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A multilevel Newton's method for the Steklov eigenvalue problem

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Abstract
This paper proposes a new type of multilevel method for solving the Steklov eigenvalue problem based on Newton's method. In this iteration method, solving the Steklov eigenvalue problem is replaced by solving a small-scale eigenvalue problem on the coarsest mesh and a sequence of augmented linear problems on refined meshes, derived by Newton step. We prove that this iteration scheme obtains the optimal convergence rate with linear complexity, which improves the overall efficiency of solving the Steklov eigenvalue problem. Moreover, an adaptive iteration scheme for multi eigenvalues based on this new multilevel method is given. Finally, some numerical experiments are provided to illustrate the efficiency of the proposed multilevel scheme.

Keywords Steklov eigenvalue problem · Finite element method · Newton's method · Multilevel iteration · Adaptive algorithm

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1 Introduction

The Steklov eigenvalue problem arises in a wide number of physical and mechanical applications. For instance, surface waves [9], stability of mechanical oscillators immersed in a viscous fluid [16], the vibration modes of a structure in contact with an incompressible fluid [10], the antiplane shearing on a system of collinear faults under a slip-dependent friction law [14], vibrations of a pendulum [2] and eigen-oscillations of mechanical systems with boundary conditions containing the frequency [22].

Because of the extensive applications, there has been a lot of research on the numerical methods for Steklov eigenvalue problems. For instance, [3, 5] and [23] studied the conforming and nonconforming finite element method, respectively. A two-grid method, multilevel/multigrid method are separately discussed in [11, 20, 24, 33]. In [28], the authors propose an HDG method for the Steklov eigenvalue problem. [4, 17, 29, 34] considered the a posteriori error estimations and adaptive algorithms. Adaptive algorithms based on the shifted inverse iteration and multilevel correction have been studied in [12] and [26], respectively. In [15], the authors consider the multiscale analysis for the Steklov eigenvalue.

Newton's method is one of the most powerful and well-known numerical methods for solving equations. With a suitable initial guess, Newton's method is guaranteed to converge and the convergence rate is quadratic under some assumptions. Newton's method has been widely used to handle minimization and maximization problems, solving transcendent equations, complex functions and nonlinear systems of equations, etc. Especially, in [18, 21, 27, 30–32, 35], Newton's method has been applied to solve eigenvalue problems.

The main aim of this paper is to propose a type of multilevel/multigrid scheme based on Newton's method for Steklov eigenvalue problems. In this iteration scheme, solving the Steklov eigenvalue problem is decomposed into solving the Steklov eigenvalue problem on the initial coarse mesh and a series of augmented boundary problems on each refined mesh. In this iteration method, we use the multilevel technique to get the initial approximation for Newton iteration, which guarantees the convergence rate. The corresponding error estimate and complexity analysis of the proposed iteration scheme for the Steklov eigenvalue problem is given in this paper, which implies that this new method obtains an optimal convergence rate with linear computational work. Besides, we also give an adaptive iteration scheme for multi eigenvalues based on this new multilevel Newton iteration method for solving the Steklov eigenvalue problem. The efficiency of this adaptive method for multi eigenvalues is checked in the numerical example.

This paper is organized as follows. Section 2 is devoted to introducing the finite element method for the Steklov eigenvalue problem and the a priori error estimation. In Section 3, we give a multilevel Newton iteration scheme for the Steklov eigenvalue problem and analyze the error estimates and computational work of the proposed scheme. A multilevel Newton iteration scheme and an adaptive multilevel Newton iteration scheme for multi eigenvalues are proposed in Section 4 and Section 5, respectively. In Section 6, some numerical examples are presented to validate the efficiency of the new type of multilevel method and our theoretical analysis. Some concluding remarks are given in the final section.

2 Preliminaries

In this section, we first introduce the Steklov eigenvalue problem considered in this paper. Then, the finite element approximation and priori error estimates for the Steklov eigenvalue problem are given.

2.1 Steklov eigenvalue problem

Let Ω be a bounded domain in $\mathbb{R}^2$, with the cone property [1] and Lipschitz-continuous boundary, and let the boundary $\Gamma = \partial\Omega$. Let Γ_0 and Γ_1 be two complementary parts of Γ such that

$$\Gamma = \Gamma_0 \cup \Gamma_1, \quad \Gamma_0 \cap \Gamma_1 = \emptyset, \quad \text{meas}(\Gamma_1) > 0.$$

We consider the following Steklov eigenvalue problem:

$$\begin{cases} -\Delta u + u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma_0, \\ \frac{\partial u}{\partial \mathbf{n}} = \lambda u, & \text{on } \Gamma_1, \end{cases} \tag{2.1}$$

where and $\frac{\partial}{\partial \mathbf{n}}$ is the outward normal derivative on Γ. Let V be a closed subspace of $H^1(\Omega)$ defined by

$$V = \{v \in H^1(\Omega) | v = 0 \text{ on } \Gamma_0\}.$$

For the finite element method, we introduce the corresponding weak form of the problem (2.1) as follows: Find $(\lambda, u) \in \mathbb{R} \times V$ such that $b(u,u) = 1$ and

$$a(u, v) = \lambda b(u, v), \quad \forall v \in V, \tag{2.2}$$

where

$$a(u, v) = \int_\Omega (\nabla u \nabla v + uv) \mathrm{d}\Omega, \tag{2.3}$$

$$b(u, v) = \int_{\Gamma_1} uv \mathrm{d}s. \tag{2.4}$$

The bilinear form $b(\cdot,\cdot)$ is symmetric, continuous and semidefinite on the space $V \times V$ and symmetric, continuous and coercive over $L^2(\Gamma) \times L^2(\Gamma)$. In this paper, for all $v \in V$, we define $\|v\|_a := \sqrt{a(v, v)}$, $\|v\|_b := \sqrt{b(v, v)}$. It is easy to know that $\|\cdot\|_a$ is nothing but the standard norm $\|\cdot\|_1$ and $\|\cdot\|_b$ is a seminorm.

From [7], we know (2.2) has an eigenvalue sequence $\{\lambda_j\}$:

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots, \quad \lim_{k\to\infty} \lambda_k = \infty,$$

and the associated eigenfunctions

$$u_1, u_2, \cdots, u_k, \cdots,$$

where $b(u_i,u_j) = \delta_{ij}$ (δ_{ij} is Kronecker notation). In the sequence $\{\lambda_j\}$, the λ_j are repeated according to their geometric multiplicity. In order to give the error estimates, let $M(\lambda_i)$ denote the eigenfunction space corresponding to the eigenvalue λ_i which is defined by

$$M\left(\lambda_i\right) = \left\{w \in V : w \text{ is an eigenfunction of (2.2) corresponding to } \lambda_i\right\}.$$

Furthermore, we consider the Dirichlet-Neumann problem associated with (2.2): for $f \in L^2(\Gamma)$, find $Tf \in V$ such that

$$a(Tf, v) = b(f, v), \quad \forall v \in V, \tag{2.5}$$

where $T : L^2(\Gamma) \to V$ is the solution operator. Since the form $a(\cdot,\cdot)$ is V-elliptic and this problem is well defined.

Now we turn to give the following regularity results.

Lemma 2.1 [5, (4.10)] and [10, Proposition 4.4] For the Dirichlet-Neumann problem (2.5), if $f \in L^2(\Gamma)$, then $Tf \in H^{1+\gamma/2}(\Omega)$ and

$$\|Tf\|_{1+\gamma/2} \lesssim \|f\|_b, \tag{2.6}$$

where $\gamma = 1$ if Ω is convex and $\gamma < \pi/\omega$ (with ω being the largest inner angle of Γ) (see, e.g., [19]). Furthermore, if $f \in H^{1/2}(\Gamma)$, we have $Tf \in H^{1+\gamma}(\Omega)$ and

$$\|Tf\|_{1+\gamma} \lesssim \|f\|_{1/2,\Gamma}, \tag{2.7}$$

Denote the Rayleigh quotient over V by $R(\cdot)$ as follows: for any $u \in V$, $\|u\|_b \neq 0$

$$R(u) = \frac{a(u,u)}{b(u,u)}.$$

Then, the eigenvalues λ can be characterized as extremum of $R(\cdot)$ (see, e.g., [6, 7, 17]) by the following *minimum-maximum principle*:

$$\lambda_i = \min_{S_i} \max_{\substack{u\in S_i \\ \|u\|_b\neq 0}} R(u) = \max_{u\in\mathcal{K}_i} R(u), \quad i = 1, 2, \cdots,$$

and *maximum-minimum principle*:

$$\lambda_i = \max_{S_{i-1}} \min_{\substack{u\in S_{i-1}^{\perp} \\ \|u\|_b\neq 0}} R(u) = \min_{\substack{u\in \mathcal{K}_{i-1}^{\perp} \\ \|u\|_b\neq 0}} R(u), \quad i = 1, 2, \cdots,$$

where S_i is a i-dimensional subspace of V, $\mathcal{K}_i$ is the space spanned by the eigenfunctions $\{u_j\}_{j=1}^{i}$ and $S_{i-1}^{\perp}$ is the orthogonal complement of S_{i-1} in V with respect to $a(\cdot,\cdot)$.

In this paper, the letter C (with or without subscripts) denotes a generic positive constant which may be different at its different occurrences through the paper. For convenience, the symbols $\lesssim$, $\gtrsim$ and $\approx$ will be used in this paper. These $x_1 \lesssim y_1, x_2 \gtrsim y_2$ and $x_3 \approx y_3$, mean that $x_1 \leq C_1 y_1$, $x_2 \geq c_2 y_2$ and $c_3 x_3 \leq y_3 \leq C_3 x_3$ for some constants C_1,c_2,c_3 and C_3 that are independent of mesh size (see, e.g., [36]).

2.2 Discretization by the finite element method and error estimates

Now, let us define the finite element approximations of the problem (2.2). First, we generate a shape-regular decomposition of the computing domain $\Omega \subset \mathbb{R}^2$ into triangles or rectangles. The diameter of a cell $K \in \mathcal{T}_h$ is denoted by h_K. The mesh diameter h describes the maximum diameter of all cells $K \in \mathcal{T}_h$. We consider the linear Lagrange conforming finite element space which is defined as follows

$$V_h = \left\{ v_h \in C(\overline{\Omega}) \mid v_h|_K \in \mathcal{P}_1(K), \ \ \forall K \in \mathcal{T}_h \right\} \cap V, \tag{2.8}$$

where $\mathcal{P}_1(K)$ denotes the space of polynomials of degree no more than 1. Assume that the finite element space V_h satisfies the following assumption:

$$\lim_{h\to 0} \inf_{v_h \in V_h} \|w - v_h\|_a = 0, \ \ \forall w \in V. \tag{2.9}$$

The finite element approximation for (2.2) is defined as follows: Find $(\bar{\lambda}_h, \bar{u}_h) \in \mathbb{R} \times V_h$ such that $b(\bar{u}_h, \bar{u}_h) = 1$ and

$$a(\bar{u}_h, v_h) = \bar{\lambda}_h b(\bar{u}_h, v_h), \quad \forall v_h \in V_h. \tag{2.10}$$

Similarly, we know from [7] the eigenvalue problem (2.10) has eigenvalues

$$0 < \bar{\lambda}_{1,h} \leq \bar{\lambda}_{2,h} \leq \cdots \leq \bar{\lambda}_{k,h} \leq \cdots \leq \bar{\lambda}_{N_h,h},$$

and the corresponding eigenfunctions

$$\bar{u}_{1,h}, \bar{u}_{2,h}, \cdots, \bar{u}_{k,h}, \cdots, \bar{u}_{N_h,h},$$

where $b(\bar{u}_{i,h}, \bar{u}_{j,h}) = \delta_{ij}$, $1 \leq i,j \leq N_h$ (N_h is the dimension of the finite element space V_h).

From [6, 7, 17], we have the following *minimum-maximum principle*:

$$\lambda_{i,h} = \min_{S_{i,h}} \max_{\substack{u_h \in S_{i,h} \\ \|u_h\|_b \neq 0}} R(u_h) = \max_{u_h \in \mathcal{K}_{i,h}} R(u_h), \ \ i = 1, 2, \cdots, N_h,$$

and *maximum-minimum principle*:

$$\lambda_{i,h} = \max_{S_{i-1,h}} \min_{\substack{u_h \in S_{i-1,h}^{\perp} \\ \|u_h\|_b \neq 0}} R(u_h) = \min_{\substack{u_h \in \mathcal{K}_{i-1,h}^{\perp} \\ \|u_h\|_b \neq 0}} R(u_h), \quad i = 1, 2, \cdots, N_h,$$

where $S_{i,h}$ is a i-dimensional subspace of V_h, $\mathcal{K}_{i,h}$ is the space spanned by the approximate eigenfunctions $\{u_{j,h}\}_{j=1}^{i}$ and $S_{i-1,h}^{\perp}$ is the orthogonal complement subspace of $S_{i-1,h}$ in V_h with respect to $a(\cdot,\cdot)$.

Then we can define $\eta_a(h)$ as

$$\eta_a(h) = \sup_{\substack{f \in L^2(\Gamma) \\ \|f\|_b = 1}} \inf_{v_h \in V_h} \|Tf - v_h\|_a. \tag{2.11}$$

There exist the following error estimates for the eigenpair approximations by finite element method.

Proposition 2.1 [6, Lemma 3.7, (3.29b)], [7, P. 699] (i) For any eigenfunction approximation $\bar{u}_{i,h}$ of (2.10) ($i = 1,2,\cdots,N_h$), there is an eigenfunction u_i ($\|u_i\|_b = 1$) of (2.2) corresponding to λ_i satisfying the following error estimates

$$\|u_i - \bar{u}_{i,h}\|_a \leq \bar{C}_i \delta_h(\lambda_i), \tag{2.12}$$

$$\|u_i - \bar{u}_{i,h}\|_b \leq \bar{C}_i \eta_a(h) \|u_i - \bar{u}_{i,h}\|_a, \tag{2.13}$$

$$\bar{\lambda}_{i,h} - \lambda_i \leq \bar{C}_i \delta_h^2(\lambda_i), \tag{2.14}$$

where

$$\delta_h(\lambda_i) := \sup_{w \in M(\lambda_i), \|w\|_b = 1} \inf_{v_h \in V_h} \|w - v_h\|_a.$$

Here and hereafter $\bar{C}_i$ is some constant depending on λ_i but independent of the mesh size h.

Corollary 2.1 Based on the regularity (2.7), if V_h is the linear finite element space, then we have the following estimates for $\eta_a(h)$ and $\delta_h(\lambda_i)$

$$\eta_a(h) = \mathcal{O}(h^{\gamma}), \tag{2.15}$$

$$\delta_h(\lambda_i) = \mathcal{O}(h^{\gamma}). \tag{2.16}$$

The following trace theorem gives the relation between $\|\cdot\|_b$ and $\|\cdot\|_a$.

Lemma 2.2 [1] There exists a constant $C_{\text{tr}} > 0$, such that

$$\|v\|_b \leq C_{\text{tr}} \|v\|_a^{1/2} \|v\|_0^{1/2} \leq C_{\text{tr}} \|v\|_a, \ \forall v \in V. \tag{2.17}$$

For the convergence analysis, we introduce the error expansion of the eigenvalue by the Rayleigh quotient formula which comes from [6, 7].

Lemma 2.3 [6, 7] Assume (λ,u) is a true solution of the eigenvalue problem (2.2), and $\widehat{u} \in V$ such that $b(\widehat{u},\widehat{u}) \neq 0$. Then we have the following expansion

$$R(\widehat{u}) - \lambda = \frac{a(\widehat{u}-u,\widehat{u}-u)}{b(\widehat{u},\widehat{u})} - \lambda\frac{b(\widehat{u}-u,\widehat{u}-u)}{b(\widehat{u},\widehat{u})}.$$

3 Multilevel Newton iteration method for Steklov eigenvalue problem

This section aims to present a type of multilevel Newton iteration scheme for solving the Steklov eigenvalue problem. We first introduce a type of one Newton iteration step to improve the accuracy of the given eigenpair approximation and then give the multilevel iteration scheme. Next, the error estimation and complexity of this new method are analyzed.

In this section, we only consider the first and simple eigenvalue.

3.1 Existence and uniqueness of solutions

This subsection introduces the main idea that deduces our numerical method in this paper. Here, we use Lagrange multiplier method to transform Steklov eigenvalue problem (2.2) into the following nonlinear problem: Find $(\lambda, u) \in \mathbb{R} \times V$ such that for any $(\mu, v) \in \mathbb{R} \times V$

$$\langle \mathcal{G}(\lambda, u), (\mu, v)\rangle := a(u, v) - \lambda b(u, v) + \frac{1}{2}\mu(1 - b(u, u)) = 0. \tag{3.1}$$

If we have an eigenpair approximation $(\widetilde{\lambda}, \widetilde{u})$ with $b(\widetilde{u}, \widetilde{u}) = 1$, Newton's method for the nonlinear problem (3.1) is to find $(\widehat{\lambda}, \widehat{u}) \in \mathbb{R} \times V$ such that for any $(\mu, v) \in \mathbb{R} \times V$

$$\langle \mathcal{G}'(\widetilde{\lambda}, \widetilde{u})(\widehat{u} - \widetilde{u}, \widehat{\lambda} - \widetilde{\lambda}), (\mu, v)\rangle = -\langle \mathcal{G}(\widetilde{\lambda}, \widetilde{u}), (\mu, v)\rangle, \tag{3.2}$$

where $\mathcal{G}'$ is the Fréchet derivation of $\mathcal{G}$. That is

$$\begin{aligned} &a(\widehat{u} - \widetilde{u}, v) - \widetilde{\lambda} b(\widehat{u} - \widetilde{u}, v) - (\widehat{\lambda} - \widetilde{\lambda}) b(\widetilde{u}, v) - \mu b(\widehat{u} - \widetilde{u}, \widetilde{u}) \\ &= -\big(a(\widetilde{u}, v) - \widetilde{\lambda} b(\widetilde{u}, v) + \tfrac{1}{2}\mu(1 - b(\widetilde{u}, \widetilde{u}))\big). \end{aligned} \tag{3.3}$$

Actually, the Newton form (3.3) can be rewritten as the following problem: Find $(\widehat{\lambda}, \widehat{u}) \in \mathbb{R} \times V$ such that

$$\begin{cases} a(\widehat{u}, v) - \widetilde{\lambda} b(\widehat{u}, v) - \widehat{\lambda} b(\widetilde{u}, v) = -\widetilde{\lambda} b(\widetilde{u}, v), & \forall v \in V, \\ -\mu b(\widehat{u}, \widetilde{u}) = -\mu, & \forall \mu \in \mathbb{R}. \end{cases} \tag{3.4}$$

In order to prove the unique solvability of (3.4), for any given approximate solution $(\widetilde{\lambda}, \widetilde{u}) \in \mathbb{R} \times V$ with $\widetilde{\lambda} = R(\widetilde{u})$ and $b(\widetilde{u}, \widetilde{u}) = 1$, we define the following bilinear forms

$$\begin{aligned} \mathcal{A}(\widetilde{\lambda}; w, v) &= a(w, v) - \widetilde{\lambda} b(w, v), \ \forall (w, v) \in V \times V, \\ \mathcal{B}(\widetilde{u}; v, \mu) &= -\mu b(\widetilde{u}, v), \qquad\qquad \forall (\mu, v) \in \mathbb{R} \times V. \end{aligned} \tag{3.5}$$

Then, we just need to consider the well-posedness of the following mixed boundary value problem: Find $(\widehat{\lambda}, \widehat{u}) \in \mathbb{R} \times V$ such that

$$\begin{cases} \mathcal{A}(\widetilde{\lambda}; \widehat{u}, v) + \mathcal{B}(\widetilde{u}; v, \widehat{\lambda}) = -\widetilde{\lambda} b(\widetilde{u}, v), \ \forall v \in V, \\ \mathcal{B}(\widetilde{u}; \widehat{u}, \mu) = -\mu, \qquad \forall \mu \in \mathbb{R}. \end{cases} \tag{3.6}$$

Before giving the well-posedness of (3.6), we define the spectral projection $E_i : V \to \mathrm{span}\{u_i\}$ as follows

$$a(E_i w, u_i) = a(w, u_i), \quad \forall w \in V. \tag{3.7}$$

Then we have the following existence and uniqueness theorem.

Theorem 3.1 Assume $(\widetilde{\lambda}, \widetilde{u}) \in \mathbb{R} \times V$, with $\widetilde{\lambda} = R(\widetilde{u})$ and $b(\widetilde{u}, \widetilde{u}) = 1$, is a given eigenpair approximation corresponding to λ_1 such that

$$\|\widetilde{u} - E_1 \widetilde{u}\|_a^2 \le \tfrac{\lambda_2 - \lambda_1}{2(1 + 2\lambda_2 C_{\mathrm{tr}}^2)}. \tag{3.8}$$

Then the bilinear forms defined in (3.5) satisfy the following conditions

a) There exists $C_{\mathcal{A}} = \frac{\lambda_2 - \lambda_1}{2\lambda_2} > 0$, such that

$$\mathcal{A}(\widetilde{\lambda}; v, v) \ge C_{\mathcal{A}} \|v\|_a^2, \quad \forall v \in V_0, \tag{3.9}$$

where $V_0 = \{v \in V : \mathscr{B}(\widetilde{u}; v, \mu) = 0, \ \forall \mu \in \mathbb{R}\} = \{v \in V : b(\widetilde{u}, v) = 0\}$.

b) There exists $C_{\mathscr{B}} = \frac{2C_{\mathrm{tr}}}{\sqrt{1 + 4\lambda_1 C_{\mathrm{tr}}^2}} > 0$ such that

$$\inf_{\mu \in \mathbb{R}} \sup_{v \in V} \frac{\mathcal{B}(\widetilde{u}; v, \mu)}{\|v\|_a |\mu|} \ge C_{\mathcal{B}}. \tag{3.10}$$

Proof 1 Decompose $\widetilde{u}$ as $\widetilde{u} = E_1 \widetilde{u} + (I - E_1)\widetilde{u}$, where I denotes the identity operator. Then, from (2.10) and (3.7) we have

$$b(E_1 \widetilde{u}, (I - E_1)\widetilde{u}) = \lambda_1^{-1} a(E_1 \widetilde{u}, (I - E_1)\widetilde{u}) = 0,$$

and

$$1 = b(\widetilde{u}, \widetilde{u}) = b(E_1 \widetilde{u}, E_1 \widetilde{u}) + b\big((I - E_1)\widetilde{u}, (I - E_1)\widetilde{u}\big) = \|E_1 \widetilde{u}\|_b^2 + \|(I - E_1)\widetilde{u}\|_b^2.$$

Combining the trace theorem (2.17), the following equalities hold

$$\|\widetilde{u} - E_1\widetilde{u}\|_b^2 \leq C_{\mathrm{tr}}^2\|\widetilde{u} - E_1\widetilde{u}\|_a^2, \quad \|E_1\widetilde{u}\|_b^2 \geq 1 - C_{\mathrm{tr}}^2\|\widetilde{u} - E_1\widetilde{u}\|_a^2. \tag{3.11}$$

According to Lemma 2.3, $b(\widetilde{u}, \widetilde{u}) = 1$ and $a(E_1\widetilde{u}, E_1\widetilde{u}) = \lambda_1 b(E_1\widetilde{u}, E_1\widetilde{u})$, we can obtain

$$\widetilde{\lambda} - \lambda_1 = \|\widetilde{u} - E_1\widetilde{u}\|_a^2 - \lambda_1\|\widetilde{u} - E_1\widetilde{u}\|_b^2 \leq \|\widetilde{u} - E_1\widetilde{u}\|_a^2,$$

which means

$$\widetilde{\lambda} \leq \lambda_1 + \|\widetilde{u} - E_1\widetilde{u}\|_a^2. \tag{3.12}$$

For any $v \in V_0$, we also do the decomposition $v = E_1 v + (I - E_1)v$. Noting

$$b(\widetilde{u}, v) = b(E_1\widetilde{u}, E_1 v) + b\big((I - E_1)\widetilde{u}, (I - E_1)v\big) = 0,$$

together with the definition of E_1 in (3.7), we have

$$b(E_1\widetilde{u}, E_1 v) = -b\big((I - E_1)\widetilde{u}, (I - E_1)v\big) = -b\big((I - E_1)\widetilde{u}, v\big).$$

Combining the above equality, the Hölder inequality and (3.11), the following estimates hold

$$\|E_1\widetilde{u}\|_b\|E_1 v\|_b = \big|b(E_1\widetilde{u}, E_1 v)\big| = \Big|-b\big((I - E_1)\widetilde{u}, v\big)\Big| \leq \|(I - E_1)\widetilde{u}\|_b\|v\|_b. \tag{3.13}$$

Setting $\varepsilon = \|(I - E_1)\widetilde{u}\|_a$, together with (3.11) and (3.13) we have

$$\|E_1 v\|_b^2 \leq \frac{\|(I-E_1)\widetilde{u}\|_b^2}{\|E_1\widetilde{u}\|_b^2}\|v\|_b^2 \leq \frac{C_{\mathrm{tr}}^2\|(I-E_1)\widetilde{u}\|_a^2}{1-C_{\mathrm{tr}}^2\|(I-E_1)\widetilde{u}\|_a^2}\|v\|_b^2 = \frac{C_{\mathrm{tr}}^2\varepsilon^2}{1-C_{\mathrm{tr}}^2\varepsilon^2}\|v\|_b^2. \tag{3.14}$$

From the *maximum-minimum principle* and $a((I - E_1)v, u_1) = 0$, there holds

$$\begin{aligned} \lambda_2 &= \min_{\substack{u\in \mathrm{span}\{u_1\}^{\perp} \\ \|u\|_b\neq 0}} R(u) \leq R((I - E_1)v) \\ &= \frac{a\big((I-E_1)v,(I-E_1)v\big)}{b\big((I-E_1)v,(I-E_1)v\big)} \leq \frac{a(v,v)}{b\big((I-E_1)v,(I-E_1)v\big)}. \end{aligned} \tag{3.15}$$

From (3.14) and (3.15), the following estimates hold

$$\begin{aligned} b(v, v) &= b(E_1 v, E_1 v) + b\big((I - E_1)v, (I - E_1)v\big) \\ &\leq b(E_1 v, E_1 v) + \frac{1}{\lambda_2}a(v, v) \\ &\leq \frac{C_{\mathrm{tr}}^2\varepsilon^2}{1-C_{\mathrm{tr}}^2\varepsilon^2}\|v\|_b^2 + \frac{1}{\lambda_2}a(v, v). \end{aligned} \tag{3.16}$$

When (3.8) hold, that is

$$\varepsilon^2 = \|(I - E_1)\widetilde{u}\|_a^2 \le \frac{\lambda_2-\lambda_1}{2(1+2\lambda_2 C_{\rm tr}^2)} \le \frac{1}{4C_{\rm tr}^2}, \tag{3.17}$$

(3.16) leads to

$$b(v,v) \le \frac{1}{\lambda_2\left(1-\frac{C_{\rm tr}^2\varepsilon^2}{1-C_{\rm tr}^2\varepsilon^2}\right)}a(v,v) \le \frac{1}{\lambda_2\left(1-\frac{4}{3}C_{\rm tr}^2\varepsilon^2\right)}a(v,v) \le \frac{1}{\lambda_2\left(1-2C_{\rm tr}^2\varepsilon^2\right)}a(v,v). \tag{3.18}$$

Then, combining (3.8), (3.12), (3.17) and (3.18), there holds

$$\begin{aligned} a(v,v) - \widetilde{\lambda}b(v,v) &\ge \left(1 - \frac{\widetilde{\lambda}}{\lambda_2(1-2C_{\rm tr}^2\varepsilon^2)}\right)a(v,v) \\ &= \frac{\lambda_2-\widetilde{\lambda}-2\lambda_2 C_{\rm tr}^2\varepsilon^2}{\lambda_2(1-2C_{\rm tr}^2\varepsilon^2)}a(v,v) \\ &\ge \frac{\lambda_2-(\lambda_1+\varepsilon^2)-2\lambda_2 C_{\rm tr}^2\varepsilon^2}{\lambda_2}a(v,v) \\ &= \frac{(\lambda_2-\lambda_1)-\varepsilon^2(1+2\lambda_2 C_{\rm tr}^2)}{\lambda_2}a(v,v) \\ &\ge \frac{\lambda_2-\lambda_1}{2\lambda_2}a(v,v), \end{aligned}$$

which is the desired result (3.9). Now, we come to consider the second desired result (3.10). Take $v = -\widetilde{u} \in V$, since $b(\widetilde{u},\widetilde{u}) = 1$ we have

$$\begin{aligned} \inf_{\mu\in\mathbb{R}} \sup_{v\in V} \frac{\mathcal{B}(\widetilde{u};v,\mu)}{\|v\|_a|\mu|} &= \inf_{\mu\in\mathbb{R}} \sup_{v\in V} \frac{-\mu b(\widetilde{u},v)}{\|v\|_a|\mu|} = \sup_{v\in V} \frac{-b(\widetilde{u},v)}{\|v\|_a} \\ &\ge \frac{-b(\widetilde{u},-\widetilde{u})}{\|-\widetilde{u}\|_a} = \frac{b(\widetilde{u},\widetilde{u})}{\|\widetilde{u}\|_a} = \frac{1}{\sqrt{\widetilde{\lambda}}}. \end{aligned} \tag{3.19}$$

From (3.8), (3.12) and (3.17), there hold

$$\widetilde{\lambda} \le \lambda_1 + \|\widetilde{u} - E_1\widetilde{u}\|_a^2 \le \lambda_1 + \frac{1}{4C_{\rm tr}^2} = \frac{1+4\lambda_1 C_{\rm tr}^2}{4C_{\rm tr}^2}.$$

Therefore, (3.19) means that (3.10) holds for $C_{\mathscr{B}} = \frac{2C_{\rm tr}}{\sqrt{1+4\lambda_1 C_{\rm tr}^2}}$, and we complete the proof. □

Remark 3.1 According to Theorem 3.1 and the theory for the mixed finite element method [13], there exists only one solution $(\widehat{\lambda},\widehat{u})$ of Newton iteration (3.6) to approximate the first eigenpair (λ_1,u_1).

Corollary 3.1 ([13]) Under the conditions of Theorem 3.1, for the approximation eigenpair $(\widehat{\lambda},\widehat{u})$ of Newton iteration (3.6), the following inequality holds

$$\|\widehat{u}\|_a + \left|\widehat{\lambda}\right| \le C_d \sup_{0\ne(\mu,v)\in\mathbb{R}\times V} \frac{\mathcal{A}(\widetilde{\lambda};\widehat{u},v)+\mathcal{B}(\widetilde{u};v,\widehat{\lambda})+\mathcal{B}(\widetilde{u};\widehat{u},\mu)}{\|v\|_a+|\mu|}, \tag{3.20}$$

where constant C_d depends on $C_{\mathcal{A}}$ and $C_{\mathscr{B}}$ defined in Theorem 3.1.

3.2 One Newton iteration step

Assume we have obtained an eigenpair approximation $(\lambda_{1,h_k}, u_{1,h_k}) \in \mathbb{R} \times V_{h_k}$ of the first eigenpair $(\lambda_1, u_1) \in \mathbb{R} \times V$ with $\|u_{1,h_k}\|_b = 1$. Now we introduce a type of one iteration step to improve the accuracy of the current eigenpair approximation based on Newton iteration. Let $V_{h_{k+1}} \subset V$ be a finer finite element space such that $V_{h_k} \subset V_{h_{k+1}}$. Based on this finer finite element space, we define the following one Newton iteration step.

Algorithm 1 One newton iteration step.

1. Solve the following augmented mixed variational equation: Find $(\widehat{\lambda}_{1,h_{k+1}}, \widehat{u}_{1,h_{k+1}}) \in \mathbb{R} \times V_{h_{k+1}}$ such that for any $(\mu, v_{h_{k+1}}) \in \mathbb{R} \times V_{h_{k+1}}$

$$\begin{cases} \mathcal{A}(\lambda_{1,h_k}; \widehat{u}_{1,h_{k+1}}, v_{h_{k+1}}) + \mathcal{B}(u_{1,h_k}; v_{h_{k+1}}, \widehat{\lambda}_{1,h_{k+1}}) = -\lambda_{1,h_k} b(u_{1,h_k}, v_{h_{k+1}}), \\ \mathcal{B}(u_{1,h_k}; \widehat{u}_{1,h_{k+1}}, \mu) = -\mu. \end{cases} \tag{3.21}$$

2. Normalize $\widehat{u}_{1,h_{k+1}}$ as

$$u_{1,h_{k+1}} = \frac{\widehat{u}_{1,h_{k+1}}}{\|\widehat{u}_{1,h_{k+1}}\|_b} \tag{3.22}$$

and compute the Rayleigh quotient for $u_{1,h_{k+1}}$

$$\lambda_{1,h_{k+1}} = R(u_{1,h_{k+1}}). \tag{3.23}$$

Then we obtain a new eigenpair approximation $(\lambda_{1,h_{k+1}}, u_{1,h_{k+1}}) \in \mathbb{R} \times V_{h_{k+1}}$. Summarize the above two steps as

$$(\lambda_{1,h_{k+1}}, u_{1,h_{k+1}}) = \texttt{Newton\_Iteration}(\lambda_{1,h_k}, u_{1,h_k}, V_{h_{k+1}}).$$

Theorem 3.2 Assume $(\lambda_{1,h_k}, u_{1,h_k})$, with $\lambda_{1,h_k} = R(u_{1,h_k})$ and $\|u_{1,h_k}\|_b = 1$, is an eigenpair approximation corresponding to λ_1 such that (3.8) holds. After performing Algorithm 1, the output $(\lambda_{1,h_{k+1}}, u_{1,h_{k+1}}) \in \mathbb{R} \times V_{h_{k+1}}$ has the following error estimates

$$\|\bar{u}_{1,h_{k+1}} - u_{1,h_{k+1}}\|_a \le C_1 \|\bar{u}_{1,h_{k+1}} - u_{1,h_k}\|_a^2, \tag{3.24}$$

$$|\bar{\lambda}_{1,h_{k+1}} - \lambda_{1,h_{k+1}}| \le C_2 \|\bar{u}_{1,h_{k+1}} - u_{1,h_k}\|_a^4, \tag{3.25}$$

where C_1 and C_2 depend on $C_{\mathcal{A}}$ and $C_{\mathscr{B}}$ but independent of the mesh sizes h_k and h_{k+1}.

Proof 2 From the variational form (2.10) and the definition of bilinear forms (3.5), we know that $(\bar{\lambda}_{1,h_{k+1}}, \bar{u}_{1,h_{k+1}})$ satisfies the following equations, for any $(\mu, v_{h_{k+1}}) \in \mathbb{R} \times V_{h_{k+1}}$

$$\begin{cases} \mathcal{A}(\lambda_{1,h_k}; \bar{u}_{1,h_{k+1}}, v_{h_{k+1}}) + \mathcal{B}(u_{1,h_k}; v_{h_{k+1}}, \bar{\lambda}_{1,h_{k+1}}) \\ \qquad = (\bar{\lambda}_{1,h_{k+1}} - \lambda_{1,h_k}) b(\bar{u}_{1,h_{k+1}}, v_{h_{k+1}}) - \bar{\lambda}_{1,h_{k+1}} b(u_{1,h_k}, v_{h_{k+1}}), \\ \mathcal{B}(u_{1,h_k}; \bar{u}_{1,h_{k+1}}, \mu) = -\mu b(\bar{u}_{1,h_{k+1}}, u_{1,h_k}). \end{cases} \tag{3.26}$$

Let us define $w_{h_{k+1}} := \bar{u}_{1,h_{k+1}} - \widehat{u}_{1,h_{k+1}}$ and $\gamma_{h_{k+1}} := \bar{\lambda}_{1,h_{k+1}} - \widehat{\lambda}_{1,h_{k+1}}$. From step 1 of Algorithm 1 and (3.26), the following equations hold, for any $(\mu, v_{h_{k+1}}) \in \mathbb{R} \times V_{h_{k+1}}$

$$\begin{cases} \mathcal{A}(\lambda_{1,h_k}; w_{h_{k+1}}, v_{h_{k+1}}) + \mathcal{B}(u_{1,h_k}; v_{h_{k+1}}, \gamma_{h_{k+1}}) \\ \qquad = (\bar{\lambda}_{1,h_{k+1}} - \lambda_{1,h_k}) b(\bar{u}_{1,h_{k+1}} - u_{1,h_k}, v_{h_{k+1}}), \\ \mathcal{B}(u_{1,h_k}; w_{h_{k+1}}, \mu) = -\mu b(\bar{u}_{1,h_{k+1}} - u_{1,h_k}, u_{1,h_k}). \end{cases} \tag{3.27}$$

Similarly to Corollary 3.1, for problem (3.27) the following estimate hold

$$\begin{aligned} &\|w_{h_{k+1}}\|_a + |\gamma_{h_{k+1}}| \\ \lesssim &\sup_{0 \neq (\mu, v_{h_{k+1}}) \in \mathbb{R} \times V_{h_{k+1}}} \frac{\mathcal{A}(\lambda_{1,h_k}; w_{h_{k+1}}, v_{h_{k+1}}) + \mathcal{B}(u_{1,h_k}; v_{h_{k+1}}, \gamma_{h_{k+1}}) + \mathcal{B}(u_{1,h_k}; w_{h_{k+1}}, \mu)}{\|v_{h_{k+1}}\|_a + |\mu|}. \end{aligned} \tag{3.28}$$

Using Lemma 2.3, trace theorem (2.17) and (3.27), we have

$$\begin{aligned} &\mathcal{A}(\lambda_{1,h_k}; w_{h_{k+1}}, v_{h_{k+1}}) + \mathcal{B}(u_{1,h_k}; v_{h_{k+1}}, \gamma_{h_{k+1}}) \\ = &(\bar{\lambda}_{1,h_{k+1}} - \lambda_{1,h_k}) b(\bar{u}_{1,h_{k+1}} - u_{1,h_k}, v_{h_{k+1}}) \\ \leq &|\bar{\lambda}_{1,h_{k+1}} - \lambda_{1,h_k}| \|\bar{u}_{1,h_{k+1}} - u_{1,h_k}\|_b \|v_{h_{k+1}}\|_b \\ \lesssim &\|\bar{u}_{1,h_{k+1}} - u_{1,h_k}\|_a^2 \|v_{h_{k+1}}\|_a. \end{aligned} \tag{3.29}$$

Noting $b(\bar{u}_{1,h_{k+1}}, \bar{u}_{1,h_{k+1}}) = 1$ and $b(u_{1,h_k}, u_{1,h_k}) = 1$

$$\begin{aligned} \mathcal{B}(u_{1,h_k}; w_{h_{k+1}}, \mu) &= -\mu b(\bar{u}_{1,h_{k+1}} - u_{1,h_k}, u_{1,h_k}) \\ &= \frac{\mu}{2} b(\bar{u}_{1,h_{k+1}} - u_{1,h_k}, \bar{u}_{1,h_{k+1}} - u_{1,h_k}) \\ &\lesssim |\mu| \|\bar{u}_{1,h_{k+1}} - u_{1,h_k}\|_a^2. \end{aligned} \tag{3.30}$$

From (3.28), (3.29) and (3.30), we have

$$\|w_{h_{k+1}}\|_a + |\gamma_{h_{k+1}}| \lesssim \|\bar{u}_{1,h_{k+1}} - u_{1,h_k}\|_a^2.$$

Then the following estimate holds

$$\|\bar{u}_{1,h_{k+1}} - \widehat{u}_{1,h_{k+1}}\|_a \lesssim \|\bar{u}_{1,h_{k+1}} - u_{1,h_k}\|_a^2. \tag{3.31}$$

Combining the trace theorem (2.17), normalization (3.22), (3.31) and $\|\bar{u}_{1,h_{k+1}}\|_b = 1$, we have the following inequalities

$$
\begin{aligned}
&\|\bar{u}_{1,h_{k+1}} - u_{1,h_{k+1}}\|_a \\
\leq\ & \left\|\bar{u}_{1,h_{k+1}} - \frac{\bar{u}_{1,h_{k+1}}}{\|\widehat{u}_{1,h_{k+1}}\|_b}\right\|_a + \left\|\frac{\bar{u}_{1,h_{k+1}}}{\|\widehat{u}_{1,h_{k+1}}\|_b} - u_{1,h_{k+1}}\right\|_a \\
=\ & \frac{\|\bar{u}_{1,h_{k+1}}\|_a}{\|\widehat{u}_{1,h_{k+1}}\|_b}\Big|\|\widehat{u}_{1,h_{k+1}}\|_b - 1\Big| + \left\|\frac{\bar{u}_{1,h_{k+1}}}{\|\widehat{u}_{1,h_{k+1}}\|_b} - \frac{\widehat{u}_{1,h_{k+1}}}{\|\widehat{u}_{1,h_{k+1}}\|_b}\right\|_a \\
=\ & \frac{\|\bar{u}_{1,h_{k+1}}\|_a}{\|\widehat{u}_{1,h_{k+1}}\|_b}\Big|\|\widehat{u}_{1,h_{k+1}}\|_b - \|\bar{u}_{1,h_{k+1}}\|_b\Big| + \frac{\|\bar{u}_{1,h_{k+1}} - \widehat{u}_{1,h_{k+1}}\|_a}{\|\widehat{u}_{1,h_{k+1}}\|_b} \\
\leq\ & \frac{\|\bar{u}_{1,h_{k+1}}\|_a}{\|\widehat{u}_{1,h_{k+1}}\|_b}\|\bar{u}_{1,h_{k+1}} - \widehat{u}_{1,h_{k+1}}\|_b + \frac{\|\bar{u}_{1,h_{k+1}} - \widehat{u}_{1,h_{k+1}}\|_a}{\|\widehat{u}_{1,h_{k+1}}\|_b} \\
\lesssim\ & \|\bar{u}_{1,h_{k+1}} - \widehat{u}_{1,h_{k+1}}\|_a \lesssim \|\bar{u}_{1,h_{k+1}} - u_{1,h_k}\|_a^2,
\end{aligned}
$$

which means (3.24) hold. Furthermore, from Lemma 2.3 and (3.24), the other two desired results (3.25) can be obtained directly. The proof is complete. □

Remark 3.2 From Theorem 3.2, we know that Newton's method has *second order* convergence rate when the initial approximation is good enough.

3.3 Multilevel Newton iteration method

This subsection is firstly devoted to introducing a type of multilevel scheme based on the *One Newton Iteration Step* given by Algorithm 1. Then, the convergence of this multilevel iteration method is considered.

Before introducing the multigrid scheme, we define a sequence of triangulations $\mathcal{T}_{h_k}$ of Ω. Suppose $\mathcal{T}_{h_1}$ is the initial mesh and let $\mathcal{T}_{h_k}$ be obtained from $\mathcal{T}_{h_{k-1}}$ via regular refinement (produce β^d subelements) such that

$$h_k = \frac{1}{\beta} h_{k-1}.$$

Based on this sequence of meshes, we construct the corresponding nested linear finite element spaces such that

$$V_{h_1} \subset V_{h_2} \subset \cdots \subset V_{h_n}, \tag{3.32}$$

and the following relation of approximation errors hold for $k = 2, \cdots, n$

$$\frac{1}{\beta}\eta_a(h_{k-1}) \leq C_\eta \eta_a(h_k), \quad \frac{1}{\beta}\delta_{h_{k-1}}(\lambda) \leq C_\delta \delta_{h_k}(\lambda), \quad \beta\delta_{h_{k+1}}(\lambda) \leq C'_\delta \delta_{h_k}(\lambda). \tag{3.33}$$

Algorithm 2 Multilevel eigenvalue iteration scheme.

1. Construct a sequence of nested finite element spaces $V_{h_1}, V_{h_2}, \cdots, V_{h_n}$ such that (3.32) and (3.33) hold.
2. Solve the following eigenvalue problem: Find $(\lambda_{1,h_1}, u_{1,h_1}) \in \mathbb{R} \times V_{h_1}$ such that $b(u_{1,h_1}, u_{1,h_1}) = 1$ and
$$a(u_{1,h_1}, v_{h_1}) = \lambda_{1,h_1} b(u_{1,h_1}, v_{h_1}), \quad \forall v_{h_1} \in V_{h_1}. \tag{3.34}$$
3. Do $k = 1, \cdots, n-1$
Obtain a new eigenpair approximation $(\lambda_{1,h_{k+1}}, u_{1,h_{k+1}}) \in \mathbb{R} \times V_{h_{k+1}}$ by a Newton iteration step
$$(\lambda_{1,h_{k+1}}, u_{1,h_{k+1}}) = \texttt{Newton\_Iteration}(\lambda_{1,k}, u_{1,h_k}, V_{h_{k+1}}).$$
End do

Finally, we obtain an eigenpair approximation $(\lambda_{1,h_n}, u_{1,h_n}) \in \mathbb{R} \times V_{h_n}$.

Theorem 3.3 Suppose h_1 is small enough such that u_{1,h_1} satisfies condition (3.8)

$$\|u_{1,h_1} - E_1 u_{1,h_1}\|_a^2 \le \frac{\lambda_2-\lambda_1}{2(1+2\lambda_2 C_{\mathrm{tr}}^2)},$$

and

$$2C_1\big(C_\delta\beta + 2\bar{C}_1^2(C_\delta\beta + C'_\delta\beta^{-1})\big)\delta_{h_1}(\lambda_1) \le 1.$$

Then, the output of Algorithm 2, $(\lambda_{1,h_n}, u_{1,h_n})$, has the following error estimates

$$\|u_{1,h_n} - \bar{u}_{1,h_n}\|_a \le \delta_{h_n}(\lambda_1), \tag{3.35}$$

$$|\lambda_{1,h_n} - \bar{\lambda}_{1,h_n}| \le C_3\delta_{h_n}^2(\lambda_1). \tag{3.36}$$

Besides, there exists an eigenfunction $u_1 \in M(\lambda_1)$ such that the final convergence results hold

$$\|u_1 - u_{1,h_n}\|_a \le C_4\delta_{h_n}(\lambda_1), \tag{3.37}$$

$$|\lambda_1 - \lambda_{1,h_n}| \le C_5\delta_{h_n}^2(\lambda_1). \tag{3.38}$$

Proof 3 Let us prove (3.35) by the method of induction. Firstly, according to (2.10) and (3.34), we have $u_{1,h_1} = \bar{u}_{1,h_1}$, which means (3.35) holds for $n = 1$. Then we assume (3.35) holds for $n = k$, that is

$$\|u_{1,h_k} - \bar{u}_{1,h_k}\|_a \le \delta_{h_k}^2(\lambda_1). \tag{3.39}$$

Now let us consider the case of $n = k+1$. Combining (3.24), (3.33), (3.39), Proposition 2.1 and the triangle inequality leads to the following estimates

$$\begin{aligned}
&\|u_{1,h_{k+1}} - \bar{u}_{1,h_{k+1}}\|_a \le C_1\|u_{1,h_k} - \bar{u}_{1,h_{k+1}}\|_a^2 \\
\le\ & 2C_1(\|u_{1,h_k} - \bar{u}_{1,h_k}\|_a^2 + \|\bar{u}_{1,h_k} - \bar{u}_{1,h_{k+1}}\|_a^2) \\
\le\ & 2C_1\big(\delta_{h_k}^2(\lambda_1) + (2\|\bar{u}_{1,h_k} - u\|_a^2 + 2\|u - \bar{u}_{1,h_{k+1}}\|_a^2)\big) \\
\le\ & 2C_1\Big(\delta_{h_k}^2(\lambda_1) + 2\big(\bar{C}_1^2\delta_{h_k}^2(\lambda_1) + \bar{C}_1^2\delta_{h_{k+1}}^2(\lambda_1)\big)\Big) \\
\le\ & 2C_1\big(C_\delta\beta\delta_{h_k}(\lambda_1)\delta_{h_{k+1}}(\lambda_1) + 2\bar{C}_1^2(C_\delta\beta + C_\delta'\beta^{-1})\delta_{h_k}(\lambda_1)\delta_{h_{k+1}}(\lambda_1)\big) \\
=\ & 2C_1\big(C_\delta\beta + 2\bar{C}_1^2(C_\delta\beta + C_\delta'\beta^{-1})\big)\delta_{h_k}(\lambda_1)\delta_{h_{k+1}}(\lambda_1).
\end{aligned}$$

This means that (3.35) also holds for $n = k + 1$ if $2C_1\big(C_\delta\beta + 2\bar{C}_1^2(C_\delta\beta + C_\delta'\beta^{-1})\big)\delta_{h_k}(\lambda_1) \le 1$. Thus we prove the desired result (3.35). From Lemma 2.3 and (3.35), we easily get the second result (3.36). Finally, (3.37) and (3.38) can be deduced from (2.12), (2.14), (3.35)–(3.36) and the triangle inequality. □

3.4 Work estimate of multilevel eigenvalue iteration scheme

In this subsection, we turn our attention to the estimate of computational work for Algorithm 2. We will show that Algorithm 2 makes the Steklov eigenvalue problem solving need almost the optimal computational work if the linear (3.21) only needs the linear computational work.

First, we investigate the dimension of each level linear finite element space as $N_k := \dim V_{h_k}$. Then the following property holds

$$N_k \approx \left(\frac{1}{\beta}\right)^{d(n-k)} N_n, \quad k = 1, 2, \cdots, n. \tag{3.40}$$

Theorem 3.4 Assume solving the eigenvalue problem in the coarse space V_{h_1} needs work $\mathcal{O}(M_{h_1})$ and the work for solving the linear equation (3.21) in each level space V_{h_k} is only $\mathcal{O}(N_k)$ for $k = 2, \cdots, n$. Then the work involved in Algorithm 2 is $\mathcal{O}(N_n + M_{h_1})$. Furthermore, the complexity will be $\mathcal{O}(N_n)$ provided $M_{h_1} \le N_n$.

Proof 4 Let W_k denote the work of the iteration step defined in Algorithm 1 in the k-th finite element space V_{h_k} for $k = 2, \cdots, n$. From the iteration definition in Algorithm 1, we have

$$W_k = \mathcal{O}(N_k), \qquad \text{for } k = 2, \cdots, n. \tag{3.41}$$

Iterating (3.41) and using the fact (3.40), the following estimates hold

$$\begin{aligned}\text{Total work } &= \sum_{k=1}^{n} W_k = \mathcal{O}\left(M_{h_1} + \sum_{k=2}^{n} N_k \right) \\ &= \mathcal{O}\left(M_{h_1} + \sum_{k=2}^{n} \left(\frac{1}{\beta}\right)^{d(n-k)} N_n \right) = \mathcal{O}(N_n + M_{h_1}).\end{aligned} \tag{3.42}$$

This means the computational work is $\mathcal{O}(N_n + M_{h_1})$ and the one $\mathcal{O}(N_n)$ can be derived with the condition $M_{h_1} \leq N_n$. □

Remark 3.3 The discrete linear system (3.21) leads to a saddle-point structure problem. The assumption that solving linear system (3.21) needs $\mathcal{O}(N_k)$ work is reasonable since [8] provides the corresponding multigrid methods.

4 Multilevel scheme for multi eigenvalues

In this section, we extend the Newton's iteration method for the first eigenvalue of the Steklov eigenvalue problem to multi eigenvalues (include simple and multiple eigenvalues). And, the efficiency of this scheme will be tested in Section 6.

4.1 Multi eigenvalues

Now, we turn to give the existence and uniqueness for multi eigenvalues similar to Theorem 3.1. Assume that $\lambda_m < \lambda_{m+1}$ and we have obtained the first m eigenpairs approximation $\{(\widetilde{\lambda}_i, \widetilde{u}_i)\}_{i=1}^{m}$ to the problem (3.1), which satisfy

$$b(\widetilde{u}_i, \widetilde{u}_j) = \delta_{ij}, \quad i,j = 1, \cdots, m,$$

where $\widetilde{\lambda}_i = R(\widetilde{u}_i)$ is the Rayleigh quotient of $\widetilde{u}_i$.

The Newton's method for multi eigenvalues of (3.1) is to find $(\Lambda_i, \widehat{u}_i) \in \mathbb{R}^m \times V$ ($i = 1,\cdots,m$) such that, for any $(X, v) \in \mathbb{R}^m \times V$

$$\begin{cases} \mathcal{A}(\widetilde{\lambda}_i; \widehat{u}_i, v) + \sum_{j=1}^{m} \mathcal{B}(\widetilde{u}_j; v, \Lambda_i^{(j)}) = -\widetilde{\lambda}_i b(\widetilde{u}_i, v), \\ \mathcal{B}(\widetilde{u}_j; \widehat{u}_i, X^{(j)}) = -X^{(j)}\delta_{ij}, \qquad \forall j = 1, \cdots, m, \end{cases} \tag{4.1}$$

where $\Lambda_i^{(j)}$ and $X^{(j)}$ is the j-th component of Λ_i and X respectively.

Now, we come to prove (4.1) has only one solution for any $i = 1,\cdots,m$. For this aim, we define the following bilinear forms

$$\widetilde{\mathcal{B}}(v,X)=\sum_{j=1}^{m}\mathcal{B}(\widetilde{u}_j;v,X^{(j)})=-\sum_{j=1}^{m}X^{(j)}b(\widetilde{u}_j,v). \tag{4.2}$$

Here and hereafter in this section $u\in V$, $v\in V$, $X=(X^{(1)},X^{(2)},\cdots,X^{(m)})^T\in W=\mathbb{R}^m$.

Assume that $f_i\in V'$, $g_i\in W'$ are defined as

$$f_i(v)=-\widetilde{\lambda}_i b(\widetilde{u}_i,v),\qquad g_i(X)=-\sum_{j=1}^{m}X^{(j)}\delta_{ij}.$$

We consider the following multi mixed problems: Find $(\Lambda_i,\widehat{u}_i)\in\mathbb{R}^m\times V$ such that

$$\begin{cases}\mathcal{A}(\widetilde{\lambda}_i;\widehat{u}_i,v)+\widetilde{\mathcal{B}}(v,\Lambda_i) = f_i(v),\ \forall v\in V,\\ \widetilde{\mathcal{B}}(\widehat{u}_i,X) = g_i(X),\ \forall X\in W.\end{cases} \tag{4.3}$$

Define $\mathcal{K}=M(\lambda_1)\cup\cdots\cup M(\lambda_m)$. About the existence and uniqueness of problem (4.3), the following theorem holds.

Theorem 4.1 Assume that there exists a decomposition of eigenfunction space $\mathcal{K}$ satisfying $\mathcal{K}=M(\lambda_1)\oplus\cdots\oplus M(\lambda_m)$ such that $\widetilde{u}_i$ is an eigenfunction approximation to $M(\lambda_i)$ $(i=1,\cdots,m)$ with

$$\|\widetilde{u}_i-E_i\widetilde{u}_i\|_a^2\le\frac{\lambda_{m+1}-\lambda_m}{(m+1)\left(1+(m+1)\lambda_{m+1}C_{\mathrm{tr}}^2\right)} \tag{4.4}$$

and $\widetilde{\lambda}_i=R(\widetilde{u}_i)$. Then the bilinear forms defined in (4.2) satisfy the following conditions

1. There exists $\widetilde{C}_{\mathcal{A}}=\frac{m(\lambda_{m+1}-\lambda_m)}{(m+1)\lambda_{m+1}}>0$ such that
$$\mathcal{A}(\widetilde{\lambda}_i;v,v)\ge\widetilde{C}_{\mathcal{A}}\|v\|_a^2,\qquad\forall v\in\widetilde{V}_0, \tag{4.5}$$
where $\widetilde{V}_0=\{v\in V:\widetilde{\mathscr{B}}(v,X)=0,\ \forall X\in W\}=\{v\in V:b(\widetilde{u}_j,v)=0,\ j=1,\cdots m\}$.
2. There exists $\widetilde{C}_{\mathscr{B}}=\frac{(m+1)C_{\mathrm{tr}}}{\sqrt{1+(m+1)^2C_{\mathrm{tr}}^2\lambda_m}}>0$ such that
$$\inf_{X\in W}\sup_{v\in V}\frac{\widetilde{\mathcal{B}}(v,X)}{\|v\|_a\|X\|}\ge\widetilde{C}_{\mathcal{B}}, \tag{4.6}$$

where $\|X\|:=\max_{j\in\{1,\cdots,m\}}|X^{(j)}|$.

Based on these two conditions, for any $i(i=1,\cdots,m)$, the multi mixed equations (4.3) have only one solution.

Proof 5 We decompose $\widetilde{u}_i$ as $\widetilde{u}_i = E_i\widetilde{u}_i + (I - E_i)\widetilde{u}_i$. Then, $\{E_1\widetilde{u}_1, \cdots, E_m\widetilde{u}_m\}$ is an orthogonal basis of eigenfunction space $\mathcal{K}$. Similarly, we have

$$b\big(E_i\widetilde{u}_i, (I - E_i)\widetilde{u}_i\big) = \widetilde{\lambda}_i^{-1} a\big(E_i\widetilde{u}_i, (I - E_i)\widetilde{u}_i\big) = 0,$$

and $\|\widetilde{u}_i\|_b^2 = \|E_i\widetilde{u}_i\|_b^2 + \|(I - E_i)\widetilde{u}_i\|_b^2$ $(i = 1, \cdots, m)$. Therefore, $(I - E_i)\widetilde{u}_i$ and $\widetilde{u}_i$ have estimates

$$\|(I - E_i)\widetilde{u}_i\|_b \le C_{\mathrm{tr}}\|(I - E_i)\widetilde{u}_i\|_a, \quad \|E_i\widetilde{u}_i\|_b^2 \ge 1 - C_{\mathrm{tr}}^2\|(I - E_i)\widetilde{u}_i\|_a^2, \quad i = 1, \cdots, m. \tag{4.7}$$

According to Lemma 2.3, $b(\widetilde{u}_i, \widetilde{u}_i) = 1$ and $a(E_i\widetilde{u}_i, E_i\widetilde{u}_i) = \lambda_i b(E_i\widetilde{u}_i, E_i\widetilde{u}_i)$, we can obtain

$$\widetilde{\lambda}_i - \lambda_i = \|\widetilde{u}_i - E_i\widetilde{u}_i\|_a^2 - \lambda_i\|\widetilde{u}_i - E_I\widetilde{u}_i\|_b^2 \le \|\widetilde{u}_i - E_i\widetilde{u}_i\|_a^2,$$

which means

$$\widetilde{\lambda}_i \le \lambda_i + \|\widetilde{u}_i - E_i\widetilde{u}_i\|_a^2. \tag{4.8}$$

Similarly, we also do decomposition $v \in \widetilde{V}_0$ as

$$v = E_1 v + \cdots + E_m v + v^* = E_i v + (I - E_i)v, \quad i = 1, \cdots, m$$

satisfying

$$v^* \bot_b \mathcal{K}, \quad E_i v \in \mathrm{span}\{E_i\widetilde{u}_i\}, \quad (I - E_i)v = \sum_{j=1, j\ne i}^{m} E_j v + v^*, \quad (I - E_i)v \bot_b \mathrm{span}\{E_i\widetilde{u}_i\}.$$

According to the definition of $\widetilde{V}_0$, we have

$$0 = b(\widetilde{u}_i, v) = b\big(E_i\widetilde{u}_i + (I - E_i)\widetilde{u}_i, E_i v + (I - E_i)v\big).$$

Therefore

$$\begin{aligned} \|E_i v\|_b \|E_i\widetilde{u}_i\|_b &= |b(E_i v, E_i\widetilde{u}_i)| = |-b\big((I - E_i)v, (I - E_i)\widetilde{u}_i\big)| \\ &= |b\big(v, (I - E_i)\widetilde{u}_i\big)| \le \|v\|_b \|(I - E_i)\widetilde{u}_i\|_b, \quad i = 1, \cdots, m. \end{aligned} \tag{4.9}$$

Setting $\varepsilon_i = \|(I - E_i)\widetilde{u}_i\|_a$ $(i = 1, \cdots, m)$, combining (4.7) and (4.9), the following estimates hold

$$\|E_i v\|_b^2 \le \frac{C_{\mathrm{tr}}^2\varepsilon_i^2}{\|E_i\widetilde{u}_i\|_b^2}\|v\|_b^2 \le \frac{C_{\mathrm{tr}}^2\varepsilon_i^2}{1 - C_{\mathrm{tr}}^2\varepsilon_i^2}\|v\|_b^2, \quad i = 1, \cdots, m. \tag{4.10}$$

Recall (4.4)

$$\varepsilon_i^2 = \|(I - E_i)\widetilde{u}_i\|_a^2 \le \frac{\lambda_{m+1} - \lambda_m}{(m+1)\big(1 + (m+1)\lambda_{m+1}C_{\mathrm{tr}}^2\big)} \le \frac{1}{(m+1)^2 C_{\mathrm{tr}}^2},$$

that is

$$\frac{1}{1-C_{\rm tr}^2\varepsilon_i^2} \le \frac{(m+1)^2}{m(m+2)} \le \frac{m+1}{m}. \tag{4.11}$$

From the *maximum-minimum principle* and $v^* \perp_a \mathcal{K}$, there holds

$$\lambda_{m+1} = \min_{\substack{u\in\mathcal{K}^\perp \\ \|u\|_b\neq 0}} R(u) \le R(v^*) \le \frac{a(v,v)}{b(v^*,v^*)}. \tag{4.12}$$

From (4.10), (4.11), (4.12) and the property $\|v\|_b^2 = \|E_1 v\|_b^2 + \cdots + \|E_m v\|_b^2 + \|v^*\|_b^2$,

$$\begin{aligned} b(v,v) &= b(E_1 v, E_1 v) + \cdots + b(E_m v, E_m v) + b(v^*, v^*) \\ &\le \sum_{i=1}^m \frac{C_{\rm tr}^2\varepsilon_i^2}{1-C_{\rm tr}^2\varepsilon_i^2} b(v,v) + b(v^*,v^*) \\ &\le \sum_{i=1}^m \frac{C_{\rm tr}^2\varepsilon_i^2}{1-C_{\rm tr}^2\varepsilon_i^2} b(v,v) + \frac{1}{\lambda_{m+1}} a(v,v) \\ &\le \frac{(m+1)C_{\rm tr}^2}{m} \sum_{i=1}^m \varepsilon_i^2 b(v,v) + \frac{1}{\lambda_{m+1}} a(v,v) \\ &\le (m+1)C_{\rm tr}^2 \varepsilon_{\max}^2 b(v,v) + \frac{1}{\lambda_{m+1}} a(v,v), \end{aligned} \tag{4.13}$$

where $\varepsilon_{\max} = \max\limits_{i=1,\cdots,m} \varepsilon_i$. Then, (4.13) means

$$b(v,v) \le \frac{1}{\lambda_{m+1}\left(1-(m+1)C_{\rm tr}^2\varepsilon_{\max}^2\right)} a(v,v). \tag{4.14}$$

Using the definition of $\mathcal{A}(\widetilde{\lambda}_i;\cdot,\cdot)$, (4.4), (4.8) and (4.14), the following inequalities hold

$$\begin{aligned} \mathcal{A}(\widetilde{\lambda}_i;v,v) &= a(v,v) - \widetilde{\lambda}_i b(v,v) \ge \left(1 - \frac{\widetilde{\lambda}_i}{\lambda_{m+1}\left(1-(m+1)C_{\rm tr}^2\varepsilon_{\max}^2\right)}\right) a(v,v) \\ &= \frac{\lambda_{m+1}\left(1-(m+1)C_{\rm tr}^2\varepsilon_{\max}^2\right) - \widetilde{\lambda}_i}{\lambda_{m+1}\left(1-(m+1)C_{\rm tr}^2\varepsilon_{\max}^2\right)} a(v,v) \\ &= \frac{\lambda_{m+1} - \widetilde{\lambda}_i - (m+1)C_{\rm tr}^2\varepsilon_{\max}^2\lambda_{m+1}}{\lambda_{m+1}\left(1-(m+1)C_{\rm tr}^2\varepsilon_{\max}^2\right)} a(v,v) \\ &\ge \frac{\lambda_{m+1} - (\lambda_i + \varepsilon_i^2) - (m+1)C_{\rm tr}^2\varepsilon_{\max}^2\lambda_{m+1}}{\lambda_{m+1}\left(1-(m+1)C_{\rm tr}^2\varepsilon_{\max}^2\right)} a(v,v) \\ &\ge \frac{\lambda_{m+1} - \lambda_i - \varepsilon_{\max}^2\left(1+(m+1)C_{\rm tr}^2\lambda_{m+1}\right)}{\lambda_{m+1}} a(v,v) \\ &\ge \frac{\lambda_{m+1} - \lambda_m - \frac{\lambda_{m+1}-\lambda_m}{m+1}}{\lambda_{m+1}} a(v,v) \\ &= \frac{m(\lambda_{m+1}-\lambda_m)}{(m+1)\lambda_{m+1}} a(v,v). \end{aligned} \tag{4.15}$$

It means (4.5) holds for $\widetilde{C}_{\mathcal{A}} = \frac{m(\lambda_{m+1}-\lambda_m)}{(m+1)\lambda_{m+1}} > 0$ when (4.4) holds.

Now, we come to prove (4.6). For any $X \in W$, assume that the index $s \in \{1,2,\cdots,m\}$ satisfies $\|X\| = |X^{(s)}|$. From $b(\widetilde{u}_i, \widetilde{u}_j) = \delta_{ij}$ $(i,j = 1,\cdots,m)$ and the definition of $\widetilde{\mathscr{B}}(\cdot,\cdot)$, taking $v = -\frac{X^{(s)}}{|X^{(s)}|}\widetilde{u}_s$ we have

$$\sup_{v\in V}\frac{\widetilde{\mathcal{B}}(v,X)}{\|v\|_a}=\sup_{v\in V}\frac{-\sum_{j=1}^{m}X^{(j)}b(\widetilde{u}_j,v)}{\|v\|_a}\geq\frac{|X^{(s)}|b(\widetilde{u}_s,\widetilde{u}_s)}{\|\widetilde{u}_s\|_a}=\frac{\|X\|}{\sqrt{\widetilde{\lambda}_s}}\geq\frac{\|X\|}{\sqrt{\widetilde{\lambda}_m}}. \tag{4.16}$$

Then combining (4.4), (4.8) and (4.16), we have

$$\sup_{v\in V}\frac{\widetilde{\mathcal{B}}(v,X)}{\|v\|_a}\geq\frac{\|X\|}{\sqrt{\lambda_m+\varepsilon_m^2}}\geq\frac{\|X\|}{\sqrt{\lambda_m+\frac{1}{(m+1)^2C_{\rm tr}^2}}}=\widetilde{C}_{\mathcal{B}}\|X\|, \tag{4.17}$$

where $\widetilde{C}_{\mathscr{B}}=\frac{(m+1)C_{\rm tr}}{\sqrt{1+(m+1)^2C_{\rm tr}^2\lambda_m}}>0$. It means that (4.6) holds.

From the theory for the mixed finite element method [13], there exists only one solution for the (4.3) for any $i=1,\cdots,m$.

4.2 Multilevel iteration for multi eigenvalues

Similarly, we first give one iteration step to improve the given approximations to the first m eigenpairs. Assume we have obtained the first m eigenpairs approximation $(\lambda_{i,h_k},u_{i,h_k})\in\mathbb{R}\times V_{h_k}$ with $\|u_{i,h_k}\|_b=1$ $(i=1,\cdots,m)$. Now we introduce a type of iteration step to improve the accuracy of the current eigenpair approximations. Let $V_{h_{k+1}}\subset V$ be a finer finite element space such that $V_{h_k}\subset V_{h_{k+1}}$.

Algorithm 3 One Newton iteration step for multi eigenvalues.

1. Do $i=1,\cdots,m$.
   Find $(\Lambda_{i,h_{k+1}},\widehat{u}_{i,h_{k+1}})\in W\times V_{h_{k+1}}$ such that, for any $(X,v_{h_{k+1}})\in W\times V_{h_{k+1}}$
   $$\begin{cases}\mathcal{A}(\lambda_{i,h_k};\widehat{u}_{i,h_{k+1}},v_{h_{k+1}})+\sum\limits_{j=1}^{m}\mathcal{B}(u_{j,h_k};v_{h_{k+1}},\Lambda_{i,h_{k+1}}^{(j)})=-\lambda_{i,h_k}b(u_{i,h_k},v_{h_{k+1}}),\\ \mathcal{B}(u_{j,h_k};\widehat{u}_{i,h_{k+1}},\mu)=X^{(j)}\delta_{ij},\qquad\forall j=1,\cdots,m,\end{cases} \tag{4.18}$$
   where $\Lambda_{i,h_{k+1}}^{(j)}$ is the j-th component of $\Lambda_{i,h_{k+1}}$.
   End Do
2. Construct a m-dimensional space $\widehat{V}_{h_{k+1}}=\text{span}\{\widehat{u}_{1,h_{k+1}},\cdots,\widehat{u}_{m,h_{k+1}}\}$ and for $i=1,2,\cdots,m$ solve the following eigenvalue problem: Find $(\lambda_{i,h_{k+1}},u_{i,h_{k+1}})\in\mathbb{R}\times\widehat{V}_{h_{k+1}}$, such that $b(u_{i,h_{k+1}},u_{i,h_{k+1}})=1$ and
   $$a(u_{i,h_{k+1}},v_{h_{k+1}})=\lambda_{i,h_{k+1}}b(u_{i,h_{k+1}},v_{h_{k+1}}),\quad\forall v_{h_{k+1}}\in\widehat{V}_{h_{k+1}}.$$

We summarize above two steps as

$$\{\lambda_{i,h_{k+1}},u_{i,h_{k+1}}\}_{i=1}^{m}=\texttt{Newton\_Iteration\_Multi}(\{\lambda_{i,h_k},u_{i,h_k}\}_{i=1}^{m},V_{h_{k+1}}).$$

Now, we are ready to give the corresponding multilevel correction method for multi eigenvalues.

Algorithm 4 Multilevel iteration for multi eigenvalues.

1. Construct a series of nested finite element spaces $V_{h_1}, V_{h_2}, \cdots, V_{h_n}$ such that (3.32) and (3.33) hold.
2. Solve the eigenvalue problem in the initial finite element space V_{h_1}: Find $(\lambda_{i,h_1}, u_{i,h_1}) \in \mathbb{R} \times V_{h_1}$ such that $b(u_{i,h_1}, u_{i,h_1}) = 1$ and
$$a(u_{i,h_1}, v_{h_1}) = \lambda_{i,h_1} b(u_{i,h_1}, v_{h_1}), \quad \forall v_{h_1} \in V_{h_1}. \tag{4.19}$$
Choose the first m eigenpairs $\{\lambda_{i,h_1}, u_{i,h_1}\}_{i=1}^m$ which approximate the desired eigenpairs.
3. Do $k = 1, \cdots, n-1$.
Obtain new eigenpair approximations $\{\lambda_{i,h_{k+1}}, u_{i,h_{k+1}}\}_{i=1}^m \in \mathbb{R} \times V_{h_{k+1}}$ by a Newton iteration step defined in Algorithm 3
$$\{\lambda_{i,h_{k+1}}, u_{i,h_{k+1}}\}_{i=1}^m = \texttt{Newton\_Iteration\_Multi}(\{\lambda_{i,h_k}, u_{i,h_k}\}_{i=1}^m, V_{h_{k+1}}).$$
End do

Finally, we get m eigenpair approximations $\{\lambda_{i,h_n}, u_{i,h_n}\}_{i=1}^m \in \mathbb{R} \times V_{h_n}$.

Remark 4.1 In Algorithm 3, computation can be used to solve linear system (4.18) for different i. Then, the work estimate of Algorithm 4 is the same as Algorithm 2 (presented in Theorem 3.4).

5 Multilevel iteration with adaptive method

In this section, based on the a posteriori error estimators we will establish an adaptive multilevel Newton iteration for the Steklov eigenvalue problem. Here, we only describe the scheme without analysis.

In the above multilevel Newton iteration method, we refine the mesh uniformly. However, this is not practical since the amount of required memory will increase very rapidly as we refine the mesh. Hence, an efficient refinement strategy is desired. On the one hand, the solution should be resolved well with the refined mesh. On the other hand, the total amount of the mesh elements should be controlled well to make the simulation efficient. Based on the above discussion, the adaptive mesh method is a competitive candidate for the refinement strategy.

A standard adaptive mesh process can be described by the following one

$$\cdots\textbf{Solve} \rightarrow \textbf{Estimate} \rightarrow \textbf{Mark} \rightarrow \textbf{Refine}\cdots.$$

More precisely, to get $\mathcal{T}_{h_{k+1}}$ from $\mathcal{T}_{h_k}$, we first solve the discrete equation on $\mathcal{T}_{h_k}$ to get the approximate solution and then calculate the a posteriori error estimator on each mesh element. Next, we mark the elements with big errors and these elements are refined in such a way that the triangulation is still shaped regular and conforming. Here, we choose the classical residual type a posteriori error estimator for (2.1). First, define the element residual $\mathcal{R}_K(u_h)$ and the jump residual $\mathcal{J}_e(u_h)$ for the eigenpair approximation (λ_h, u_h) as follows (see, e.g., [4, 17]):

$$\mathcal{R}_K(u_h) := -\Delta u_h + u_h, \quad \text{in } K \in \mathcal{T}_h,$$
$$\mathcal{J}_e(u_h) := \begin{cases} \frac{1}{2}(\nabla u_h^+ \cdot \nu^+ + \nabla u_h^- \cdot \nu^-) := \frac{1}{2}[[\nabla u_h]]_e \cdot \nu_e, & \text{for } e \in \mathcal{E}_h, \\ \nabla u_h \cdot \nu - \lambda_h u_h, & \text{for } e \in \mathcal{E}_\Gamma, \end{cases}$$

where e is the common edge of elements K^+ and K^- with outward normals ν^+ and ν^- respectively, $\nu_e = \nu^+$, $\mathcal{E}_h$ is the set of all inner edges of $\mathcal{T}_h$ and $\mathcal{E}_\Gamma$ is the set of all boundary edges of $\mathcal{T}_h$.

For each element $K \in \mathcal{T}_h$, we define the local error indicator $\eta_h(u_h, K)$

$$\eta_h(u_h, K) := \left(h_K^2 \|\mathcal{R}_K(u_h)\|_{0,K} + \sum_{e \in \mathcal{E}_h, e \subset \partial K} h_e \|\mathcal{J}_e(u_h)\|_{0,e} \right)^{1/2}, \tag{5.1}$$

and the error indicator for a subdomain $\omega \subset \Omega$ by

$$\eta_h(u_h, \omega) := \left(\sum_{K \in \mathcal{T}_h, K \subset \omega} \eta_h^2(u_h, K) \right)^{1/2}. \tag{5.2}$$

Based on the error indicator (5.2), we choose the Dörfler's marking strategy for m approximations $u_{1,h}, \cdots, u_{m,h}$ to construct subset $\mathscr{M}_h$ for local refinement.

Algorithm 5 Dörfler's marking strategy.

1. Given a parameter $\theta \in (0, 1)$.
2. Construct a minimal subset $\mathcal{M}_h$ from $\mathcal{T}_h$ by selecting some elements in $\mathcal{T}_h$ such that
$$\sum_{i=1}^{m} \eta_h(u_{i,h}, \mathcal{M}_h) \geq \theta \sum_{i=1}^{m} \eta_h(u_{i,h}, \mathcal{T}_h).$$
3. Mark all the elements in $\mathcal{M}_h$.

Now we state the multilevel iteration scheme with an adaptive method for the Steklov eigenvalue problem. Based on the adaptive refinement method described above, and one Newton iteration step for multi eigenvalues defined by Algorithm 3, the multilevel iteration method is given in the following algorithm.

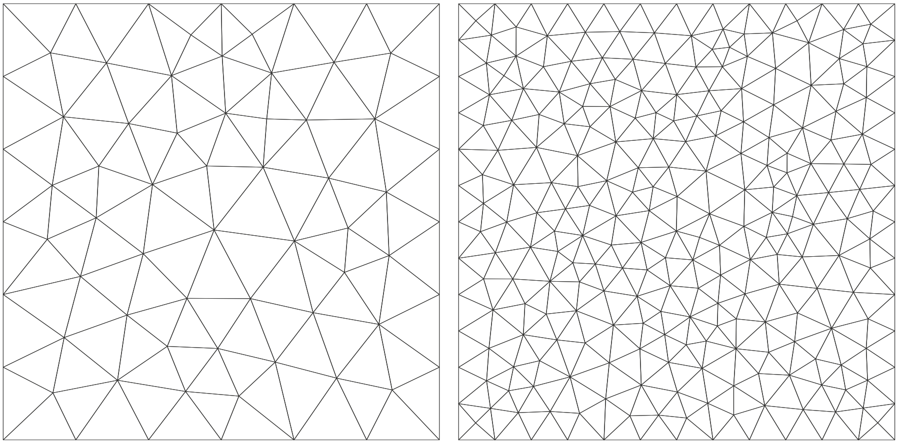

Fig. 1 The initial meshes for Example 1

Algorithm 6 Adaptive multilevel newton iteration for multi eigenvalues.

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1. Generate a coarse triangulation $\mathcal{T}_{h_1}$ for the computing domain Ω and choose a mark parameter $\theta \in (0, 1)$.
2. Build the initial finite element space V_{h_1} on the triangulation $\mathcal{T}_{h_1}$, and solve the Steklov eigenvalue problem on V_{h_1}: Find $(\lambda_{i,h_1}, u_{i,h_1}) \in \mathbb{R} \times V_{h_1}$ such that $b(u_{i,h_1}, u_{i,h_1}) = 1$ and

$$a(u_{i,h_1}, v_{h_1}) = \lambda_{i,h_1} b(u_{i,h_1}, v_{h_1}), \quad \forall v_{h_1} \in V_{h_1}.$$

   Choose the first m eigenpairs $\{\lambda_{i,h_1}, u_{i,h_1}\}_{i=1}^{m}$ which approximate the desired eigenpairs and set $k = 1$
3. Compute the local error indicators $\eta_{h_k}(u_{i,h_k}, K),\ i = 1, \cdots, m$ on each element of $\mathcal{T}_{h_k}$ according to (5.1).
4. Construct $\mathcal{M}_{h_k} \subset \mathcal{M}_{h_k}$ by Algorithm 5 and refine $\mathcal{M}_{h_k}$ to get a new conforming mesh $\mathcal{T}_{h_{k+1}}$, and construct finite element space $V_{h_{k+1}}$.
5. Obtain new eigenpair approximations $\{\lambda_{i,h_{k+1}}, u_{i,h_{k+1}}\}_{i=1}^{m} \in \mathbb{R} \times V_{h_{k+1}}$ by Algorithm 3 on $\mathcal{T}_{h_{k+1}}$:

$$\{\lambda_{i,h_{k+1}}, u_{i,h_{k+1}}\}_{i=1}^{m} = \texttt{Newton\_Iteration\_Multi}(\{\lambda_{i,h_k}, u_{j,h_k}\}_{i=1}^{m}, V_{h_{k+1}}).$$

6. Let $k = k + 1$ and go to step 3.

Finally, we get m eigenpair approximations $\{\lambda_{i,h_n}, u_{i,h_n}\}_{i=1}^{m} \in \mathbb{R} \times V_{h_n}$.

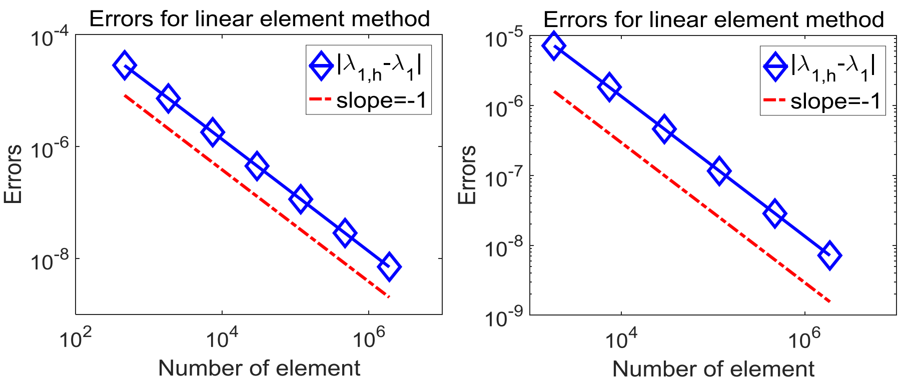

Fig. 2 The errors of the multilevel iteration algorithm for the first eigenvalue λ_1 on the unit square. (The left subfigure is for the coarse initial mesh in Fig. 1 and the right one for the fine initial mesh in Fig. 1)

6 Numerical results

In this section, some numerical examples are presented to illustrate the efficiency of the multilevel iteration scheme proposed in Algorithms 2, 4 and 6, respectively. For simplicity, we consider the situation that $\Gamma_0 = \emptyset$ and $\Gamma_1 = \Gamma$. In this paper, all schemes are running on the same machine ThinkPad T570 (Matlab, R2016b). The machine is equipped with Intel Core i7-7500U (2.90GHz) CPU with 8G memory.

6.1 Steklov eigenvalue problem on unit square

We first consider Steklov eigenvalue problem defined on unit square $\Omega = (0,1) \times (0,1)$. The sequence of linear finite element spaces is constructed on the series of meshes which are produced by the regular refinement with $\beta = 2$ (producing β^2 subelements). In this example, we choose two meshes that are generated by the Delaunay method as the initial mesh $\mathcal{T}_{h_1}$ to produce two sequences of finite element spaces for investigating the convergence behaviors. Figure 1 shows this two initial meshes, the left is coarse mesh ($h_1 = 1/6$) and the right is fine mesh ($h_1 = 1/12$).

Since the exact eigenvalue is unknown for this problem, we use an accurate enough approximation [0.240079083080045, 1.492303119894411, 1.492303120006201, 2.082647034280811] given by the extrapolation method (see, e.g., [25]) as the first four exact eigenvalues to investigate the error. Algorithm 2 is applied to solve the eigenvalue problem. Figure 2 gives the corresponding numerical results for the first eigenvalue $\lambda_1 = 0.2400790830800452$. From Fig. 2, we find that the multilevel iteration scheme can obtain the optimal error estimates as the expected one for the direct finite element method, which confirms with the convergence Theorem 3.3 for multilevel Newton's method. To show the high efficiencies of Algorithm 2, we also give the CPU time of our multigrid method and the standard finite element method to solve the problem for coarse initial mesh in Table 1.

Table 1 The CPU time of Algorithm 2 and the direct finite element method for first eigenvalue of Example 1 (coarse initial mesh)

Level	Number of degrees of freedom	Time of Algorithm 2 (s)	Time of direct FEM (s)
1	464	0.034874	0.042280
2	1856	0.059542	0.096778
3	7424	0.218019	0.339148
4	29696	0.767854	1.349842
5	118784	4.421342	10.641550
6	475136	28.759214	65.702863
7	1900544	169.821724	365.798662

We also check the convergence behavior for multi eigenvalue approximations with Algorithm 4. Here the first four eigenvalues are investigated. Similarly, we use the coarse and fine initial meshes shown in Fig. 1, respectively. The corresponding numerical results are given in Fig. 3 and Table 2, which also exhibit the optimal convergence orders and high efficiencies of the multilevel iteration scheme.

6.2 Steklov eigenvalue problem on dumbbell-shaped domain

In order to show our multilevel Newton iteration method can work well with adaptive method (Algorithm 6), we discuss the Steklov eigenvalue problem defined on a dumbbell-shaped domain $\Omega = (0,\pi)^2 \cup \left[\pi, \frac{5}{4}\pi\right] \times \left(\frac{3}{8}\pi, \frac{5}{8}\pi\right) \cup \left(\frac{5}{4}\pi, \frac{9}{4}\pi\right) \times (0,\pi)$. The initial mesh for this dumbbell-shaped domain is given in Fig. 4.

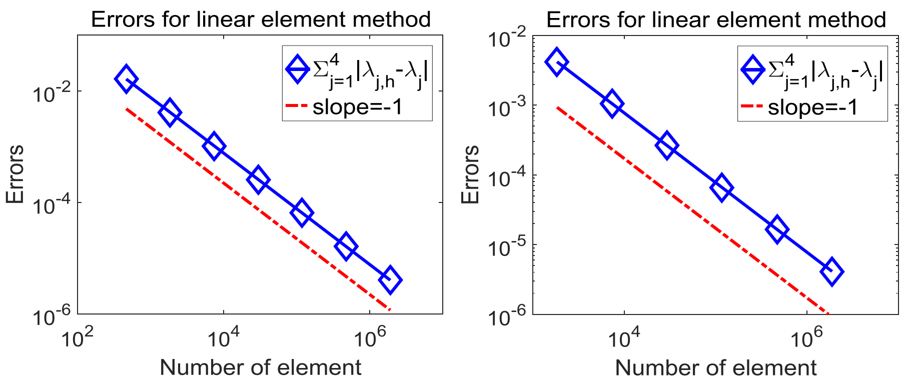

Fig. 3 The errors of the multilevel iteration algorithm for the first four eigenvalues on the unit square (The left subfigure is for the coarse initial mesh in Fig. 1 and the right one for the fine initial mesh in Fig. 1)

Table 2 The CPU time of Algorithm 4 and the direct finite element method for the first four eigenvalues of Example 1 (coarse initial mesh)

Level	Number of degrees of freedom	Time of Algorithm 4 (s)	Time of direct FEM (s)
1	464	0.035578	0.061425
2	1856	0.078948	0.301429
3	7424	0.276420	0.708873
4	29696	1.027417	2.007012
5	118784	5.675398	12.800388
6	475136	36.104498	68.077419
7	1900544	221.599882	447.711348

It is easy to know that reentrant corners of the dumbbell domain result in the singularities of the eigenfunctions. The convergence order for eigenfunction approximations is less than 1 by the linear finite element method, which is the order predicted by the theory for regular eigenfunctions. We consider using the adaptive Algorithm 6 to solve this problem. Figure 5 shows the mesh after 9 adaptive refinements.

Since the exact solution is unknown, we use the accurate enough approximation [0.580124563836536,0.606949611404787,0.767254752494938,0.767868169277588, 0.771144654505056] given by the extrapolation method (see, e.g., [25]) as the first five exact eigenvalues to investigate the error. First, we investigate the convergent rate of the adaptive posterior error estimator $\eta_h(u_h, \mathcal{T}_h)$ defined in (5.1). Figure 6 presents the corresponding numerical results for the first five eigenfunction approximations. Here, we use $\eta_h(u_{i,h})$ to denote the i-th error estimator $\eta_h(u_{i,h}, \mathcal{T}_h)$. The error estimates of eigenvalues are also given in Fig. 6 which shows that our multilevel iteration method combines well with the adaptive finite element method naturally and Algorithm 6 has the optimal convergence rate.

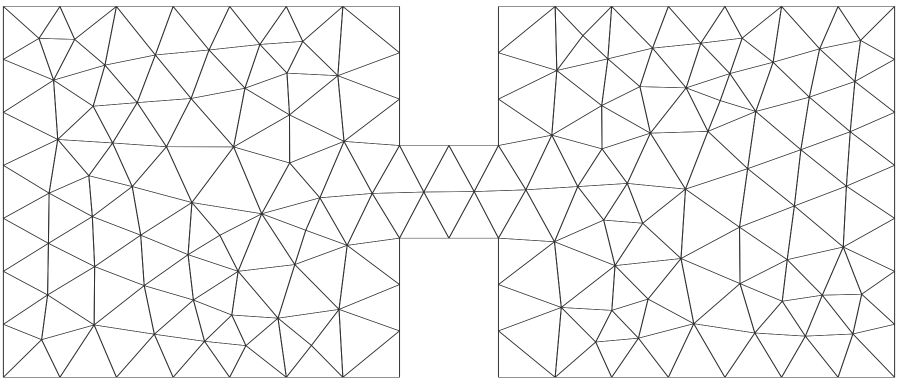

Fig. 4 The initial meshes for Example 2

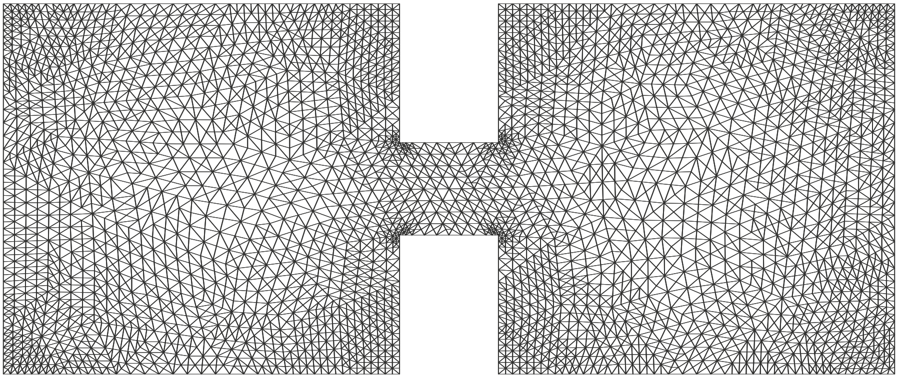

Fig. 5 The mesh after 9 adaptive refinements

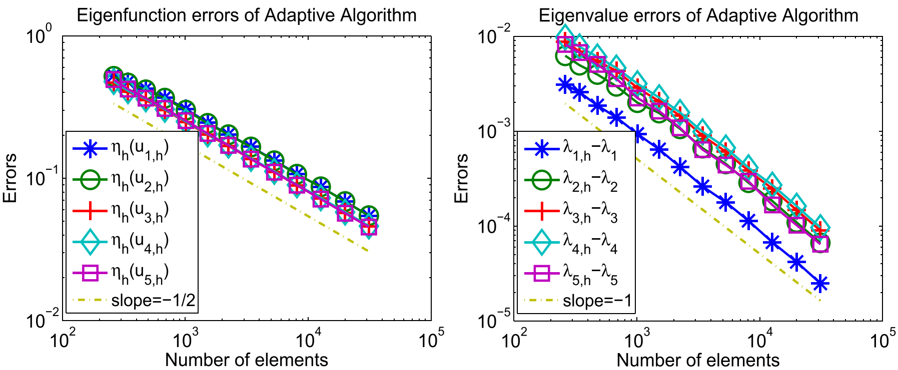

Fig. 6 The errors of the adaptive multilevel iteration algorithm for the first five eigenfunction approximations, where $\eta_h(u_{i,h})$ $(i=1,2,\cdots,5)$ denote the i-th posterior error estimator and $\lambda_{i,h}$ $(i=1,2,\cdots,5)$ denote the i-th eigenvalue approximation of λ_i $(i=1,2,\cdots,5)$

7 Concluding remarks

In this paper, a type of multilevel method for the Steklov eigenvalue problem based on Newton iteration is proposed. With this iteration method, solving the Steklov eigenvalue problem on the finest finite element space can be substituted by solving a small-scale Steklov eigenvalue problem in the coarsest space and solving a sequence of augmented linear problems in the corresponding sequence of finite element spaces, derived by Newton iteration step. We use the current approximate solution as the start solution of the next level and the quadratic convergence rate of Newton's method ensures the accuracy of the numerical solution. Then, the proposed scheme improves the overall efficiency of Steklov eigenvalue problem solving by the finite element method. We prove that our multilevel method obtains an optimal convergence rate with linear complexity. Some numerical examples express the efficiency of this iteration method. What's more, this type of multilevel iteration method works well with the adaptive finite element method for multi eigenvalues, which also be tested in numerical examples.

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Declarations

Conflict of interest The authors declare no competing interests.

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