticity eigenvalue problem (3.1) on the basis of our recent advances in the multilevel correction method, multigrid iteration and adaptive refinement technique.

Following the definition of the element residual $\widehat{\mathscr{R}}_T(\mathbf{u}_k)$ and jump $\widehat{\mathscr{J}}_E(\mathbf{u}_k)$ for the elasticity boundary value problem, for the elasticity eigenvalue problem (3.2), we define:

$$\mathscr{R}_T(\lambda_k, \mathbf{u}_k) := \mathrm{div}\sigma(\mathbf{u}_k) + \lambda_k \mathbf{u}_k, \quad \text{for } T \in \mathscr{T}_k,$$
$$\mathscr{J}_E(\mathbf{u}_k) := -\sigma(\mathbf{u}_k^+) \cdot \nu^+ - \sigma(\mathbf{u}_k^-) \cdot \nu^- := [\sigma(\mathbf{u}_k) \cdot \nu_E], \quad \text{for } E \in \mathscr{E}_k.$$

On the basis of the two definitions above, we define the local error estimator for the elasticity eigenvalue problem (3.2) on each mesh element $T\ in \mathscr{T}_k$ by:

$$\eta_k^2(\lambda_k, \mathbf{u}_k; T) := h_T^2 \|\mathscr{R}_T(\lambda_k, \mathbf{u}_k)\|_{0,T}^2 + \sum_{E \in \mathscr{E}_k, E \subset \partial T} h_E \|\mathscr{J}_E(\mathbf{u}_k)\|_{0,E}^2,$$
$$\mathrm{osc}_k^2(\lambda_k, \mathbf{u}_k; T) := h_T^2 \|(I - P_T)\mathscr{R}_T(\lambda_k, \mathbf{u}_k)\|_{0,T}^2 + \sum_{E \in \mathscr{E}_k, E \subset \partial T} h_E \|(I - P_E)\mathscr{J}_E(\mathbf{u}_k)\|_{0,E}^2.$$

Then for a subset $\mathscr{T}' \subset \mathscr{T}_k$, the global error estimators are defined by

$$\eta_k^2(\lambda_k, \mathbf{u}_k; \mathscr{T}') := \sum_{T \in \mathscr{T}'} \eta_k^2(\lambda_k, \mathbf{u}_k; T), \quad \mathrm{osc}_k^2(\lambda_k, \mathbf{u}_k; \mathscr{T}') := \sum_{T \in \mathscr{T}'} \mathrm{osc}_k^2(\lambda_k, \mathbf{u}_k; T).$$

Similarly, we introduce the following notations for the elasticity eigenvalue problem:

- $(\lambda_k, \mathbf{u}_k) = \mathrm{EEG\_SOLVE}(V_k)$: Solve the elasticity eigenvalue problem in V_k and return the finite element solution $(\lambda_k, \mathbf{u}_k) \in \mathbb{R} \times V_k$ for the desired eigenpair.
- $\{\eta_k(\lambda_k, \mathbf{u}_k; T)\}_{T \in \mathscr{T}_k} = \mathrm{EEG\_ESTIMATE}(\lambda_k, \mathbf{u}_k, \mathscr{T}_k)$: Compute $\eta_k(\lambda_k, \mathbf{u}_k; T)$ on each mesh element $T \in \mathscr{T}_k$.
- $\mathscr{M}_k = \mathrm{EEG\_MARK}(\theta, \eta_k(\lambda_k, \mathbf{u}_k; T), \mathscr{T}_k)$: Choose a minimal subset $\mathscr{M}_k$ from $\mathscr{T}_k$ satisfying

$$\eta_k(\lambda_k, \mathbf{u}_k; \mathscr{M}_k) \geq \theta \eta_k(\lambda_k, \mathbf{u}_k; \mathscr{T}_k). \tag{3.7}$$

Subsequently, we design the novel adaptive multigrid method for the elasticity eigenvalue problem (3.2) in Algorithm 3.1, which is the main component of this paper.

Algorithm 3.1 Multilevel Correction Adaptive Multigrid Method
Construct a coarse mesh $\mathscr{T}_H$ and a coarse finite element space V_H on the computing domain Ω. Select an initial mesh $\mathscr{T}_1$ and an initial finite element space V_1 through refining $\mathscr{T}_H$ using the regular method such that $V_H \subseteq V_1$. Set $k := 1$ and execute the following loops:

1. $(\lambda_k, \mathbf{u}_k) = \begin{cases} \mathrm{EEG\_SOLVE}(V_1), & \text{when } k = 1; \\ \mathrm{EEG\_SOLVE}(V_H \oplus \mathrm{span}\{\check{\mathbf{u}}_k\}), & \text{when } k > 1; \end{cases}$
2. $\{\eta_k(\lambda_k, \mathbf{u}_k; T)\}_{T \in \mathscr{T}_k} = \mathrm{EEG\_ESTIMATE}(\lambda_k, \mathbf{u}_k, \mathscr{T}_k)$;

3. $\mathscr{M}_k = \text{EEG\_MARK}(\theta, \eta_k(\lambda_k, \mathbf{u}_k; T), \mathscr{T}_k)$;
4. $(\mathscr{T}_{k+1}, V_{k+1}) = \text{REFINE}(\mathscr{T}_k, \mathscr{M}_k)$;
5. (a) set $\mathbf{u}_{k+1}^{(0)} = \mathbf{u}_k$;
   (b) For $\ell = 0, \ldots, p-1$:

$$\mathbf{u}_{k+1}^{(\ell+1)} = \text{MGEBVP\_SOLVE}(\lambda_k \mathbf{u}_k, \mathbf{u}_{k+1}^{(\ell)}, V_{k+1}),$$

   End For.
   (c) Set $\breve{\mathbf{u}}_{k+1} = \mathbf{u}_{k+1}^{(p)}$;
6. Set $k := k+1$ and proceed to step 1.

Remark 3.1 In Algorithm 3.1, we need to solve p-times the elasticity boundary value problem and a small-scale elasticity eigenvalue problem during each iteration. The key point of Algorithm 3.1 is the addition of a correction step after solving the elasticity boundary value problems in adaptive spaces. In fact, if we remove these correction steps, the algorithm will become an inverse power method and the corresponding convergence rate will then depend on the eigenvalue gaps (see e.g. [1, 11]). Therefore, such a strategy is not sufficiently stable and may fail to converge to the desired solutions when the eigenvalue gap is small. In this case, the correction step can help to guarantee the convergence. Hence, we actually obtain two drivers for the convergence in Algorithm 3.1 by adding the correction step; thus a good convergence rate can be achieved even when the eigenvalue gap is small.

Although we need to additionally solve an elasticity eigenvalue problem in $V_H \oplus \text{span}\{\breve{\mathbf{u}}_k\}$, the dimension and sparsity of such elasticity eigenvalue problem will remain unchanged; thus the computational time is negligible compared to that of elasticity boundary value problem defined in adaptive space.

The main computational work of Algorithm 3.1 is spent on the elasticity boundary value problems. However, it should be noted that the optimal complexity of AFEM means only that the discretization scale is optimal but not that the computational work is optimal. This is because we still need to solve the elasticity boundary value problem in each level of the adaptive finite element space. To improve the efficiency of the elasticity boundary value problem in each adaptive space, the adaptive multigrid method is further involved in our algorithm which can solve the boundary value problem with linear computational work.

3.2 The efficiency and reliability of the a posteriori error estimator

In this section, we prove that the a posteriori error estimator defined for the elasticity eigenvalue problem has the efficiency and reliability property. The proof is mainly based on the connections between the elasticity boundary value problem and the elasticity eigenvalue problem. Such connections will also play an important role in the proof of the convergence property.

For the purpose of theoretical analysis, let us define an elasticity boundary value problem as follows: Find $\mathbf{w}^k \in V$ such that

$$a(\mathbf{w}^k, \mathbf{v}) = (\lambda_k \mathbf{u}_k, \mathbf{v}) \quad \forall \mathbf{v} \in V. \tag{3.8}$$

Denote

$$\widetilde{\mathbf{u}}_k = P_k \mathbf{w}^{k-1}. \tag{3.9}$$

Then, we can establish the following connections between the elasticity boundary value problem and the elasticity eigenvalue problem.

Theorem 3.1 *Assume that the adaptive multigrid iteration for the elasticity boundary value problem*

$$\mathbf{u}_k^{(\ell+1)} = \text{MGBVP\_SOLVE}(\lambda_{k-1}\mathbf{u}_{k-1}, \mathbf{u}_k^{(\ell)}, V_k) \tag{3.10}$$

satisfies the following reduction property:

$$\|\widetilde{\mathbf{u}}_k - \mathbf{u}_k^{(\ell+1)}\|_{a,\Omega} \leq \nu \|\widetilde{\mathbf{u}}_k - \mathbf{u}_k^{(\ell)}\|_{a,\Omega}. \tag{3.11}$$

Let u be the exact solution of (3.1), and u_k be approximate solution produced by Algorithm 3.1. Then, the following connections between the elasticity boundary value problem and the elasticity eigenvalue problem hold:

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_k\|_{a,\Omega} &= \|\mathbf{w}^k - P_k \mathbf{w}^k\|_{a,\Omega} \\ &\quad + \mathscr{O}(r(V_H, \nu))(\|\mathbf{u} - \mathbf{u}_{k-1}\|_{a,\Omega} + \|\mathbf{u} - \mathbf{u}_k\|_{a,\Omega}), \qquad (3.12) \\ \|\mathbf{u} - \mathbf{u}_k\|_{a,\Omega} &= \|\mathbf{w}^{k-1} - P_k \mathbf{w}^{k-1}\|_{a,\Omega} \\ &\quad + \mathscr{O}(r(V_H, \nu))(\|\mathbf{u} - \mathbf{u}_{k-1}\|_{a,\Omega} + \|\mathbf{u} - \mathbf{u}_k\|_{a,\Omega}), \qquad (3.13) \end{aligned}$$

where $r(V_H, \nu) = \eta_a(V_H) + \nu^p$ and the symbol $a = b + \mathscr{O}(r(V_H, \nu))c$ means $|a - b| \lesssim r(V_H, \nu)c$.

Proof $\mathbf{u} - \mathbf{u}_k$ can be decomposed into the following four parts

$$\mathbf{u} - \mathbf{u}_k = (\mathbf{u} - \mathbf{w}^k) + (\mathbf{w}^k - P_k \mathbf{w}^k) + (P_k \mathbf{w}^k - P_k \mathbf{w}^{k-1}) + (P_k \mathbf{w}^{k-1} - \mathbf{u}_k). \tag{3.14}$$

For the first part of (3.14), we have the following estimates by using (3.1), Lemma 3.1 and (3.8):

$$\begin{aligned} \|\mathbf{u} - \mathbf{w}^k\|_{a,\Omega}^2 &= a(\mathbf{u} - \mathbf{w}^k, \mathbf{u} - \mathbf{w}^k) \\ &= \left(\lambda \mathbf{u} - \lambda_k \mathbf{u}_k, \mathbf{u} - \mathbf{w}^k\right) \\ &\leq \|\lambda \mathbf{u} - \lambda_k \mathbf{u}_k\|_{0,\Omega} \|\mathbf{u} - \mathbf{w}^k\|_{0,\Omega} \\ &= \|(\lambda - \lambda_k)\mathbf{u}_k + \lambda(\mathbf{u} - \mathbf{u}_k)\|_0 \|\mathbf{u} - \mathbf{w}^k\|_0 \\ &\lesssim (|\lambda - \lambda_k| + \|\mathbf{u} - \mathbf{u}_k\|_{0,\Omega}) \|\mathbf{u} - \mathbf{w}^k\|_{0,\Omega} \\ &\lesssim (|\lambda - \lambda_k| + \|\mathbf{u} - \mathbf{u}_k\|_{0,\Omega}) \|\mathbf{u} - \mathbf{w}^k\|_{a,\Omega} \\ &\lesssim \eta_a(V_H) \|\mathbf{u} - \mathbf{u}_k\|_{a,\Omega} \|\mathbf{u} - \mathbf{w}^k\|_{a,\Omega}, \end{aligned} \tag{3.15}$$

which leads to

$$\|\mathbf{u}-\mathbf{w}^k\|_{a,\Omega} \lesssim \eta_a(V_H)\|\mathbf{u}-\mathbf{u}_k\|_{a,\Omega}. \tag{3.16}$$

The third part of (3.14) can be estimated as follows by using (2.9) and the proved result (3.16)

$$\begin{aligned}\|P_k(\mathbf{w}^k-\mathbf{w}^{k-1})\|_{a,\Omega} &\le \|\mathbf{u}-\mathbf{w}^k\|_{a,\Omega}+\|\mathbf{u}-\mathbf{w}^{k-1}\|_{a,\Omega}\\ &\lesssim \eta_a(V_H)\big(\|\mathbf{u}-\mathbf{u}_k\|_{a,\Omega}+\|\mathbf{u}-\mathbf{u}_{k-1}\|_{a,\Omega}\big).\end{aligned} \tag{3.17}$$

For the last part of (3.14), because $\breve{\mathbf{u}}_k-\mathbf{u}_k \in V_H \oplus \mathrm{span}\{\breve{\mathbf{u}}_k\}$, we can derive

$$\begin{aligned}&\|P_k\mathbf{w}^{k-1}-\mathbf{u}_k\|_{a,\Omega}^2\\ &\quad= a(P_k\mathbf{w}^{k-1}-\mathbf{u}_k,\, P_k\mathbf{w}^{k-1}-\mathbf{u}_k)\\ &\quad= a(P_k\mathbf{w}^{k-1}-\mathbf{u}_k,\, P_k\mathbf{w}^{k-1}-\breve{\mathbf{u}}_k)+a(P_k\mathbf{w}^{k-1}-\mathbf{u}_k, \breve{\mathbf{u}}_k-\mathbf{u}_k)\\ &\quad= a(P_k\mathbf{w}^{k-1}-\mathbf{u}_k,\, P_k\mathbf{w}^{k-1}-\breve{\mathbf{u}}_k)+(\lambda_{k-1}\mathbf{u}_{k-1}-\lambda_k\mathbf{u}_k, \breve{\mathbf{u}}_k-\mathbf{u}_k)\\ &\quad\lesssim \|P_k\mathbf{w}^{k-1}-\mathbf{u}_k\|_{a,\Omega}\|P_k\mathbf{w}^{k-1}-\breve{\mathbf{u}}_k\|_{a,\Omega}\\ &\qquad+\eta_a(V_H)(\|\mathbf{u}-\mathbf{u}_k\|_{a,\Omega}+\|\mathbf{u}-\mathbf{u}_{k-1}\|_{a,\Omega})\|\breve{\mathbf{u}}_k-\mathbf{u}_k\|_{a,\Omega}\\ &\quad\lesssim \|P_k\mathbf{w}^{k-1}-\mathbf{u}_k\|_{a,\Omega}\|P_k\mathbf{w}^{k-1}-\breve{\mathbf{u}}_k\|_{a,\Omega}+\eta_a(V_H)(\|\mathbf{u}-\mathbf{u}_k\|_{a,\Omega}+\\ &\qquad\|\mathbf{u}-\mathbf{u}_{k-1}\|_{a,\Omega})(\|P_k\mathbf{w}^{k-1}-\breve{\mathbf{u}}_k\|_{a,\Omega}+\|\mathbf{u}_k-P_k\mathbf{w}^{k-1}\|_{a,\Omega}).\end{aligned} \tag{3.18}$$

From (3.11), $P_k\mathbf{w}^{k-1}-\breve{\mathbf{u}}_k$ can be estimated as follows

$$\begin{aligned}&\|P_k\mathbf{w}^{k-1}-\breve{\mathbf{u}}_k\|_{a,\Omega}\\ &\quad\le \nu^p\|P_k\mathbf{w}^{k-1}-\mathbf{u}_{k-1}\|_{a,\Omega}\\ &\quad\le \nu^p(\|P_k\mathbf{w}^{k-1}-P_k\mathbf{u}\|_{a,\Omega}+\|P_k\mathbf{u}-\mathbf{u}\|_{a,\Omega}+\|\mathbf{u}-\mathbf{u}_{k-1}\|_{a,\Omega})\\ &\quad\le \nu^p(\|\mathbf{u}-\mathbf{w}^{k-1}\|_{a,\Omega}+\|\mathbf{u}-\mathbf{u}_k\|_{a,\Omega}+\|\mathbf{u}-\mathbf{u}_{k-1}\|_{a,\Omega})\\ &\quad\le \nu^p(1+C\eta_a(V_H))(\|\mathbf{u}-\mathbf{u}_k\|_{a,\Omega}+\|\mathbf{u}-\mathbf{u}_{k-1}\|_{a,\Omega}).\end{aligned} \tag{3.19}$$

Combining (3.18) and (3.19) leads to the following estimate

$$\begin{aligned}&\|P_k\mathbf{w}^{k-1}-\mathbf{u}_k\|_{a,\Omega}^2\\ &\quad\lesssim (\nu^p+\eta_a(V_H))\big(\|\mathbf{u}-\mathbf{u}_k\|_{a,\Omega}+\|\mathbf{u}-\mathbf{u}_{k-1}\|_{a,\Omega}\big)\|P_k\mathbf{w}^{k-1}-\mathbf{u}_k\|_{a,\Omega}\\ &\qquad+\nu^p\eta_a(V_H)\big(\|\mathbf{u}-\mathbf{u}_k\|_{a,\Omega}+\|\mathbf{u}-\mathbf{u}_{k-1}\|_{a,\Omega}\big)^2,\end{aligned} \tag{3.20}$$

which further yields

$$\|P_k\mathbf{w}^{k-1}-\mathbf{u}_k\|_{a,\Omega} \lesssim (\nu^p+\eta_a(V_H))\big(\|\mathbf{u}-\mathbf{u}_k\|_{a,\Omega}+\|\mathbf{u}-\mathbf{u}_{k-1}\|_{a,\Omega}\big). \tag{3.21}$$

Based on (3.16), (3.17) and (3.21), we can obtain the desired result (3.12).

The second connection (3.13) can be proved in the same way by decomposing $\mathbf{u}-\mathbf{u}_k$ into the following three parts

$$\mathbf{u}-\mathbf{u}_k=(\mathbf{u}-\mathbf{w}^{k-1})+(\mathbf{w}^{k-1}-P_k\mathbf{w}^{k-1})+(P_k\mathbf{w}^{k-1}-\mathbf{u}_k).$$

Then we complete the proof. □

Remark 3.2 To improve the efficiency for the elasticity boundary value problem in each adaptive space, the adaptive multigrid method is adopted in Algorithm 3.1, and we can get a convergence rate for the adaptive multigrid iteration which only depends on the multilevel mesh sequence [13, 51].

In Theorem 3.1, we have established connections between the elasticity boundary value problem and the elasticity eigenvalue problem, which implies that the difference is a high-order term. Therefore, we can prove the theoretical conclusions for the elasticity eigenvalue problem by following the procedure for the elasticity boundary value problem.

Similarly, the following two theorems can be proved in the same way as Theorem 3.1 by combining the definitions of the error estimators, inverse inequality and trace theorem.

Theorem 3.2 *Let u be the exact solution of* (3.1)*, and u_k be approximate solution produced by Algorithm* 3.1*. Then, we have the following connections for the a posteriori error estimators between the elasticity boundary value problem and the elasticity eigenvalue problem:*

$$\begin{aligned}\eta_k(\lambda_k,\mathbf{u}_k;\mathscr{T}_k)&=\widehat{\eta}_k(P_k\mathbf{w}^{k-1};\mathscr{T}_k)\\&\quad+\mathscr{O}\big(r(V_H,\nu)\big)(\|\mathbf{u}-\mathbf{u}_{k-1}\|_{a,\Omega}+\|\mathbf{u}-\mathbf{u}_k\|_{a,\Omega}),\quad(3.22)\\\eta_k(\lambda_k,\mathbf{u}_k;\mathscr{T}_k)&=\widehat{\eta}_k(P_k\mathbf{w}^{k};\mathscr{T}_k)\\&\quad+\mathscr{O}\big(r(V_H,\nu)\big)(\|\mathbf{u}-\mathbf{u}_{k-1}\|_{a,\Omega}+\|\mathbf{u}-\mathbf{u}_k\|_{a,\Omega}).\quad(3.23)\end{aligned}$$

Proof Using the triangle inequality and the definition of error estimators leads to

$$\begin{aligned}&|\eta_k(\lambda_k,\mathbf{u}_k;T)-\widehat{\eta}_k(P_k\mathbf{w}^{k-1};T)|\\&=\Bigg|\Big(h_T^2\|\lambda_k\mathbf{u}_k+\mathrm{div}\sigma(\mathbf{u}_k)\|_{0,T}^2+\sum_{E\in\mathscr{E}_k,E\subset\partial T}h_E\|[\mathrm{div}\sigma(\mathbf{u}_k)]\cdot\nu_E\|_{0,E}^2\Big)^{1/2}\\&\quad-\Big(h_T^2\|\lambda_{k-1}\mathbf{u}_{k-1}+\mathrm{div}\sigma(\widetilde{\mathbf{u}}_k)\|_{0,T}^2+\sum_{E\in\mathscr{E}_k,E\subset\partial T}h_E\|[\mathrm{div}\sigma(\widetilde{\mathbf{u}}_k)]\cdot\nu_E\|_{0,E}^2\Big)^{1/2}\Bigg|\\&\le\Bigg\{\Big(h_T^2\|\lambda_k\mathbf{u}_k-\lambda_{k-1}\mathbf{u}_{k-1}+\mathrm{div}\sigma(\mathbf{u}_k-\widetilde{\mathbf{u}}_k)\|_{0,T}\Big)^2\\&\quad+h_E\sum_{E\in\mathscr{E}_k,E\subset\partial T}\Big(\|[\mathrm{div}\sigma(\mathbf{u}_k)]\cdot\nu_E-[\mathrm{div}\sigma(\widetilde{\mathbf{u}}_k)]\cdot\nu_E\|_{0,E}\Big)^2\Bigg\}^{1/2}.\quad(3.24)\end{aligned}$$

From the inverse estimate, we have

$$\|\mathrm{div}\sigma(\mathbf{v_k})\|_{0,T} \lesssim h_T^{-1}\|\sigma(\mathbf{v_k})\|_{0,T}, \quad \forall T \in \mathscr{T}_h,\ \mathbf{v_k} \in V_k. \tag{3.25}$$

From the inverse estimate and the trace inequality

$$\|\mathbf{v}\|_{0,\partial T} \lesssim h_T^{-1/2}\|\mathbf{v}\|_{0,T} + h_T^{s-1/2}\|\mathbf{v}\|_{s,T} \quad \forall s > 1/2,\ \mathbf{v} \in H^s(T),\ T \in \mathscr{T}_k,$$

we have

$$h_E\|[\mathrm{div}\sigma(\mathbf{v_k})]\cdot\nu_E\|_{0,E}^2 \lesssim \|\sigma(\mathbf{v_k})\|_{0,T}^2, \quad \forall \mathbf{v_k} \in V_k. \tag{3.26}$$

Using (3.24)–(3.26), we can derive

$$|\eta_k(\lambda_k, \mathbf{u}_k; T) - \widehat{\eta}_k(P_k\mathbf{w}^{k-1}; T)| \lesssim h_T\|\lambda_k\mathbf{u}_k - \lambda_{k-1}\mathbf{u}_{k-1}\|_{0,T} + \|\sigma(\mathbf{u}_k - \widetilde{\mathbf{u}}_k)\|_{0,T}. \tag{3.27}$$

From Lemma 3.1, (3.21) and (3.27), there holds

$$\begin{aligned}
&|\eta_k(\lambda_k, \mathbf{u}_k; \mathscr{T}_k) - \widehat{\eta}_k(P_k\mathbf{w}^{k-1}; \mathscr{T}_k)| \\
&= \left|\left(\sum_{T\in\mathscr{T}_k}\eta_k^2(\lambda_k, \mathbf{u}_k; T)\right)^{1/2} - \left(\sum_{T\in\mathscr{T}_k}\widehat{\eta}_k^2(P_k\mathbf{w}^{k-1}; T)\right)^{1/2}\right| \\
&\lesssim \left(\sum_{T\in\mathscr{T}_k}\big(\eta_k(\lambda_k, \mathbf{u}_k; T) - \widehat{\eta}_k(P_k\mathbf{w}^{k-1}; T)\big)^2\right)^{1/2} \\
&\lesssim r(V_H, \nu)(\|\mathbf{u} - \mathbf{u}_k\|_{a,\Omega} + \|\mathbf{u} - \mathbf{u}_{k-1}\|_{a,\Omega}).
\end{aligned}$$

This is the desired result (3.22). The second result (3.23) can be derived similarly. □

Theorem 3.3 *Let u be the exact solution of (3.1), and u_k be approximate solution produced by Algorithm 3.1. Then, we have the following connections for oscillations between the elasticity boundary value problem and the elasticity eigenvalue problem:*

$$\begin{aligned}
osc_k(\lambda_k, \mathbf{u}_k; \mathscr{T}_k) &= \widehat{osc}_k(P_k\mathbf{w}^{k-1}; \mathscr{T}_k) \\
&\quad + \mathscr{O}\big(r(V_H, \nu)\big)(\|\mathbf{u} - \mathbf{u}_{k-1}\|_{a,\Omega} + \|\mathbf{u} - \mathbf{u}_k\|_{a,\Omega}), \qquad (3.28)\\
osc_k(\lambda_k, \mathbf{u}_k; \mathscr{T}_k) &= \widehat{osc}_k(P_k\mathbf{w}^{k}; \mathscr{T}_k) \\
&\quad + \mathscr{O}\big(r(V_H, \nu)\big)(\|\mathbf{u} - \mathbf{u}_{k-1}\|_{a,\Omega} + \|\mathbf{u} - \mathbf{u}_k\|_{a,\Omega}). \qquad (3.29)
\end{aligned}$$

Based on Theorems 3.1–3.3, we can prove the efficiency and reliability of the a posterior error estimator through Lemma 2.1.

Theorem 3.4 *Let u be the exact solution of (3.1), and u_k be approximate solution produced by Algorithm 3.1. Then, two constants C_{u} and C_ℓ that are independent of the mesh size exist such that, when $r(V_H, \nu)$ is sufficiently small, the following efficiency and reliability hold:*

$$\|\mathbf{u} - \mathbf{u}_k\|_{a,\Omega}^2 \leq C_{\mathrm{u}}\eta_k^2(\lambda_k, \mathbf{u}_k; \mathcal{T}_k) + \mathcal{O}(r^2(V_H, \nu))\|\mathbf{u} - \mathbf{u}_{k-1}\|_{a,\Omega}^2$$

and

$$C_\ell\eta_k^2(\lambda_k, \mathbf{u}_k; \mathcal{T}_k) \leq \|\mathbf{u} - \mathbf{u}_k\|_{a,\Omega}^2 + \mathrm{osc}_k^2(\lambda_k, \mathbf{u}_k; \mathcal{T}_k) + \mathcal{O}(r^2(V_H, \nu))\|\mathbf{u} - \mathbf{u}_{k-1}\|_{a,\Omega}^2.$$

Proof Since $\mathbf{w}^{k-1}$ is the exact solution of the elasticity boundary value problem, we can derive the following reliability and efficiency by using Lemma 2.1

$$\|\mathbf{w}^{k-1} - P_k\mathbf{w}^{k-1}\|_{a,\Omega} \leq \widehat{C}_u\widehat{\eta}_k(P_k\mathbf{w}^{k-1}, \mathcal{T}_k)$$

and

$$\widehat{C}_\ell\widehat{\eta}_k^2(P_k\mathbf{w}^{k-1}, \mathcal{T}_k) \leq \|\mathbf{w}^{k-1} - P_k\mathbf{w}^{k-1}\|_{a,\Omega}^2 + \widehat{osc}_k^2(P_k\mathbf{w}^{k-1}, \mathcal{T}_k).$$

Then we can get the desired results by combining the above estimates and Theorems 3.1–3.3. □

4 Convergence of multilevel correction adaptive multigrid algorithm for the elasticity eigenvalue problem

4.1 Convergence of multilevel correction adaptive multigrid algorithm

This section provides the convergence estimates of Algorithm 3.1 on the basis of existing results for the elasticity boundary value problem presented in Sect. 2 and the connections presented in Theorems 3.1–3.3.

Theorem 4.1 *When $r(V_H, \nu)$ is sufficiently small, there exist constants γ, $\alpha_0 > 0$ and $\alpha \in (0, 1)$ which depend only on the mesh refinement parameter θ and the shape regularity of the mesh, such that the approximate solution $(\lambda_k, \mathbf{u}_k)$ produced by Algorithm 3.1 satisfies*

$$\|\mathbf{u} - \mathbf{u}_k\|_{a,\Omega}^2 + \gamma\eta_k^2(\lambda_k, \mathbf{u}_k; \mathcal{T}_k) \leq \alpha^2\big(\|\mathbf{u} - \mathbf{u}_{k-1}\|_{a,\Omega}^2 + \gamma\eta_{k-1}^2(\lambda_{k-1}, \mathbf{u}_{k-1}; \mathcal{T}_{k-1})\big) + \alpha_0^2 r^2(V_H, \nu)\|\mathbf{u} - \mathbf{u}_{k-2}\|_{a,\Omega}^2. \quad (4.1)$$

Proof From (3.7) and Theorem 3.2, there holds

$$\widehat{\eta}_{k-1}(P_{k-1}\mathbf{w}^{k-1}; \mathcal{M}_{k-1}) \geq \theta\widehat{\eta}_{k-1}(P_{k-1}\mathbf{w}^{k-1}; \mathcal{T}_{k-1}) - C_\eta r(V_H, \nu)(\|\mathbf{u} - \mathbf{u}_{k-1}\|_{a,\Omega} + \|\mathbf{u} - \mathbf{u}_{k-2}\|_{a,\Omega}). \quad (4.2)$$

Then using (4.2) and the proof procedure of Lemma 2.3 in [9], there exist constants $\hat{\gamma} > 0$ and $\hat{\xi} \in (0, 1)$ such that

$$
\begin{aligned}
&\|\mathbf{w}^{k-1} - P_k \mathbf{w}^{k-1}\|_{a,\Omega}^2 + \hat{\gamma}\widehat{\eta}_k^2(P_k \mathbf{w}^{k-1}; \mathscr{T}_k) \\
&\quad \le \hat{\xi}^2\big(\|\mathbf{w}^{k-1} - P_{k-1}\mathbf{w}^{k-1}\|_{a,\Omega}^2 + \hat{\gamma}\widehat{\eta}_{k-1}^2(P_{k-1}\mathbf{w}^{k-1}; \mathscr{T}_{k-1})\big) \\
&\qquad + Cr^2(V_H, \nu)\big(\|\mathbf{u} - \mathbf{u}_{k-1}\|_{a,\Omega}^2 + \|\mathbf{u} - \mathbf{u}_{k-2}\|_{a,\Omega}^2\big).
\end{aligned} \tag{4.3}
$$

On the one hand, from (3.13), (3.22), (4.3) and Young inequality, we can derive

$$
\begin{aligned}
&\|\mathbf{u} - \mathbf{u}_k\|_{a,\Omega}^2 + \hat{\gamma}\eta_k^2(\lambda_k, \mathbf{u}_k; \mathscr{T}_k) \\
&\quad \le (1+\delta_1)\big(\|\mathbf{w}^{k-1} - P_k \mathbf{w}^{k-1}\|_{a,\Omega}^2 + \hat{\gamma}\widehat{\eta}_k^2(P_k \mathbf{w}^{k-1}; \mathscr{T}_k)\big) \\
&\qquad + C\delta_1^{-1} r^2(V_H, \nu)\big(\|\mathbf{u} - \mathbf{u}_k\|_{a,\Omega}^2 + \|\mathbf{u} - \mathbf{u}_{k-1}\|_{a,\Omega}^2\big) \\
&\quad \le (1+\delta_1)\hat{\xi}^2\big(\|\mathbf{w}^{k-1} - P_{k-1}\mathbf{w}^{k-1}\|_{a,\Omega}^2 + \hat{\gamma}\widehat{\eta}_{k-1}^2(P_{k-1}\mathbf{w}^{k-1}; \mathscr{T}_{k-1})\big) \\
&\qquad + C\delta_1^{-1} r^2(V_H, \nu)\big(\|\mathbf{u} - \mathbf{u}_k\|_{a,\Omega}^2 + \hat{\gamma}\eta_k^2(\lambda_k, \mathbf{u}_k; \mathscr{T}_k)\big) \\
&\qquad + C\delta_1^{-1} r^2(V_H, \nu)\big(\|\mathbf{u} - \mathbf{u}_{k-1}\|_{a,\Omega}^2 + \|\mathbf{u} - \mathbf{u}_{k-2}\|_{a,\Omega}^2\big),
\end{aligned}
$$

which yields the following estimates

$$
\begin{aligned}
&\|\mathbf{u} - \mathbf{u}_k\|_{a,\Omega}^2 + \hat{\gamma}\eta_k^2(\lambda_k, \mathbf{u}_k; \mathscr{T}_k) \\
&\quad \le \frac{(1+\delta_1)\hat{\xi}^2}{1 - C\delta_1^{-1} r^2(V_H, \nu)}\big(\|\mathbf{w}^{k-1} - P_{k-1}\mathbf{w}^{k-1}\|_{a,\Omega}^2 + \hat{\gamma}\widehat{\eta}_k^2(P_{k-1}\mathbf{w}^{k-1}; \mathscr{T}_{k-1})\big) \\
&\qquad + \frac{C\delta_1^{-1}}{1 - C\delta_1^{-1} r^2(V_H, \nu)} r^2(V_H, \nu)\big(\|\mathbf{u} - \mathbf{u}_{k-1}\|_{a,\Omega}^2 + \|\mathbf{u} - \mathbf{u}_{k-2}\|_{a,\Omega}^2\big).
\end{aligned} \tag{4.4}
$$

On the other hand, using the similar technique on the right side of (4.4), we can obtain

$$
\begin{aligned}
&\|\mathbf{w}^{k-1} - P_{k-1}\mathbf{w}^{k-1}\|_{a,\Omega}^2 + \hat{\gamma}\widehat{\eta}_{k-1}^2(P_{k-1}\mathbf{w}^{k-1}; \mathscr{T}_{k-1}) \\
&\quad \le \big(1+\delta_1\big)\big(\|\mathbf{u} - \mathbf{u}_{k-1}\|_{a,\Omega}^2 + \hat{\gamma}\eta_{k-1}^2(\lambda_{k-1}, \mathbf{u}_{k-1}; \mathscr{T}_{k-1})\big) \\
&\qquad + C\delta_1^{-1} r^2(V_H, \nu)\big(\|\mathbf{u} - \mathbf{u}_{k-1}\|_{a,\Omega}^2 + \|\mathbf{u} - \mathbf{u}_{k-2}\|_{a,\Omega}^2\big).
\end{aligned} \tag{4.5}
$$

Finally, combining (4.4) and (4.5) leads to

$$
\begin{aligned}
\|\mathbf{u} - \mathbf{u}_k\|_{a,\Omega}^2 + \gamma\eta_k^2(\lambda_k, \mathbf{u}_k; \mathscr{T}_k) &\le \alpha^2\big(\|\mathbf{u} - \mathbf{u}_{k-1}\|_{a,\Omega}^2 + \gamma\eta_{k-1}^2(\lambda_{k-1}, \mathbf{u}_{k-1}; \mathscr{T}_{k-1})\big) \\
&\quad + \alpha_0^2 r^2(V_H, \nu)\|\mathbf{u} - \mathbf{u}_{k-2}\|_{a,\Omega}^2.
\end{aligned}
$$

with

$$
\alpha^2 := \frac{(1+\delta_1)(1+\delta_1 + C\delta_1^{-1} r^2(V_H, \nu))\hat{\xi}^2 + C\delta_1^{-1} r^2(V_H, \nu)}{1 - C\delta_1^{-1} r^2(V_H, \nu)},
$$

$$\alpha_0^2 := \frac{(1+\delta_1)C\delta_1^{-1}\hat{\xi}^2 + C\delta_1^{-1}}{1 - C\delta_1^{-1}r^2(V_H,\nu)}, \quad \gamma := \hat{\gamma}.$$

The contraction property (4.1) can be proved through choosing δ_1 small enough such that $\alpha < 1$. Then we complete the proof. □

Based on Theorem 4.1, the final convergence result is provided in Theorem 4.2.

Theorem 4.2 *When $r(V_H,\nu)$ is sufficiently small, two constants $\beta > 0$ and $\widehat{\alpha} \in (0,1)$ exist, that depend only on the parameter θ and the shape regularity of the mesh, such that*

$$E_k^2 + \beta^2 r^2(V_H,\nu)E_{k-1}^2 \le \widehat{\alpha}^2(E_{k-1}^2 + \beta^2 r^2(V_H,\nu)E_{k-2}^2), \tag{4.6}$$

where $E_k^2 = \|\mathbf{u} - \mathbf{u}_k\|_{a,\Omega}^2 + \gamma\eta_k^2(\lambda_k,\mathbf{u}_k;\mathscr{T}_k)$.

Proof From Theorem 4.1, it is obvious that the following estimate holds

$$E_k^2 \le \alpha^2 E_{k-1}^2 + \alpha_0^2 r^2(V_H,\nu)\|\mathbf{u} - \mathbf{u}_{k-2}\|_{a,\Omega}^2.$$

Chosen $\widehat{\alpha}$ and β such that

$$\widehat{\alpha}^2 - \beta^2 r^2(V_H,\nu) = \alpha^2, \quad \widehat{\alpha}^2\beta^2 = \alpha_0^2,$$

which leads to

$$\widehat{\alpha}^2 = \frac{\alpha^2 + \sqrt{\alpha^4 + 4\alpha_0^2 r^2(V_H,\nu)}}{2} \quad \text{and} \quad \beta^2 = \frac{2\alpha_0^2}{\alpha^2 + \sqrt{\alpha^4 + 4\alpha_0^2 r^2(V_H,\nu)}}.$$

Thus, there holds $\widehat{\alpha} < 1$ when $r(V_H,\nu)$ is small enough. Then we complete the proof. □

4.2 Optimal complexity

In the last of this section, we also would like to briefly analyze the complexity of Algorithm 3.1 as [14]. Similar to the normal analysis of AFEM, to analyze the optimal complexity of Algorithm 3.1, we study a class of functions:

$$\mathscr{A}^s := \left\{\mathbf{v} \in V : |\mathbf{v}|_s < \infty\right\},$$

where

$$|\mathbf{v}|_s = \sup_{\varepsilon>0} \varepsilon \inf_{\{\mathscr{T}_1 \le \mathscr{T}_\varepsilon : \inf_{(\lambda_\varepsilon,\mathbf{u}_\varepsilon)} (\|\mathbf{v}-\mathbf{u}_\varepsilon\|_{a,\Omega}^2 + \mathrm{osc}_\varepsilon^2(\lambda_\varepsilon,\mathbf{u}_\varepsilon;\mathscr{T}_\varepsilon))^{1/2} \le \varepsilon\}} (\#\mathscr{T}_\varepsilon - \#\mathscr{T}_1)^s.$$

Herein, $\mathscr{T}_1 \leq \mathscr{T}_\varepsilon$ implies that $\mathscr{T}_\varepsilon$ is refined from $\mathscr{T}_1$, and $\#\mathscr{T}$ denotes the number of mesh elements of $\mathscr{T}$. Hence the functions belong to $\mathscr{A}^s$ can be approximated to a tolerance ε by piecewise polynomials on $\mathscr{T}_\varepsilon$ with $\#\mathscr{T}_\varepsilon - \#\mathscr{T}_1 \lesssim \varepsilon^{-1/s}|\mathbf{v}|_s^{1/s}$.

First, the initial mesh size is assumed to be sufficiently small such that

$$r(V_H,\nu)\|\mathbf{u}-\mathbf{u}_k\|_{a,\Omega}^2 \leq \|\mathbf{u}-\mathbf{u}_{k+1}\|_{a,\Omega}^2, \quad \text{for } k \geq 2. \tag{4.7}$$

Then, the following convergence can be deduced from Theorem 4.1:

$$\|\mathbf{u}-\mathbf{u}_k\|_{a,\Omega}^2 + \gamma\eta_k^2(\lambda_k,\mathbf{u}_k;\mathscr{T}_k) \leq \bar{\alpha}^2\big(\|\mathbf{u}-\mathbf{u}_{k-1}\|_{a,\Omega}^2 + \gamma\eta_{k-1}^2(\lambda_{k-1},\mathbf{u}_{k-1};\mathscr{T}_{k-1})\big),$$

with $\bar{\alpha}^2 = \alpha^2 + \alpha_0^2 r(V_H,\nu)$.

Remark 4.1 It is worth mentioning that the assumption (4.7) implies the initial mesh should be fine enough such that the consecutive approximate solution doesn't change significantly. If $\mathbf{u}$ is not a piecewise linear polynomial, a sharp lower and upper bound exist for $\|\mathbf{u}-\mathbf{u}_k\|_{a,\Omega}$ (see [36], etc). Then the assumption (4.7) is reasonable generally.

In (4.7) and the convergence analysis, we all give some constraints on $\mathscr{T}_H$ and the multigrid iteration time p to derive the theoretical conclusions. But we will observe from the numerical experiments that a coarse mesh $\mathscr{T}_H$ and two or three times multigrid iteration steps are enough to derive the optimal numerical results.

Lemma 4.1 [9] *Let $\mathscr{T}_s$ and $\mathscr{T}_t$ be two conforming refinements of $\mathscr{T}_1$. Then, the smallest common refinement of two meshes $\mathscr{T}_s$ and $\mathscr{T}_t$, in other words, $\mathscr{T} := \mathscr{T}_s \oplus \mathscr{T}_t$, is conforming and satisfies*

$$\#\mathscr{T} \leq \#\mathscr{T}_s + \#\mathscr{T}_t - \#\mathscr{T}_1. \tag{4.8}$$

Lemma 4.2 *Let $(\lambda_k,\mathbf{u}_k) \in \mathbb{R}\times V_k$ be the approximate eigenpair produced by Algorithm 3.1. Let $\mathscr{T}_k \leq \mathscr{T}_{k,*}$, and $(\lambda_{k,*},\mathbf{u}_{k,*}) = \mathrm{EEG\_SOLVE}(V_H \oplus \mathrm{span}\{P_{k,*}\mathbf{w}^k\})$. Suppose that we have:*

$$\|\mathbf{u}-\mathbf{u}_{k,*}\|_{a,\Omega}^2 + \mathrm{osc}_{k,*}^2(\lambda_{k,*},\mathbf{u}_{k,*};\mathscr{T}_{k,*}) \leq \beta_*^2\big(\|\mathbf{u}-\mathbf{u}_k\|_{a,\Omega}^2 + \mathrm{osc}_k^2(\lambda_k,\mathbf{u}_k;\mathscr{T}_k)\big).$$

Then the following relationship about the projections $P_k\mathbf{w}^k$ and $P_{k,*}\mathbf{w}^k$ holds:

$$\begin{aligned}&\|\mathbf{w}^k - P_{k,*}\mathbf{w}^k\|_{a,\Omega}^2 + \widehat{\mathrm{osc}}_{k,*}^2(P_{k,*}\mathbf{w}^k;\mathscr{T}_{k,*})\\ &\quad\leq \widehat{\beta}_*^2\big(\|\mathbf{w}^k - P_k\mathbf{w}^k\|_{a,\Omega}^2 + \widehat{\mathrm{osc}}_k^2(P_k\mathbf{w}^k;\mathscr{T}_k)\big)\end{aligned} \tag{4.9}$$

with

$$\widehat{\beta}_*^2 := \frac{(1+\delta_2)\big((1+\delta_2+C_*\delta_2^{-1}r^2(V_H,\nu))\beta_*^2 + C_*\delta_2^{-1}r^2(V_H,\nu)\big)}{1-C_2\delta_2^{-1}r(V_H,\nu)}. \tag{4.10}$$

Proof From Theorem 3.1 and 3.3, there exists a constant $C_* > 0$ such that

$$\begin{aligned}
&\|\mathbf{w}^k - P_{k,*}\mathbf{w}^k\|_{a,\Omega}^2 + \widehat{\mathrm{osc}}_{k,*}^2(P_{k,*}\mathbf{w}^k; \mathcal{T}_{k,*}) \\
&\quad \le (1+\delta_2)\big(\|\mathbf{u} - \mathbf{u}_{k,*}\|_{a,\Omega}^2 + \mathrm{osc}_{k,*}^2(\lambda_{k,*}, \mathbf{u}_{k,*}; \mathcal{T}_{k,*})\big) \\
&\qquad + C_*\delta_2^{-1} r^2(V_H, \nu)\big(\|\mathbf{u} - \mathbf{u}_{k,*}\|_{a,\Omega}^2 + \|\mathbf{u} - \mathbf{u}_k\|_{a,\Omega}^2\big) \\
&\quad \le (1+\delta_2)\beta_*^2\big(\|\mathbf{u} - \mathbf{u}_k\|_{a,\Omega}^2 + \mathrm{osc}_k^2(\lambda_k, \mathbf{u}_k; \mathcal{T}_k)\big) \\
&\qquad + C_*\delta_2^{-1} r^2(V_H, \nu)\beta_*^2\big(\|\mathbf{u} - \mathbf{u}_k\|_{a,\Omega}^2 + \mathrm{osc}_k^2(\lambda_k, \mathbf{u}_k; \mathcal{T}_k)\big) \\
&\qquad + C_*\delta_2^{-1} r^2(V_H, \nu)\|\mathbf{u} - \mathbf{u}_k\|_{a,\Omega}^2 \\
&\quad \le C_r\big(\|\mathbf{u} - \mathbf{u}_k\|_{a,\Omega}^2 + \mathrm{osc}_k^2(\lambda_k, \mathbf{u}_k; \mathcal{T}_k)\big),
\end{aligned} \tag{4.11}$$

where $C_r = \big(1 + \delta_2 + C_*\delta_2^{-1} r^2(V_H, \nu)\big)\beta_*^2 + C_*\delta_2^{-1} r^2(V_H, \nu)$.

Similarly, we have the following estimates for the right side of (4.11)

$$\|\mathbf{u} - \mathbf{u}_k\|_{a,\Omega}^2 + \mathrm{osc}_k^2(\lambda_k, \mathbf{u}_k; \mathcal{T}_k) \le \frac{(1+\delta_2)\big(\|\mathbf{w}^k - P_k\mathbf{w}^k\|_{a,\Omega}^2 + \widehat{\mathrm{osc}}_k^2(P_k\mathbf{w}^k; \mathcal{T}_k)\big)}{1 - C_2\delta_2^{-1} r(V_H, \nu)}. \tag{4.12}$$

Combining (4.11) and (4.12) leads to the desired result (4.9). □

The following corollary is a direct consequence of Lemmas 2.4 and 4.2.

Corollary 4.1 *Let $(\lambda_k, \mathbf{u}_k) \in \mathbb{R} \times V_k$ and $(\lambda_{k,*}, \mathbf{u}_{k,*}) \in \mathbb{R} \times V_{k,*}$ be as in Lemma 4.2. Suppose that the following estimate holds*

$$\|\mathbf{u} - \mathbf{u}_{k,*}\|_{a,\Omega}^2 + \mathrm{osc}_{k,*}^2(\lambda_{k,*}, \mathbf{u}_{k,*}; \mathcal{T}_{k,*}) \le \beta_*^2\big(\|\mathbf{u} - \mathbf{u}_k\|_{a,\Omega}^2 + \mathrm{osc}_k^2(\lambda_k, \mathbf{u}_k; \mathcal{T}_k)\big),$$

where the constant $\beta_*^2 \in (0, 1/2)$. Then there holds:

$$\eta_k(\lambda_k, \mathbf{u}_k; \mathcal{T}_k \backslash (\mathcal{T}_{k,*} \cap \mathcal{T}_k)) \ge \hat{\theta}\eta_k(\lambda_k, \mathbf{u}_k; \mathcal{T}_k),$$

where $\hat{\theta} = \theta_*(1 - 2\widehat{\beta}_*^2)^{\frac{1}{2}} - Cr^{\frac{1}{2}}(V_H, \nu)$, and θ_* and $\widehat{\beta}_*$ are defined in Lemmas 2.4 and 4.2.

Lemma 4.3 *Let the exact eigenfunction $\mathbf{u} \in \mathcal{A}^s$ and $\mathcal{T}_k$ be the mesh produced by Algorithm 3.1 with the refinement parameter $\theta \in \big(0, \theta_*\big)$. Then the marked set $\mathcal{M}_k$ satisfies*

$$\#\mathcal{M}_k \lesssim \big(\|\mathbf{u} - \mathbf{u}_k\|_{a,\Omega}^2 + \mathrm{osc}_k^2(\lambda_k, \mathbf{u}_k; \mathcal{T}_k)\big)^{-1/(2s)} |\mathbf{u}|_s^{1/s}, \tag{4.13}$$

where the hidden coefficient depends on the discrepancy between θ and θ_*.

Proof Let ε be a small constant which will be defined later and let $\mathcal{T}_\varepsilon$ be the refinement from the initial mesh $\mathcal{T}_1$ which has the minimum mesh elements satisfying

$$\|\mathbf{u}-\mathbf{u}_\varepsilon\|_{a,\Omega}^2+\mathrm{osc}_\varepsilon^2(\lambda_\varepsilon,\mathbf{u}_\varepsilon;\mathcal{T}_\varepsilon)\le\varepsilon^2, \tag{4.14}$$

where $(\lambda_\varepsilon,\mathbf{u}_\varepsilon)$ is the standard finite element solution for the elasticity eigenvalue problem on $\mathcal{T}_\varepsilon$. Then based on the definition of $\mathcal{A}^s$, there holds

$$\#\mathcal{T}_\varepsilon-\#\mathcal{T}_1\lesssim\varepsilon^{-1/s}|\mathbf{u}|_s^{1/s}.$$

Let $\mathcal{T}_{k,+}=\mathcal{T}_k\oplus\mathcal{T}_\varepsilon$. Then from Lemma 4.1, we can derive

$$\#\mathcal{T}_{k,+}-\#\mathcal{T}_k\le\#\mathcal{T}_\varepsilon-\#\mathcal{T}_1.$$

Follow the definition (3.8), we denote $\mathbf{w}^\varepsilon$ as the exact solution of the elasticity boundary value problem with the right hand term $\lambda_\varepsilon\mathbf{u}_\varepsilon$, then from Lemma 2.2, we have

$$\begin{aligned}&\|\mathbf{w}^\varepsilon-P_{k,+}\mathbf{w}^\varepsilon\|_{a,\Omega}^2+\widehat{\mathrm{osc}}_{k,+}^2(P_{k,+}\mathbf{w}^\varepsilon;\mathcal{T}_{k,+})\\&\le\hat{C}\big(\|\mathbf{w}^\varepsilon-P_\varepsilon\mathbf{w}^\varepsilon\|_{a,\Omega}^2+\widehat{\mathrm{osc}}_\varepsilon^2(P_\varepsilon\mathbf{w}^\varepsilon;\mathcal{T}_\varepsilon)\big).\end{aligned} \tag{4.15}$$

Set $(\lambda_{k,+},\mathbf{u}_{k,+})=\mathrm{EEG\_SOLVE}(V_H\oplus\mathrm{span}\{P_{k,+}\mathbf{w}^k,\,P_{k,+}\mathbf{w}^\varepsilon\})$. Perform the similar procedure for (4.15) as that for Theorem 4.1, we can obtain

$$\begin{aligned}&\|\mathbf{u}-\mathbf{u}_{k,+}\|_{a,\Omega}^2+\mathrm{osc}_{k,+}^2(\lambda_{k,+},\mathbf{u}_{k,+};\mathcal{T}_{k,+})\\&\le\beta_0^2\big(\|\mathbf{u}-\mathbf{u}_\varepsilon\|_{a,\Omega}^2+\mathrm{osc}_\varepsilon^2(\lambda_\varepsilon,\mathbf{u}_\varepsilon;\mathcal{T}_\varepsilon)\big)\le\beta_0^2\varepsilon^2,\end{aligned} \tag{4.16}$$

where

$$\beta_0^2=\frac{(1+\delta_1)\big(1+\delta_1+C\delta_1^{-1}r(V_H,\nu)\big)\hat{C}}{1-C\delta_1^{-1}r(V_H,\nu)}.$$

Let β_* that appears in Corollary 4.1 be small enough such that $\hat{\theta}\ge\theta$ and set

$$\varepsilon=\frac{\beta_*}{\beta_0}\big(\|\mathbf{u}-\mathbf{u}_k\|_{a,\Omega}^2+\mathrm{osc}_k^2(\lambda_k,\mathbf{u}_k;\mathcal{T}_k)\big)^{\frac{1}{2}}.$$

Then from (4.16) and Corollary 4.1, we can obtain that $\mathcal{T}_{k,+}$ satisfies

$$\eta_k(\lambda_k,\mathbf{u}_k;\mathcal{T}_k\backslash(\mathcal{T}_{k,+}\cap\mathcal{T}_k))\ge\theta\eta_k(\lambda_k,\mathbf{u}_k;\mathcal{T}_k). \tag{4.17}$$

Because Dörfler's marking strategy chooses a minimum set $\mathcal{M}_k$ satisfying

$$\eta_k(\lambda_k,\mathbf{u}_k;\mathcal{M}_k)\ge\theta\eta_k(\lambda_k,\mathbf{u}_k;\mathcal{T}_k),$$

thus $\mathscr{M}_k$ satisfies

$$\begin{aligned}\#\mathscr{M}_k &\leq \#\big(\mathscr{T}_k\backslash(\mathscr{T}_{k,+}\cap\mathscr{T}_k)\big) \leq \#\mathscr{T}_{k,+} - \#\mathscr{T}_k \leq \#\mathscr{T}_\varepsilon - \#\mathscr{T}_1 \\ &\lesssim \left(\frac{\beta_*}{\beta_0}\right)^{-1/s} \big(\|\mathbf{u}-\mathbf{u}_k\|^2_{a,\Omega} + \mathrm{osc}_k^2(\lambda_k, \mathbf{u}_k; \mathscr{T}_k)\big)^{-1/(2s)} |\mathbf{u}|_s^{1/s},\end{aligned}$$

which is (4.13) with the coefficient depending on the discrepancy between θ and θ_*. □

From Lemma 4.3, we can obtain the optimal complexity analysis of Algorithm 3.1. Actually, we derive the same upper bound in Lemma 4.3 as that in [14]; hence, we can obtain the optimal complexity of Algorithm 3.1 using the same method. Herein, we only present the conclusion in the following theorem.

Theorem 4.3 *Let* $\mathbf{u} \in \mathscr{A}^s$ *be the exact eigenfunction of* (3.1) *and* $\{(\lambda_k, \mathbf{u}_k)\}$ *be the approximate eigenpairs produced by Algorithm* 3.1. *Then, the* ℓ*-th approximate eigenpair satisfies the following optimal bound:*

$$\|\mathbf{u}-\mathbf{u}_\ell\|^2_{a,\Omega} + \mathrm{osc}_\ell^2(\lambda_\ell, \mathbf{u}_\ell; \mathscr{T}_\ell) \lesssim (\#\mathscr{T}_\ell - \#\mathscr{T}_1)^{-2s}.$$

Remark 4.2 We want to emphasize that Theorem 4.3 gives the optimal complexity of Algorithm 3.1, which is the same the classical AFEM for eigenvalue problems. But the optimal complexity means the discretization scale (or adaptive mesh) is optimal, not the computational work. Actually, the computational work directly reflects the solving efficiency. Now, we make a brief estimate about the computational work of Algorithm 3.1. which has an essential difference from that of the classical AFEM and multilevel AFEM for eigenvalue problems. Herein, we need to use additionally, that the sequence of unknowns belongs to a geometric progression (see e.g. [6]):

$$N_k < \sigma_0 N_k \leq N_{k+1} < \sigma_1 N_k, \quad k = 1, 2, \ldots \tag{4.18}$$

Theorem 4.4 *Assume the computational work for the elasticity eigenvalue problem in* V_H *and* V_1 *is* M_H *and* M_1*, and the computational work of the adaptive multigrid iteration for elasticity boundary value problem in* V_k *is* $O(N_k)$ *for* $k = 2, \ldots, n$. *Then the total computational work of Algorithm* 3.1 *is* $O\big(M_1 + M_H\log(N_n) + N_n\big)$. *Further, if* M_H *and* M_1 *is small enough, a linear computational work* $O\big(N_n\big)$ *can be derived.*

Proof Let W_k denote the computational work of Algorithm 3.1 on V_k, and W denote the whole computational work. Then we have

$$\begin{aligned} W = \sum_{k=1}^n W_k &= \mathscr{O}\left(M_1 + \sum_{k=2}^n (N_k + M_H)\right) \\ &= \mathscr{O}\left(M_1 + M_H(n-1) + N_n \sum_{k=2}^n \sigma_0^{(k-n)}\right) \\ &= \mathscr{O}\big(M_1 + M_H\log(N_n) + N_n\big). \end{aligned} \tag{4.19}$$

Further, a linear computational work $\mathcal{O}(N_n)$ for Algorithm 3.1 can be derived from (4.19) if M_H and M_1 are small enough. □

5 Numerical experiments

In this section, we present some numerical experiments conducted using Algorithm 3.1 for the elasticity eigenvalue problem. In these numerical experiments, we set the parameter $p = 2$, and each adaptive multigrid method involved in Algorithm 3.1 for the elasticity boundary value problems is executed by performing one multigrid V-cycle iteration using the conjugate gradient smoother twice [13]. For the small-scale elasticity eigenvalue problems, we adopt the implicitly restarted Lanczos method, which is included in the popular package ARPACK. The adaptive finite element spaces are constructed by the linear finite element space on the meshes through adaptive refinement.

5.1 Example 1

In the first example, we solved the elasticity eigenvalue problem with Lamé constants $\underline{\mu} = 1$ and $\underline{\lambda} = 1$ in the three dimensional non-convex domain $\Omega = (-1, 1)^3 \backslash [0, 1)^3$. Owing to the non-convex property, singularity of the eigenfunction is expected. Therefore, we used the multilevel correction adaptive multigrid method presented in Algorithm 3.1 to solve the smallest eigenvalue of this example with the refinement parameter $\theta = 0.4$.

As the exact eigenpair is unknown, an approximate solution derived on a fine mesh is selected as the exact solution in our numerical experiments. In this example, we set $H = h_1 = 1/8$ and $V_1 = V_H$. Figure 1 shows the initial mesh and the mesh after 10 adaptive refinements. For the adaptive mesh, the convergence rate is described according to the number of degrees of freedom because the local refinement leads to different mesh sizes. The mesh size h is equivalent to $N^{-1/d}$ for uniform refinement, where d denotes the dimension of the space. Then the optimal convergence rate of adaptive finite element method for eigenfunction and eigenvalue can reach $N^{-1/d}$ and $N^{-2/d}$, respectively, and the same conclusions can be found in [8, 48], etc. Figure 2 shows the errors of the approximate solutions derived by Algorithm 3.1. From Fig. 2, we found that the approximate solutions derived by Algorithm 3.1 have the optimal convergence rate.

In addition, we analyzed the CPU times of Algorithm 3.1 and the standard AFEM (i.e., the elasticity eigenvalue problem is solved directly in each adaptive space) to verify the efficiency of Algorithm 3.1. The corresponding results are presented in Fig. 3, which shows that Algorithm 3.1 is more efficient than the standard AFEM.

Besides, we also test Algorithm 3.1 for the 10 smallest eigenvalues. Figures 4 and 5 demonstrate the corresponding error estimates and computational time. From Fig. 4, we can still find that the approximate solutions derived by Algorithm 3.1 have the optimal convergence rate. From Fig. 5, we can still find that Algorithm 3.1 is more efficient than the standard AFEM.

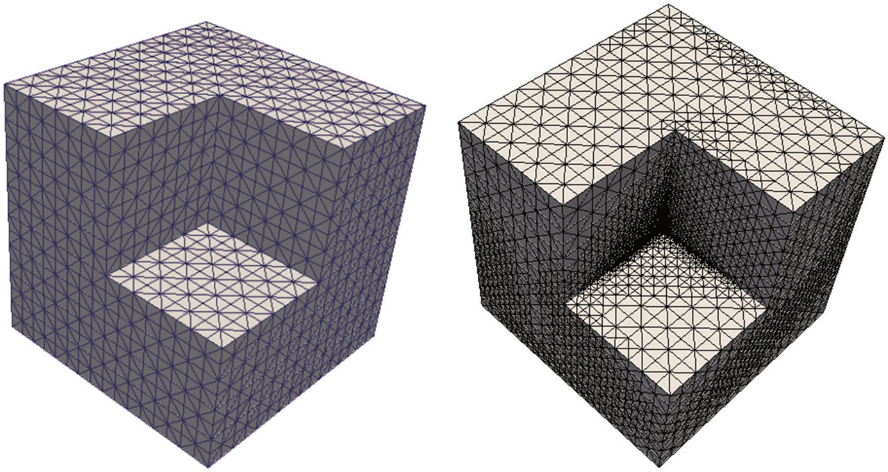
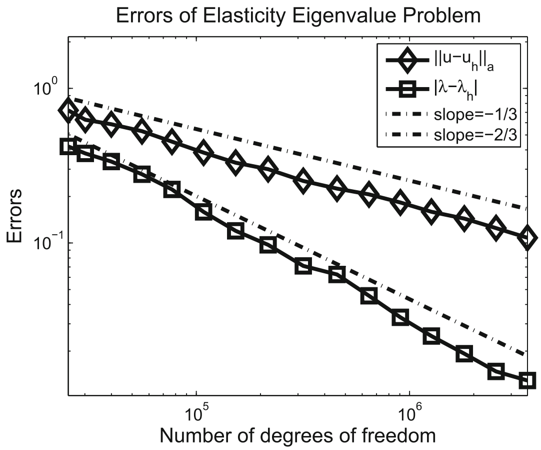


Fig. 1 Initial mesh and the adaptive mesh of Algorithm 3.1 for Example 1

Fig. 2 Errors of Algorithm 3.1 for Example 1



5.2 Example 2

In the second example, we solved the elasticity eigenvalue problem on the domain $\Omega = (0, 1)^3$ with Lamé constants $\underline{\mu} = 1$ and $\underline{\lambda} = 1$ on $\Omega = (0, 1)^3 \backslash [1/2, 1)^3$, $\underline{\mu} = 1$ and $\underline{\lambda} = 100$ on $\Omega = (1/2, 1)^3$. Since the discontinuity of the Lame constants also leads to low regularity of eigenfunctions, so we use Algorithm 3.1 to solve the smallest eigenvalue of this example with refinement parameter $\theta = 0.4$.

As the exact eigenpair is unknown, an approximate solution derived on a finer mesh is selected as the exact solution. In this example, we set $H = h_1 = 1/16$ and $V_1 = V_H$. Figure 6 shows the initial mesh. Figure 7 shows adaptive mesh and the cross section of the adaptive mesh after 10 adaptive refinements. Figure 8 depicts the error of the approximate solution derived by Algorithm 3.1. From Fig. 8, we found that the approximate solution derived by Algorithm 3.1 has the optimal convergence rate.

Fig. 3 Computational time (in s) of Algorithm 3.1 for Example 1

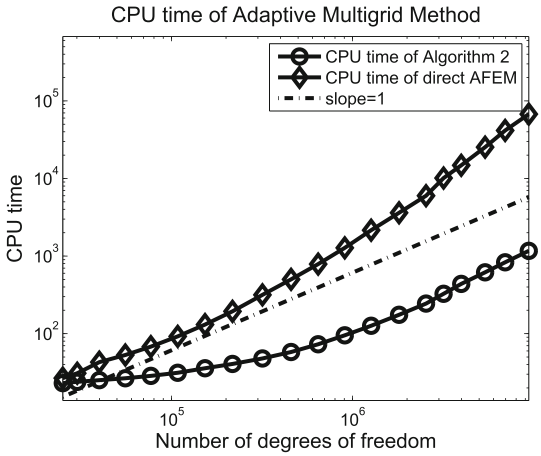


Fig. 4 Errors of the ten smallest eigenvalues for Example 1

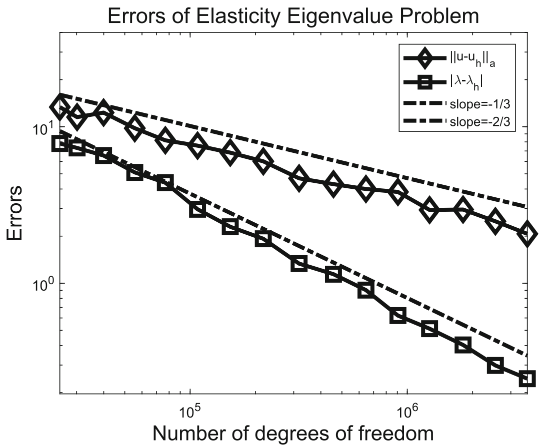
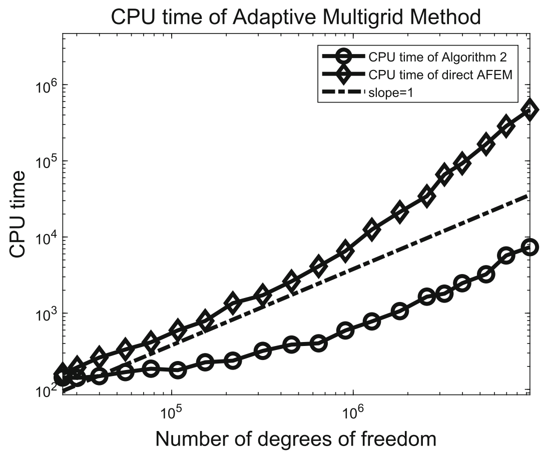


Fig. 5 Computational time (in s) of the ten smallest eigenvalues for Example 1



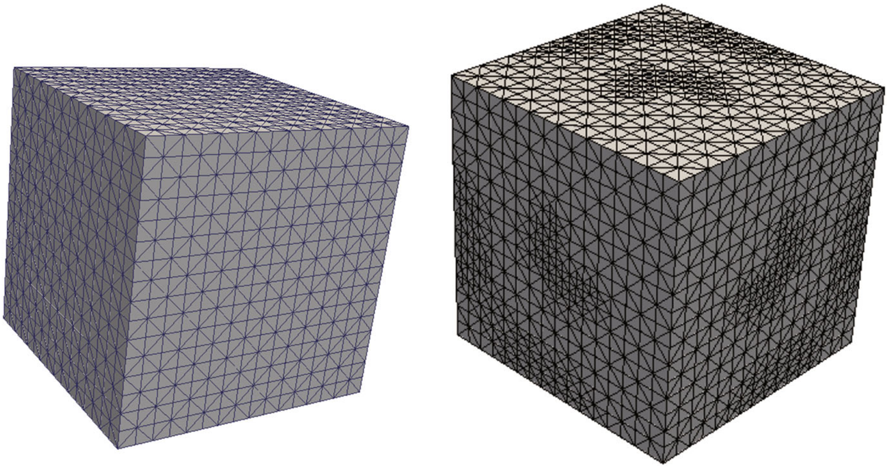


Fig. 6 Initial mesh and the adaptive mesh of Algorithm 3.1 for Example 2

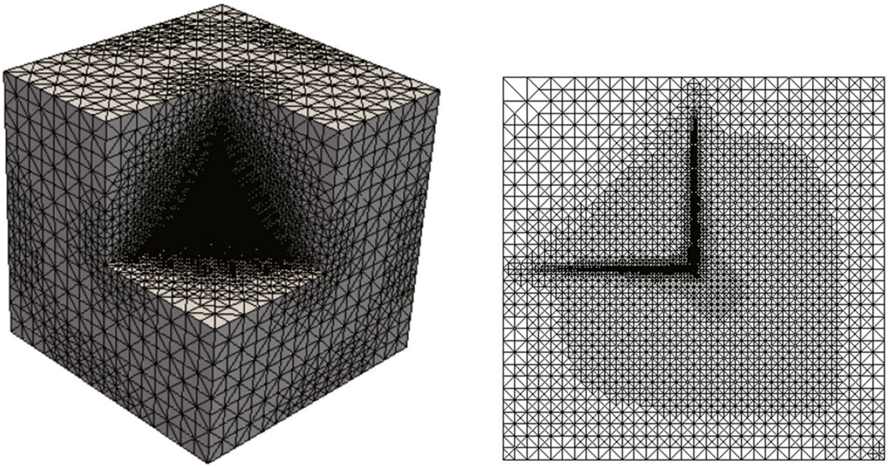


Fig. 7 The cross section along coordinate axis and xy plane of the adaptive mesh for Example 2

In addition, we also analyzed the CPU times of Algorithm 3.1 and the standard AFEM to show the efficiency of Algorithm 3.1. The corresponding results are presented in Fig. 9, which shows that Algorithm 3.1 is more efficient than the standard AFEM.

6 Concluding remarks

In this paper, we proposed a novel multilevel correction adaptive multigrid method for solving the elasticity eigenvalue problem on the basis of the adaptive multigrid method and multilevel correction method. The key point of our approach is to transform the

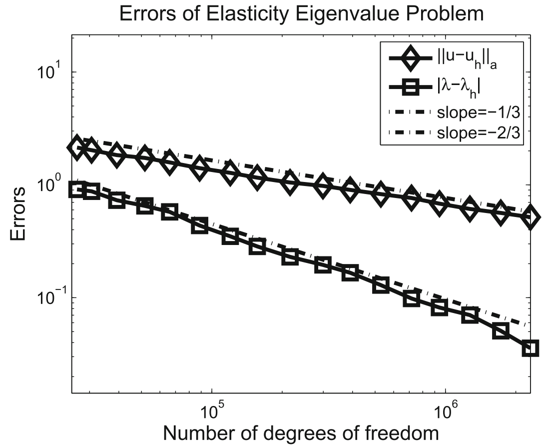


Fig. 8 Errors of Algorithm 3.1 for Example 2

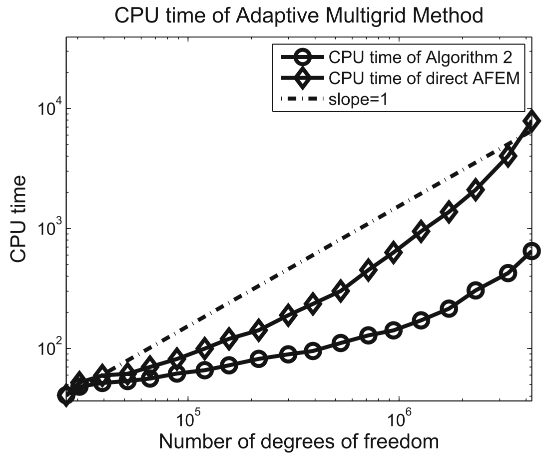


Fig. 9 Computational time (in s) of Algorithm 3.1 for Example 2

elasticity eigenvalue problem into a series of elasticity boundary value problems in a sequence of adaptive finite element spaces and some small-scale elasticity eigenvalue problems in a low-dimensional space. Further, the involved elasticity boundary value problems were solved using the adaptive multigrid method. In addition, we proved the convergence of the proposed algorithm rigorously. In the future, we plan to extend the proposed algorithm to other linear and nonlinear eigenvalue problems.

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Declarations

Conflict of interest The authors declared that they have no conflicts of interest to this work.

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