Hamilton cycles in almost distance-hereditary graphs

Bing Chen[a] and Bo Ning[b*]

[a]Department of Applied Mathematics, School of Science,

Xi'an University of Technology, Xi'an, Shaanxi 710048, P.R. China

[b]Center for Applied Mathematics,

Tianjin University, Tianjin, 300072, P.R. China

Abstract

Let G be a graph on $n \geq 3$ vertices. A graph G is almost distance-hereditary if each connected induced subgraph H of G has the property $d_H(x,y) \leq d_G(x,y)+1$ for any pair of vertices $x,y \in V(H)$. Adopting the terminology introduced by Broersma et al. and Čada, a graph G is called 1-heavy if at least one of the end vertices of each induced subgraph of G isomorphic to $K_{1,3}$ (a claw) has degree at least $n/2$, and is called claw-heavy if each claw of G has a pair of end vertices with degree sum at least n. In this paper we prove the following two theorems: (1) Every 2-connected, claw-heavy and almost distance-hereditary graph is Hamiltonian. (2) Every 3-connected, 1-heavy and almost distance-hereditary graph is Hamiltonian. The first result improves a previous theorem of Feng and Guo [J.-F. Feng and Y.-B. Guo, Hamiltonian cycle in almost distance-hereditary graphs with degree condition restricted to claws, *Optimazation* **57** (2008), no. 1, 135–141]. For the second result, its connectedness condition is sharp since Feng and Guo constructed a 2-connected 1-heavy graph which is almost distance-hereditary but not Hamiltonian.

Keywords: Hamilton cycle; Almost distance-hereditary graph; Claw-free graph; 1-heavy graph; 2-heavy graph; Claw-heavy graph

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1 Introduction

In this paper, we only consider the graphs which are finite, undirected and without multi-edges and loops. For terminology and notation not defined here, we refer to Bondy and Murty [1].

*Corresponding Author: Bo Ning: Center for Applied Mathematics, Tianjin University, Tianjin, 300072, China, E-mail: bo.ning@tju.edu.cn.

Let G be a graph with vertex set $V(G)$ and H be a subgraph of G. For a vertex $v \in V(G)$, we denote by $N_H(v)$ the set of vertices which are adjacent to v in H, and by $d_H(v) = |N_H(v)|$ the *degree* of v in H. For two vertices $x, y \in V(G)$, an (x, y)-*path* in H is a path starting from x to y with all vertices in H. The *distance* of x and y in H, denoted by $d_H(x, y)$, is defined as the length of a shortest (x, y)-path in H. When there is no danger of ambiguity, we use $N(v)$, $d(v)$ and $d(x, y)$ instead of $N_G(v)$, $d_G(v)$ and $d_G(x, y)$, respectively.

A graph is called *Hamiltonian* if it contains a Hamilton cycle, i.e., a cycle passing through all the vertices of the graph. The study of cycles, especially Hamilton cycles, may be one of the most important and most studied areas of graph theory. It is well-known that to determine whether a given graph contains a Hamilton cycle is $\mathcal{NP}$-complete, shown by Karp [2]. However, if we only consider some restricted graph classes, then the situation is completely changed. A graph G is called *distance-hereditary* if each connected induced subgraph H has the property that $d_H(x, y) = d_G(x, y)$ for any pair of vertices x, y in H. This concept was introduced by Howorka [3] and a complete characterization of distance-hereditary graphs could be found in [3]. In 2002, Hsieh, Ho, Hsu and Ko [4] obtained an $O(|V| + |E|)$-time algorithm to solve the Hamiltonian problem on distance-hereditary graphs. Some other optimization problems can also be solved in linear time for distance-hereditary graphs although they are proved to be $\mathcal{NP}$-hard for general graphs. For references in this direction, we refer to [5, 6].

A graph G is called *almost distance-hereditary* if each connected induced subgraph H of G has the property $d_H(x, y) \leq d_G(x, y) + 1$ for any pair of vertices $x, y \in V(H)$. For some properties and a characterization of almost-distance hereditary graphs, we refer to [7].

Let G be a graph. An induced subgraph of G isomorphic to $K_{1,3}$ is called a *claw*. The vertex of degree 3 in the claw is called its *center* and the other vertices are its *end vertices*. G is called *claw-free* if G contains no claw. Throughout this paper, whenever the vertices of a claw are listed, its center is always the first one.

Many results about the existence of Hamilton cycles in claw-free graphs have been obtained. In particular, Feng and Guo [8] gave the following result on Hamiltonicity of almost distance-hereditary claw-free graphs.

Theorem 1 (Feng and Guo [8]). *Let G be a 2-connected claw-free graph. If G is almost distance-hereditary, then G is Hamiltonian.*

Let G be a graph on n vertices. A vertex v of G is called *heavy* if $d(v) \geq n/2$. Broersma et al. [9] introduced the concepts of 1-heavy graph and 2-heavy graph. Later, Fujisawa and

Yamashita [10] and Čada [11] introduced the concept of claw-heavy graphs, independently. Following [9, 10, 11], we say that a claw in G is *1-heavy* (*2-heavy*) if at least one (two) of its end vertices is (are) heavy. G is called 1-heavy (*2-heavy*) if every claw of it is 1-heavy (2-heavy), and called *claw-heavy* if every claw of it has two end vertices with degree sum at least n. It is easily seen that every claw-free graph is 1-heavy (2-heavy, claw-heavy), every 2-heavy graph is claw-heavy and every claw-heavy graph is 1-heavy. But not every claw-heavy graph is 2-heavy, and not every 1-heavy graph is claw-heavy.

In [12], Feng and Guo extended Theorem 1 to a larger graph class of 2-heavy graphs.

Theorem 2 (Feng and Guo [12])**.** *Let G be a 2-connected 2-heavy graph. If G is almost distance-hereditary, then G is Hamiltonian.*

Feng and Guo [12] also constructed a 2-connected 1-heavy graph which is almost distance-hereditary but not Hamiltonian. Thus it is natural to ask which is the minimum connectivity for a 1-heavy almost distance-hereditary graph under this connectivity condition to be Hamiltonian.

Motivated by [9, 13, 14, 15], in this paper we obtain the following two theorems which extend Theorem 1 and Theorem 2. In particular, Theorem 3 improves Theorem 2, and Theorem 4 answers the problem proposed above.

Theorem 3. *Let G be a 2-connected claw-heavy graph. If G is almost distance-hereditary, then G is Hamiltonian.*

Theorem 4. *Let G be a 3-connected 1-heavy graph. If G is almost distance-hereditary, then G is Hamiltonian.*

We emphasize that our technique of proofs is different from Feng and Guo [12]. One of our main tools is the so called "Ore-cycle" (motivated by Lemma 3 in [13]) introduced by Li et al. [16].

Remark 1. The graph in Fig.1 shows that the result in Theorem 3 indeed strengthen that in Theorem 2. As shown in [13, Fig.2], let $n \geq 10$ be an even integer and $K_{n/2} + K_{n/2-3}$ denote the join of two complete graphs $K_{n/2}$ and $K_{n/2-3}$. Choose a vertex $y \in V(K_{n/2})$ and construct a graph G with $V(G) = V(K_{n/2} + K_{n/2-3}) \cup \{v, u, x\}$ and $E(G) = E(K_{n/2} + K_{n/2-3}) \cup \{uv, uy, ux\} \cup \{vw, xw : w \in V(K_{n/2-3})\}$. It is easy to see that G is a Hamiltonian graph satisfying the condition of Theorem 3, but not the condition of Theorem 2.

We postpone the proofs of Theorem 3 and 4 to Section 3.

2 Preliminaries

Let G be a graph on n vertices and $k \geq 3$ be an integer. Recall that a vertex of degree at least $n/2$ in G is a *heavy vertex*; otherwise it is *light*. A claw in G is called a *light claw* if all its end vertices are light, and is called an *o-light claw* if any pair of end vertices has degree sum less than n. A cycle in G is called a *heavy cycle* if it contains all heavy vertices of G. Following [16], we use $\widetilde{E}(G)$ to denote the set $\{uv : uv \in E(G)$ or $d(u) + d(v) \geq n, u, v \in V(G)\}$. A sequence of vertices $C = v_1v_2 \ldots v_kv_1$ is called an *Ore-cycle* or briefly, *o-cycle* of G, if we have $v_iv_{i+1} \in \widetilde{E}(G)$ for every $i \in \{1, 2, \cdots, k\}$, where $v_1 = v_{k+1}$.

Let G be a graph and k be a nonnegative integer. For a cycle C of G and a vertex $u \in V(G)\backslash V(C)$, a subgraph F of G is said to be a $(u, C; k)$-*fan* if F is a union of paths $P_1, P_2, \ldots, P_k$, where P_i is a (u, w_i)-path ($1 \leq i \leq k$), $P_i \cap C = \{w_i\}$ ($1 \leq i \leq k$) and $P_i \cap P_j = \{u\}$ for $1 \leq i < j \leq k$. In the following, we use $F = (u; P_1, P_2, \ldots, P_k)$ to denote the fan. The vertices in $V(F)\backslash\{w_1, w_2, \ldots, w_k\}$ are called *internal vertices of* F.

We need some notations from [13]. Let H be a path or a cycle with a given orientation. We denote by $\overleftarrow{H}$ the same graph as H but with the reverse orientation. For a vertex $v \in V(H)$, we use v_H^+ to denote the successor of v on H, and v_H^- to denote its predecessor. If $S \subseteq V(H)$, then define $S_H^+ = \{v_H^+ : v \in S\}$ and $S_H^- = \{v_H^- : v \in S\}$. If there is no danger of ambiguity, we denote v_H^+, v_H^-, S_H^+ and S_H^- by v^+, v^-, S^+ and S^-, respectively. For two vertices $u, v \in V(H)$, we denote by $H[u, v]$ the segment of H from u to v, and denote $H(u, v)$, $H[u, v)$ and $H(u, v]$ by the paths $H[u, v] - \{u, v\}$, $H[u, v] - \{v\}$ and $H[u, v] - \{u\}$, respectively.

To prove Theorems 3 and 4, the following six lemmas are needed. In particular, similar proofs of the facts in Lemma 3 can be found in [13, 16] (for example, see Claims 1-4 of Theorem 8 in [16]). For the convenience of the readers, we write the detailed proofs here.

Lemma 1 (Bollobás and Brightwell [17], Shi [18]). *Every 2-connected graph contains a heavy cycle.*

Lemma 2 (Li, Wang, Ryjáček, Zhang [16]). *Let G be graph and let C' be an o-cycle of G. Then there exists a cycle C of G such that $V(C') \subseteq V(C)$.*

Lemma 3. *Let G be a non-Hamiltonian graph on n vertices, C be a longest cycle (a longest heavy cycle) of G, R a component of $G - V(C)$, and $A = \{v_1, v_2, \ldots, v_k\}$ the set of neighbors of R on C. Let $u \in V(R)$ and $v_i, v_j \in A$. Then there hold*
(a) $uv_i^- \notin \widetilde{E}(G)$, $uv_i^+ \notin \widetilde{E}(G)$; (b) $v_i^-v_j^- \notin \widetilde{E}(G)$, $v_i^+v_j^+ \notin \widetilde{E}(G)$; and (c) if $v_i^-v_i^+ \in \widetilde{E}(G)$, then $v_iv_j^- \notin \widetilde{E}(G)$, $v_iv_j^+ \notin \widetilde{E}(G)$.

Furthermore, if G is a 2-connected claw-heavy graph, then
(d) $v_i^-v_i^+ \in \widetilde{E}(G)$ *and* $v_j^-v_j^+ \in \widetilde{E}(G)$*; (e)* $v_i^-v_i^+ \in E(G)$ *or* $v_j^-v_j^+ \in E(G)$.

Proof. (a) Suppose that $uv_i^- \in \widetilde{E}(G)$. Then $C' = uv_iC[v_i, v_i^-]v_i^-u$ is an o-cycle of length longer than C, a contradiction. The other assertion can be proved similarly.

(b) Suppose that $v_i^-v_j^- \in \widetilde{E}(G)$. Then $C' = uv_jC[v_j, v_i^-]v_i^-v_j^-\overleftarrow{C}[v_j^-, v_i]v_iu$ is an o-cycle of length longer than C, a contradiction. The other assertion can be proved similarly.

(c) Suppose that $v_iv_j^- \in \widetilde{E}(G)$. Then $C' = v_iuv_jC[v_j, v_i^-]v_i^-v_i^+C[v_i^+, v_j^-]v_j^-v_i$ is an o-cycle of length longer than C, a contradiction. The other assertion can be proved similarly.

(d) If $v_i^-v_i^+ \notin E(G)$, then by (a), we know $uv_i^- \notin E(G)$, $uv_i^+ \notin E(G)$. Thus $\{v_i, u, v_i^-, v_i^+\}$ induces a claw. Since G is claw-heavy, by (a), we have $d(v_i^-) + d(v_i^+) \geq n$. This implies that $v_i^-v_i^+ \in \widetilde{E}(G)$. If $v_i^-v_i^+ \in E(G)$, then obviously $v_i^-v_i^+ \in \widetilde{E}(G)$. The other assertion can be proved similarly.

(e) Suppose that $v_i^-v_i^+ \notin E(G)$ and $v_j^-v_j^+ \notin E(G)$. By (d), we have $d(v_i^-)+d(v_i^+) \geq n$ and $d(v_j^-) + d(v_j^+) \geq n$. This implies that $d(v_i^-) + d(v_j^-) \geq n$ or $d(v_i^+) + d(v_j^+) \geq n$. Thus $v_i^-v_j^- \in \widetilde{E}(G)$ or $v_i^+v_j^+ \in \widetilde{E}(G)$, a contradiction to (b). □

Lemma 4. *Let G be a non-Hamiltonian graph, C be a longest cycle (a longest heavy cycle) of G, R a component of $G - V(C)$, and $A = \{v_1, v_2, \dots, v_k\}$ the set of neighbors of R on C. Let $v_i, v_j \in A$. Then there hold*
(a) for $l \in V(C(v_i, v_j^-])$*, if* $v_i^-l \in \widetilde{E}(G)$*, then* $l^-v_j^+ \notin \widetilde{E}(G)$ *and* $l^+v_j^+ \notin \widetilde{E}(G)$*;*
(b) for $l \in V(C[v_i, v_j^-]) \cap N(v_i)$*, if* $v_i^-v_i^+ \in \widetilde{E}(G)$*, then* $l^-v_j^- \notin \widetilde{E}(G)$ *and* $l^+v_j^+ \notin \widetilde{E}(G)$*; and*
(c) for $l \in V(C[v_i, v_j^-]) \cap N(v_i) \cap N(v_j^-)$*, if* $v_i^-v_i^+ \in \widetilde{E}(G)$*, then* $l^-l^+ \notin \widetilde{E}(G)$*.*

Proof. Let P be a (v_i, v_j)-path with all internal vertices in R.

(a) Suppose $l^-v_j^+ \in \widetilde{E}(G)$. Then $C' = \overleftarrow{P}C[v_i, l^-]l^-v_j^+C[v_j^+, v_i^-]v_i^-lC[l, v_j]$ is an o-cycle such that $V(C) \subset V(C')$. By Lemma 2, there is a longer cycle C'' containing all vertices in C, that is, a longer cycle (a longer heavy cycle) in G, contradicting the choice of C. Suppose $l^+v_j^+ \in \widetilde{E}(G)$. Then $C' = P\overleftarrow{C}[v_j, l^+]l^+v_j^+C[v_j^+, v_i^-]v_i^-l\overleftarrow{C}[l, v_i]$ is an o-cycle such that $V(C) \subset V(C')$, a contradiction.

(b) Suppose $l^-v_j^- \in \widetilde{E}(G)$. Then $C' = PC[v_j, v_i^-]v_i^-v_i^+C[v_i^+, l^-]l^-v_j^-\overleftarrow{C}[v_j^-, l]lv_i$ is an o-cycle such that $V(C) \subset V(C')$, a contradiction. Suppose $l^+v_j^+ \in \widetilde{E}(G)$. Then $C' = P\overleftarrow{C}[v_j, l^+]l^+v_j^+C[v_j^+, v_i^-]v_i^-v_i^+C[v_i^+, l]lv_i$ is an o-cycle such that $V(C) \subset V(C')$, a contradiction.

(c) Suppose $l^-l^+ \in \widetilde{E}(G)$. Then $C' = PC[v_j, v_i^-]v_i^-v_i^+C[v_i^+, l^-]l^-l^+C[l^+, v_j^-]v_j^-lv_i$ is an o-cycle such that $V(C) \subset V(C')$, a contradiction. □

Lemma 5. *Let G be a 3-connected 1-heavy non-Hamiltonian graph and C be a longest heavy cycle of G, R a component of $G-V(C)$. Let $u \in V(R)$, and v_0, v_1, v_2 be three neighbors of u which are in the order around C and v_1^-, v_1^+ are light. Let $l_i \in C[v_i^+, v_{i+1}^-)$ such that $v_{i+1}^- l_i \in E(G)$ and $l_i v_i \in E(G)$, $s_i \in C(v_{i-1}^+, v_i^-]$ such that $v_{i-1}^+ s_i \in E(G)$ and $s_i v_i \in E(G)$ (where the indices are taken modulo 3).*
(a) If $v_2^- v_2^+ \notin E(G)$ and $v_0^- v_0^+ \notin E(G)$, then (i) $v_1^- v_2 \notin E(G)$, (ii) $v_1 l_1^- \in E(G)$ and $v_1 s_1^+ \in E(G)$, (iii) $v_1, l_1^+, l_1^-, s_1^-, s_1^+, l_1, s_1$ are light;
(b) If $v_2^- v_2^+ \in E(G)$ and $v_0^- v_0^+ \in E(G)$, then $\{v_1^+, l_0^+, s_2^+\}$ induces an independent set.

Proof. By Lemma 3 (a), $uv_1^- \notin E(G)$ and $uv_1^+ \notin E(G)$. If $v_1^- v_1^+ \notin E(G)$, then since u, v_1^-, v_1^+ are light, $\{v_1, u, v_1^-, v_1^+\}$ induces a light claw, a contradiction. Thus $v_1^- v_1^+ \in E(G)$.

(a) Since $v_2^- v_2^+ \notin E(G)$ and $v_0^- v_0^+ \notin E(G)$, we have v_2^-, v_0^+ are heavy or v_2^+, v_0^- are heavy by Lemma 3 (b).

(i) Suppose $v_1^- v_2 \in E(G)$ and v_2^-, v_0^+ are heavy. Let $C' = v_1^- v_2 C[v_2, v_0] v_0 u v_1 C[v_1, v_2^-] v_2^- v_0^+ C[v_0^+, v_1^-]$. Then C' is an o-cycle such that $V(C) \subset V(C')$, a contradiction.

Suppose $v_1^- v_2 \in E(G)$ and v_2^+, v_0^- are heavy. Now $\{v_2, v_2^-, u, v_1^-\}$ induces a light claw, a contradiction.

(ii) Suppose $v_1 l_1^- \notin E(G)$. Note that $v_1 l_1^+ \notin \widetilde{E}(G)$ and $l_1^- l_1^+ \notin \widetilde{E}(G)$ by Lemma 4 (b) and (c). Since v_2^-, v_0^+ are heavy or v_2^+, v_0^- are heavy, by Lemma 3 (c) and Lemma 4 (b), v_1, l_1^+, l_1^- are light. Now $\{l_1, l_1^+, v_1, l_1^-\}$ induces a light claw, a contradiction. Similarly, we can prove that $v_1 s_1^+ \in E(G)$.

(iii) By Lemma 3 (c) and Lemma 4 (b), $v_1, l_1^+, l_1^-, s_1^-, s_1^+$ are light. Since $v_1 l_1^- \in E(G)$, we obtain $v_2^+ l_1 \notin \widetilde{E}(G)$ and $v_0^+ l_1 \notin \widetilde{E}(G)$ by Lemma 4 (b). Note that either v_0^+ or v_2^+ is a heavy vertex. This implies l_1 is a light vertex. The other assertion that s_1 is light can be proved similarly.

(b) Since $v_0 l_0 \in E(G)$ and $v_2 s_2 \in E(G)$, $v_1^+ l_0^+ \notin \widetilde{E}(G)$ and $v_1^+ s_2^+ \notin \widetilde{E}(G)$ by Lemma 4 (b). Furthermore, we can prove that $l_0^+ s_2^+ \notin E(G)$. (Otherwise, $C' = v_0 u v_2 s_2 \overleftarrow{C}[s_2, {l_0}^+] {l_0}^+ s_2^+ C[s_2^+, v_2^-] v_2^- v_2^+ C[v_2^+, v_0^-] v_0^- v_0^+ C[v_0^+, l_0] l_0 v_0$ is an o-cycle such that $V(C) \subset V(C')$, a contradiction.) □

Lemma 6. *Let G be a non-Hamiltonian almost distance-hereditary graph, C be a longest cycle (a longest heavy cycle) of G, R a component of $G-V(C)$. If there exists a vertex $u \in V(R)$ such that $N_C(u) = \{v_1, v_2, \ldots, v_r\}$, then there hold*
(a) for any induced (u,v)-path P, where $v \in N_C^-(u)$ or $v \in N_C^+(u)$, the length of P is at most 3;
(b) if $v_i^- v_i^+ \in \widetilde{E}(G)$, then there exists a vertex $l_i \in C[v_i^+, v_{i+1}^-)$ such that $v_{i+1}^- l_i \in E(G)$

and $l_iv_i \in E(G)$*, and there exists a vertex* $s_i \in C(v_{i-1}^+, v_i^-]$ *such that* $v_{i-1}^+ s_i \in E(G)$ *and* $s_iv_i \in E(G)$*;*
(c) *if* $v_i^- v_i^+ \in E(G)$ *and* $v_{i+1}^- v_{i+1}^+ \in E(G)$*, then* $v_{i+1}^+ l_i \in E(G)$*; and*
(d) *if* $v_i^- v_i^+ \in E(G)$ *and* $v_{i+1}^- v_{i+1}^+ \in E(G)$*, then both* $\{l_i, l_i^-, v_i, v_{i+1}^-\}$ *and* $\{l_i, l_i^+, v_i, v_{i+1}^+\}$ *induce claws.*

Proof. (a) Suppose there exists an induced (u, v)-path P such that the length of P is at least 4. It follows that $d_P(u, v) \geq 4$, contradicting the fact that $d_G(u, v) = 2$ and G is almost distance-hereditary.

(b) Let $H = G[\{u\} \cup V(C[v_i, v_{i+1}])] - \{v_{i+1}\}$. Since $d_G(v_{i+1}^-, u) = 2$ and G is almost distance-hereditary, we have $d_H(v_{i+1}^-, u) \leq 3$. Since $v_i^- v_i^+ \in \widetilde{E}(G)$, by Lemma 3 (c), we have $v_iv_{i+1}^- \notin E(G)$ and $d_H(v_{i+1}^-, u) = 3$. It follows that $d_H(v_{i+1}^-, v_i) = 2$. So there exists a vertex $l_i \in C[v_i^+, v_{i+1}^-)$ such that $v_{i+1}^- l_i \in E(G)$ and $l_iv_i \in E(G)$. The other assertion can be proved similarly.

(c) Suppose $v_{i+1}^+ l_i \notin E(G)$. Let $H = G[\{v_{i+1}^+, v_{i+1}^-, l_i, v_i, u\}]$. By Lemma 3 (c), $v_iv_{i+1}^- \notin E(G)$ and $v_iv_{i+1}^+ \notin E(G)$. We can see H is an induced (u, v_{i+1}^+)-path of length 4 in G, contradicting Lemma 6 (a).

(d) By Lemma 3 (c) and Lemma 4 (b), we have $v_iv_{i+1}^- \notin \widetilde{E}(G)$ and $l_i^- v_{i+1}^- \notin \widetilde{E}(G)$. By Lemma 6 (c) and Lemma 4 (b), we have $v_il_i^- \notin E(G)$. So $\{l_i, l_i^-, v_i, v_{i+1}^-\}$ induces a claw. The other assertion can be proved similarly. □

3 Proofs of Theorems 3 and 4

Proof of Theorem 3

Let G be a graph satisfying the condition of Theorem 3. Let C be a longest cycle of G and assign an orientation to it. Suppose G is not Hamiltonian. Then $V(G)\backslash V(C) \neq \emptyset$. Let R be a component of $G - C$, and $A = \{v_1, v_2, \ldots, v_k\}$ be the set of neighbors of R on C. Since G is 2-connected, there exists a (v_i, v_j)-path $P = v_iu_1 \ldots u_rv_j$ with all internal vertices in R, and $v_i, v_j \in A$. Choose P such that:
(1) $|V(C(v_i, v_j))|$ is as small as possible;
(2) $|V(P)|$ is as small as possible subject to (1).

Claim 1. There is no o-cycle C' in G such that $V(C) \subset V(C')$.

Proof. Otherwise, C' is an o-cycle such that $V(C) \subset V(C')$. By Lemma 2, there exists a cycle containing all vertices in C' and longer than C, contradicting the choice of C. □

By Lemma 3 (e), without loss of generality, assume that $v_i^- v_i^+ \in E(G)$.

Claim 2. $r = 1$, that is, $V(P) = \{v_i, u_1, v_j\}$.

Proof. Suppose $r \geq 2$. Consider $H = G[V(P) \cup V(C[v_i, v_j])] - \{v_j\}$. Since $v_i^- v_i^+ \in E(G)$, $v_i v_j^- \notin E(G)$ by Lemma 3 (c). Thus $d_H(v_j^-, v_i) \geq 2$. By the choice condition of P and Lemma 3 (a), we have $d_P(v_i, u_r) \geq 2$ and $d_H(v_j^-, u_r) = d_H(v_j^-, v_i) + d_P(v_i, u_r) \geq 4$, which yields a contradiction to the fact G is almost distance-hereditary and $d_G(v_j^-, u_r) = 2$. Hence $V(P) = \{v_i, u_1, v_j\}$. □

Claim 3. $|V(C[v_i, v_j])| \geq 5$.

Proof. Suppose $|V(C[v_i, v_j])| = 4$ or $|V(C[v_i, v_j])| = 3$. This means $C[v_i, v_j] = v_i v_i^+ v_j^- v_j$ or $C[v_i, v_j] = v_i v_j^- v_j$. Let $C' = v_i u_1 v_j v_j^- v_j^+ C[v_j^+, v_i^-] v_i^- v_i^+ v_i$ or $C' = v_i u_1 v_j v_j^- v_j^+ C[v_j^+, v_i]$. Then C' is an o-cycle such that $V(C) \subset V(C')$ by Lemma 3 (d), contradicting Claim 1. □

Recall that $v_i^- v_i^+ \in E(G)$. Let $H = G[\{u_1, v_i^-\} \cup V(C[v_i, v_j])] - \{v_i\}$. Since $d_G(v_i^-, u_1) = 2$ and G is almost distance-hereditary, $d_H(v_i^-, u_1) \leq 3$. By Lemma 3 (c) and (d), we have $v_i^- v_j \notin E(G)$. By the choice of P, $u_1 v \notin E(G)$, where $v \in C[v_i^+, v_j^-]$. It follows that $d_H(v_i^-, u_1) = 3$ and $d_H(v_i^-, v_j) = 2$. By Lemma 3 (b), (c) and (d), $v_i^- v_j^- \notin E(G)$ and $v_i^+ v_j \notin E(G)$. Thus there exists a vertex $w \in C(v_i^+, v_j^-)$ such that $v_i^- w \in E(G)$ and $w v_j \in E(G)$. Note that w is well-defined.

Claim 4. $w v_j^+ \notin \widetilde{E}(G)$.

Proof. Suppose $w v_j^+ \in \widetilde{E}(G)$. By Lemma 4 (a), we obtain $v_i^- w^+ \notin \widetilde{E}(G)$. Since $v_i^- w \in E(G)$, we have $v_j w^+ \notin \widetilde{E}(G)$ by Lemma 4 (b) and by symmetry. Note that $v_j v_i^- \notin \widetilde{E}(G)$ by Lemma 3 (c). Thus $\{w, w^+, v_j, v_i^-\}$ induces an o-light claw in G, a contradiction. □

Next we will show that $\{v_j, u_1, w, v_j^+\}$ induces an o-light claw and get a contradiction. Before proving this fact, the following claim is needed.

Claim 5. $u_1 w \notin \widetilde{E}(G)$.

Proof. First we will show that $w^- v_i^- \notin \widetilde{E}(G)$. Since $v_j^- v_j^+ \in \widetilde{E}(G)$ and $v_j w \in E(G)$, we have $w^- v_i^- \notin \widetilde{E}(G)$ by Lemma 4 (b) and symmetry.

Next we will show that $w^- v_j \in \widetilde{E}(G)$. Suppose not. Consider the subgraph induced by $\{w, w^-, v_j, v_i^-\}$. Note that $v_j v_i^- \notin \widetilde{E}(G)$ by Lemma 3 (c) and $w^- v_i^- \notin \widetilde{E}(G)$ by the analysis above. Then $\{w, w^-, v_j, v_i^-\}$ induces an o-light claw, a contradiction.

Now we will show that $u_1 w \notin \widetilde{E}(G)$, since otherwise, $C' = u_1 w C[w, v_j^-] v_j^- v_j^+ C[v_j^+, w^-] w^- v_j u_1$ is an o-cycle such that $V(C) \subset V(C')$, contradicting Claim 1. □

By Claims 4, 5 and Lemma 3 (a), $\{v_j, u_1, w, v_j^+\}$ induces an o-light claw, contradicting the fact G is claw-heavy. The proof of Theorem 3 is complete. $\square$

Proof of Theorem 4.

Let G be a graph satisfying the condition of Theorem 4. By Lemma 1, there exists a heavy cycle in G. Now choose a longest heavy cycle C of G and assign an orientation to it. Suppose G is not Hamiltonian. Then $V(G)\backslash V(C) \neq \emptyset$. Let R be a component of $G - C$ and $A = \{w_1, w_2, \ldots, w_k\}$ be the set of neighbors of R on C. Since G is 3-connected, for any vertex u of R, there exists a $(u, C; 3)$-fan F such that $F = (u; Q_1, Q_2, Q_3)$, where $Q_1 = ux_1 \ldots x_{r_1} w_i$, $Q_2 = uy_1 \ldots y_{r_2} w_j$ and $Q_3 = uz_1 \ldots z_{r_3} w_k$, and w_i, w_j, w_k are in the order of the orientation of C.

By the choice of C, all internal vertices of F are not heavy. By Lemma 3 (b), there is at most one heavy vertex in $N_C^+(R)$ and at most one heavy vertex in $N_C^-(R)$. Without loss of generality, assume that w_i^-, w_i^+ are light. Hence $w_i^- w_i^+ \in E(G)$, otherwise $\{w_i, w_i^-, w_i^+, x_{r_1}\}$ induces a light claw, contradicting G is 1-heavy.

Claim 1. There exists a (u,C;3)-fan F such that $V(F) = \{u, w_i, w_j, w_k\}$.

Proof. Now we choose the fan F in such a way that:

(1) $Q_1 = uw_i$;

(2) $|V(C[w_i, w_j])|$ is as small as possible subject to (1);

(3) $|V(Q_2)|$ is as small as possible subject to (1) and (2);

(4) $|V(C[w_k, w_i])|$ is as small as possible subject to (1), (2) and (3);

(5) $|V(Q_3)|$ is as small as possible subject to (1), (2), (3) and (4).

Since G is 3-connected, for any neighbor of C in R, say u (with $uw_i \in E(G)$, where $w_i \in V(C)$), there are three disjoint paths from u to C. Obviously, we can choose one such path as uw_i. Thus (1) is well-defined, and furthermore, the choice condition of F is well-defined.

Claim 1.1. $V(Q_2) = \{u, w_j\}$.

Proof. Suppose $V(Q_2)\backslash\{u, w_j\} \neq \emptyset$. Without loss of generality, set $y = y_{r_2}$. Let $H = G[V(Q_1) \cup V(Q_2) \cup V(C[w_i, w_j])] - \{w_j\}$. Note that $w_i^- w_i^+ \in E(G)$. By Lemma 3 (c), it is easy to see that $w_i w_j^- \notin E(G)$, so $d_H(w_j^-, w_i) \geq 2$. Meantime, the choice condition (2) implies that $N(V(Q_2)\backslash\{w_j\}) \cap V(C(w_i, w_j)) = \emptyset$. This means that $d_H(w_j^-, y) = d_H(w_j^-, w_i) + d_H(w_i, y) \geq 2 + d_H(w_i, y)$. Since G is almost distance-hereditary and $d_G(w_j^-, y) = 2$, we have $d_H(w_j^-, y) = 3$ and $yw_i \in E(G)$. Let $F' = (y; Q_1', Q_2', Q_3')$ such that $Q_1' = yw_i$, $Q_2' = yw_j$ and $Q_3' = Q_2[y, u]Q_3[u, w_k]$. Then F' is a $(y, C; 3)$-fan satisfying (1), (2) and $|V(Q_2')| = 2$, contradicting the choice condition (3), a contradiction. $\square$

Claim 1.2. $V(Q_3) = \{u, w_k\}$.

Proof. Suppose $V(Q_3)\backslash\{u, w_k\} \neq \emptyset$. Without loss of generality, set $z = z_{r_3}$.

If $zw_i \notin E(G)$, then set $H = G[V(Q_1) \cup V(Q_3) \cup V(C[w_k, w_i])] - \{w_k\}$. Since $w_i^- w_i^+ \in E(G)$, we obtain $w_k^+ w_i \notin E(G)$ by Lemma 3 (c). This means $d_H(w_k^+, w_i) \geq 2$. By the choice condition (4), we have $N(V(Q_3)\backslash\{w_k\}) \cap V(C(w_k, w_i)) = \emptyset$, and hence $d_H(w_k^+, z) = d_H(w_k^+, w_i) + d_H(w_i, z)$. Since $zw_i \notin E(G)$, $d_H(w_i, z) \geq 2$ and we get $d_H(w_k^+, z) \geq 4$. It yields a contradiction to the fact G is almost distance-hereditary and $d_G(w_k^+, z) = 2$.

If $zw_i \in E(G)$, then set $H = G[V(C[w_i, w_j]) \cup V(Q_3[u, z])] - \{w_i\}$. Note that $N_C(z) \cap V(C(w_i, w_j]) = \emptyset$ (by the choice conditions (2), (5)) and $N_C(V(Q_3)\backslash\{z, w_k\}) \cap V(C(w_i, w_j)) = \emptyset$ (by the choice condition (2)). Since $d_G(w_i^+, z) = 2$ and G is almost distance-hereditary, $d_H(w_i^+, z) \leq 3$. But if $w_i^+ w_j \notin E(G)$, then the distance from z to w_i^+ in H is at least 4, where in such a shortest path, the path $Q_3[z, u]$ contributes at least 1, the path $Q_2[u, w_j]$ contributes 1, a contradiction. Thus we have $w_i^+ w_j \in E(G)$, and hence $w_j^- w_j^+ \notin \widetilde{E}(G)$ by Lemma 3 (c). Consider the subgraph induced by $\{w_j, w_i^+, w_j^+, u\}$. Since G is 1-heavy and w_i^+, u are light, w_j^+ is heavy. Now let $H = G[\{w_i^-\} \cup V(C[w_i, w_j]) \cup V(Q_3[u, z])] - \{w_i\}$. Similarly, since $d_G(w_i^-, z) = 2$, we have $d_H(w_i^-, z) = 3$, and $w_i^- w_j \in E(G)$. Consider the subgraph induced by $\{w_j, w_i^-, w_j^-, u\}$. Similarly, we can see w_j^- is heavy, and hence $w_j^- w_j^+ \in \widetilde{E}(G)$, a contradiction. Thus $V(Q_3) = \{u, w_k\}$. □

By Claims 1.1 and 1.2, the proof of Claim 1 is complete. □

By Claim 1, there exists a $(u, C; 3)$-fan F such that $V(F)\backslash V(C) = \{u\}$. Suppose that $N_C(u) = \{v_1, v_2, \ldots, v_r\}$ ($r \geq 3$) and $v_1, v_2, \ldots, v_r$ are in the order of the orientation of C. In the following, all the subscripts of v are taken modulo r, and $v_0 = v_r$.

By Lemma 3 (b), there is at most one heavy vertex in $N_C^+(u)$ and at most one heavy vertex in $N_C^-(u)$. Since $r \geq 3$, we know that there exists $v_j \in N_C(u)$, such that v_j^-, v_j^+ are light, and hence $v_j^- v_j^+ \in E(G)$ by the fact G is 1-heavy. Without loss of generality, assume that $v_1^- v_1^+ \in E(G)$ and v_1^-, v_1^+ are light. By Lemma 6 (b), there exists a vertex $l_1 \in C[v_1^+, v_2^-)$ such that $v_2^- l_1 \in E(G)$ and $l_1 v_1 \in E(G)$, and there exists a vertex $s_1 \in C(v_0^+, v_1^-]$ such that $v_0^+ s_1 \in E(G)$ and $s_1 v_1 \in E(G)$.

We divide the proof into two cases.

Case 1. $v_2^- v_2^+ \notin E(G)$ and $v_0^- v_0^+ \notin E(G)$.

Both $\{v_2, v_2^-, v_2^+, u\}$ and $\{v_0, v_0^-, v_0^+, u\}$ induce claws. By Lemma 3 (b) and the fact G is 1-heavy, v_2^- and v_0^+ are heavy or v_2^+ and v_0^- are heavy.

Claim 2. $v_1^- l_1 \in E(G)$ and $l_1 v_2 \in E(G)$.

Proof. Suppose $v_1^- l_1 \notin E(G)$. Note that $uv_1^- \notin \widetilde{E}(G)$ by Lemma 3 (a). By Lemma 5 (a), l_1 is light. Now $\{v_1, l_1, u, v_1^-\}$ induces a light claw, a contradiction.

Suppose $l_1 v_2 \notin E(G)$. Let $H = G[\{v_1^-, l_1, v_2^-, v_2, u\}]$. By Lemma 3, we get $uv_2^- \notin E(G)$, $uv_1^- \notin E(G)$ and $v_1^- v_2^- \notin E(G)$. Note that $v_2 v_1^- \notin E(G)$ by Lemma 5 (a). Now $G[\{v_1^-, l_1, v_2^-, v_2, u\}]$ is an induced path of length 4 in G, contradicting Lemma 6 (a). $\square$

Now we consider the following two subcases.

Subcase 1.1. v_2^-, v_0^+ are heavy vertices.

By Lemma 3 (a), $uv_2^+ \notin \widetilde{E}(G)$. By Lemma 5 (a), we have $v_1 l_1^- \in E(G)$. Note that $l' := l_1^- \in N(v_1)$ and $v_1^- v_1^+ \in E(G)$. By Lemma 4 (b), $l'^+ v_2^+ = l_1 v_2^+ \notin \widetilde{E}(G)$. By Lemma 5 (a) and Lemma 3 (b), l_1 and v_2^+ are light. Now $\{v_2, l_1, u, v_2^+\}$ induces a light claw, a contradiction.

Subcase 1.2. v_2^+, v_0^- are heavy vertices.

Consider the subgraph induced by $\{v_0^+, s_1, l_1, v_2, u\}$. It is easily to check that $v_0^+ s_1 \in E(G)$, $l_1 v_2 \in E(G)$ (by Claim 2) and $v_2 u \in E(G)$. By Lemma 3 (a), $v_0^+ u \notin E(G)$. By Lemma 5 (a) and Lemma 4 (b), we know that $v_1 l_1^- \in E(G)$ and $v_0^+ l_1 \notin E(G)$. Now we obtain $s_1 l_1 \notin E(G)$ or $v_0^+ v_2 \in E(G)$ or $s_1 v_2 \in E(G)$ (Otherwise, $G[\{v_0^+, s_1, l_1, v_2, u\}]$ is an induced path of length 4, contradicting Lemma 6 (a)).

Suppose $s_1 l_1 \notin E(G)$. By Lemma 5 (a), l_1, s_1 are light. Now $\{v_1, l_1, s_1, u\}$ induces a light claw, contradicting G is 1-heavy.

Suppose $v_0^+ v_2 \in E(G)$. Consider the subgraph induced by $\{v_2, v_2^-, v_0^+, u\}$. By Lemma 3 (a), we have $uv_2^- \notin E(G)$ and $uv_0^+ \notin E(G)$. Since v_2^-, v_0^+, u are light and G is 1-heavy, $v_0^+ v_2^- \in E(G)$. By Lemma 5 (a) and Claim 2, $v_1 l_1^- \in E(G)$ and $l_1 v_2 \in E(G)$. Now $C' = v_1 l_1^- \overleftarrow{C}[l_1^-, v_1^+] v_1^+ v_1^- \overleftarrow{C}[v_1^-, v_0^+] v_0^+ v_2^- \overleftarrow{C}[v_2^-, l_1] l_1 v_2 C[v_2, v_0] v_0 u v_1$ is an o-cycle such that $V(C) \subset V(C')$, a contradiction.

Suppose $s_1 v_2 \in E(G)$. Consider the subgraph induced by $\{v_2, v_2^-, s_1, u\}$. By Lemma 5 (a) and Lemma 3 (b), s_1 and v_2^- are light. Since G is 1-heavy, $s_1 v_2^- \in E(G)$. Now $C' = v_1 u v_2 C[v_2, s_1] s_1 v_2^- \overleftarrow{C}[v_2^-, v_1^+] v_1^+ s_1^+ C[s_1^+, v_1]$ is an o-cycle such that $V(C) \subset V(C')$, a contradiction. (First, we can prove $s_1 v_1^+ \in E(G)$. Otherwise, $\{v_1, s_1, v_1^+, u\}$ induces a light claw, a contradiction. Note that $v_0^+ v_1^+ \notin E(G)$ and $v_0^+ s_1^+ \notin E(G)$ by Lemma 3 (b) and Lemma 4 (b). Then we obtain $v_1^+ s_1^+ \in E(G)$ since otherwise $\{s_1, v_0^+, v_1^+, s_1^+\}$ induces a light claw, a contradiction.)

Case 2. $v_2^- v_2^+ \in E(G)$ or $v_0^- v_0^+ \in E(G)$.

Without loss of generality (by symmetry), assume that $v_2^- v_2^+ \in E(G)$.

Subcase 2.1. $v_0^- v_0^+ \notin E(G)$.

By Lemma 3 (a), $uv_0^- \notin E(G)$ and $uv_0^+ \notin E(G)$. Now $\{v_0, v_0^-, v_0^+, u\}$ induces a claw. Since G is 1-heavy and u is light, v_0^- is heavy or v_0^+ is heavy.

Suppose v_0^- is heavy. By Lemma 3 (b), (c) and Lemma 4 (b), v_2^-, v_1 and l_1^- are light. By Lemma 6 (d), $\{l_1, l_1^-, v_1, v_2^-\}$ induces a light claw, a contradiction.

Suppose v_0^+ is heavy. By Lemma 3 (b), (c) and Lemma 4 (b), we can see v_2^+, v_1 and l_1^+ are light. By Lemma 6 (d), $\{l_1, l_1^+, v_1, v_2^+\}$ induces a light claw, a contradiction. (Note that $v_1^- v_1^+ \in E(G)$ and $v_2^- v_2^+ \in E(G)$. By Lemma 6 (c), $l_1 v_2^+ \in E(G)$.)

Subcase 2.2. $v_0^- v_0^+ \in E(G)$.

By Lemma 6 (b), there exists a vertex $s_2 \in C(v_1^+, v_2^-]$ such that $v_1^+ s_2 \in E(G)$ and $s_2 v_2 \in E(G)$.

Claim 3. (i) v_1 is heavy, (ii) $l_0^-, l_0^+, s_2^-, s_2^+$ are light.

Proof. Recall that the definition of l_0 occurred in the condition of Lemma 5 before. Let $l_0 \in C[v_0^+, v_1^-)$ such that $v_0^- l_0 \in E(G)$ and $l_0 v_0 \in E(G)$.

(i) By Lemma 6 (d), each of $\{l_0, l_0^-, v_0, v_1^-\}$ and $\{l_1, l_1^-, v_1, v_2^-\}$ induces a claw. Since G is 1-heavy, at least one vertex of $\{l_1^-, v_1, v_2^-\}$ is heavy.

Suppose v_2^- is heavy. By Lemma 3 (b), (c) and Lemma 4 (b), v_1^-, v_0 and l_0^- are light. Now $\{l_0, l_0^-, v_0, v_1^-\}$ induces a light claw, contradicting G is 1-heavy.

Suppose l_1^- is heavy. By Lemma 6 (c), $v_2^+ l_1 \in E(G)$. By Lemma 4 (a) and (b), $v_1^- l_1^- \notin \widetilde{E}(G)$ and $v_0 l_1^- \notin \widetilde{E}(G)$. This implies that v_0, v_1^- are light. At the same time, we can prove that $l_0{}^-$ is light. (Otherwise, $C' = v_0 u v_2 \overleftarrow{C}[v_2, l_1] l_1 v_2^+ C[v_2^+, v_0^-] v_0^- v_0^+ C[v_0^+, l_0^-] l_0^- l_1^- \overleftarrow{C}[l_1^-, l_0] l v_0$ is an o-cycle such that $V(C) \subset V(C')$, a contradiction.) Now $\{l_0, l_0^-, v_0, v_1^-\}$ induces a light claw, contradicting G is 1-heavy.

Note that $\{l_1, l_1^-, v_1, v_2^-\}$ induces a claw and v_2^-, l_1^- are light. Since G is 1-heavy, v_1 is heavy.

(ii) Note that v_1 is heavy. If l_0^+ is heavy, then $C' = v_0 u v_1 l_0^+ C[l_0^+, v_1^-] v_1^- v_1^+ C[v_1^+, v_0^-] v_0^- v_0^+ C[v_0^+, l_0] l_0 v_0$ is an o-cycle such that $V(C) \subset V(C')$, a contradiction. If l_0^- is heavy, then $C' = v_0 l_0 C[l_0, v_1^-] v_1^- v_1^+ C[v_1^+, v_0^-] v_0^- v_0^+ C[v_0^+, l_0^-] l_0^- v_1 u v_0$ is an o-cycle such that $V(C) \subset V(C')$, a contradiction. Similarly, by symmetry, we can prove that s_2^-, s_2^+ are light. □

Claim 4. $v_1^- l_0^+ \in E(G)$ and $v_1^- s_2^+ \in E(G)$.

Proof. Suppose $v_1^- l_0^+ \notin E(G)$. By Lemma 4 (b) and (c), $v_1^- l_0^- \notin \widetilde{E}(G)$ and $l_0^- l_0^+ \notin \widetilde{E}(G)$. By Claim 3, l_0^+, l_0^- are light. Now $\{l_0, l_0^-, l_0^+, v_1^-\}$ induces a light claw, a contradiction.

By Lemma 6 (c) and by symmetry, we obtain $v_1^- s_2 \in E(G)$. Suppose $v_1^- s_2^+ \notin E(G)$. By Lemma 4 (b) and (c), $v_1^- s_2^- \notin \widetilde{E}(G)$ and $s_2^- s_2^+ \notin \widetilde{E}(G)$. By Claim 3, s_2^+, s_2^- are light. Now $\{s_2, s_2^-, s_2^+, v_1^-\}$ induces a light claw, a contradiction. □

By Claim 3, Lemma 5 (b) and Claim 4, l_0^+, s_2^+ are light, $\{v_1^+ l_0^+, v_1^+ s_2^+, l_0^+ s_2^+\} \cap E(G) = \emptyset$ and $\{v_1^- l_0^+, v_1^- s_2^+\} \subset E(G)$. It is proved that $\{v_1^-, v_1^+, l_0^+, s_2^+\}$ induces a light claw, a contradiction.

The proof of Theorem 4 is complete. □

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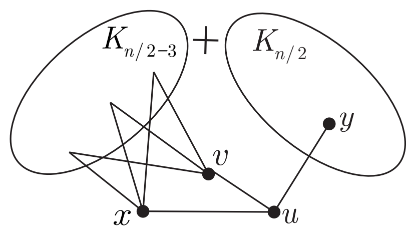

Fig.1 A graph which shows the result in Theorem 3 strengthens that in Theorem 2