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Color degree sum conditions for rainbow triangles in edge-colored graphs

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Abstract Let G be an edge-colored graph and v a vertex of G. The color degree of v is the number of colors appearing on the edges incident to v. A rainbow triangle in G is one in which all edges have distinct colors. In this paper, we first prove that an edge-colored graph on n vertices contains a rainbow triangle if the color degree sum of any two adjacent vertices is at least $n+1$. Afterwards, we characterize the edge-colored graphs on n vertices containing no rainbow triangles but satisfying that each pair of adjacent vertices has color degree sum at least n.

Keywords edge-colored graphs · rainbow triangles

1 Introduction

All graphs considered in this paper are finite, simple and undirected. Let $G=(V(G),E(G))$ be a graph, where $V(G)$ and $E(G)$ are the vertex set and the edge set of G, respectively. An *edge-coloring* of G is a mapping $C: E(G) \rightarrow \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. Denote by $C(e)$ the color of an edge e in G. An edge-coloring is *proper* if adjacent edges receive distinct colors. When $E(G)$ is assigned an edge-coloring, we call G an *edge-colored graph* (or briefly, a *colored graph*). Let H be a subgraph of G. If each two edges in H have distinct colors, then H is called *rainbow*. For a vertex v of G, denote by $N_G(v)$ and $d_G(v)$ the neighbor set and the degree of v in G,

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respectively. The *color degree* of v in G with respect to the edge-coloring C, denoted by $d^c_G(v)$, is the number of colors appearing on the edges incident to v. Denote by $\delta^c(G)$ the minimum color-degree of vertices in G. Let r be a color. We use $d^r_G(v)$ to denote the number of edges incident to v and receiving the color r. When there is no ambiguity, we write $N(v)$ for $N_G(v)$, $d(v)$ for $d_G(v)$, $d^c(v)$ for $d^c_G(v)$ and $d^r(v)$ for $d^r_G(v)$. A triangle is a cycle of length 3. If G contains no triangles, then we say that G is *triangle-free*. For terminology and notations not defined here, we refer the reader to [2].

The topic of rainbow subgraphs has been well studied, such as rainbow matchings and rainbow cycles, see the survey paper [3]. Here we mainly focus on the existence of rainbow triangles in colored graphs.

Let G be a graph on n vertices. We know from Mantel's Theorem that G contains a triangle if $|E(G)| > \lfloor n^2/4 \rfloor$. As a corollary, G contains a triangle if $d(v) \geq (n+1)/2$ for every vertex $v \in V(G)$.

In order to generalize Mantel's Theorem to a colored graph G with order n, Li and Wang [6] conjectured in 2006 that G contains a rainbow triangle if $d^c(v) \geq (n+1)/2$ for every vertex $v \in V(G)$. This conjecture was formally published in [7] in 2012 and confirmed by Li [4] in 2013.

Theorem 1 (Li [4]) *Let G be a colored graph on n vertices. If $d^c(v) \geq (n+1)/2$ for every vertex $v \in V(G)$, then G contains a rainbow triangle.*

Independently, Li et al. [5] proved a stronger result, obtaining Theorem 1 as a corollary.

Theorem 2 (Li et al. [5]) *Let G be a colored graph on n vertices. If $\sum_{v\in V(G)} d^c(v) \geq n(n+1)/2$, then G contains a rainbow triangle.*

Li et al. [5] also proved that the bound of color-degree in Theorem 1 is tight for the existence of rainbow triangles, but can be lowered to $n/2$ with some simple exceptions.

Theorem 3 (Li et al. [5]) *Let G be a colored graph on n vertices. If $d^c(v) \geq n/2$ for every vertex $v \in V(G)$ and G contains no rainbow triangles, then n is even and G is a properly colored $K_{n/2,n/2}$, unless $G = K_4 - e$ or K_4 when $n = 4$.*

Motivated by the relation between the classic Dirac's condition and Ore's condition for long cycles, we wonder whether a graph G contains a rainbow triangle when

$$d^c(u) + d^c(v) \geq |V(G)| + 1 \tag{1}$$

for any nonadjacent vertices $u, v \in V(G)$.

In fact, Bondy [1] proved that a graph G on n vertices is pancyclic if $d(u)+d(v) \geq n+1$ for any nonadjacent vertices $u, v \in V(G)$. Certainly, G contains a triangle when G is pancyclic.

However, when we study the existence of rainbow triangles in a colored graph G under the color degree sum condition (1), we find a class of counterexamples.

Example 1 *Construct a colored graph G as follows:*

$$V(G) = \{v_1, v_2, \ldots, v_n\},$$

$$E(G) = \{v_i v_j : 1 \le i < j \le n, 1 \le i \le \lceil c/2 \rceil\},$$

and

$$C(v_i v_j) = \min\{i, j\},$$

where $c \in [n+1, 2n-2]$ *is a constant integer. Obviously,* G *satisfies that* $d^c(u) + d^c(v) \ge c \ge n+1$ *for every pair of nonadjacent vertices* $u, v \in V(G)$ *but contains no rainbow triangles.*

Oppositely, motivated by the fact that a graph G contains a triangle if there is an edge $uv \in E(G)$ satisfying $d(u) + d(v) \ge |G| + 1$, we show that the color degree sum condition for adjacent vertices is able to guarantee the existence of rainbow triangles in colored graphs.

Theorem 4 *Let* G *be a colored graph on* n *vertices and* $E(G) \ne \emptyset$*. If* $d^c(u) + d^c(v) \ge n+1$ *for every edge* $uv \in E(G)$*, then* G *contains a rainbow triangle.*

In fact, the color degree sum "$n+1$" is sharp for the existence of rainbow triangles. This can be shown by the following two kinds of colored graphs.

Example 2 *A properly colored complete bipartite graph* $K_{k,n-k}$ *with* $1 \le k \le n/2$.

Example 3 *Let* D_n *be a colored graph defined as follows:*

$$V(D_n) = \{u_1, u_2, v_1, v_2, \ldots, v_{n-2}\},$$

$$E(D_n) = \{u_1 u_2\} \cup \{u_i v_j : i = 1, 2;\ j = 1, 2, \ldots, n-2\},$$

$$C(u_1 u_2) = 0,\ C(u_i v_j) = j,\ (i = 1, 2;\ j = 1, 2, \ldots, n-2).$$

It is easy to check that both examples satisfy that $d^c(u) + d^c(v) \ge n$ for every edge uv but contain no rainbow triangles. Let $\mathscr{G}_1^c$ be the set of all properly colored complete bipartite graphs and $\mathscr{G}_2^c$ be the set of all D_n-type graphs.

With more efforts, we can prove that $\mathscr{G}_1^c$ and $\mathscr{G}_2^c$ are the only classes of extremal graphs when lowering the bound of "$n+1$" to "n".

Theorem 5 *Let* G *be a colored graph on* $n \ge 5$ *vertices and* $E(G) \ne \emptyset$*. If* $d^c(u) + d^c(v) \ge n$ *for every edge* $uv \in E(G)$ *and* G *contains no rainbow triangles, then* $G \in \mathscr{G}_1^c \cup \mathscr{G}_2^c$.

Here the condition that $E(G) \ne \emptyset$ in above theorems is necessary. If $E(G)$ is empty, then the restrictions on the color degree sum of adjacent vertices are meaningless.

2 Two lemmas

Before presenting the proofs of the main results, we first prove the following lemmas.

Lemma 1 *Let G be a colored graph on n vertices and $E(G) \neq \emptyset$. If G is triangle-free and $d^c(u)+d^c(v) \geq n$ for every edge $uv \in E(G)$, then G is a complete bipartite graph with a proper edge-coloring.*

Proof Since G contains no triangles, for every edge $uv \in E(G)$, we have $N(u) \cap N(v) = \emptyset$. So $d(u)+d(v) \leq n$. Also, $d(u)+d(v) \geq d^c(u)+d^c(v) \geq n$. Hence $d(u)+d(v) = d^c(u)+d^c(v) = n$. This implies that G is properly colored.

Let xy be an edge in G and $N(x) = A$. Then $N(y) = V(G)\backslash A$. Let $N(y) = B$. Then $y \in A$ and $x \in B$. Since G is triangle-free, $G[A]$ and $G[B]$ are empty graphs. For any vertex $a \in A$, we have $ax \in E(G)$ and $N(a) \subseteq B$. Thus

$$|B| \geq d(a) \geq d^c(a) = n - d^c(x) = n - d(x) = n - |A| = |B|.$$

This implies that $N(a) = B$. Similarly, for any vertex $b \in B$, we have $N(b) = A$.

Hence $G = (A,B)$ is a complete bipartite graph with a proper edge-coloring. □

Lemma 2 *Let G be a colored graph on $n \geq 6$ vertices such that $d^c(u)+d^c(v) \geq n$ for every edge $uv \in E(G)$. Let x be a vertex in G such that $d^c(x) = \delta^c(G)$ and let $G' = G-x$. If G' is a properly colored complete bipartite graph and G is not triangle-free, then G contains a rainbow triangle.*

Proof Let G' be a properly colored $K_{k,n-1-k} = (A,B)$. Then for any vertices $a_0 \in A$ and $b_0 \in B$, we have $d^c_{G'}(a_0) = n-k-1$ and $d^c_{G'}(b_0) = k$. Let $A' = N(x) \cap A$ and $B' = N(x) \cap B$. Since G is not triangle-free, we have $A' \neq \emptyset$ and $B' \neq \emptyset$.

Claim 1 *For any $a \in A'$ and $b \in B'$, $d^c_{G'}(a) \geq n/2-1$ and $d^c_{G'}(b) \geq n/2-1$.*

Proof Since $d^c(a) \geq d^c(x) \geq n - d^c(a)$ and $d^c(b) \geq d^c(x) \geq n - d^c(b)$, we have $d^c(a) \geq n/2$ and $d^c(b) \geq n/2$. So we obtain $d^c_{G'}(a) \geq d^c(a)-1 \geq n/2-1$ and $d^c_{G'}(b) \geq d^c(b)-1 \geq n/2-1$.

Claim 2 $d^c(x) \geq 3$.

Proof Choose $a \in A'$ and $b \in B'$. Then

$$d^c_{G'}(a) + d^c_{G'}(b) = n-1. \tag{2}$$

If n is odd, then $n \geq 7$. By Claim 1 and (2), $d^c_{G'}(a) = d^c_{G'}(b) = (n-1)/2$. Thus $d^c(b) \leq d^c_{G'}(b)+1 = (n+1)/2$. So $d^c(x) \geq n - d^c(b) \geq (n-1)/2 \geq 3$.

If n is even. By Claim 1 and (2), we have $\min\{d^c_{G'}(a), d^c_{G'}(b)\} = n/2-1$. Thus $\min\{d^c(a), d^c(b)\} \leq \min\{d^c_{G'}(a), d^c_{G'}(b)\}+1 = n/2$. So $d^c(x) \geq n - \min\{d^c(a), d^c(b)\} \geq n/2 \geq 3$.

Claim 2 implies that there exist $a_1 \in A'$ and $b_1 \in B'$ such that $C(xa_1) \neq C(xb_1)$. Let $C(xa_1) = 1$ and $C(xb_1) = 2$. Now, we will prove by contradiction.

Suppose that G contains no rainbow triangles. Then $C(a_1b_1) \in \{1,2\}$. Without loss of generality, let $C(a_1b_1) = 1$. Then $d^c(a_1) = d^c_{G'}(a_1)$. Hence, for any $b \in B$, we get $d^c(b) \geq n - d^c(a_1) = n - d^c_{G'}(a_1) = d^c_{G'}(b) + 1$. Thus $B' = B$ and $d^{C(xb)}(b) = 1$.

Since $|B'| = |B| = d^c_{G'}(a_1) \geq n/2 - 1 \geq 2$, we have $B' \backslash \{b_1\} \neq \emptyset$. Let b be a vertex in $B' \backslash \{b_1\}$. Consider the triangle xa_1b. Since $d^{C(xb)}(b) = 1$ and G' is properly colored, we have $C(xb) = C(xa_1) = 1$. This means that $C(xb) = 1$ for every vertex $b \in B' \backslash \{b_1\}$.

Furthermore, by Claim 2, there is a vertex $a_2 \in A'$ such that $C(xa_2) \notin \{1,2\}$. Let $C(xa_2) = 3$. Let b_2 be a vertex in $B' \backslash \{b_1\}$. Then $C(xb_2) = 1$. Since the triangle xa_2b_1 is not rainbow and $d^{C(xb_1)}(b_1) = 1$, we have $C(a_2b_1) = 3$. Similarly, consider the triangle xa_2b_2 and the fact that $d^{C(xb_2)}(b_2) = 1$. We get $C(a_2b_2) = 3$. This contradicts that G' is a properly colored graph. □

3 Proofs of Theorems

Proof of Theorem 4. Suppose the contrary. Let G be a counterexample with $|V(G)| + |E(G)|$ as small as possible. Let xy be an edge of G. Then

$$n - 1 \geq \max\{d^c(x), d^c(y)\} \geq (d^c(x) + d^c(y))/2 \geq (n+1)/2$$

This implies that $n \geq 3$. If $\delta^c(G) \geq (n+1)/2$, then by Theorem 1, G contains a rainbow triangle, a contradiction. So there must be a vertex $x \in V(G)$ such that $d^c(x) < (n+1)/2$. Let $G' = G - x$.

Claim 1 *$E(G')$ is nonempty.*

Proof If $d(x) = 0$, then there is nothing to prove. If $d(x) > 0$, then there exists a vertex $y \in N(x)$ and $d(y) \geq d^c(y) \geq n + 1 - d^c(x) > (n+1)/2 \geq 2$. So $d_{G'}(y) = d(y) - 1 > 1$. This shows that $E(G')$ is nonempty.

Claim 2 *For any edge $uv \in E(G')$, $d^c_{G'}(u) + d^c_{G'}(v) \geq n$.*

Proof If $u \notin N(x)$ or $v \notin N(x)$, then $d^c_{G'}(u) + d^c_{G'}(v) \geq d^c(u) + d^c(v) - 1 \geq n$. If $u, v \in N(x)$, then $d^c(u) > (n+1)/2$ and $d^c(v) > (n+1)/2$. Thus $d^c_{G'}(u) + d^c_{G'}(v) \geq d^c(u) + d^c(v) - 2 > n - 1$. Hence, $d^c_{G'}(u) + d^c_{G'}(v) \geq n$.

By Claims 1 and 2, G' is a smaller counterexample, a contradiction. □

Proof of Theorem 5.

Case 1 $n = 5$.

If G is triangle-free, then by Lemma 1, G is a properly colored complete bipartite graph, thus $G \in \mathscr{G}^c_1$. Now, suppose that G contains a triangle. Let $S = \{v : d^c(v) \leq 2\}$ and $T = \{v : d^c(v) \geq 3\}$.

Claim 1 *S is an independent set and T is a clique with $|T| \geq 2$.*

Proof Since $d^c(u)+d^c(v)\geq 5$ for every edge $uv\in E(G)$, S is an independent set. Furthermore, we have $|T|\geq 1$ by the fact that $E(G)\neq\emptyset$. If $|T|=1$, then G is a bipartite graph, this contradicts that G contains a triangle. So we have $|T|\geq 2$. Now we will prove that T is a clique by contradiction.

Assume that there are $u,v\in T$ such that $uv\notin E(G)$. Then $d(u)=d(v)=3$ and $d^c(u)=d^c(v)=3$. Let $\{x,y,z\}=V(G)\backslash\{u,v\}$. Let $C(ux)=1, C(uy)=2$ and $C(uz)=3$. Since G is not a bipartite graph, the edge-set of $G[\{x,y,z\}]$ is nonempty. So there exists a vertex in $\{x,y,z\}$, say x, satisfying that $d^c(x)\geq 3$. Furthermore, there is a vertex $s\in\{y,z\}$ such that $xs\in E(G)$ and $C(xs)\neq 1$. Without loss of generality, let $s=y$. Then $C(xy)=2$. Now consider the triangle vxy. We have $C(xv)=2$ or $C(yv)=2$.

If $C(xv)=2$, then $xz\in E(G)$ and $C(xz)=3$. Now, xzv is a triangle, and $C(zv)\neq C(xv)$. So $C(vz)=C(xz)=3$. Note that $d^c(z)\geq 5-d^c(v)=2$. So $yz\in E(G)$ and $C(yz)\neq 3$. Since xyz is a triangle but not rainbow, we have $C(yz)=2$, thus $d^c(y)\leq 2$ and for the edge yz we have $d^c(y)+d^c(z)\leq 4<5$, a contradiction.

If $C(yv)=2$, then $d^c(y)\leq 2$. Furthermore, we have $d^c(y)\geq 5-d^c(u)=2$. So $d^c(y)=2$. This implies that $yz\in E(G)$ and $C(yz)=3$. Since $d^c(z)\geq 5-d^c(y)=3$, we have $C(vz)\neq 3$. Consider the triangle yzv. We have $C(zv)=2$. However, this contradicts that $C(vy)\neq C(vz)$.

In summary, $|T|$ is a clique.

Claim 2 $|T|=2$.

Proof By contradiction.

If $|T|=5$. By Theorem 1, G contains a rainbow triangle, a contradiction.

If $|T|=4$. By Claim 1, $G[T]\cong K_4$. We first prove that $d^c_{G[T]}(v)=2$ for every vertex $v\in T$. Since $3\geq d^c_{G[T]}(v)\geq d^c(v)-1\geq 2$, it is sufficient to show that $d^c_{G[T]}(v)\neq 3$ for every vertex $v\in T$. Suppose that this is not true. Then there is a vertex $v_0\in T$ such that $d^c_{G[T]}(v_0)=3$. Let $T=\{v_0,v_1,v_2,v_3\}$. Without loss of generality, let $C(v_0v_i)=i$ $(i=1,2,3)$ and let $C(v_1v_2)=1$. To guarantee that $d^c_{G[T]}(v_i)\geq 2$ $(i=1,3)$, we have $C(v_1v_3)=3$ and $C(v_3v_2)=2$ by considering triangles $v_0v_1v_3$ and $v_0v_2v_3$. Thus, we obtain a rainbow triangle $v_1v_2v_3$, a contradiction. So for every vertex $v\in T$, $d^c_{G[T]}(v)=2$. Let $\{x\}=V(G)\backslash T$. We have $C(xv_i)\in E(G)$ and $d^{C(xv_i)}_G(v_i)=1$ $(i=0,1,2,3)$. Since G contains no rainbow triangles, $C(xv_i)=C(xv_j)$ $(i,j=0,1,2,3)$. Thus $d^c(x)=1$ and $d^c(x)+d^c(v_0)=4<5$, a contradiction.

If $|T|=3$. Let $T=\{x,y,z\}$ and $S=\{u,v\}$. By Claim 1, xyz is a triangle and $uv\notin E(G)$. Furthermore, we can assume that $C(xy)=C(xz)=1$, $C(ux)=2$ and $C(vx)=3$. This implies that $d^c(x)=3$ and $d^c(u)=d^c(v)=2$. Thus, there exists a vertex $s\in\{y,z\}$ such that $C(us)\neq C(ux)$. Combining this with the fact that $C(ux)\neq C(xy)$ and $C(ux)\neq C(xz)$, we have $C(us)=C(xs)$. Without loss of generality, let $s=y$. Then $C(uy)=1$. Now, consider that $d^c(y)\geq 3$ and $d^c(v)=2$. We have $C(yv)=C(xv)=3$ and $C(vz)=C(xz)=1$. Note that the edge yz is contained in the triangle vyz. So $C(yz)=1$ or 3. However, this implies that $d^c(y)\leq 2$, a contradiction.

Thus we have $|T|\leq 2$. By Claim 1, we get $|T|=2$.

Now, let $T=\{u,v\}$ and $S=\{x,y,z\}$. By Claim 1, $uv\in E(G)$ and S is an independent set. If $d^c(x)=d^c(y)=d^c(z)=1$, then $d^c(u)=d^c(v)=4$. Thus, obviously,

$G \in \mathscr{G}_2^c$. If there is a vertex in S, say x, satisfying $d^c(x) = 2$, then $C(xu) \neq C(xv)$. Since xuv is not a rainbow triangle, we can assume that $C(xu) = C(uv)$. Thus we have $yu, zu \in E(G)$, $d^c(u) = 3$, $C(yu) \neq C(uv)$, $C(zu) \neq C(uv)$ and $d^c(y) = d^c(z) = 2$. Since yuv and zuv are not rainbow triangles, we have $C(yv) = C(zv) = C(uv)$. This implies that $d^c(v) \leq 2$, a contradiction.

Case 2 $n \geq 6$.

We prove by induction. Note that Theorem 5 is ture for graphs on 5 vertices. Assume that it is true for graphs of order $n-1$ ($n \geq 6$). We will prove that it is also true for graphs of order n.

Let G be a graph on $n \geq 6$ vertices. Since G contains no rainbow triangles, by Theorem 1, we have $\delta^c(G) \leq n/2$. If G is triangle-free, by Lemma 1, G is a complete bipartite graph with a proper edge-coloring. If $\delta^c(G) = n/2$, by Theorem 3, n is even and G is a properly colored $K_{n/2,n/2}$. In both cases, we have $G \in \mathscr{G}_1^c$.

Now, consider the case that $\delta^c(G) < n/2$ and G is not triangle-free. Let x be a vertex in G such that $d^c(x) = \delta^c(G)$. Let $G' = G - x$. Similar to the proof of Theorem 4, we have $E(G') \neq \emptyset$ and $d^c_{G'}(u) + d^c_{G'}(v) \geq n-1$ for every edge $uv \in E(G')$. This implies that G' satisfies the conditions in Theorem 5. By assumption, $G' \in \mathscr{G}_1^c \cup \mathscr{G}_2^c$. However, by Lemma 2, G' is not a properly colored bipartite graph. Hence, $G' \in \mathscr{G}_2^c$. Now, we will prove that $G \in \mathscr{G}_2^c$. Without loss of generality, let

$$V(G') = \{u_1, u_2, v_1, v_2, \ldots, v_{n-3}\},$$

$$E(G') = \{u_1u_2\} \cup \{u_iv_j : i = 1,2;\ j = 1,2,\ldots,n-3\},$$

$$C(u_1u_2) = 0,\ C(u_iv_j) = j,\ (i = 1,2;\ j = 1,2,\ldots,n-3).$$

Thus we have

$$d^c_{G'}(u_1) = d^c_{G'}(u_2) = n-2,$$

$$d^c_{G'}(v_i) = 1, (i = 1,2,\ldots,n-3).$$

Since

$$d^c(x) + d^c(v_i) \leq 2d^c(v_i) \leq 2d^c_{G'}(v_i) + 2 = 4 < n\ (i = 1,2,\ldots,n-3),$$

we have

$$N(x) \subseteq \{u_1, u_2\},$$

$$d^c(v_i) = d^c_{G'}(v_i) = 1,\ (i = 1,2,\ldots,n-3).$$

Furthermore, we get

$$n \leq d^c(u_j) + d^c(v_1) \leq d^c_{G'}(u_j) + 2 = n, (j = 1,2).$$

This implies that

$$d^c(u_j) = d^c_{G'}(u_j) + 1,\ (j = 1,2).$$

Thus

$$\{u_1, u_2\} \subseteq N(x),$$

$$1 \leq d^c(x) \leq d^c(v_1) = 1,$$

Now, $N(x) = \{u_1, u_2\}$, $d^c(u_1) = d^c(u_2) = n-1$ and $d^c(x) = d^c(v_i) = 1$ for $i = 1,2,\ldots,n-3$. This implies that $G \in \mathscr{G}_2^c$. The proof is complete. □

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