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Computable Error Estimates for a Nonsymmetric Eigenvalue Problem

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Abstract. We provide some computable error estimates in solving a nonsymmetric eigenvalue problem by general conforming finite element methods on general meshes. Based on the complementary method, we first give computable error estimates for both the original eigenfunctions and the corresponding adjoint eigenfunctions, and then we introduce a generalised Rayleigh quotient to deduce a computable error estimate for the eigenvalue approximations. Some numerical examples are presented to illustrate our theoretical results.

AMS subject classifications: 65N30, 65N25, 65L15, 65B99

Key words: Nonsymmetric eigenvalue problem, computable error estimates, asymptotical exactness, finite element method, complementary method.

1. Introduction

The numerical solution of the nonsymmetric eigenvalue problem we discuss here is important in scientific and engineering computation — e.g. convection-diffusion problems in fluid mechanics and environmental applications [9, 12, 13]. Classical *a priori* error estimates only give the asymptotic convergence order in the standard Galerkin finite element method for the nonsymmetric eigenvalue problem [4], but *a posteriori* error estimates are of great importance for the adaptive finite element method in particular. More discussion of *a posteriori* error estimates can be found in Refs. [2, 5–7, 10, 12, 13, 15, 16] and other references therein.

Here we consider computable *a posteriori* error estimates for the eigenpair approximation of the nonsymmetric eigenvalue problem, solved by the conforming finite element method on general meshes. Our approach is based on the complementary energy

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method [11, 15–18]. Recently, the complementary energy method has been applied to derive the *a posteriori* error estimates for symmetric eigenvalue problems [21] and nonlinear eigenvalue problems [20]. It is well known that the nonsymmetric eigenvalue problem is always associated with an adjoint eigenvalue problem. Using the complementary energy method, we first derive asymptotic upper bounds for the error estimates of the original eigenfunction approximation and the adjoint eigenfunction approximation. Based on the *a posteriori* error estimates for the eigenfunction approximations and a generalised Rayleigh quotient, we then obtain asymptotic upper bounds for the error estimates of the eigenvalues by the conforming finite element method. This means we can provide a computable range of eigenvalues in the complex plane. Furthermore, the error estimates proposed here have both efficiency and reliability properties, which is necessary for the *a posteriori* error estimator.

The finite element method and corresponding error estimates for the nonsymmetric eigenvalue problem are given in Section 2. Asymptotic upper-bound computable error estimates of the original eigenfunction approximation and the adjoint eigenfunction approximation are proposed in Section 3. Based on the results in Section 3, in Section 4 we provide an upper bound for the error estimate of the eigenvalue approximations of the nonsymmetric eigenvalue problem. Some numerical examples are presented in Section 5 to illustrate the theoretical analysis, and our concluding remarks are made in Section 6.

2. Finite Element Method

We use the standard notation $W^{s,p}(\Omega)$ for Sobolev spaces, and $\|\cdot\|_{s,p,\Omega}$ and $|\cdot|_{s,p,\Omega}$ for their associated norms and seminorms, respectively — e.g. see Ref. [1]. For $p=2$, we denote $H^s(\Omega)=W^{s,2}(\Omega)$ and $H_0^1(\Omega)=\{v\in H^1(\Omega):\ v|_{\partial\Omega}=0\}$, where $v|_{\partial\Omega}=0$ is in the sense of trace, and $\|\cdot\|_{s,\Omega}=\|\cdot\|_{s,2,\Omega}$. Here we consider the complex Hilbert space $H_0^1(\Omega)$, and abbreviate $\|\cdot\|_{s,\Omega}$ as $\|\cdot\|_s$.

2.1. Nonsymmetric eigenvalue problem

For simplicity, we choose to consider the following nonsymmetric eigenvalue problem: Find $\lambda\in\mathscr{C}$ and u such that

$$\begin{cases} -\boldsymbol{\Delta} u+\mathbf{b}\cdot\boldsymbol{\nabla} u+u=\lambda u & \text{in } \Omega\,, \\ u=0 \quad \text{on } \partial\Omega\,, \end{cases} \tag{2.1}$$

where $\Omega\subset\mathscr{R}^d$ $(d=2,3)$ is a bounded polygonal domain with boundary $\partial\Omega$, $\boldsymbol{\Delta}$ and $\boldsymbol{\nabla}$ respectively denote the Laplacian and gradient operator, and $\mathbf{b}=\mathbf{b}(\mathbf{x})\in(W^{1,\infty}(\Omega))^d$ is a bounded real or complex vector function on Ω.

To address the finite element discretisation, we invoke the following variational form for the problem (2.1): Find $(\lambda,u)\in\mathscr{C}\times V$ such that

$$a(u,v)=\lambda(u,v)\,,\quad \forall\, v\in V\,, \tag{2.2}$$

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where $V := H_0^1(\Omega)$ and

$$a(u,v) := (\nabla u, \nabla v) + (\mathbf{b}\cdot\nabla u, v) + (u,v),$$
$$(u,v) := \int_\Omega u\overline{v}\, d\Omega,$$

with $\overline{v}$ denoting the complex conjugate of v. From the above definitions, it is evident that the inner products $a(u,v)$ and (u,v) are linear for the first variable and conjugate linear for the second variable, and we note that $(\cdot,\cdot)$ is the standard $L^2(\Omega)$ inner product.

For the nonsymmetric eigenvalue problem (2.2), there exists the following corresponding adjoint eigenvalue problem [4]. Find $\lambda^* \in \mathscr{C}$ and u^* such that

$$\begin{cases} -\Delta u^* - \nabla\cdot(\bar{\mathbf{b}}u^*) + u^* = \lambda^* u^* & \text{in } \Omega, \\ u^* = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Here, (2.1) and (2.3) connect with each other according to $\lambda^* = \bar{\lambda}$. Using the unified notation, we have the following variational form for the problem (2.3). Find $(\lambda, u^*) \in \mathscr{C}\times V$ such that

$$a(v,u^*) = (v, \lambda^* u^*) = \lambda(v,u^*), \quad \forall\, v \in V. \tag{2.4}$$

The conjugate bilinear form $a(\cdot,\cdot)$ is assumed to satisfy [22]

$$\|w\|_1 \lesssim \sup_{v\in V}\frac{|a(w,v)|}{\|v\|_1} \text{ and } \|w\|_1 \lesssim \sup_{v\in V}\frac{|a(v,w)|}{\|v\|_1}, \quad \forall\, w \in V; \tag{2.5}$$

and we suppose $a(\cdot,\cdot)$ is V-elliptic — i.e.

$$\|v\|_1^2 \lesssim \text{Re } a(v,v), \quad \forall\, v \in V, \tag{2.6}$$

where Re denotes the real part of a complex number.

For simplicity, we only consider nondefective eigenvalues (with ascent equal to 1) of the nonsymmetric eigenvalue problem. Thus the algebraic multiplicity equals the geometric multiplicity and the generalised eigenspace is the same as the eigenspace. More detail on nonsymmetric eigenvalue problems may be found in Ref. [4].

2.2. Finite element method

We now demonstrate the finite element method for the nonsymmetric eigenvalue problem (2.2) and its corresponding adjoint problem (2.4) — cf. Refs. [4,7,8]. The computing domain $\Omega \subset \mathscr{R}^d$ $(d = 2,3)$ is decomposed into shape-regular triangles or rectangles for $d = 2$ and tetrahedrons or hexahedrons for $d = 3$. The diameter of a cell $K \in \mathscr{T}_h$ is denoted by h_K, with the mesh diameter h describing the maximum diameter of all such cells. Based on the mesh $\mathscr{T}_h$, the conforming finite element space denoted by $V_h \subset V$ is then constructed. For simplicity, we only consider the Lagrange-type conforming finite element space

$$V_h = \left\{ v_h \in C(\bar{\Omega}) \;\middle|\; v_h|_K \in \mathscr{P}_k, \quad \forall K \in \mathscr{T}_h \right\} \cap H_0^1(\Omega), \tag{2.7}$$

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where $\mathscr{P}_k$ denotes the space of polynomials of degree at most k. Here we also assume that the finite element space V_h satisfies the following conditions corresponding to the inequalities (2.5):

$$\|w_h\|_1 \lesssim \sup_{v_h\in V_h}\frac{|a(w_h,v_h)|}{\|v_h\|_1} \quad \text{and} \quad \|w_h\|_1 \lesssim \sup_{v_h\in V_h}\frac{|a(v_h,w_h)|}{\|v_h\|_1}, \quad \forall\, w_h \in V_h\,. \tag{2.8}$$

The standard finite element method for (2.2) is to solve the following eigenvalue problem: Find $(\lambda_h,u_h)\in \mathscr{C}\times V_h$ such that

$$a(u_h,v_h)=\lambda_h(u_h,v_h), \quad \forall\, v_h\in V_h\,. \tag{2.9}$$

The discretisation of the adjoint problem (2.4) is taken in the same finite element space: Find $(\lambda_h,u_h^*)\in \mathscr{C}\times V_h$ such that

$$a(v_h,u_h^*)=\lambda_h(v_h,u_h^*), \quad \forall\, v_h\in V_h\,. \tag{2.10}$$

Assume λ is an eigenvalue of the variational forms (2.2) and (2.4) with multiplicity m. According to the spectral theories of compact operators, there exist m eigenvalues $\lambda_{1,h},\cdots,\lambda_{m,h}$ of (2.9) and (2.10) converging to λ, and we denote the respective corresponding eigenvectors by $u_{1,h},\cdots,u_{m,h}$ and $u_{1,h}^*,\cdots,u_{m,h}^*$. The two eigenspaces corresponding to the eigenvalue λ of (2.2) and (2.4) are respectively:

$$M(\lambda)=\big\{u\in V: u \text{ is an eigenfunction of (2.2) corresponding to } \lambda\big\}, \tag{2.11}$$
$$M^*(\lambda)=\big\{u^*\in V: u^* \text{ is an eigenfunction of (2.4)corresponding to } \lambda\big\}. \tag{2.12}$$

We also define

$$M_h(\lambda)=\operatorname{span}\big\{u_{1,h},\cdots,u_{m,h}\big\}, \tag{2.13}$$
$$M_h^*(\lambda)=\operatorname{span}\big\{u_{1,h}^*,\cdots,u_{m,h}^*\big\}. \tag{2.14}$$

For two linear subspaces A and B of V, we denote

$$\widehat{\Theta}(A,B)=\sup_{w\in A,\|w\|_1=1}\inf_{v\in B}\|w-v\|_1, \qquad \widehat{\Phi}(A,B)=\sup_{w\in A,\|w\|_0=1}\inf_{v\in B}\|w-v\|_0;$$

and define the gaps between A and B in $\|\cdot\|_1$ as

$$\Theta(A,B)=\max\big\{\widehat{\Theta}(A,B),\widehat{\Theta}(B,A)\big\}, \tag{2.15}$$

and in $\|\cdot\|_0$ as

$$\Phi(A,B)=\max\big\{\widehat{\Phi}(A,B),\widehat{\Phi}(B,A)\big\}. \tag{2.16}$$

Then we introduce the following notation for error estimation:

$$\delta_h(\lambda) := \sup_{u\in M(\lambda),\|u\|_0=1} \inf_{v_h\in V_h} \|u-v_h\|_1\,, \tag{2.17}$$

$$\delta_h^*(\lambda) := \sup_{u^*\in M^*(\lambda),\|u^*\|_0=1} \inf_{v_h\in V_h} \|u^*-v_h\|_1\,, \tag{2.18}$$

$$\eta_a(h) := \sup_{f\in V,\|f\|_0=1} \inf_{v_h\in V_h} \|Tf-v_h\|_1\,, \tag{2.19}$$

$$\eta_a^*(h) := \sup_{f\in V,\|f\|_0=1} \inf_{v_h\in V_h} \|T_*f-v_h\|_1\,, \tag{2.20}$$

where the operators T and $T_* \in \mathscr{L}(V)$ are defined as

$$a(Tu,v)=(u,v)=a(u,T_*v)\,,\quad \forall\, u,v\in V\,. \tag{2.21}$$

Noting that the ascent of the nonsymmetric eigenvalue problem considered is equal to 1, we have the error estimates in the following theorem.

Theorem 2.1 (cf. Refs. [4, Section 8], [22]). *When the mesh size h is small enough, we have the following error estimates:*

$$\Theta\big(M(\lambda),M_h(\lambda)\big) \le C_\lambda\delta_h(\lambda)\,, \tag{2.22}$$

$$\Theta\big(M^*(\lambda),M_h^*(\lambda)\big) \le C_\lambda\delta_h^*(\lambda)\,, \tag{2.23}$$

$$\Phi\big(M(\lambda),M_h(\lambda)\big) \le C_\lambda\eta_a^*(h)\delta_h(\lambda)\,, \tag{2.24}$$

$$\Phi\big(M^*(\lambda),M_h^*(\lambda)\big) \le C_\lambda\eta_a(h)\delta_h^*(\lambda)\,, \tag{2.25}$$

$$|\lambda-\lambda_{i,h}| \le C_\lambda\delta_h(\lambda)\delta_h^*(\lambda)\,,\quad i=1,\cdots,m\,, \tag{2.26}$$

where $\lambda_{1,h},\cdots,\lambda_{m,h}$ are the eigenvalue approximations converging to λ. Here and hereafter C_λ denotes some constant depending on the eigenvalue λ but independent of the mesh size h.

3. Complementarity-based Error Estimate

We now derive computable error estimates for the eigenfunction approximations of the nonsymmetric eigenvalue problem and its corresponding adjoint problem, based on the complementary method. For simplicity, (λ,u) and (λ,u^*) hereafter denote the solution of the variational problems (2.2) and (2.4), respectively. Let (λ_h,u_h) and (λ_h,u_h^*) be the corresponding finite element eigenpair approximations in $\mathscr{C}\times V_h$. We first recall the following theorem.

Lemma 3.1. *Let $\Omega\subset\mathscr{R}^d$ $(d=2,3)$ be a bounded Lipschitz domain with unit outward normal ν to $\partial\Omega$. Then the following Green's formula holds:*

$$(v,\nabla\cdot\mathbf{p})+(\nabla v,\mathbf{p})=(v,\mathbf{p}\cdot\nu)_{\partial\Omega}\,,\quad \forall\, v\in H^1(\Omega) \text{ and } \forall\,\mathbf{p}\in\mathbf{W}\,, \tag{3.1}$$

where $(\cdot,\cdot)_{\partial\Omega}$ is the $L^2(\partial\Omega)$ inner product on the boundary $\partial\Omega$ and $\mathbf{W}:=H(\mathrm{div};\Omega)=\{\mathbf{p}\in(L^2(\Omega))^d:\nabla\cdot\mathbf{p}\in L^2(\Omega)\}$. In particular,

$$(v,\nabla\cdot\mathbf{p})+(\nabla v,\mathbf{p})=0\,,\quad \forall\, v\in V \text{ and } \forall\,\mathbf{p}\in\mathbf{W}\,. \tag{3.2}$$

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Theorem 3.1. *Assume that the mesh size h is small enough, and that we have the finite element eigenpair approximations* $(\lambda_h, u_h) \in \mathscr{C} \times V_h$ *and* $(\lambda_h, u_h^*) \in \mathscr{C} \times V_h$ *corresponding to the exact eigenvalue* λ*. Then there exist exact eigenfunctions* $u \in M(\lambda)$ *and* $u^* \in M^*(\lambda)$ *such that the following inequalities hold:*

$$\|u - u_h\|_1 \le \min_{\mathbf{p}\in\mathbf{W}} \frac{1}{1-\alpha\eta_a^*(h)} \eta(\lambda_h, u_h, \mathbf{p}), \tag{3.3}$$

$$\|u^* - u_h^*\|_1 \le \min_{\mathbf{p}\in\mathbf{W}} \frac{1}{1-\alpha^*\eta_a(h)} \eta^*(\lambda_h, u_h^*, \mathbf{p}), \tag{3.4}$$

where

$$\alpha = C_\lambda^2 \|u\|_0 \delta_h^*(\lambda) + C_\lambda^2 |\lambda_h| \eta_a^*(h) + C_\lambda \|\mathbf{b}\|_{L^\infty(\Omega)},$$

$$\alpha^* = C_\lambda^2 \|u^*\|_0 \delta_h(\lambda) + C_\lambda^2 |\lambda_h| \eta_a(h) + C_\lambda \|\mathbf{b}\|_{L^\infty(\Omega)},$$

and the error estimators $\eta(\lambda_h, u_h, \mathbf{p})$ *and* $\eta^*(\lambda_h, u_h^*, \mathbf{p})$ *are defined as*

$$\eta(\lambda_h, u_h, \mathbf{p}) = \left(\|\lambda_h u_h - \mathbf{b}\cdot\nabla u_h - u_h + \nabla\cdot\mathbf{p}\|_0^2 + \|\mathbf{p} - \nabla u_h\|_0^2\right)^{1/2}, \tag{3.5}$$

$$\eta^*(\lambda_h, u_h^*, \mathbf{p}) = \left(\|\bar{\lambda}_h u_h^* + \nabla\cdot(\bar{\mathbf{b}} u_h^*) - u_h^* + \nabla\cdot\mathbf{p}\|_0^2 + \|\mathbf{p} - \nabla u_h^*\|_0^2\right)^{1/2}. \tag{3.6}$$

Proof. From inequality (2.24) in Theorem 2.1, for any $u_h \in V_h$ we can choose $u \in M(\lambda)$ such that $\|u - u_h\|_0 \le C_\lambda \eta_a^*(h) \|u - u_h\|_1$. Setting $w = u - u_h \in V$ and combining (2.2), (2.9) and (3.2), for any $\mathbf{p} \in \mathbf{W}$ we have

$$\begin{aligned}
\|u - u_h\|_1^2 =& \big(\nabla(u - u_h), \nabla w\big) + (u - u_h, w) \\
=& a(u, w) - (\mathbf{b}\cdot\nabla u, w) - (\nabla u_h, \nabla w) - (u_h, w) \\
=& \lambda(u, w) - (\mathbf{b}\cdot\nabla u, w) - (\nabla u_h, \nabla w) - (u_h, w) + (\nabla\cdot\mathbf{p}, w) + (\mathbf{p}, \nabla w) \\
=& (\lambda_h u_h - \mathbf{b}\cdot\nabla u_h - u_h + \nabla\cdot\mathbf{p}, w) + (\mathbf{p} - \nabla u_h, \nabla w) \\
&+ \big(\lambda u - \lambda_h u_h - \mathbf{b}\cdot\nabla(u - u_h), w\big) \\
\le& \|\lambda_h u_h - \mathbf{b}\cdot\nabla u_h - u_h + \nabla\cdot\mathbf{p}\|_0 \|w\|_0 + \|\mathbf{p} - \nabla u_h\|_0 \|\nabla w\|_0 \\
&+ \big(\|\lambda u - \lambda_h u_h\|_0 + \|\mathbf{b}\cdot\nabla(u - u_h)\|_0\big)\|w\|_0 \\
\le& \big(\|\lambda_h u_h - \mathbf{b}\cdot\nabla u_h - u_h + \nabla\cdot\mathbf{p}\|_0^2 + \|\mathbf{p} - \nabla u_h\|_0^2\big)^{1/2} \|w\|_1 \\
&+ \big(|\lambda - \lambda_h| \|u\|_0 + |\lambda_h| \|u - u_h\|_0 + \|\mathbf{b}\|_{L^\infty(\Omega)} \|\nabla(u - u_h)\|_0\big)\|w\|_0 .
\end{aligned} \tag{3.7}$$

Combining (2.24), (2.26) in Theorem 2.1 and (3.7) leads to the estimate

$$\begin{aligned}
\|u - u_h\|_1^2 \le& \eta(\lambda_h, u_h, \mathbf{p}) \|w\|_1 + \big(C_\lambda \delta_h^*(\lambda) \|u - u_h\|_1 \|u\|_0 \\
&+ C_\lambda |\lambda_h| \eta_a^*(h) \|u - u_h\|_1 + \|\mathbf{b}\|_{L^\infty} \|u - u_h\|_1\big) C_\lambda \eta_a^*(h) \|w\|_1 \\
=& \eta(\lambda_h, u_h, \mathbf{p}) \|w\|_1 + \alpha \eta_a^*(h) \|u - u_h\|_1 \|w\|_1 .
\end{aligned}$$

The desired result (3.3) immediately follows from the arbitrariness of $\mathbf{p} \in \mathbf{W}$ and $\eta_a^*(h) \to 0$ as $h \to 0$.

Similarly, Eq. (3.4) can be proved with the complementary approach. For $u_h^* \in V_h$ we can choose $u^* \in M^*(\lambda)$ satisfying $\|u^* - u_h^*\|_0 \le C_\lambda \eta_a^*(h)\|u^* - u_h^*\|_1$. Then setting $w = u^* - u_h^* \in V$ and using (2.4), (2.10) and (3.2), for any $\mathbf{p} \in \mathbf{W}$ we have

$$
\begin{aligned}
\|u^* - u_h^*\|_1^2 =& \big(\nabla w, \nabla(u^* - u_h^*)\big) + (w, u^* - u_h^*) \\
=& a(w, u^*) - (\mathbf{b}\cdot\nabla w, u^*) - (\nabla w, \nabla u_h^*) - (w, u_h^*) \\
=& \lambda(w, u^*) - (\mathbf{b}\cdot\nabla w, u^*) - (\nabla w, \nabla u_h^*) - (w, u_h^*) + (w, \nabla\cdot\mathbf{p}) + (\nabla w, \mathbf{p}) \\
=& (w, \bar\lambda u^*) + \big(w, \nabla\cdot(\bar{\mathbf{b}}u^*)\big) - (\nabla w, \nabla u_h^*) - (w, u_h^*) + (w, \nabla\cdot\mathbf{p}) + (\nabla w, \mathbf{p}) \\
=& \big(w, \bar\lambda_h u_h^* + \nabla\cdot(\bar{\mathbf{b}}u_h^*) - u_h^* + \nabla\cdot\mathbf{p}\big) + (\nabla w, \mathbf{p} - \nabla u_h^*) \\
&+ \Big(w, \bar\lambda u^* - \bar\lambda_h u_h^* + \nabla\cdot\big(\bar{\mathbf{b}}(u^* - u_h^*)\big)\Big) \\
=& \big(w, \bar\lambda_h u_h^* + \nabla\cdot(\bar{\mathbf{b}}u_h^*) - u_h^* + \nabla\cdot\mathbf{p}\big) + (\nabla w, \mathbf{p} - \nabla u_h^*) \\
&+ \big(w, \bar\lambda u^* - \bar\lambda_h u_h^*\big) - \big(\nabla w, \bar{\mathbf{b}}(u^* - u_h^*)\big) \\
\le& \big\|\bar\lambda_h u_h^* + \nabla\cdot(\bar{\mathbf{b}}u_h^*) - u_h^* + \nabla\cdot\mathbf{p}\big\|_0 \|w\|_0 + \|\mathbf{p} - \nabla u_h^*\|_0 \|\nabla w\|_0 \\
&+ \|\bar\lambda u^* - \bar\lambda_h u_h^*\|_0 \|w\|_0 + \big\|\bar{\mathbf{b}}(u^* - u_h^*)\big\|_0 \|\nabla w\|_0 \\
\le& \Big(\big\|\bar\lambda_h u_h^* + \nabla\cdot(\bar{\mathbf{b}}u_h^*) - u_h^* + \nabla\cdot\mathbf{p}\big\|_0^2 + \|\mathbf{p} - \nabla u_h^*\|_0^2\Big)^{1/2} \|w\|_1 \\
&+ \big(|\lambda - \lambda_h|\|u^*\|_0 + |\lambda_h|\|u^* - u_h^*\|_0\big)\|w\|_0 + \|\mathbf{b}\|_{L^\infty(\Omega)}\|u^* \\
&- u_h^*\|_0\|w\|_1 \, .
\end{aligned} \tag{3.8}
$$

From inequalities (2.25) and (2.26) in Theorem 2.1, we therefore have

$$
\begin{aligned}
\|u^* - u_h^*\|_1^2 \le& \eta^*(\lambda_h, u_h^*, \mathbf{p})\|w\|_1 + \big(C_\lambda \delta_h(\lambda)\|u^* - u_h^*\|_1 \|u^*\|_0 + C_\lambda |\lambda_h| \eta_a(h)\|u^* - u_h^*\|_1 \\
&+ \|\mathbf{b}\|_{L^\infty(\Omega)}\|u^* - u_h^*\|_1\big) C_\lambda \eta_a(h)\|w\|_1 \\
=& \eta^*(\lambda_h, u_h^*, \mathbf{p})\|w\|_1 + \alpha^* \eta_a(h)\|u^* - u_h^*\|_1 \|w\|_1 \, ,
\end{aligned}
$$

so the second result (3.4) follows from the arbitrariness of $\mathbf{p} \in \mathbf{W}$ and the fact $\eta_a(h) \to 0$ as $h \to 0$. □

The consequent natural problems are to seek the minimisation of $\eta(\lambda_h, u_h, \mathbf{p})$ over $\mathbf{W}$ for the given eigenpair approximation (λ_h, u_h), and the minimisation of $\eta^*(\lambda_h, u_h^*, \mathbf{p})$ over $\mathbf{W}$ for the given eigenpair approximation (λ_h, u_h^*). We define the following two optimisation problems:

$$
\text{Find } \hat{\mathbf{p}} \in \mathbf{W} \text{ such that } \eta(\lambda_h, u_h, \hat{\mathbf{p}}) = \min_{\mathbf{p}\in\mathbf{W}} \eta(\lambda_h, u_h, \mathbf{p}); \quad \text{and} \tag{3.9}
$$

$$
\text{Find } \hat{\mathbf{p}}^* \in \mathbf{W} \text{ such that } \eta^*(\lambda_h, u_h^*, \hat{\mathbf{p}}^*) = \min_{\mathbf{p}\in\mathbf{W}} \eta^*(\lambda_h, u_h^*, \mathbf{p}). \tag{3.10}
$$

The following lemma shows that these two optimisation problems are equivalent to problems involving certain partial differential equations.

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Lemma 3.2. *The optimisation problems* (3.9) *and* (3.10) *are respectively equivalent to the following problems:*

$$\textit{Find } \hat{\mathbf{p}} \in \mathbf{W} \textit{ such that } \hat{a}(\hat{\mathbf{p}},\mathbf{q}) = \mathscr{F}(\mathbf{q}), \quad \forall\, \mathbf{q} \in \mathbf{W}; \quad \textit{and} \tag{3.11}$$

$$\textit{Find } \hat{\mathbf{p}}^* \in \mathbf{W} \textit{ such that } \hat{a}(\hat{\mathbf{p}}^*,\mathbf{q}) = \mathscr{F}^*(\mathbf{q}), \quad \forall\, \mathbf{q} \in \mathbf{W}, \tag{3.12}$$

where

$$\begin{aligned}
&\hat{a}(\mathbf{p},\mathbf{q}) = (\nabla\cdot\mathbf{p}, \nabla\cdot\mathbf{q}) + (\mathbf{p},\mathbf{q}),\\
&\mathscr{F}(\mathbf{q}) = (-\lambda_h u_h + \mathbf{b}\cdot\nabla u_h, \nabla\cdot\mathbf{q}),\\
&\mathscr{F}^*(\mathbf{q}) = \big(-\bar{\lambda}_h u_h^* - \nabla\cdot(\bar{\mathbf{b}} u_h^*), \nabla\cdot\mathbf{q}\big).
\end{aligned}$$

Proof. We first prove that the optimisation problems (3.9) and (3.11) are equivalent. Let $\hat{\mathbf{p}} \in \mathbf{W}$ solve the optimisation problem (3.9) and $\mathbf{q} \in \mathbf{W}$ be arbitrary. Since the function $J(t) := \eta^2(\lambda_h, u_h, \hat{\mathbf{p}} + t\mathbf{q})$ has a minimum at $t = 0$, we have in particular

$$J'(0) = \lim_{t\to 0} \frac{J(t) - J(0)}{t - 0} = \lim_{t\to 0} \frac{1}{t}\big(\eta^2(\lambda_h, u_h, \hat{\mathbf{p}} + t\mathbf{q}) - \eta^2(\lambda_h, u_h, \hat{\mathbf{p}})\big) = 0\,. \tag{3.13}$$

According to Lemma 3.1 and the definition of the estimator (3.5),

$$\begin{aligned}
&\eta^2(\lambda_h, u_h, \hat{\mathbf{p}} + t\mathbf{q}) - \eta^2(\lambda_h, u_h, \hat{\mathbf{p}})\\
=&\big\|\lambda_h u_h - \mathbf{b}\cdot\nabla u_h - u_h + \nabla\cdot(\hat{\mathbf{p}} + t\mathbf{q})\big\|_0^2 + \big\|(\hat{\mathbf{p}} + t\mathbf{q}) - \nabla u_h\big\|_0^2\\
&- \|\lambda_h u_h - \mathbf{b}\cdot\nabla u_h - u_h + \nabla\cdot\hat{\mathbf{p}}\|_0^2 - \|\hat{\mathbf{p}} - \nabla u_h\|_0^2\\
=&t^2\big((\nabla\cdot\mathbf{q}, \nabla\cdot\mathbf{q}) + (\mathbf{q},\mathbf{q})\big) + t\big((\lambda_h u_h - \mathbf{b}\cdot\nabla u_h - u_h + \nabla\cdot\hat{\mathbf{p}}, \nabla\cdot\mathbf{q})\\
&+ (\nabla\cdot\mathbf{q}, \lambda_h u_h - \mathbf{b}\cdot\nabla u_h - u_h + \nabla\cdot\hat{\mathbf{p}}) + (\hat{\mathbf{p}} - \nabla u_h, \mathbf{q}) + (\mathbf{q}, \hat{\mathbf{p}} - \nabla u_h)\big)\\
=&t^2\big((\nabla\cdot\mathbf{q}, \nabla\cdot\mathbf{q}) + (\mathbf{q},\mathbf{q})\big) + 2t\ \mathrm{Re}\big((\lambda_h u_h - \mathbf{b}\cdot\nabla u_h + \nabla\cdot\hat{\mathbf{p}}, \nabla\cdot\mathbf{q}) + (\hat{\mathbf{p}},\mathbf{q})\big).
\end{aligned}$$

Then Eq. (3.13) may be rewritten

$$J'(0) = 2\mathrm{Re}\big((\lambda_h u_h - \mathbf{b}\cdot\nabla u_h, \nabla\cdot\mathbf{q}) + (\nabla\cdot\hat{\mathbf{p}}, \nabla\cdot\mathbf{q}) + (\hat{\mathbf{p}},\mathbf{q})\big) = 0\,. \tag{3.14}$$

We can replace $\mathbf{q}$ in Eq. (3.14) with $\mathbf{iq}$ to obtain

$$\mathrm{Im}\big((\lambda_h u_h - \mathbf{b}\cdot\nabla u_h, \nabla\cdot\mathbf{q}) + (\nabla\cdot\hat{\mathbf{p}}, \nabla\cdot\mathbf{q}) + (\hat{\mathbf{p}},\mathbf{q})\big) = 0\,, \tag{3.15}$$

where Im denotes the imaginary part. Thus from Eqs. (3.14) and (3.15) we have that $\hat{\mathbf{p}}$ satisfies (3.11).

On the other hand, if $\hat{\mathbf{p}} \in \mathbf{W}$ is the solution of the problem (3.11) we can show that $\hat{\mathbf{p}}$ is also the solution of the optimisation problem (3.9). For any $\mathbf{q} \in \mathbf{W}$, using the Green formula (3.2) we have

$$\begin{aligned}
\eta^2(\lambda_h,u_h,\mathbf{q}) =&\eta^2\big(\lambda_h,u_h,\hat{\mathbf{p}}+(\mathbf{q}-\hat{\mathbf{p}})\big)\\
=&\Big\|\lambda_h u_h-\mathbf{b}\cdot\nabla u_h-u_h+\nabla\cdot\big(\hat{\mathbf{p}}+(\mathbf{q}-\hat{\mathbf{p}})\big)\Big\|_0^2+\Big\|\big(\hat{\mathbf{p}}+(\mathbf{q}-\hat{\mathbf{p}})\big)-\nabla u_h\Big\|_0^2\\
=&\|\lambda_h u_h-\mathbf{b}\cdot\nabla u_h-u_h+\nabla\cdot\hat{\mathbf{p}}\|_0^2+\|\hat{\mathbf{p}}-\nabla u_h\|_0^2\\
&+\big(\nabla\cdot(\mathbf{q}-\hat{\mathbf{p}}),\nabla\cdot(\mathbf{q}-\hat{\mathbf{p}})\big)+(\mathbf{q}-\hat{\mathbf{p}},\mathbf{q}-\hat{\mathbf{p}})\\
&+\ 2\mathrm{Re}\Big(\big(\lambda_h u_h-\mathbf{b}\cdot\nabla u_h-u_h+\nabla\cdot\hat{\mathbf{p}},\nabla\cdot(\mathbf{q}-\hat{\mathbf{p}})\big)+(\hat{\mathbf{p}}-\nabla u_h,\mathbf{q}-\hat{\mathbf{p}})\Big)\\
=&\eta^2(\lambda_h,u_h,\hat{\mathbf{p}})+\big(\nabla\cdot(\mathbf{q}-\hat{\mathbf{p}}),\nabla\cdot(\mathbf{q}-\hat{\mathbf{p}})\big)+(\mathbf{q}-\hat{\mathbf{p}},\mathbf{q}-\hat{\mathbf{p}})\\
&+\ 2\mathrm{Re}\Big(\big(\lambda_h u_h-\mathbf{b}\cdot\nabla u_h+\nabla\cdot\hat{\mathbf{p}},\nabla\cdot(\mathbf{q}-\hat{\mathbf{p}})\big)+(\hat{\mathbf{p}},\mathbf{q}-\hat{\mathbf{p}})\Big)\\
=&\eta^2(\lambda_h,u_h,\hat{\mathbf{p}})+\big(\nabla\cdot(\mathbf{q}-\hat{\mathbf{p}}),\nabla\cdot(\mathbf{q}-\hat{\mathbf{p}})\big)+(\mathbf{q}-\hat{\mathbf{p}},\mathbf{q}-\hat{\mathbf{p}})\\
\geq&\eta^2(\lambda_h,u_h,\hat{\mathbf{p}}),
\end{aligned}\tag{3.16}$$

so the optimisation problem (3.9) follows. Thus we have the equivalence of (3.9) and (3.11), and similarly we can prove that (3.10) and (3.12) are equivalent to complete the proof. □

It is obvious that $\hat{a}(\cdot,\cdot)$ defines the standard inner product in the Hilbert space $\mathbf{W}=H(\mathrm{div};\Omega)$ and induces the standard norm $\|\mathbf{q}\|_{\mathbf{W}}=\sqrt{\hat{a}(\mathbf{q},\mathbf{q})}$ for any $\mathbf{q}\in\mathbf{W}$. From the Riesz representation theorem, we know that both the dual problems (3.11) and (3.12) have unique solutions. The equivalence described in Lemma 3.2 guarantees the well-posedness of the optimisation problems (3.9) and (3.10).

We now state some properties of the estimators $\eta(\lambda_h,u_h,\mathbf{p})$ and $\eta^*(\lambda_h,u_h^*,\mathbf{p})$.

Lemma 3.3. *Assume* $\hat{\mathbf{p}}$ *and* $\hat{\mathbf{p}}^*$ *are the solutions of the problems* (3.11) *and* (3.12)*, respectively. Then for any* $\mathbf{p}\in\mathbf{W}$*, the following equalities hold:*

$$\eta^2(\lambda_h,u_h,\mathbf{p})=\eta^2(\lambda_h,u_h,\hat{\mathbf{p}})+\|\hat{\mathbf{p}}-\mathbf{p}\|_{\mathbf{W}}^2,\tag{3.17}$$

$$\eta^{*2}(\lambda_h,u_h^*,\mathbf{p})=\eta^{*2}(\lambda_h,u_h^*,\hat{\mathbf{p}}^*)+\|\hat{\mathbf{p}}^*-\mathbf{p}\|_{\mathbf{W}}^2.\tag{3.18}$$

Lemma 3.3 can be deduced readily from (3.16) and the definition of the norm $\|\cdot\|_{\mathbf{W}}$. Choosing some certain approximate solutions $\mathbf{p}_h\in\mathbf{W}$ and $\mathbf{p}_h^*\in\mathbf{W}$ of (3.11) and (3.12), respectively, we can give computable asymptotic upper bounds of the error estimates for the eigenfunction approximations u_h and u_h^* as follows.

Corollary 3.1. *There exist eigenfunctions* $u\in M(\lambda)$ *and* $u^*\in M^*(\lambda)$ *such that the error estimators* $\eta(\lambda_h,u_h,\mathbf{p}_h)$ *and* $\eta^*(\lambda_h,u_h^*,\mathbf{p}_h^*)$ *have the asymptotic upper bounds*

$$\|u-u_h\|_1\leq\frac{1}{1-\alpha\eta_a^*(h)}\eta(\lambda_h,u_h,\mathbf{p}_h),\tag{3.19}$$

$$\|u^*-u_h^*\|_1\leq\frac{1}{1-\alpha^*\eta_a(h)}\eta^*(\lambda_h,u_h^*,\mathbf{p}_h^*).\tag{3.20}$$

Now we proceed to discuss the efficiency and reliability of the estimators $\eta(\lambda_h, u_h, \hat{\mathbf{p}})$, $\eta^*(\lambda_h, u_h^*, \hat{\mathbf{p}}^*)$ and $\eta(\lambda_h, u_h, \mathbf{p}_h)$, $\eta^*(\lambda_h, u_h^*, \mathbf{p}_h^*)$.

Theorem 3.2. *Assume $\hat{\mathbf{p}}$ and $\hat{\mathbf{p}}^*$ are the solutions of the optimisation problems* (3.9) *and* (3.10)*, respectively. Under the conditions of Theorem* 3.1*, there exist exact eigenfunctions $u \in M(\lambda)$ and $u^* \in M^*(\lambda)$ satisfying the inequalities*

$$\theta_1 \|u - u_h\|_1 \leq \eta(\lambda_h, u_h, \hat{\mathbf{p}}) \leq \theta_2 \|u - u_h\|_1 , \tag{3.21}$$

$$\theta_1^* \|u^* - u_h^*\|_1 \leq \eta^*(\lambda_h, u_h^*, \hat{\mathbf{p}}^*) \leq \theta_2^* \|u^* - u_h^*\|_1 , \tag{3.22}$$

where

$$\begin{aligned}
&\theta_1 = 1 - \alpha\eta_a^*(h), \quad \theta_1^* = 1 - \alpha^*\eta_a(h), \\
&\theta_2 = \left(3C_\lambda^2 \|u_h\|_0^2 (\delta_h^*(\lambda))^2 + 3C_\lambda^2 |\lambda - 1|^2 (\eta_a^*(h))^2 + 3\|\mathbf{b}\|_{L^\infty(\Omega)}^2 + 1\right)^{1/2}, \\
&\theta_2^* = \left(3C_\lambda^2 \|u_h^*\|_0^2 \delta_h^2(\lambda) + 3C_\lambda^2 |\lambda - 1|^2 \eta_a^2(h) + 3\|\mathbf{b}\|_{W^{1,\infty}(\Omega)}^2 + 1\right)^{1/2}.
\end{aligned}$$

Proof. The left-hand inequalities in (3.21) and (3.22) are respectively the direct consequences of (3.3) and (3.4), so we proceed to consider the right-hand inequalities. For u_h in (3.21), we can choose $u \in M(\lambda)$ such that $\|u - u_h\|_0 \leq C_\lambda \eta_a^*(h) \|u - u_h\|_1$. From (2.2), (3.5) and that $\nabla u \in \mathbf{W}$, we have

$$\begin{aligned}
\eta^2(\lambda_h, u_h, \nabla u) &= \|\lambda_h u_h - \mathbf{b}\cdot\nabla u_h - u_h + \Delta u\|_0^2 + \|\nabla u - \nabla u_h\|_0^2 \\
&= \left\|\lambda_h u_h - \mathbf{b}\cdot\nabla u_h - u_h - (\lambda u - \mathbf{b}\cdot\nabla u - u)\right\|_0^2 + \|\nabla u - \nabla u_h\|_0^2 .
\end{aligned}$$

Combining (3.9) and Theorem 2.1, we obtain

$$\begin{aligned}
\eta^2(\lambda_h, u_h, \hat{\mathbf{p}}) \leq& \eta^2(\lambda_h, u_h, \nabla u) \\
=& \left\|\lambda_h u_h - \mathbf{b}\cdot\nabla u_h - u_h - (\lambda u - \mathbf{b}\cdot\nabla u - u)\right\|_0^2 + \|\nabla u - \nabla u_h\|_0^2 \\
=& \left\|(\lambda_h - \lambda)u_h + (\lambda - 1)(u_h - u) + \mathbf{b}\cdot\nabla(u - u_h)\right\|_0^2 + \|\nabla(u - u_h)\|_0^2 \\
\leq& 3|\lambda_h - \lambda|^2 \|u_h\|_0^2 + 3|\lambda - 1|^2 \|u - u_h\|_0^2 + \left(3\|\mathbf{b}\|_{L^\infty(\Omega)}^2 + 1\right)\left\|\nabla(u - u_h)\right\|_0^2 \\
\leq& \theta_2^2 \|u - u_h\|_1^2 ,
\end{aligned}$$

which is the right-hand inequality in (3.21). For the right-hand inequality in (3.22), we choose $u^* \in M^*(\lambda)$ such that $\|u^* - u_h^*\|_0 \leq C_\lambda \eta_a(h) \|u^* - u_h^*\|_1$. Then using (2.4), (3.6) and that $\nabla u^* \in \mathbf{W}$ we have

$$\begin{aligned}
\left(\eta^*(\lambda_h, u_h^*, \nabla u^*)\right)^2 =& \left\|\bar{\lambda}_h u_h^* + \nabla\cdot(\bar{\mathbf{b}} u_h^*) - u_h^* + \Delta u^*\right\|_0^2 + \|\nabla u^* - \nabla u_h^*\|_0^2 \\
=& \left\|\bar{\lambda}_h u_h^* + \nabla\cdot(\bar{\mathbf{b}} u_h^*) - u_h^* + u^* - \nabla\cdot(\bar{\mathbf{b}} u^*) - \bar{\lambda} u^*\right\|_0^2 \\
&+ \left\|\nabla(u^* - u_h^*)\right\|_0^2 ,
\end{aligned}$$

on noting that u^* satisfies (2.3). Then combining (3.10) and Theorem 2.1, we obtain

$$\begin{aligned}\big(\eta^*(\lambda_h,u_h^*,\hat{\mathbf{p}}^*)\big)^2 \le&\big(\eta^*(\lambda_h,u_h^*,\nabla u^*)\big)^2\\ =&\left\|\bar{\lambda}_h u_h^* + \nabla\cdot(\bar{\mathbf{b}}u_h^*) - u_h^* + u^* - \nabla\cdot(\bar{\mathbf{b}}u^*) - \bar{\lambda}u^*\right\|_0^2 + \left\|\nabla(u^*-u_h^*)\right\|_0^2\\ =&\left\|(\bar{\lambda}_h-\bar{\lambda})u_h^* + (\bar{\lambda}-1)(u_h^*-u^*) + \nabla\cdot(\bar{\mathbf{b}}(u_h^*-u^*))\right\|_0^2\\ &+\left\|\nabla(u^*-u_h^*)\right\|_0^2\\ \le&3|\lambda_h-\lambda|^2\|u_h^*\|_0^2 + 3|\lambda-1|^2\|u^*-u_h^*\|_0^2\\ &+\big(3\|\mathbf{b}\|^2_{W^{1,\infty}(\Omega)}+1\big)\|u^*-u_h^*\|_1^2\\ \le&(\theta_2^*)^2\|u^*-u_h^*\|_1^2,\end{aligned}$$

which is the right-hand inequality in (3.22). □

Corollary 3.2. *Assume the conditions of Theorem* 3.2 *hold and two constants* $\gamma_1 > 0$ *and* $\gamma_1^* > 0$ *exist such that* $\|\hat{\mathbf{p}}-\mathbf{p}_h\|_{\mathbf{W}} \le \gamma_1\|u-u_h\|_1$ *and* $\|\hat{\mathbf{p}}^*-\mathbf{p}_h^*\|_{\mathbf{W}} \le \gamma_1^*\|u^*-u_h^*\|_1$. *Then the following inequalities hold:*

$$\eta(\lambda_h,u_h,\mathbf{p}_h) \le \gamma_2\|u-u_h\|_1, \tag{3.23}$$

$$\eta^*(\lambda_h,u_h^*,\mathbf{p}_h^*) \le \gamma_2^*\|u^*-u_h^*\|_1, \tag{3.24}$$

where $\gamma_2 = \sqrt{\theta_2^2+\gamma_1^2}$ *and* $\gamma_2^* = \sqrt{(\theta_2^*)^2+(\gamma_1^*)^2}$.

Proof. From Eq. (3.17) and inequality (3.21) we have

$$\begin{aligned}\eta^2(\lambda_h,u_h,\mathbf{p}_h) &= \eta^2(\lambda_h,u_h,\hat{\mathbf{p}}) + \|\hat{\mathbf{p}}-\mathbf{p}_h\|^2_{\mathbf{W}}\\ &\le \theta_2^2\|u-u_h\|_1^2 + \gamma_1^2\|u-u_h\|_1^2\\ &= (\theta_2^2+\gamma_1^2)\|u-u_h\|_1^2, \end{aligned}\tag{3.25}$$

which implies (3.23). Inequality (3.24) similarly follows from (3.18) and (3.22). □

Remark 3.1. In our computations (cf. Section 5), we use higher order finite element methods to solve the problems (3.11) and (3.12) such that their approximations $\mathbf{p}_h$ and $\mathbf{p}_h^*$ satisfy the conditions $\|\hat{\mathbf{p}}-\mathbf{p}_h\|_{\mathbf{W}} \le \gamma_1\|u-u_h\|_1$ and $\|\hat{\mathbf{p}}^*-\mathbf{p}_h^*\|_{\mathbf{W}} \le \gamma_1^*\|u^*-u_h^*\|_1$ in Corollary 3.2 with two bounded constants γ_1 and γ_1^*.

4. Computable Error Bound for the Eigenvalue Approximation

Given the upper bound for the error estimates of the eigenfunction approximations presented in Section 3, we proceed to obtain a computable error estimate for eigenvalue approximations. The process is direct, from the following Rayleigh quotient expansion for the nonsymmetric eigenvalue problem.

Lemma 4.1. *Assume* $(\lambda,u)\in\mathscr{C}\times V$ *and* $(\lambda,u^*)\in\mathscr{C}\times V$ *satisfy* (2.2) *and* (2.4) *respectively, and suppose* $\psi,\ \psi^*\in V$ *are such that* $(\psi,\psi^*)\neq 0$. *If we define*

$$\widehat{\lambda}=\frac{a(\psi,\psi^*)}{(\psi,\psi^*)},$$

then we have the expansion

$$\widehat{\lambda}-\lambda \;=\; \frac{a(\psi-u,\psi^*-u^*)-\lambda(\psi-u,\psi^*-u^*)}{(\psi,\psi^*)}. \tag{4.1}$$

Proof. From the variational forms (2.2) and (2.4),

$$\begin{aligned}
&a(\psi-u,\psi^*-u^*)-\lambda(\psi-u,\psi^*-u^*)\\
=&\big(a(\psi,\psi^*)-\lambda(\psi,\psi^*)\big)+\big(a(u,u^*)-\lambda(u,u^*)\big)-\big(a(u,\psi^*)-\lambda(u,\psi^*)\big)\\
&-\big(a(\psi,u^*)-\lambda(\psi,u^*)\big)\\
=&a(\psi,\psi^*)-\lambda(\psi,\psi^*)\\
=&\big(a(\psi,\psi^*)-\widehat{\lambda}(\psi,\psi^*)\big)+(\widehat{\lambda}-\lambda)(\psi,\psi^*)\\
=&(\widehat{\lambda}-\lambda)(\psi,\psi^*),
\end{aligned} \tag{4.2}$$

hence the desired result Eq. (4.1). □

Theorem 4.1. *Suppose* $(\lambda_h,u_h)\in\mathscr{C}\times V_h$ *and* $(\lambda_h,u_h^*)\in\mathscr{C}\times V_h$ *are the solutions corresponding to* λ *of the discrete problems* (2.9) *and* (2.10), *respectively. Then if* $(u_h,u_h^*)=1$, *we have the error estimate*

$$|\lambda_h-\lambda|\le\theta(h)\eta(\lambda_h,u_h,\hat{\mathbf{p}})\eta^*(\lambda_h,u_h^*,\hat{\mathbf{p}}^*), \tag{4.3}$$

where

$$\theta(h)=\frac{1+C_\lambda\|\mathbf{b}\|_{L^\infty(\Omega)}\eta_a(h)+C_\lambda^2|\lambda-1|\eta_a(h)\eta_a^*(h)}{(1-\alpha\eta_a^*(h))(1-\alpha^*\eta_a(h))} \tag{4.4}$$

and

$$\lim_{h\to 0}\theta(h)=1\,. \tag{4.5}$$

Furthermore, assuming that $\mathbf{p}_h$ *and* $\mathbf{p}_h^*$ *are the respective approximate solutions of the optimisation problems* (3.11) *and* (3.12), *when* h *is small enough the following explicit asymptotic result holds:*

$$|\lambda_h-\lambda|\le 2\eta(\lambda_h,u_h,\mathbf{p}_h)\eta^*(\lambda_h,u_h^*,\mathbf{p}_h^*)\,. \tag{4.6}$$

Proof. We can choose $u \in M(\lambda)$ such that $\|u-u_h\|_0 \le C_\lambda \eta_a^*(h)\|u-u_h\|_1$, and also choose $u^* \in M^*(\lambda)$ satisfying $\|u^*-u_h^*\|_0 \le C_\lambda \eta_a(h)\|u^*-u_h^*\|_1$. From Lemma 4.1,

$$
\begin{aligned}
|\lambda_h-\lambda| =& \Big|a(u_h-u,u_h^*-u^*)-\lambda(u_h-u,u_h^*-u^*)\Big| \\
=& \Big|\big(\nabla(u_h-u),\nabla(u_h^*-u^*)\big)+\big(\mathbf{b}\cdot\nabla(u_h-u),u_h^*-u^*\big) \\
& -(\lambda-1)(u_h-u,u_h^*-u^*)\Big| \\
\le& \|u_h-u\|_1\|u_h^*-u^*\|_1+\|\mathbf{b}\|_{L^\infty(\Omega)}\big\|\nabla(u_h-u)\big\|_0\|u_h^*-u^*\|_0 \\
& +|\lambda-1|\,\|u_h-u\|_0\|u_h^*-u^*\|_0\,.
\end{aligned} \tag{4.7}
$$

Then combining Theorem 2.1, (3.3), (3.4) and (4.7) we have

$$
\begin{aligned}
|\lambda_h-\lambda| \le& \big(1+C_\lambda\|\mathbf{b}\|_{L^\infty(\Omega)}\eta_a(h)+C_\lambda^2|\lambda-1|\eta_a(h)\eta_a^*(h)\big)\|u_h-u\|_1\|u_h^*-u^*\|_1 \\
\le& \frac{1+C_\lambda\|\mathbf{b}\|_{L^\infty(\Omega)}\eta_a(h)+C_\lambda^2|\lambda-1|\eta_a(h)\eta_a^*(h)}{(1-\alpha\eta_a^*(h))(1-\alpha^*\eta_a(h))}\eta(\lambda_h,u_h,\hat{\mathbf{p}})\eta^*(\lambda_h,u_h^*,\hat{\mathbf{p}}^*) \\
=& \theta(h)\eta(\lambda_h,u_h,\hat{\mathbf{p}})\eta^*(\lambda_h,u_h^*,\hat{\mathbf{p}}^*),
\end{aligned} \tag{4.8}
$$

which is the desired result (4.3). □

Obviously, since $\eta_a(h),\eta_a^*(h)\to 0$ as $h\to 0$ we know that $\lim_{h\to 0}\theta(h)=1$, and the explicit asymptotic result (4.6) can be deduced directly from (3.9), (3.10) and (4.5).

5. Numerical Results

We illustrate the efficiency of the computable *a posteriori* error estimates for the proposed eigenpair approximations in three examples, where $\mathbf{b}=[b_1,b_2]^T\in\mathscr{C}^2$ in the nonsymmetric term is some constant vector. The nonsymmetric eigenvalue problems (2.2) and (2.4) were solved on the unit square $\Omega=(0,1)\times(0,1)$ with the real vector $\mathbf{b}=[1,1/2]^T$ and then the complex vector $\mathbf{b}=[1+2\mathrm{i},1/2-1\mathrm{i}]^T$, as discussed in Subsections 5.1 and 5.2. In Subsection 5.3, we discuss the solution of the nonsymmetric eigenvalue problem on an L-shaped domain $\Omega=(-1,1)\times(-1,1)/[0,1)\times(-1,0]$ with $\mathbf{b}=[1,1/2]^T$, using the adaptive finite element method. All of these three examples drawn from Refs. [12,13,22] are nondefective, and we computed their eigenvalues and corresponding eigenfunctions.

In the first two examples, the exact solutions of the nonsymmetric eigenvalue problem and its adjoint are [12,13]

$$\lambda_{k,\ell}=\frac{b_1^2+b_2^2}{4}+(k^2+\ell^2)\pi^2+1\,, \tag{5.1}$$

$$u_{k,\ell}=\exp\Big(\frac{b_1x_1+b_2x_2}{2}\Big)\sin(k\pi x_1)\sin(\ell\pi x_2)\,, \tag{5.2}$$

$$u_{k,\ell}^*=\exp\Big(-\frac{\overline{b_1}x_1+\overline{b_2}x_2}{2}\Big)\sin(k\pi x_1)\sin(\ell\pi x_2)\,, \tag{5.3}$$

https://doi.org/10.4208/eajam.140317.250517a

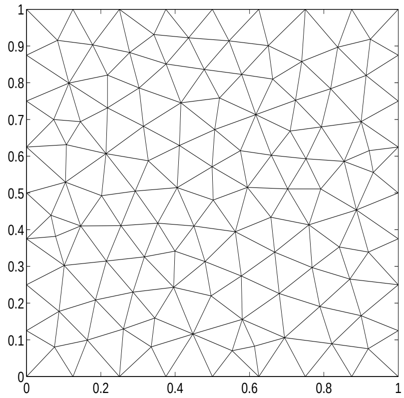

Figure 1: The initial mesh for the unit square.

where $k,\ell \in \mathcal{N}^+$. The *a posteriori* error estimators $\eta(\lambda_h,u_h,\hat{\mathbf{p}}_h)$ and $\eta^*(\lambda_h,u_h^*,\hat{\mathbf{p}}_h^*)$ were obtained using the finite element method to solve the dual problems (3.11) and (3.12). In all examples, the dual problems were solved on the same mesh $\mathcal{T}_h$, with the $H(\mathrm{div};\Omega)$ conforming finite element space [3]

$$\mathbf{W}_h^p = \left\{\mathbf{w}\in\mathbf{W}\,\middle|\, \mathbf{w}|_K \in \mathrm{RT}_p\,,\ \forall\, K\in\mathcal{T}_h\right\}$$

where $\mathrm{RT}_p = (\mathcal{P}_p)^d + \mathbf{x}\mathcal{P}_p$, to produce the finite element method for the dual problems (3.11) and (3.12) as follows:

$$\text{Find } \hat{\mathbf{p}}_h \in \mathbf{W}_h^p \text{ such that } \hat{a}(\hat{\mathbf{p}}_h,\mathbf{q}_h) = \mathcal{F}(\mathbf{q}_h)\,,\ \forall\,\mathbf{q}_h\in\mathbf{W}_h^p\,;\quad \text{and} \tag{5.4}$$

$$\text{Find } \hat{\mathbf{p}}_h^* \in \mathbf{W}_h^p \text{ such that } \hat{a}(\hat{\mathbf{p}}_h^*,\mathbf{q}_h) = \mathcal{F}^*(\mathbf{q}_h)\,,\ \forall\,\mathbf{q}_h\in\mathbf{W}_h^p\,. \tag{5.5}$$

To illustrate the efficiency of (4.3) in Theorem 4.1, we compare the error estimate $|\lambda-\lambda_h|$ with $\xi_h := \eta(\lambda_h,u_h,\hat{\mathbf{p}}_h)\,\eta^*(\lambda_h,u_h^*,\hat{\mathbf{p}}_h^*)$ in the numerical results.

5.1. Nonsymmetric eigenvalue problem for real constant vector b

In the first example, where vector $\mathbf{b} = [1,1/2]^T$ and the nonsymmetric eigenvalue problems (2.2) and (2.4) are defined on the unit square $\Omega = (0,1)\times(0,1)$, we produced a sequence of finite element spaces on a sequence of meshes obtained by regular refinement (connecting the midpoint of each edge) from an initial mesh generated by the Delaunay method — cf. Fig. 1.

We first used the linear conforming finite element method to solve the nonsymmetric eigenvalue problems (2.2) and (2.4), and then solved both dual problems (5.4) and (5.5) in the finite element spaces $\mathbf{W}_h^0$ and $\mathbf{W}_h^1$. Fig. 2 depicts our numerical results for the first eigenpair. From the left and middle subfigures, we see the computable error estimates are efficient when the corresponding dual problems are solved with $\mathbf{W}_h^1$, in agreement with Corollary 3.2 — the efficiency indices $\eta(\lambda_h,u_h,\mathbf{p}_h)/\|u-u_h\|_1$ are [0.997985259387722,

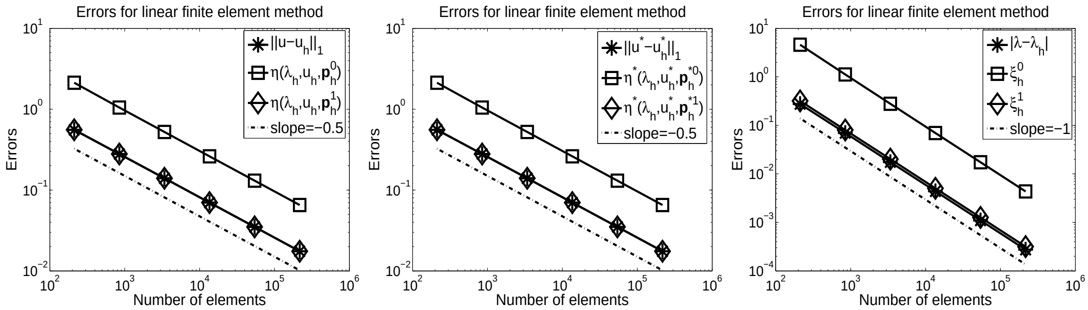

Figure 2: The errors of the first eigenpair for the nonsymmetric eigenvalue problems on the unit square domain with a real constant vector $\mathbf{b}$ and solved by the linear finite element method, where $\eta(\lambda_h,u_h,\mathbf{p}_h^0)$ and $\eta(\lambda_h,u_h,\mathbf{p}_h^1)$ in the left subfigure denote the *a posteriori* error estimates $\eta(\lambda_h,u_h,\hat{\mathbf{p}}_h)$ when the dual problem is solved by $\mathbf{W}_h^0$ and $\mathbf{W}_h^1$ respectively, and $\eta^*(\lambda_h,u_h^*,\mathbf{p}_h^{*0})$ and $\eta^*(\lambda_h,u_h^*,\mathbf{p}_h^{*1})$ in the middle subfigure denote the corresponding quantities of the adjoint problem, and in the right subfigure $\xi_h^0=\eta(\lambda_h,u_h,\mathbf{p}_h^0)\eta^*(\lambda_h,u_h^*,\mathbf{p}_h^{*0})$ and $\xi_h^1=\eta(\lambda_h,u_h,\mathbf{p}_h^1)\eta^*(\lambda_h,u_h^*,\mathbf{p}_h^{*1})$.

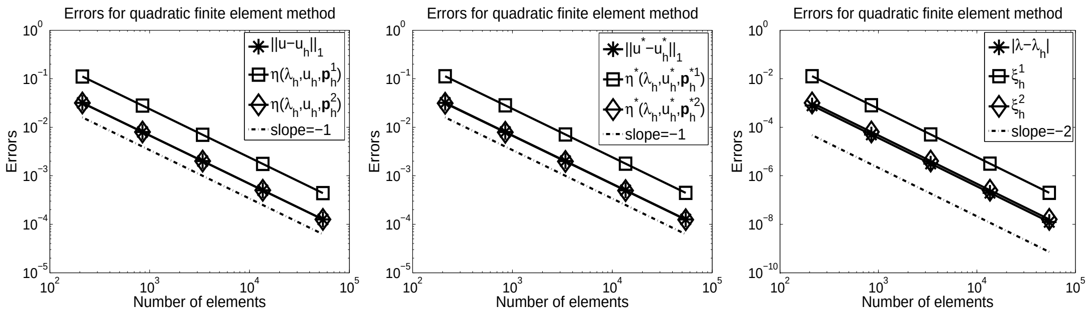

Figure 3: The errors of the first eigenpair for the nonsymmetric eigenvalue problems on the unit square domain with a real constant vector $\mathbf{b}$ and solved by the quadratic finite element method, where $\eta(\lambda_h,u_h,\mathbf{p}_h^1)$ and $\eta(\lambda_h,u_h,\mathbf{p}_h^2)$ in the left subfigure denote the *a posteriori* error estimates $\eta(\lambda_h,u_h,\hat{\mathbf{p}}_h)$ when the dual problem is solved by $\mathbf{W}_h^1$ and $\mathbf{W}_h^2$ respectively, and $\eta^*(\lambda_h,u_h^*,\mathbf{p}_h^{*1})$ and $\eta^*(\lambda_h,u_h^*,\mathbf{p}_h^{*2})$ in the middle subfigure denote the corresponding quantities of the adjoint problem, and in the right subfigure $\xi_h^1=\eta(\lambda_h,u_h,\mathbf{p}_h^1)\eta^*(\lambda_h,u_h^*,\mathbf{p}_h^{*1})$ and $\xi_h^2=\eta(\lambda_h,u_h,\mathbf{p}_h^2)\eta^*(\lambda_h,u_h^*,\mathbf{p}_h^{*2})$.

0.999528187698220, 0.999884580446569, 0.999971303308288, 0.999992834043646, 0.999998208728180]. The right subfigure reflects the bound for the eigenvalue error estimate $|\lambda-\lambda_h|$ in Theorem 4.1.

The quadratic finite element method was also applied to solve the nonsymmetric eigenvalue problems (2.2) and (2.4), and we chose the finite element spaces $\mathbf{W}_h^1$ and $\mathbf{W}_h^2$ for the dual problems (5.4) and (5.5). The corresponding numerical results for the first eigenpair are presented in Fig. 3. The left and middle subfigures show that the computable *a posteriori* error estimates are efficient when the corresponding dual problems are solved by $\mathbf{W}_h^2$, as expected from Corollary 3.2. The right subfigure reflects the bound of the eigenvalue error $|\lambda-\lambda_h|$ in Theorem 4.1.

We also checked the efficiency of the computable bound for the summation of the first 6 eigenvalues — i.e. $21/16+[2\pi^2,5\pi^2,5\pi^2,8\pi^2,10\pi^2,10\pi^2]$ in this example. Fig. 4 gives

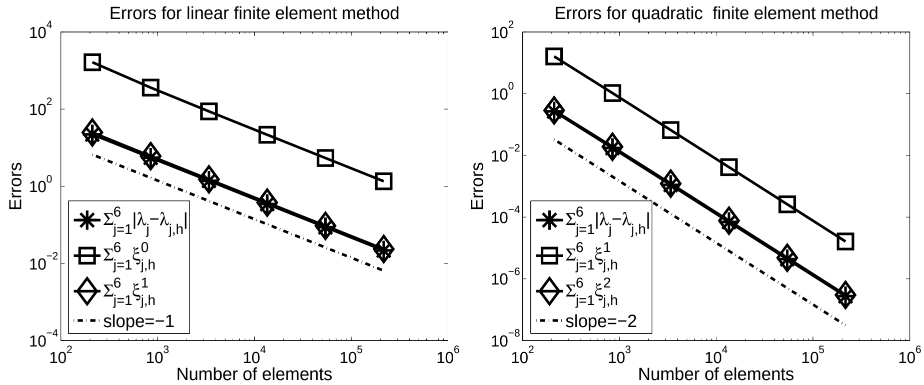

Figure 4: The errors for the error summation of the first 6 eigenvalues on the unit square domain with a real constant vector $\mathbf{b}$. In the left subfigure, the nonsymmetric eigenvalue problems are solved by the linear finite element method, where $\xi^0_{j,h} = \eta(\lambda_{j,h}, u_{j,h}, \mathbf{p}^0_{j,h})\eta^*(\lambda_{j,h}, u^*_{j,h}, \mathbf{p}^{*0}_{j,h})$ and $\xi^1_{j,h} = \eta(\lambda_{j,h}, u_{j,h}, \mathbf{p}^1_{j,h})\eta^*(\lambda_{j,h}, u^*_{j,h}, \mathbf{p}^{*1}_{j,h})$. In the right subfigure the eigenvalue problems are solved by quadratic finite element method, where $\xi^1_{j,h} = \eta(\lambda_{j,h}, u_{j,h}, \mathbf{p}^1_{j,h})\eta^*(\lambda_{j,h}, u^*_{j,h}, \mathbf{p}^{*1}_{j,h})$ and $\xi^2_{j,h} = \eta(\lambda_{j,h}, u_{j,h}, \mathbf{p}^2_{j,h})\eta^*(\lambda_{j,h}, u^*_{j,h}, \mathbf{p}^{*2}_{j,h})$.

the corresponding numerical results. The left subfigure resulted from using the linear finite element method to solve the eigenvalue problems (2.2) and (2.4), and the dual problems (5.4) and (5.5) were solved in $\mathbf{W}^0_h$ and $\mathbf{W}^1_h$. The right subfigure resulted from using the quadratic finite element method to solve the eigenvalue problems (2.2) and (2.4), and the dual problems (5.4) and (5.5) were solved in $\mathbf{W}^1_h$ and $\mathbf{W}^2_h$. Fig. 4 also reveals that the computable error estimates are efficient when the errors of the dual problems are small compared to the errors of the original nonsymmetric eigenvalue problems, consistent with Corollary 3.2.

5.2. Nonsymmetric eigenvalue problem for complex constant vector b

In the second example, where the nonsymmetric term is assumed to be a complex constant vector $\mathbf{b} = [1+2\mathrm{i}, 1/2-1\mathrm{i}]^T$, again using the initial mesh shown in Fig. 1 we solved the nonsymmetric eigenvalue problems (2.2) and (2.4) on the unit square $\Omega = (0,1)\times(0,1)$ by the linear finite element method, and the corresponding dual problems (5.4) and (5.5) were both solved in the finite element spaces $\mathbf{W}^0_h$ and $\mathbf{W}^1_h$. Fig. 5 presents the numerical results for the first eigenpair. The left and middle subfigures show that the computable *a posteriori* error estimate is efficient when the corresponding dual problems are solved by $\mathbf{W}^1_h$, in agreement with Corollary 3.2. The right subfigure also confirms the result presented in Theorem 4.1.

The quadratic finite element method was also chosen to solve the nonsymmetric eigenvalue problems (2.2) and (2.4), and the corresponding dual problems (5.4) and (5.5) were both solved in the finite element spaces $\mathbf{W}^1_h$ and $\mathbf{W}^2_h$. Fig. 6 presents the corresponding numerical results for the first eigenpair. The left and middle subfigure show that the computable *a posteriori* error estimate is efficient when the corresponding dual problems are

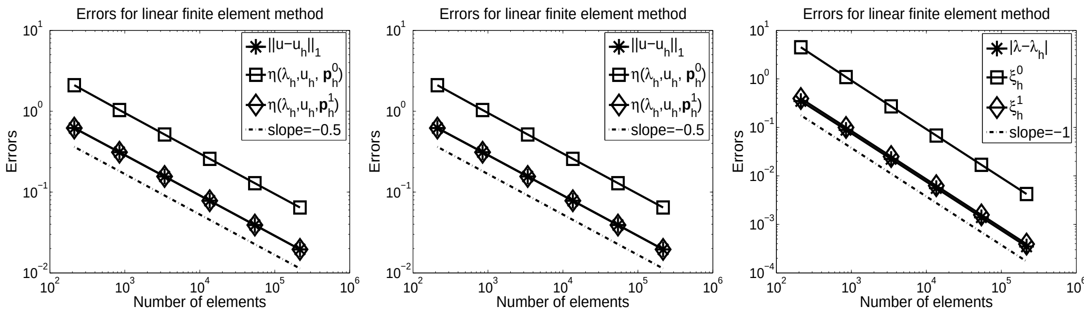

Figure 5: The errors for the first eigenpair for the nonsymmetric eigenvalue problems on the unit square domain with a complex constant vector $\mathbf{b}$ solved by the linear finite element method, where $\eta(\lambda_h, u_h, \mathbf{p}_h^0)$ and $\eta(\lambda_h, u_h, \mathbf{p}_h^1)$ in the left subfigure denote the *a posteriori* error estimates $\eta(\lambda_h, u_h, \hat{\mathbf{p}}_h)$ when the dual problem is solved by both $\mathbf{W}_h^0$ and $\mathbf{W}_h^1$, and $\eta^*(\lambda_h, u_h^*, \mathbf{p}_h^{*0})$ and $\eta^*(\lambda_h, u_h^*, \mathbf{p}_h^{*1})$ in the middle subfigure denote the corresponding quantities of the adjoint problem, and in the right subfigure $\xi_h^0 = \eta(\lambda_h, u_h, \mathbf{p}_h^0)\eta^*(\lambda_h, u_h^*, \mathbf{p}_h^{*0})$ and $\xi_h^1 = \eta(\lambda_h, u_h, \mathbf{p}_h^1)\eta^*(\lambda_h, u_h^*, \mathbf{p}_h^{*1})$.

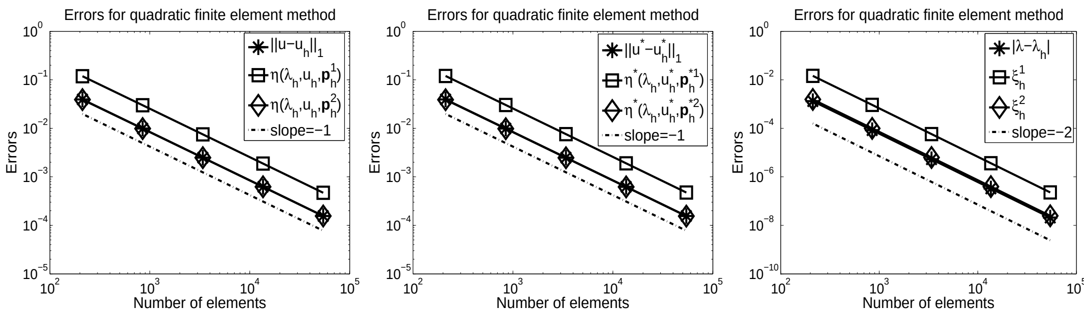

Figure 6: The errors for the first eigenpair for the nonsymmetric eigenvalue problems on the unit square domain with a complex constant vector $\mathbf{b}$ solved by the quadratic finite element method, where $\eta(\lambda_h, u_h, \mathbf{p}_h^1)$ and $\eta(\lambda_h, u_h, \mathbf{p}_h^2)$ in the left subfigure denote the *a posteriori* error estimates $\eta(\lambda_h, u_h, \hat{\mathbf{p}}_h)$ when the dual problem is solved by both $\mathbf{W}_h^0$ and $\mathbf{W}_h^1$, and $\eta^*(\lambda_h, u_h^*, \mathbf{p}_h^{*1})$ and $\eta^*(\lambda_h, u_h^*, \mathbf{p}_h^{*2})$ in the middle subfigure denote corresponding quantities of the adjoint problem, and in the right subfigure $\xi_h^1 = \eta(\lambda_h, u_h, \mathbf{p}_h^1)\eta^*(\lambda_h, u_h^*, \mathbf{p}_h^{*1})$ and $\xi_h^2 = \eta(\lambda_h, u_h, \mathbf{p}_h^2)\eta^*(\lambda_h, u_h^*, \mathbf{p}_h^{*2})$.

solved by $\mathbf{W}_h^2$, which is also consistent with Corollary 3.2. The right subfigure shows the efficiency of the bound for the error $|\lambda - \lambda_h|$ of the first eigenvalue presented in Theorem 4.1.

5.3. Nonsymmetric eigenvalue problem on the L-shaped domain

Finally, we considered the nonsymmetric eigenvalue problems (2.2) and (2.4) defined on the L-shaped domain $\Omega = (-1,1) \times (-1,1)/[0,1) \times (-1,0]$ with $\mathbf{b} = [1, 1/2]^T$. Since Ω has a re-entrant corner, the first eigenfunction is expected to be singular. For the first eigenvalue approximation, the convergence order by the linear finite element method red proved to be less than 2, the order predicted by the theory for regular eigenfunctions. Since the exact eigenvalue is unknown, we chose an adequately accurate approximation

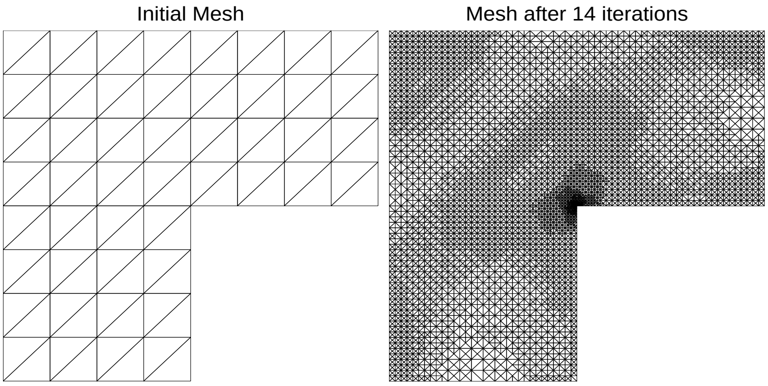

Figure 7: The initial mesh (left) and the subfigure after 14 adaptive iterations (right) for the L-shaped domain.

$\lambda = 10.95240442893276$ by the extrapolation method [14] to represent the exact first eigenvalue for our numerical tests.

We produced a sequence of finite element spaces on the sequence of meshes generated by the adaptive refinement (c.f. [19, 21]). The initial mesh and the subfigure after 14 adaptive iterations are shown in Fig. 7. The ZZ recovery method [23] was adopted as the *a posteriori* error estimator in the adaptive refinement for the eigenfunction and the corresponding adjoint eigenfunction approximations $\sqrt{\|u-u_h\|_1^2+\|u^*-u_h^*\|_1^2}$. The nonsymmetric eigenvalue problems (2.2) and (2.4) were solved by the linear finite element method, and the corresponding dual problems (5.4) and (5.5) were both solved in the finite element spaces $\mathbf{W}_h^0$ and $\mathbf{W}_h^1$. In order to test the efficiency of the computable *a posteriori* error estimate for the eigenfunction approximations, we defined an average quantity $\zeta_h := \sqrt{\eta^2(\lambda_h, u_h, \hat{\mathbf{p}}_h)+\eta^{*2}(\lambda_h, u_h^*, \hat{\mathbf{p}}_h^*)}$ and compared it with the ZZ error estimator. Fig. 8 gives the corresponding numerical results for the adaptive iterations. From the numerical results, we see that the computable *a posteriori* error estimate works well on the adaptive meshes for both eigenvalue and eigenfunction when the dual problems are solved in $\mathbf{W}_h^1$.

6. Concluding Remarks

We have obtained computable error estimates for nondefective eigenpair approximations of the nonsymmetric eigenvalue problem, solved by general conforming finite element methods on the general meshes. The computable error estimate of the eigenvalue approximation can provide a computable range, including the exact eigenvalue in the complex plane. Some numerical examples demonstrate the efficiency of the proposed computable error estimates. In future, we envisage extending our approach to Stokes and other eigenvalue problems that involve the Kohn-Sham or Hartree-Fock equations, and to compute the band gaps in photonic crystals.

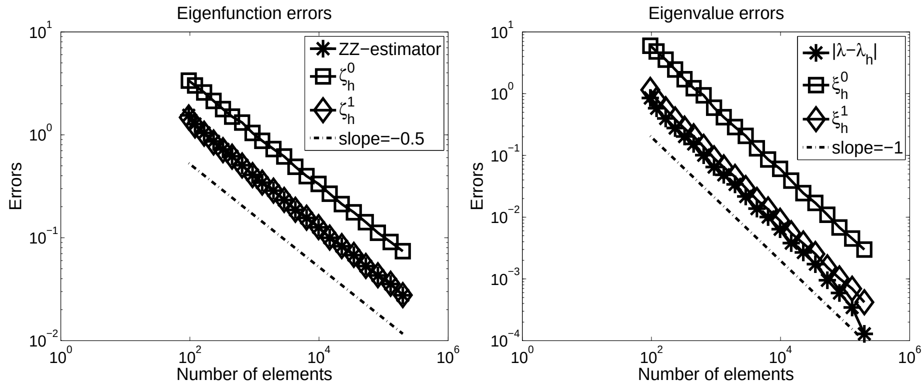

Figure 8: The errors for the first eigenpair when the nonsymmetric eigenvalue problems are defined on the L-shaped domain solved by the linear finite element method. In the left subfigure, $\zeta_h^j := \sqrt{\eta^2(\lambda_h, u_h, \mathbf{p}_h^j) + \eta^{*2}(\lambda_h, u_h^*, \mathbf{p}_h^{*j})}$ $(j = 0,\ 1)$, $\eta(\lambda_h, u_h, \mathbf{p}_h^0)$ and $\eta(\lambda_h, u_h, \mathbf{p}_h^1)$ denote the *a posteriori* error estimate $\eta(\lambda_h, u_h, \hat{\mathbf{p}}_h)$ when the dual problems are solved by both $\mathbf{W}_h^0$ and $\mathbf{W}_h^1$, $\eta^*(\lambda_h, u_h^*, \mathbf{p}_h^{*0})$ and $\eta^*(\lambda_h, u_h^*, \mathbf{p}_h^{*1})$ are the corresponding quantities of the adjoint problem, and in the right subfigure $\xi_h^0 = \eta(\lambda_h, u_h, \mathbf{p}_h^0)\eta^*(\lambda_h, u_h^*, \mathbf{p}_h^{*0})$ and $\xi_h^1 = \eta(\lambda_h, u_h, \mathbf{p}_h^1)\eta^*(\lambda_h, u_h^*, \mathbf{p}_h^{*1})$.

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