BILINEAR EXOTIC CALDERÓN-ZYGMUND OPERATORS

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ABSTRACT. We introduce a bilinear extension of the so-called exotic Calderón-Zygmund operators. These kernels arise naturally from the bilinear singular integrals associated with Zygmund dilations. We show that such a class of operators satisfy a $T1$ theorem in the same form as the standard Calderón-Zygmund operators. However, one-parameter weighted estimates may fail in general, and unlike the linear case, we are not able to provide the end-point estimates in full generality.

1. INTRODUCTION AND MAIN RESULTS

We work in $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, and if $x \in \mathbb{R}^2$, we use (x^1, x^2) to denote its coordinates. Let $K : \mathbb{R}^2 \times \mathbb{R}^2 \setminus \{x^1 = y^1 \text{ or } x^2 = y^2\} \to \mathbb{C}$ satisfy the size estimate

$$|K(x,y)| \lesssim \frac{1}{|x^1 - y^1|} \frac{1}{|x^2 - y^2|} D_{\theta_2}(x,y) \tag{1.1}$$

and the mixed Hölder and size estimate

$$|K(x,y) - K((w^1,x^2),y)| \lesssim \frac{|x^1 - w^1|^{\theta_1}}{|x^1 - y^1|^{1+\theta_1}} \frac{1}{|x^2 - y^2|} D_{\theta_2}(x,y)$$

whenever $|x^1 - w^1| \le |x^1 - y^1|/2$, where $\theta_1, \theta_2 \in (0,1]$ and

$$D_{\theta_2}(x,y) := \left(\frac{|x^1 - y^1|}{|x^2 - y^2|} + \frac{|x^2 - y^2|}{|x^1 - y^1|} \right)^{-\theta_2} < 1. \tag{1.2}$$

We also demand K to satisfy the other three symmetric mixed Hölder and size estimates. Then we call K a linear exotic Calderón-Zygmund kernel (see [10]).

The singularity of such kernels lie in between the standard Calderón-Zygmund kernels and product Calderón-Zygmund kernels [5, 12, 18]. Indeed, this can be seen directly by the fact that for all $\theta_2 \in (0,1]$,

$$\frac{1}{|x^1 - y^1|^2 + |x^2 - y^2|^2} \le \frac{1}{|x^1 - y^1|} \frac{1}{|x^2 - y^2|} D_{\theta_2}(x,y) < \frac{1}{|x^1 - y^1|} \frac{1}{|x^2 - y^2|}.$$

So the standard Calderón-Zygmund kernels satisfy (1.1) automatically, and the linear exotic Calderón-Zygmund kernels always satisfy the size estimate of product Calderón-Zygmund kernels as well.

The study of this class of kernels is motivated by the recent work [11], where the singular integrals associated with Zygmund dilations are systematically studied. In $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ Zygmund dilations are

$$(x_1, x_2, x_3) \mapsto (\delta_1 x_1, \delta_2 x_2, \delta_1 \delta_2 x_3).$$

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A key feature in the singular integrals associated with Zygmund dilations (see [8, 9, 11]) is that, in the kernel estimates there is an extra decay factor of the form

$$(t+\frac{1}{t})^{-\theta}$$

comparing with the standard product Calderón-Zygmund kernels on $\mathbb{R}^3$. This motivates the authors in [10] to consider the above exotic Calderón-Zygmund operators, which can be seen as a counterpart of Zygmund singular integrals on $\mathbb{R}^2$. In fact, the definition of exotic Calderón-Zygmund operators follows exactly the same logic, that is, comparing with the standard product setting in $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, there is such an extra decay factor $D_{\theta_2}(x, y)$.

Extending linear results to the bilinear setting is an important topic in harmonic analysis, see [6] and [16] for instance, where the linear one-parameter and bi-parameter Calderón-Zygmund theory are extended to the bilinear setting, respectively. Recently in [1], Airta, Martikainen and the third named author have extended the result in [11] to the bilinear setting. In particular, using a bilinear variant of the Fefferman-Pipher multiplier (see [4]) as a model, they derived the natural kernel estimates in the bilinear setting, and the $T1$ theorem for paraproduct free bilinear singular integrals associated with Zygmund dilations is also presented. The point we would like to emphasize here is that, similar to the linear case, in the kernel estimates there is also an extra decay factor of the form $(t+t^{-1})^{-\theta}$ comparing with the standard bilinear product theory. This motivates us to consider the decay factor

$$D_\theta(x,y,z) := \left(\frac{|x^1-y^1|+|x^1-z^1|}{|x^2-y^2|+|x^2-z^2|}+\frac{|x^2-y^2|+|x^2-z^2|}{|x^1-y^1|+|x^1-z^1|}\right)^{-\theta}, \tag{1.3}$$

where $|x^1-y^1|+|x^1-z^1| \neq 0$ and $|x^2-y^2|+|x^2-z^2| \neq 0$ and $\theta \in (0, 2]$. Here the range of θ coincides with that in [1]. Then combining with the standard bilinear product kernel estimates we get the bilinear exotic Calderón-Zygmund (bilinear CZX) kernel estimates. The detailed definition will be given in Section 2.

On the other hand, weighted estimates are also a core problem in singular integral theory. In the linear case, weighted estimates are inequalities of the form

$$\int |Tf|^p w \leq C \int |f|^p w, \qquad 1<p<\infty, \tag{1.4}$$

where w is a non-negative locally integrable function. In [10] the authors proved that any linear exotic Calderón-Zygmund operator T satisfies (1.4) when w is a bi-parameter Muckenhoupt A_p weight, and one can use one-parameter Muckenhoupt A_p weight if and only if $\theta_2 = 1$ (we refer the readers to Section 2 for the related notations). One of the main goals of the paper is to extend this result to the bilinear setting.

In the multilinear setting, Lerner et al. [14] first introduced the multilinear $A_{\vec{p}}$ weights, which are the natural extension of Muckenhoupt A_p weights since they characterize the weighted boundedness of the multilinear maximal function and multilinear Riesz transforms. For this reason, weighted estimates for multilinear (multi-parameter) singular integrals using multilinear (multi-parameter) $A_{\vec{p}}$ weights will be referred as genuinely weighted estimates. In [14], the authors proved the genuinely weighted estimates of general multilinear Calderón-Zygmund operators. However, the counterpart in the product setting (i.e. multi-parameter setting) are only formulated recently in [16]. In this paper we focus on the bilinear setting. Recall that we say $\vec{w} = (w_1, w_2)$ is a bilinear $A_{\vec{p}}$ weight in $\mathbb{R}^2$ with $\vec{p} = (p_1, p_2)$ and $1/p = 1/p_1 + 1/p_2$, where $1 \le p_1, p_2 \le \infty$, if

$$[\vec{w}]_{A_{\vec{p}}} = \sup_{Q\colon \text{cubes in } \mathbb{R}^2} \langle w\rangle_{p,Q} \langle w_1^{-1}\rangle_{p_1',Q} \langle w_2^{-1}\rangle_{p_2',Q} < \infty, \qquad w := w_1 w_2.$$

Here if a non-negative function $f \in L^p_{\text{loc}}$, we use $\langle f\rangle_{p,Q}$ to denote its L^p average over Q, namely,

$$\langle f\rangle_{p,Q} = \Big(\frac{1}{|Q|}\int_Q f^p\Big)^{\frac{1}{p}}.$$

Note that in the product setting, the weights are defined similar as above, but just with cubes replaced by rectangles. One may denote the related weighted class by $A^*_{\vec{p}}$. Similar to the linear case, we show that the value of θ is critical for the weighted estimates.

1.5. **Theorem.** *Let T be a bounded operator from $L^2 \times L^2 \to L^1$ with a bilinear CZX kernel.*

(1) *If $\theta < 2$ in (1.3), then for every $\vec{p} = (p_1, p_2)$ with $1 < p_1, p_2 \le \infty$ and $1/p = 1/p_1 + 1/p_2 > 0$, and for every $\vec{w} \in A^*_{\vec{p}}$*

$$\|T(f,g)w\|_{L^p} \le C([\vec{w}]_{A^*_{\vec{p}}})\|fw_1\|_{L^{p_1}}\|gw_2\|_{L^{p_2}}.$$

Moreover, if $\vec{w} \in A_{\vec{p}}$, then the operator T may fail to be bounded from $L^{p_1}(w_1^{p_1}) \times L^{p_2}(w_2^{p_2}) \to L^p(w^p)$ in general.

(2) *If $\theta = 2$ in (1.3), then for every $\vec{p} = (p_1, p_2)$ with $1 < p_1, p_2 \le \infty$ and $1/p = 1/p_1 + 1/p_2 > 0$, and for every $\vec{w} \in A_{\vec{p}}$, the operator T extends boundedly from $L^{p_1}(w_1^{p_1}) \times L^{p_2}(w_2^{p_2}) \to L^p(w^p)$.*

Theorem 1.5 demonstrates that in most cases the operators associated with bilinear CZX kernels behave like bilinear bi-parameter singular integrals when considering weighted estimates, mainly because the kernel estimates are more singular than the one-parameter bilinear Calderón-Zygmund kernels (see Section 2). Nevertheless, they still behave like standard one-parameter bilinear Calderón-Zygmund operators in many ways. In particular, we have the following $T1$ theorem.

1.6. **Theorem.** *Let T be a bilinear operator defined initially on finite linear combinations of characteristic functions of cubes of $\mathbb{R}^2$, and such that*

$$T(f,g)(x) = \iint K(x,y,z)f(y)g(z)\,\mathrm{d}y\,\mathrm{d}z$$

whenever $x \notin \operatorname{supp} f \cap \operatorname{supp} g$, where $K(x,y,z)$ is a bilinear CZX kernel. Suppose that the $T1$ conditions

$$S(1,1) \in \mathrm{BMO} \ \textit{for all } S \in \{T, T^{*1}, T^{*2}\} \tag{1.7}$$

and the weak boundedness property

$$\sup_{Q\colon \textit{cubes in } \mathbb{R}^2} \frac{1}{|Q|}|\langle T(1_Q, 1_Q), 1_Q\rangle| \lesssim 1 \tag{1.8}$$

hold, then T extends to a bounded operator from $L^p \times L^q \to L^r$ for any $1 < p, q \le \infty$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} > 0$. Conversely, if T is a bounded operator from $L^p \times L^q \to L^r$ for some $1 < p, q \le \infty$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} > 0$, then the $T1$ conditions (1.7) *and the weak boundedness property* (1.8) *must* *hold. Moreover, under the above assumptions, the operator T extends to a bounded operator from $L^1 \times L^1 \to L^{\frac{1}{2},\infty}$ if $\theta \in (1,2]$.*

Here, note that the adjoints T^{*1} and T^{*2} are defined via

$$\langle T(f,g),h\rangle = \langle T^{*1}(h,g),f\rangle = \langle T^{*2}(f,h),g\rangle,$$

and the notation $S(1,1)$ for $S \in \{T, T^{*1}, T^{*2}\}$ is defined in the usual way, see e.g. [17]. In below, if T is a bilinear operator associated with a bilinear CZX kernel, and if it satisfies the two conditions in Theorem 1.6 (or equivalently, T is bounded from $L^2 \times L^2 \to L^1$), then we say T is a bilinear exotic Calderón-Zygmund operator.

The quantitative weighted estimates have received a lot of attention in singular integral theory, which is also the main motivation to study the sparse bound of singular integrals. Our last result is concerned with the quantitative weighted estimates when $\theta = 2$, which can be viewed as a completion to the statement (2) of Theorem 1.5 and has independent interest.

1.9. **Theorem.** *Let T be a bilinear exotic Calderón-Zygmund operator with $\theta = 2$ in the decay factor. Then for every $\vec{p} = (p_1, p_2)$ with $1 < p_1, p_2 \le \infty$ and $1/p = 1/p_1 + 1/p_2 > 0$, and for every $\vec{w} \in A_{\vec{p}}$,*

$$\|T(f,g)w\|_{L^p} \lesssim [\vec{w}]_{A_{\vec{p}}}^{3\max\{p,p_1',p_2'\}} \|fw_1\|_{L^{p_1}} \|gw_2\|_{L^{p_2}}.$$

This paper is organized as the following. Section 2 is devoted to providing basic definitions. In Section 3 we give a representation theorem for bilinear exotic Calderón-Zygmund operators, and prove the weighted boundedness of the involved model operators. In particular this gives a proof for the first part of statement (1) of Theorem 1.5 (see Theorem 3.9). In Section 4 we first prove Theorem 1.6 and the second part of statement (1) of Theorem 1.5, then we prove Theorem 1.9, which also serves as a proof of the statement (2) of Theorem 1.5.

Throughout this paper we write $A \lesssim B$ if there is some absolute constant $C > 0$ such that $A \le CB$. Moreover, $A \lesssim_\tau B$ means that the constant C can also depend on some relevant given parameter $\tau > 0$. We also write $A \sim B$ if simultaneously $A \lesssim B$ and $B \lesssim A$. Sometimes we also use $C(\tau)$ to mean a constant depending on τ.

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2. Preliminaries

2.1. **CZX kernel.** Let $\theta \in (0,2]$, $\alpha \in (0,1]$. For $x = (x^1, x^2)$, $y = (y^1, y^2)$, $z = (z^1, z^2) \in \mathbb{R}^2$, we define the decay factor

$$D_\theta(x,y,z) := \left(\frac{|x^1 - y^1| + |x^1 - z^1|}{|x^2 - y^2| + |x^2 - z^2|} + \frac{|x^2 - y^2| + |x^2 - z^2|}{|x^1 - y^1| + |x^1 - z^1|} \right)^{-\theta}$$

whenever $(x,y,z) \in E$, where

$$E := \{(x,y,z) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 : |x^1 - y^1| + |x^1 - z^1| \ne 0 \text{ and } |x^2 - y^2| + |x^2 - z^2| \ne 0\}.$$

We assume that the kernel $K(x,y,z): E \to \mathbb{C}$ satisfies size estimate

$$|K(x,y,z)| \lesssim \frac{1}{\prod_{i=1}^{2}(|x^i-y^i|+|x^i-z^i|)^2} D_\theta(x,y,z)$$

and the mixed Hölder and size estimate

$$|K(x,y,z)-K((\omega^1,x^2),y,z)| \lesssim \left(\frac{|x^1-\omega^1|}{|x^1-y^1|+|x^1-z^1|}\right)^{\alpha} \frac{D_\theta(x,y,z)}{\prod_{i=1}^{2}(|x^i-y^i|+|x^i-z^i|)^2}$$

whenever $|x^1-\omega^1| \le \frac{1}{2}\max\{|x^1-y^1|,|x^1-z^1|\}$, together with other five symmetric mixed Hölder and size estimates. We will denote by $K \in BCZX(\mathbb{R}^2)$ if K satisfies all the above assumptions.

Recall that in the linear case, when $\theta_2 = 1$ one can consider an extra logarithmic factor, that is, instead of (1.2) one can define

$$D_1(x,y) = \left(\frac{|x^1-y^1|}{|x^2-y^2|}+\frac{|x^2-y^2|}{|x^1-y^1|}\right)^{-1} \log\left(\frac{|x^1-y^1|}{|x^2-y^2|}+\frac{|x^2-y^2|}{|x^1-y^1|}\right)$$

and the related results still hold, see [10, Remark 4.10]. This logarithmic factor is relevant from the point of view of Fefferman-Pipher multipliers (an important class of Zygmund type singular integrals, see [4] and [11]). However, currently the understanding to the bilinear Zygmund type singular integrals (see [1]) are not at the same level of the linear case. Hence we do not have strong motivation to discuss the logarithmic factor when $\theta = 2$, we just leave it to the interested readers.

2.2. **Dyadic lattices and Haar functions.** Given a dyadic grid $\mathcal{D}$ in $\mathbb{R}$ (or $\mathbb{R}^2$), $I \in \mathcal{D}$ and $k \in \mathbb{N}$, we define

- $\ell(I)$ is the length of I;
- $I^{(k)} \in \mathcal{D}$ is the k-th parent of I, i.e., $I \subset I^{(k)}$ and $\ell(I^{(k)}) = 2^k \ell(I)$;
- $\mathrm{ch}(I)$ is the collection of the children of I, i.e., $\mathrm{ch}(I) = \{J \in \mathcal{D} : J^{(1)} = I\}$;
- $E_I f = \langle f \rangle_I 1_I$, where $\langle f \rangle_I = \frac{1}{|I|}\int_I f$;
- $\Delta_I f$ is the martingale difference $\Delta_I f = \sum_{J \in \mathrm{ch}(I)} E_J f - E_I f$.

Let $\mathcal{D}_0$ be the standard dyadic grid in $\mathbb{R}$. We define the shifted lattice

$$\mathcal{D}(\omega) := \left\{ L + \omega := L + \sum_{i:\, 2^{-i} < \ell(L)} 2^{-i}\omega_i : L \in \mathcal{D}_0 \right\},$$

where $\omega = (\omega_i)_{i\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}} =: \Omega$. Let $\mathbb{P}_\omega$ be the product probability measure on Ω. Recall the notion of k-good cubes (denoted by $\mathcal{D}(\omega,k)$) introduced in [7]. We say that $G \in \mathcal{D}(\omega,k)$, $k \ge 2$, if $G \in \mathcal{D}(\omega)$ and

$$d(G,\partial G^{(k)}) \ge \frac{\ell(G^{(k)})}{4} = 2^{(k-2)}\ell(G).$$

Observe that for all $L \in \mathcal{D}_0$ and $k \ge 2$ we have

$$\mathbb{P}_\omega(\{\omega : L+\omega \in \mathcal{D}(\omega,k)\}) = \frac{1}{2}. \tag{2.1}$$

We also inherit the notation in [10]. For $\sigma = (\sigma^1, \sigma^2) \in \{0,1\}^{\mathbb{Z}} \times \{0,1\}^{\mathbb{Z}}$ and dyadic $\lambda > 0$ define

$$\begin{aligned} \mathcal{D}(\sigma) &:= \mathcal{D}(\sigma^1) \times \mathcal{D}(\sigma^2), \\ \mathcal{D}_\lambda(\sigma) &:= \{I^1 \times I^2 \in \mathcal{D}(\sigma) \colon \ell(I^1) = \lambda \ell(I^2)\}, \\ \mathcal{D}_\square(\sigma) &:= \mathcal{D}_1(\sigma). \end{aligned}$$

Let $\mathbb{P}_\sigma := \mathbb{P}_{\sigma^1} \times \mathbb{P}_{\sigma^2}$. For $k = (k^1, k^2)$, $k^1, k^2 \geq 2$, we define $\mathcal{D}(\sigma, k) = \mathcal{D}(\sigma^1, k^1) \times \mathcal{D}(\sigma^2, k^2)$.

Given an interval $I \subset \mathbb{R}$, let I_l and I_r be the left and right halves of I. Define

$$h_I^0 = |I|^{-\frac{1}{2}} 1_I \qquad \text{and} \qquad h_I^1 = |I|^{-\frac{1}{2}} (1_{I_l} - 1_{I_r}).$$

Now if $I = I^1 \times I^2$ is a cube, we define the Haar function h_I^η, $\eta = (\eta_1, \eta_2) \in \{0,1\}^2$, via

$$h_I^\eta = h_{I^1}^{\eta^1} \otimes h_{I^2}^{\eta^2}.$$

For notational convenience, we denote $h_I^0 = h_{I^1}^0 \otimes h_{I^2}^0$, and for $\eta \neq (0,0)$, we simply denote h_I^η by h_I. Note that h_I is cancellative. It is well-known that

$$\Delta_I f = \sum_{\eta \neq (0,0)} \langle f, h_I^\eta \rangle h_I^\eta.$$

In below we may abuse of notation to simply write

$$\Delta_I f = \langle f, h_I \rangle h_I$$

since the sum over η does not affect the main results of the paper.

2.3. BMO **functions and weights.** We say a locally integrable function $b \in \mathrm{BMO}(\mathbb{R}^2)$ if

$$\|b\|_{\mathrm{BMO}} = \|b\|_{\mathrm{BMO}(\mathbb{R}^2)} := \sup_{I \colon \text{cubes in } \mathbb{R}^2} \frac{1}{|I|} \int_I |b - \langle b \rangle_I| < \infty.$$

Given $\vec{p} = (p_1, p_2)$ with $1 \leq p_1, p_2 \leq \infty$ and $1/p = 1/p_1 + 1/p_2$. We say $\vec{w} = (w_1, w_2)$ is a bilinear $A_{\vec{p}}$ weight in $\mathbb{R}^2$ if

$$[\vec{w}]_{A_{\vec{p}}} = \sup_{Q \colon \text{cubes in } \mathbb{R}^2} \langle w \rangle_{p,Q} \langle w_1^{-1} \rangle_{p_1',Q} \langle w_2^{-1} \rangle_{p_2',Q} < \infty, \qquad w := w_1 w_2.$$

Here if a non-negative function $f \in L^p_{\mathrm{loc}}$, we use $\langle f \rangle_{p,Q}$ to denote its L^p average over Q, namely,

$$\langle f \rangle_{p,Q} = \Big(\frac{1}{|Q|} \int_Q f^p \Big)^{\frac{1}{p}}.$$

Note that the definition above is a reformulation of the initial definition given in [14]. We also define the bilinear strong $A_{\vec{p}}^*$ weights. Given $\vec{p} = (p_1, p_2)$ with $1 \leq p_1, p_2 \leq \infty$ and $1/p = 1/p_1 + 1/p_2$. We say $\vec{w} = (w_1, w_2)$ is a bilinear strong $A_{\vec{p}}^*$ weight in $\mathbb{R}^2$ if

$$[\vec{w}]_{A_{\vec{p}}^*} = \sup_{R \colon \text{rectangles in } \mathbb{R}^2} \langle w \rangle_{p,R} \langle w_1^{-1} \rangle_{p_1',R} \langle w_2^{-1} \rangle_{p_2',R} < \infty, \qquad w := w_1 w_2.$$

3. Dyadic representation and $T1$ theorem

In this section, we will provide a representation formula for the bilinear exotic Calderón-Zygmund operators, and then use it to deduce the $T1$ theorem. That is, we prove that in the expectation sense, a bilinear exotic Calderón-Zygmund operator can be decomposed into model operators such as bilinear shifts and bilinear paraproducts. And then we prove the $T1$ theorem via the boundedness of the model operators.

We begin with the definition and boundedness properties of the model operators.

3.1. **Definition.** For $k=(k^1,k^2)$, $k^i\ge 0$, we define that the bilinear shift $Q_{k,\sigma}$ has either the form

$$\langle Q_{k,\sigma}(f_1,f_2),f_3\rangle=\sum_{K\in\mathcal{D}_{2^{k^1-k^2}}(\sigma)}\sum_{\substack{I_j\in\mathcal{D}_\square(\sigma)\\ I_j^{(k)}=K}}a_{I_j,K}\big(\langle f_1,h^0_{I_1}\rangle\langle f_2,h^0_{I_2}\rangle-\langle f_1,h^0_{I_3}\rangle\langle f_2,h^0_{I_3}\rangle\big)\langle f_3,h_{I_3}\rangle$$

or the symmetric forms (i.e. the role of f_3 is replaced by f_1 or f_2), or

$$\langle Q_{k,\sigma}(f_1,f_2),f_3\rangle=\sum_{K\in\mathcal{D}_{2^{k^1-k^2}}(\sigma)}\sum_{\substack{I_j\in\mathcal{D}_\square(\sigma)\\ I_j^{(k)}=K}}a_{I_j,K}\prod_{i=1}^{3}\langle f_i,\widetilde{h}_{I_j}\rangle,\tag{3.2}$$

where there exist two indices $j_0,j_1\in\{1,2,3\}$, $j_0\neq j_1$, so that $\widetilde{h}_{I_{j_0}}=h_{I_{j_0}}$, $\widetilde{h}_{I_{j_1}}=h_{I_{j_1}}$ and for the remaining index $j\notin\{j_0,j_1\}$ we have $\widetilde{h}_{I_j}\in\{h_{I_j},h^0_{I_j}\}$. Here no matter in which form, $I_j^{(k)}=I_j^{(k^1,k^2)}=(I_j^1)^{(k^1)}\times(I_j^2)^{(k^2)}$ and

$$|a_{I_j,K}|\le\frac{|I_1|^{\frac{3}{2}}}{|K|^2}.$$

Note that if K is a cube (i.e. $k^1=k^2$), then the boundedness of $Q_{k,\sigma}$ is well-known, see e.g. [2]. It is not surprising that in general $Q_{k,\sigma}$ is still bounded. To establish the boundedness of the model operators, we introduce the definition of general shifts in our set-up: we say $S_{k_1,k_2,k_3,\sigma}$ is a shift with complexity (k_1,k_2,k_3), where $k_j=(k_j^1,k_j^2)\in\mathbb{N}^2$, $j=1,2,3$, if

$$\langle S_{k_1,k_2,k_3,\sigma}(f_1,f_2),f_3\rangle=\sum_{K\in\mathcal{D}_{2^{k_3^1-k_3^2}}(\sigma)}\sum_{\substack{I_3\in\mathcal{D}_\square(\sigma)\\ I_j^{(k_j)}=K}}a_{I_j,K}\langle f_3,h_{I_3}\rangle\prod_{i=1}^{2}\langle f_i,\widetilde{h}_{I_j}\rangle,$$

where there exists $j_0\in\{1,2\}$ with $\widetilde{h}_{I_{j_0}}=h_{I_{j_0}}$ and for $j_1\in\{1,2\}\setminus\{j_0\}$ we have $\widetilde{h}_{I_j}\in\{h_{I_j},h^0_{I_j}\}$. The symmetric form, i.e. the role of f_3 is replaced by f_1 or f_2, is also a general shift. Notice that the key is, among I_j, $j=1,2,3$, at least one of them is a cube.

3.3. **Lemma.** *Let $(w_1,w_2)\in A^*_{(4,4)}$. We have*

$$\big|\langle Q_{k,\sigma}(f_1,f_2),f_3\rangle\big|\lesssim_{[w]_{A^*_{(4,4)}}}(k^1+k^2)^2\|f_1w_1\|_{L^4}\|f_2w_2\|_{L^4}\|f_3w^{-1}\|_{L^2}.$$

Proof. We first show that if $Q_{k,\sigma}$ has the form

$$\langle Q_{k,\sigma}(f_1,f_2),f_3\rangle=\sum_{K\in\mathcal{D}_{2^{k^1-k^2}}(\sigma)}\sum_{\substack{I_j\in\mathcal{D}_\square(\sigma)\\ I_j^{(k)}=K}}a_{I_j,K}\big(\langle f_1,h_{I_1}^0\rangle\langle f_2,h_{I_2}^0\rangle-\langle f_1,h_{I_3}^0\rangle\langle f_2,h_{I_3}^0\rangle\big)\langle f_3,h_{I_3}\rangle,$$

then it can be rewritten as sum of general shifts. This process will be quite similar as [2, Lemma 2.18] (where the case $k^1=k^2$ is addressed). By symmetry, we may assume $k^1>k^2$. We have

$$|I_1|^{-1}\Big[\langle f_1,h_{I_1}^0\rangle\langle f_2,h_{I_2}^0\rangle-\langle f_1,h_{I_3}^0\rangle\langle f_2,h_{I_3}^0\rangle\Big]=\langle f_1\rangle_{I_1}\langle f_2\rangle_{I_2}-\langle f_1\rangle_{I_3}\langle f_2\rangle_{I_3}.$$

Then write

$$\begin{aligned}(3.4)\quad &\langle f_1\rangle_{I_1}\langle f_2\rangle_{I_2}-\langle f_1\rangle_{I_3}\langle f_2\rangle_{I_3}\\ &=\big(\langle f_1\rangle_{I_1}-\langle f_1\rangle_K\big)\langle f_2\rangle_{I_2}+\langle f_1\rangle_K\big(\langle f_2\rangle_{I_2}-\langle f_2\rangle_K\big)+\langle f_1\rangle_K\langle f_2\rangle_K-\langle f_1\rangle_{I_3}\langle f_2\rangle_{I_3}.\end{aligned}$$

Since

$$\langle f_1\rangle_{I_1}-\langle f_1\rangle_K=\sum_{i=1}^{k^2}\langle\Delta_{I_1^{(i)}}f_1\rangle_{I_1}+\sum_{i=k^2+1}^{k^1}\langle\Delta^1_{(I_1^1)^{(i)}}E^2_{K^2}f_1\rangle_{I_1},$$

we have

$$\begin{aligned}&\sum_{K\in\mathcal{D}_{2^{k^1-k^2}}(\sigma)}\sum_{\substack{I_j\in\mathcal{D}_\square(\sigma)\\ I_j^{(k)}=K}}a_{I_j,K}|I_1|\big(\langle f_1\rangle_{I_1}-\langle f_1\rangle_K\big)\langle f_2\rangle_{I_2}\langle f_3,h_{I_3}\rangle\\
&=\sum_{i=1}^{k^2}\sum_{K\in\mathcal{D}_{2^{k^1-k^2}}(\sigma)}\sum_{\substack{I_j\in\mathcal{D}_\square(\sigma)\\ I_j^{(k)}=K\\ j=2,3}}\sum_{L_1^{(k^1-i,k^2-i)}=K}\sum_{\substack{I_1\in\mathcal{D}_\square(\sigma)\\ I_1^{(i)}=L_1}}a_{I_j,K}|I_1|^{\frac12}\langle f_1,h_{L_1}\rangle\\
&\qquad\times\langle h_{L_1}\rangle_{I_1}\langle f_2,h_{I_2}^0\rangle\langle f_3,h_{I_3}\rangle\\
&+\sum_{i=k^2+1}^{k^1}\sum_{K\in\mathcal{D}_{2^{k^1-k^2}}(\sigma)}\sum_{\substack{I_j\in\mathcal{D}_\square(\sigma)\\ I_j^{(k)}=K\\ j=2,3}}\sum_{(L_1^1)^{(k^1-i)}=K^1}\sum_{\substack{I_1\in\mathcal{D}_\square(\sigma)\\ (I_1^1)^{(i)}=L_1^1\\ (I_1^2)^{(k^2)}=K^2}}a_{I_j,K}\frac{|I_1|^{\frac12}}{|K^2|^{\frac12}}\\
&\qquad\times\langle f_1,h_{L_1^1}\otimes h^0_{K^2}\rangle\langle h_{L_1^1}\rangle_{I_1^1}\langle f_2,h_{I_2}^0\rangle\langle f_3,h_{I_3}\rangle.\end{aligned}$$

Now note that

$$\Big|\sum_{\substack{I_1\in\mathcal{D}_\square(\sigma)\\ I_1^{(i)}=L_1}}a_{I_j,K}|I_1|^{\frac12}\langle h_{L_1}\rangle_{I_1}\Big|\le\sum_{\substack{I_1\in\mathcal{D}_\square(\sigma)\\ I_1^{(i)}=L_1}}\frac{|I_2|^{\frac12}|I_3|^{\frac12}}{|K|^2}|I_1||L_1|^{-\frac12}=\frac{|L_1|^{\frac12}|I_2|^{\frac12}|I_3|^{\frac12}}{|K|^2}$$

and

$$\Big|\sum_{\substack{I_1\in\mathcal{D}_\square(\sigma)\\ (I_1^1)^{(i)}=L_1^1\\ (I_1^2)^{(k^2)}=K^2}}a_{I_j,K}\frac{|I_1|^{\frac12}}{|K^2|^{\frac12}}\langle h_{L_1^1}\rangle_{I_1^1}\Big|\le\sum_{\substack{I_1\in\mathcal{D}_\square(\sigma)\\ (I_1^1)^{(i)}=L_1^1\\ (I_1^2)^{(k^2)}=K^2}}\frac{|I_2|^{\frac12}|I_3|^{\frac12}}{|K|^2}|I_1||L_1^1|^{-\frac12}|K^2|^{-\frac12}$$

$$= \frac{|L_1^1 \times K^2|^{\frac{1}{2}} |I_2|^{\frac{1}{2}} |I_3|^{\frac{1}{2}}}{|K|^2}.$$

So the object involved with the first term in the RHS of (3.4) can be decomposed into

$$\sum_{i=1}^{k^2} \langle S_{(k^1-i,k^2-i),k,k,\sigma}(f_1, f_2), f_3\rangle + \sum_{i=k^2+1}^{k^1} \langle S_{(k^1-i,0),k,k,\sigma}(f_1, f_2), f_3\rangle.$$

Similarly the object involved with the second term in the RHS of (3.4) can be decomposed into

$$\sum_{i=1}^{k^2} \langle S_{(0,0),(k^1-i,k^2-i),k,\sigma}(f_1, f_2), f_3\rangle + \sum_{i=k^2+1}^{k^1} \langle S_{(0,0),(k^1-i,0),k,\sigma}(f_1, f_2), f_3\rangle.$$

For the last term, since

$$\begin{aligned} \langle f_1\rangle_K \langle f_2\rangle_K - \langle f_1\rangle_{I_3} \langle f_2\rangle_{I_3} = &\sum_{i=k^2+1}^{k^1} \big(\langle f_1\rangle_{I_3^{(i,k^2)}} \langle f_2\rangle_{I_3^{(i,k^2)}} - \langle f_1\rangle_{I_3^{(i-1,k^2)}} \langle f_2\rangle_{I_3^{(i-1,k^2)}}\big) \\ &+ \sum_{i=1}^{k^2} \big(\langle f_1\rangle_{I_3^{(i)}} \langle f_2\rangle_{I_3^{(i)}} - \langle f_1\rangle_{I_3^{(i-1)}} \langle f_2\rangle_{I_3^{(i-1)}}\big). \end{aligned}$$

This is similar as well since for instance, we can write

$$\langle f_1\rangle_{I_3^{(i)}} \langle f_2\rangle_{I_3^{(i)}} - \langle f_1\rangle_{I_3^{(i-1)}} \langle f_2\rangle_{I_3^{(i-1)}} = -\langle \Delta_{I_3^{(i)}} f_1\rangle_{I_3} \langle f_2\rangle_{I_3^{(i-1)}} - \langle f_1\rangle_{I_3^{(i)}} \langle \Delta_{I_3^{(i)}} f_2\rangle_{I_3}.$$

Then we use $I_3^{(i)}$ as the new 'parent', for example, we have

$$\begin{aligned} &\sum_{K\in\mathcal{D}_{2^{k^1-k^2}}(\sigma)} \sum_{\substack{I_j\in\mathcal{D}_\square(\sigma)\\ I_j^{(k)}=K}} a_{I_j,K} |I_1| \langle \Delta_{I_3^{(i)}} f_1\rangle_{I_3} \langle f_2\rangle_{I_3^{(i-1)}} \langle f_3, h_{I_3}\rangle \\ &= \sum_{L\in\mathcal{D}_\square(\sigma)} \sum_{\substack{L_2,I_3\in\mathcal{D}_\square(\sigma)\\ L_2^{(1)}=I_3^{(i)}=L}} \Big(\sum_{\substack{I_1,I_2\in\mathcal{D}_\square(\sigma)\\ I_1^{(k)}=I_2^{(k)}=I_3^{(k)}}} a_{I_j,I_3^{(k)}} |I_1| \langle h_L\rangle_{I_3} |L_2|^{-\frac{1}{2}} \delta(I_3, L_2)\Big) \\ &\qquad \times \langle f_1, h_L\rangle \langle f_2, h^0_{L_2}\rangle \langle f_3, h_{I_3}\rangle, \end{aligned} \tag{3.5}$$

where $\delta(I_3, L_2) = 1$ if $I_3 \subset L_2$ and otherwise $\delta(I_3, L_2) = 0$. One can check that

$$\begin{aligned} &\Big| \sum_{\substack{I_1,I_2\in\mathcal{D}_\square(\sigma)\\ I_1^{(k)}=I_2^{(k)}=L^{(k^1-i,k^2-i)}}} a_{I_j,I_3^{(k)}} |I_1| \langle h_L\rangle_{I_3} |L_2|^{-\frac{1}{2}} \delta(I_3, L_2)\Big| \\ &\quad \le \sum_{\substack{I_1,I_2\in\mathcal{D}_\square(\sigma)\\ I_1^{(k)}=I_2^{(k)}=I_3^{(k)}}} \frac{|I_1||I_2||I_3|^{\frac{1}{2}}}{|I_3^{(k)}|^2} \frac{1}{|L|^{\frac{1}{2}}|L_2|^{\frac{1}{2}}} = \frac{|I_3|^{\frac{1}{2}}}{|L|^{\frac{1}{2}}|L_2|^{\frac{1}{2}}} = \frac{4|L|^{\frac{1}{2}}|L_2|^{\frac{1}{2}}|I_3|^{\frac{1}{2}}}{|L|^2}. \end{aligned}$$

Hence the RHS of (3.5) is $4\langle S_{(0,0),(1,1),(i,i),\sigma}(f_1, f_2), f_3\rangle$. The rest terms can be handled similarly. In summary, we have reduced the problem to general shifts. To finish the proof we invoke the following sparse domination result, which is essentially proved in [17, Section 5].

3.6. **Proposition.** *Let $\eta \in (0,1)$ and $S_{k_1,k_2,k_3,\sigma}$ be a shift with complexity (k_1,k_2,k_3) with $k_j = (k_j^1, k_j^2) \in \mathbb{N}^2$, $j = 1,2,3$ of the form*

$$\langle S_{k_1,k_2,k_3,\sigma}(f_1,f_2), f_3\rangle = \sum_{K \in \mathcal{D}_{2^{k_3^1-k_3^2}}(\sigma)} \sum_{\substack{I_3 \in \mathcal{D}_\square(\sigma) \\ I_j^{(k_j)} = K}} a_{I_j,K} \langle f_3, h_{I_3}\rangle \prod_{i=1}^{2} \langle f_i, \widetilde{h}_{I_j}\rangle.$$

Then there exists an η-sparse family $\mathcal{S} \subset \mathcal{D}_{2^{k_3^1-k_3^2}}(\sigma)$ such that

$$\left|\langle S_{k_1,k_2,k_3,\sigma}(f_1,f_2), f_3\rangle\right| \lesssim \max_{i,j}\{k_j^i\} \sum_{S \in \mathcal{S}} |S| \prod_{j=1}^{3} \langle |f_j|\rangle_S.$$

Here recall that we say $\mathcal{S} \subset \mathcal{D}_\lambda(\sigma)$ for some dyadic $\lambda > 0$ is an η-sparse family if for any $S \in \mathcal{S}$ there exists a measurable $E_S \subset S$ with $|E_S| \geq \eta|S|$ such that $\{E_S\}_{S\in\mathcal{S}}$ are pairwise disjoint.

With Proposition 3.6 at hand, note that by our decomposition process we have $\max_{i,j}\{k_j^i\} = \max\{k^1, k^2\}$, then we have

$$\left|\langle Q_{k,\sigma}(f_1,f_2), f_3\rangle\right| \lesssim (k^1+k^2)^2 \sup_{\mathcal{S}} \sum_{S\in\mathcal{S}} |S| \prod_{j=1}^{3} \langle |f_j|\rangle_S.$$

Let $(w_1,w_2) \in A^*_{(4,4)}$ and $w = w_1 w_2$. Then of course $(w_1,w_2) \in A_{(4,4)}(\mathcal{D}_{2^{k_3^1-k_3^2}}(\sigma))$ and by standard Carleson embedding theorem we have

$$\begin{aligned}
\sum_{S\in\mathcal{S}} |S| \prod_{j=1}^{3} \langle |f_j|\rangle_S &= \sum_{S\in\mathcal{S}} \Big(\prod_{j=1}^{2} \langle |f_j|\rangle_S\Big)\Big(\frac{1}{w^2(S)}\int_S |f_3|\Big) w^2(S) \\
&\leq \sum_{S\in\mathcal{S}} \left(\frac{1}{w^2(S)}\int_S \big[M_{\mathcal{D}_{2^{k_3^1-k_3^2}}(\sigma)}(f_1,f_2) M^{w^2}_{\mathcal{D}_{2^{k_3^1-k_3^2}}(\sigma)}(f_3 w^{-2})\big]^{\frac{1}{2}} w^2\right)^2 w^2(S) \\
&\lesssim_{[w]_{A^*_{(4,4)}}} \int M_{\mathcal{D}_{2^{k_3^1-k_3^2}}(\sigma)}(f_1,f_2) M^{w^2}_{\mathcal{D}_{2^{k_3^1-k_3^2}}(\sigma)}(f_3 w^{-2}) w^2 \\
&\leq \|M_{\mathcal{D}_{2^{k_3^1-k_3^2}}(\sigma)}(f_1,f_2)\|_{L^2(w^2)} \|M^{w^2}_{\mathcal{D}_{2^{k_3^1-k_3^2}}(\sigma)}(f_3 w^{-2})\|_{L^2(w^2)} \\
&\lesssim_{[w]_{A^*_{(4,4)}}} \|f_1 w_1\|_{L^4} \|f_2 w_2\|_{L^4} \|f_3 w^{-1}\|_{L^2},
\end{aligned}$$

where

$$M_{\mathcal{D}_{2^{k_3^1-k_3^2}}(\sigma)}(f_1,f_2)(x) := \sup_{Q\in\mathcal{D}_{2^{k_3^1-k_3^2}}(\sigma)} 1_Q(x) \prod_{j=1}^{2} \langle |f_j|\rangle_Q$$

and

$$M^{w^2}_{\mathcal{D}_{2^{k_3^1-k_3^2}}(\sigma)}(h)(x) := \sup_{Q\in\mathcal{D}_{2^{k_3^1-k_3^2}}(\sigma)} 1_Q(x) \frac{1}{w^2(Q)} \int_Q |h| w^2,$$

whose boundedness are well-known. Then it follows that

$$\left|\langle Q_{k,\sigma}(f_1,f_2), f_3\rangle\right| \lesssim_{[w]_{A^*_{(4,4)}}} (k^1+k^2)^2 \|f_1 w_1\|_{L^4} \|f_2 w_2\|_{L^4} \|f_3 w^{-1}\|_{L^2}.$$

□

3.7. **Definition.** We define the bilinear (one-parameter) paraproduct $\pi_{b,\sigma}$ if it has the form

$$\langle \pi_{b,\sigma}(f_1, f_2), f_3 \rangle = \sum_{I \in \mathcal{D}_\square(\sigma)} \langle b, h_I \rangle \langle f_1 \rangle_I \langle f_2 \rangle_I \langle f_3, h_I \rangle$$

or the symmetric forms (i.e., either f_1 or f_2 is pairing with h_I), where $b \in \mathrm{BMO}$.

The (weighted) boundedness of $\pi_{b,\sigma}$ is well-known. See e.g. [17, Section 5] for the sparse bound of $\pi_{b,\sigma}$ and from which the corresponding (weighted) boundedness follows immediately. One can also use the $H^1 - \mathrm{BMO}$ duality to obtain the boundedness of $\pi_{b,\sigma}$ directly.

3.8. **Lemma.** *Let $(w_1, w_2) \in A_{(4,4)}$. Then*

$$\big|\langle \pi_{b,\sigma}(f_1, f_2), f_3 \rangle\big| \lesssim_{[w]_{A_{(4,4)}}} \|f_1 w_1\|_{L^4} \|f_2 w_2\|_{L^4} \|f_3 w^{-1}\|_{L^2}.$$

Now we are ready to present our representation theorem.

3.9. **Theorem.** *Let T be a bilinear operator associated with a bilinear CZX kernel satisfying the weak boundedness property and the $T1$ conditions. Then we have*

$$\langle T(f_1, f_2), f_3 \rangle = \mathbb{E}_\sigma \Big[\sum_{k^1, k^2 \ge 0} \varphi(k) \langle Q_{k,\sigma}(f_1, f_2), f_3 \rangle + \sum_{i=1}^{3} \langle \pi_{b_i,\sigma}(f_1, f_2), f_3 \rangle \Big], \tag{3.10}$$

where $\varphi(k) \lesssim 2^{-\alpha k_{\min}} 2^{-\theta(k_{\max} - k_{\min})}$ and $k_{\max} = \max_{i=1,2} k^i$, $k_{\min} = \min_{i=1,2} k^i$, and $\{b_1, b_2, b_3\} = \{T(1,1), T^{*1}(1,1), T^{*2}(1,1)\}$. In particular, we have for all $1 < p, q \le \infty$ with $1/p + 1/q = 1/r > 0$

$$\|T(f_1, f_2) w\|_{L^r} \lesssim C([(w_1, w_2)]_{A^*_{(p,q)}}) \|f_1 w_1\|_{L^p} \|f_2 w_2\|_{L^q}$$

holds for every $(w_1, w_2) \in A^*_{(p,q)}$.

Proof. Assuming (3.10) momentarily, by Lemmata 3.3 and 3.8 we have

$$\begin{aligned} \big|\langle T(f_1, f_2), f_3 \rangle\big| &\lesssim C([(w_1, w_2)]_{A^*_{(4,4)}}) \|f_1 w_1\|_{L^4} \|f_2 w_2\|_{L^4} \|f_3 w^{-1}\|_{L^2} \sum_{k^1, k^2 \ge 0} (k^1 + k^2)^2 \varphi(k) \\ &\lesssim C([(w_1, w_2)]_{A^*_{(4,4)}}) \|f_1 w_1\|_{L^4} \|f_2 w_2\|_{L^4} \|f_3 w^{-1}\|_{L^2}. \end{aligned}$$

Hence by duality

$$\|T(f_1, f_2) w\|_{L^2} \lesssim C([(w_1, w_2)]_{A^*_{(4,4)}}) \|f_1 w_1\|_{L^4} \|f_2 w_2\|_{L^4}.$$

Then the desired estimate follows from extrapolation (see e.g. [16, Theorem 3.12]).

It remains to prove (3.10). First of all, note that

$$\begin{aligned} \langle T(f_1, f_2), f_3 \rangle &= \sum_{I_1, I_2, I_3 \in \mathcal{D}_\square(\sigma)} \langle T(\Delta_{I_1} f_1, \Delta_{I_2} f_2), \Delta_{I_3} f_3 \rangle \\ &= \sum_{\ell(I_1), \ell(I_2) > \ell(I_3)} + \sum_{\ell(I_2), \ell(I_3) > \ell(I_1)} + \sum_{\ell(I_3), \ell(I_1) > \ell(I_2)} + \sum_{\ell(I_1) > \ell(I_2) = \ell(I_3)} \\ &\quad + \sum_{\ell(I_2) > \ell(I_3) = \ell(I_1)} + \sum_{\ell(I_3) > \ell(I_1) = \ell(I_2)} + \sum_{\ell(I_1) = \ell(I_2) = \ell(I_3)} \\ &= \Sigma_1 + \Sigma_2 + \cdots + \Sigma_7. \end{aligned}$$

The general philosophy here is "contraction produces cancellation". For example, for Σ_1 we have

$$\begin{aligned}\Sigma_1 &= \sum_{\ell(I_1),\ell(I_2)>\ell(I_3)} \langle T(\Delta_{I_1} f_1, \Delta_{I_2} f_2), \Delta_{I_3} f_3\rangle \\ &= \sum_{\ell(I_1)=\ell(I_2)=\ell(I_3)} \langle T(E_{I_1} f_1, E_{I_2} f_2), \Delta_{I_3} f_3\rangle \\ &= \sum_{\ell(I_1)=\ell(I_2)=\ell(I_3)} \langle T(h^0_{I_1}, h^0_{I_2}), h_{I_3}\rangle \langle f_1, h^0_{I_1}\rangle\langle f_2, h^0_{I_2}\rangle\langle f_3, h_{I_3}\rangle \\ &= \sum_{\ell(I_1)=\ell(I_2)=\ell(I_3)} \langle T(h^0_{I_1}, h^0_{I_2}), h_{I_3}\rangle \big(\langle f_1, h^0_{I_1}\rangle\langle f_2, h^0_{I_2}\rangle\langle f_3, h_{I_3}\rangle \\ &\qquad - \langle f_1, h^0_{I_3}\rangle\langle f_2, h^0_{I_3}\rangle\langle f_3, h_{I_3}\rangle + \langle f_1, h^0_{I_3}\rangle\langle f_2, h^0_{I_3}\rangle\langle f_3, h_{I_3}\rangle\big),\end{aligned}$$

where the last term is readily a paraproduct. For the first two terms, we denote $I_3 = I, I_2 = I \dotplus n_2, I_1 = I \dotplus n_1$, where $I \dotplus n = I + n\ell(I)$, then we have

$$\begin{aligned}&\sum_{\ell(I_1)=\ell(I_2)=\ell(I_3)} \langle T(h^0_{I_1}, h^0_{I_2}), h_{I_3}\rangle \big[\langle f_1, h^0_{I_1}\rangle\langle f_2, h^0_{I_2}\rangle - \langle f_1, h^0_{I_3}\rangle\langle f_2, h^0_{I_3}\rangle\big]\langle f_3, h_{I_3}\rangle \\ &= \sum_{\substack{n_1,n_2\in\mathbb{Z}^2 \\ \max(|n_1^1|,|n_2^1|)\neq 0 \\ \text{or} \max(|n_1^2|,|n_2^2|)\neq 0}} \sum_{I\in\mathcal{D}_\square(\sigma)} \langle T(h^0_{I\dotplus n_1}, h^0_{I\dotplus n_2}), h_I\rangle \\ &\qquad\qquad \times \big[\langle f_1, h^0_{I\dotplus n_1}\rangle\langle f_2, h^0_{I\dotplus n_2}\rangle - \langle f_1, h^0_I\rangle\langle f_2, h^0_I\rangle\big]\langle f_3, h_I\rangle\end{aligned}$$

Here we can reduce to $\max(|n_1^1|, |n_2^1|) \neq 0$ or $\max(|n_1^2|, |n_2^2|) \neq 0$ since otherwise $n_1, n_2 = (0, 0)$ and then

$$\langle f_1, h^0_{I_1}\rangle\langle f_2, h^0_{I_2}\rangle - \langle f_1, h^0_{I_3}\rangle\langle f_2, h^0_{I_3}\rangle = 0.$$

Now write

$$\sum_{\substack{n_1,n_2\in\mathbb{Z}^2 \\ \max(|n_1^1|,|n_2^1|)\neq 0 \\ \text{or} \max(|n_1^2|,|n_2^2|)\neq 0}} = \sum_{\substack{n_1,n_2\in\mathbb{Z}^2 \\ \max(|n_1^1|,|n_2^1|)\neq 0 \\ \max(|n_1^2|,|n_2^2|)\neq 0}} + \sum_{\substack{n_1,n_2\in\mathbb{Z}^2 \\ \max(|n_1^1|,|n_2^1|)\neq 0 \\ n_1^2=n_2^2=0}} + \sum_{\substack{n_1,n_2\in\mathbb{Z}^2 \\ n_1^1=n_2^1=0 \\ \max(|n_1^2|,|n_2^2|)\neq 0}} \tag{3.11}$$

We are in the position to invoke the goodness of I. Indeed, for the first term, using (2.1) we have

$$\begin{aligned}\sum_{\substack{n_1,n_2\in\mathbb{Z}^2 \\ \max(|n_1^1|,|n_2^1|)\neq 0 \\ \max(|n_1^2|,|n_2^2|)\neq 0}} \sum_{I\in\mathcal{D}_\square(\sigma)} &= 4\mathbb{E}_\sigma \sum_{k^1,k^2=2}^{\infty} \sum_{\substack{n_1,n_2\in\mathbb{Z}^2 \\ \max(|n_1^1|,|n_2^1|)\in(2^{k^1-3},2^{k^1-2}] \\ \max(|n_1^2|,|n_2^2|)\in(2^{k^2-3},2^{k^2-2}]}} \sum_{I\in\mathcal{D}(\sigma,k)} \\ &= 4\mathbb{E}_\sigma \sum_{k^1,k^2=2}^{\infty} \sum_{\substack{n_1,n_2\in\mathbb{Z}^2 \\ \max(|n_1^1|,|n_2^1|)\in(2^{k^1-3},2^{k^1-2}] \\ \max(|n_1^2|,|n_2^2|)\in(2^{k^2-3},2^{k^2-2}]}} \sum_{K\in\mathcal{D}_{2^{k^1-k^2}}(\sigma)} \sum_{\substack{I\in\mathcal{D}(\sigma,k) \\ I^{(k)}=K}}\end{aligned}$$

Note that when $\max(|n_1^1|,|n_2^1|) \in (2^{k^1-3}, 2^{k^1-2}]$ and $\max(|n_1^2|,|n_2^2|) \in (2^{k^2-3}, 2^{k^2-2}]$ we have

$$(I \dotplus n_1)^{(k)} = (I \dotplus n_2)^{(k)} = K.$$

Therefore, the first term in the RHS of (3.11) can be written as

$$\begin{aligned} 4\mathbb{E}_\sigma \sum_{k^1,k^2=2}^{\infty} \sum_{K \in \mathcal{D}_{2^{k^1-k^2}}(\sigma)} \sum_{\substack{I_j \in \mathcal{D}^2(\sigma,k) \\ I_j^{(k)}=K}} & \gamma_{I_j} \langle T(h^0_{I_1}, h^0_{I_2}), h_{I_3}\rangle \\ & \times \big[\langle f_1, h^0_{I_1}\rangle\langle f_2, h^0_{I_2}\rangle - \langle f_1, h^0_{I_3}\rangle\langle f_2, h^0_{I_3}\rangle\big]\langle f_3, h_{I_3}\rangle, \end{aligned}$$

where $\gamma_{I_j} = 1$ if there is $n_1, n_2 \in \mathbb{Z}^2$ with $\max(|n_1^1|,|n_2^1|) \in (2^{k^1-3}, 2^{k^1-2}]$ and $\max(|n_1^2|,|n_2^2|) \in (2^{k^2-3}, 2^{k^2-2}]$ such that $I_1 = I_3 \dotplus n_1$ and $I_2 = I_3 \dotplus n_2$, otherwise $\gamma_{I_j} = 0$. The second and third terms in the RHS of (3.11) can be handled similarly.

Note that Σ_2 and Σ_3 are symmetrical to Σ_1. Regarding Σ_4, we have

$$\begin{aligned} \Sigma_4 &= \sum_{\ell(I_1)>\ell(I_2)=\ell(I_3)} \langle T(\Delta_{I_1} f_1, \Delta_{I_2} f_2), \Delta_{I_3} f_3\rangle = \sum_{\ell(I_1)=\ell(I_2)=\ell(I_3)} \langle T(E_{I_1} f_1, \Delta_{I_2} f_2), \Delta_{I_3} f_3\rangle \\ &= \sum_{\ell(I_1)=\ell(I_2)=\ell(I_3)} \langle T(h^0_{I_1}, h_{I_2}), h_{I_3}\rangle\langle f_1, h^0_{I_1}\rangle\langle f_2, h_{I_2}\rangle\langle f_3, h_{I_3}\rangle. \end{aligned}$$

Then everything can be handled just as above, the only difference is now we allow the case $n_1 = n_2 = 0$. Then again Σ_5 and Σ_6 are symmetrical to Σ_4. Finally, Σ_7 is similar as well, note that we even have more cancellative Haar functions than we need.

Now the main problem is to estimate $\langle T(h^0_{I\dotplus n_1}, h^0_{I\dotplus n_2}), h_I\rangle$. We remark that the estimate works also for e.g. $\langle T(h^0_{I\dotplus n_1}, h_{I\dotplus n_2}), h_I\rangle$, we will only use that $|h_{I\dotplus n_2}| \le |I|^{-\frac{1}{2}} 1_{I\dotplus n_2}$. Set $m^i = \max_{j=1,2} |n_j^i|$, then the analysis of the coefficients splits to the following three cases :

$$\begin{cases} m^i \in (2^{k^i-3}, 2^{k^i-2}] \text{ for some } i \in \{1,2\} \text{ and } k^i \ge 3, & \text{(Separated)} \\ \max_{i=1,2} m^i = 1 & \text{(Adjacent)} \\ m^1 = m^2 = 0 & \text{(Identical)} \end{cases}$$

In below we assume that $m^i = n_1^i$ for $i = 1, 2$ since the other cases are similar.

3.1. **Separated.** This case is split into three sub-cases.

Case I. $|n_1^i| \ge 2$, $i = 1, 2$. In this case there exist some k^1, k^2 such that

$$2^{k^i-3}\ell(I) \le (|n_1^i| - 1)\ell(I) \le |x^i - y^i| \le (|n_1^i| + 1)\ell(I) \le 2^{k^i}\ell(I),$$

$i = 1, 2$, so we have

$$\begin{aligned} D_\theta(x,y,z) &= \left(\frac{|x^1-y^1| + |x^1-z^1|}{|x^2-y^2| + |x^2-z^2|} + \frac{|x^2-y^2| + |x^2-z^2|}{|x^1-y^1| + |x^1-z^1|} \right)^{-\theta} \\ &\sim \left(\frac{|x^1-y^1|}{|x^2-y^2|} + \frac{|x^2-y^2|}{|x^1-y^1|} \right)^{-\theta} \sim \left(2^{k^1-k^2} + 2^{k^2-k^1} \right)^{-\theta}. \end{aligned}$$

Let $c_I = (c_I^1, c_I^2)$, then

$$|\langle T(h^0_{I\dotplus n_1}, h^0_{I\dotplus n_2}), h_I\rangle|$$

$$
\begin{aligned}
&=\Big|\iiint (K(x,y,z)-K(c_I,y,z))h^0_{I\dot{+}n_1}(y)h^0_{I\dot{+}n_2}(z)h_I(x)dxdydz\Big|\\
&\lesssim \iiint \Bigg|\Big(\frac{|x^1-c^1|}{|x^1-y^1|+|x^1-z^1|}+\frac{|x^2-c^2|}{|x^2-y^2|+|x^2-z^2|}\Big)^{\alpha}\\
&\quad\times (2^{k^1-k^2}+2^{k^2-k^1})^{-\theta}\frac{h^0_{I\dot{+}n_1}(y)h^0_{I\dot{+}n_2}(z)h_I(x)}{\prod_{i=1}^{2}(|x^i-y^i|+|x^i-z^i|)^2}\Bigg|dxdydz\\
&\lesssim 2^{-k_{\min}\alpha}2^{-\theta(k_{\max}-k_{\min})}\frac{|I|^{\frac{3}{2}}}{|K|^2}.
\end{aligned}
$$

Case II. $|n_1^1|<2\le|n_1^2|$. In this case we have $|x^2-y^2|\sim 2^{k^2}\ell(I)$, then

$$
\begin{aligned}
D_\theta(x,y,z)&=\Big(\frac{|x^1-y^1|+|x^1-z^1|}{|x^2-y^2|+|x^2-z^2|}+\frac{|x^2-y^2|+|x^2-z^2|}{|x^1-y^1|+|x^1-z^1|}\Big)^{-\theta}\\
&\sim\Big(\frac{2^{k^2}\ell(I)}{|x^1-y^1|+|x^1-z^1|}\Big)^{-\theta}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
|\langle T(h^0_{I\dot{+}n_1},h^0_{I\dot{+}n_2}),h_I\rangle|&\le\iiint |K(x,y,z)|h^0_{I\dot{+}n_1}(y)h^0_{I\dot{+}n_2}(z)h^0_I(x)\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
&\lesssim\iiint\frac{1}{|K^2|^2}\Big(\frac{2^{k^2}\ell(I)}{|x^1-y^1|+|x^1-z^1|}\Big)^{-\theta}\frac{h^0_{I\dot{+}n_1}(y)h^0_{I\dot{+}n_2}(z)h^0_I(x)}{(|x^1-y^1|+|x^1-z^1|)^2}\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
&\lesssim\frac{\ell(I)^{\frac{3}{2}}}{|K^2|^2}\iiint(2^{k^2}\ell(I))^{-\theta}\frac{h^0_{I^1\dot{+}n_1^1}(y^1)h^0_{I^1\dot{+}n_2^1}(z^1)h^0_{I^1}(x^1)}{|(y^1,z^1)-(x^1,x^1)|^{2-\theta}}\mathrm{d}y^1\mathrm{d}z^1\mathrm{d}x^1\\
&\lesssim\frac{\ell(I)^{\frac{1}{2}}}{|K^2|^2}(2^{k^2}\ell(I))^{-\theta}\ell(I)^{\theta}\int h^0_{I^1}(x^1)\,\mathrm{d}x^1\\
&=2^{-k^2\theta}\frac{\ell(I)}{|K^2|^2}\sim 2^{-k_{\min}\alpha}2^{-\theta(k_{\max}-k_{\min})}\frac{|I|^{\frac{3}{2}}}{|K|^2}.
\end{aligned}
$$

Case III. $|n_1^2|<2\le|n_1^1|$. This case is symmetrical to Case II. Hence the estimates are similar and we omit the details.

3.2. **Adjacent.** Observe that

$$
\begin{aligned}
&\Big(\frac{|x^1-y^1|+|x^1-z^1|}{|x^2-y^2|+|x^2-z^2|}+\frac{|x^2-y^2|+|x^2-z^2|}{|x^1-y^1|+|x^1-z^1|}\Big)^{-\theta}\\
&\qquad\lesssim\Big(\frac{|x^1-y^1|+|x^1-z^1|}{|x^2-y^2|}+\frac{|x^2-y^2|}{|x^1-y^1|+|x^1-z^1|}\Big)^{-\theta}\\
&\qquad\quad+\Big(\frac{|x^1-y^1|+|x^1-z^1|}{|x^2-z^2|}+\frac{|x^2-z^2|}{|x^1-y^1|+|x^1-z^1|}\Big)^{-\theta}.
\end{aligned}
$$

Then by symmetry we may reduce to the case $|n_1^1|=1,|n_1^2|\le 1$ and bound

$$
\iiint\Big(\frac{|x^1-y^1|+|x^1-z^1|}{|x^2-y^2|}+\frac{|x^2-y^2|}{|x^1-y^1|+|x^1-z^1|}\Big)^{-\theta}
$$

$$\times \frac{h^0_{I\dot{+}n_1}(y)h^0_{I\dot{+}n_2}(z)h^0_I(x)}{\prod_{i=1,2}(|x^i-y^i|+|x^i-z^i|)^2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Write

$$\begin{aligned}&\Big(\frac{|x^1-y^1|+|x^1-z^1|}{|x^2-y^2|}+\frac{|x^2-y^2|}{|x^1-y^1|+|x^1-z^1|}\Big)^{-\theta}\\ &\le\Big(\frac{|x^1-y^1|+|x^1-z^1|}{|x^2-y^2|}\Big)^{-\theta}\chi_{\{|x^1-y^1|+|x^1-z^1|\ge|x^2-y^2|\}}\\ &\quad+\Big(\frac{|x^2-y^2|}{|x^1-y^1|+|x^1-z^1|}\Big)^{-\theta}\chi_{\{|x^1-y^1|+|x^1-z^1|<|x^2-y^2|\}}\end{aligned}$$

Then the estimate related with the second term is easy, one just integrate over z^2, y^2, x^2 and z^1 in order, it will arrive at

$$\ell(I)|I|^{-\frac{3}{2}}\iint\frac{1_{I^1\dot{+}n^1_1}(y^1)1_{I^1}(x^1)}{|x^1-y^1|}\,\mathrm{d}x^1\,\mathrm{d}y^1\lesssim|I|^{-\frac{1}{2}}\sim\frac{|I|^{\frac{3}{2}}}{|K|^2}. \tag{3.12}$$

For the estimate related with the first term, integrating over z^2 and z^1, we arrive at

$$|I|^{-\frac{3}{2}}\iint\frac{|x^2-y^2|^\theta}{(|x^1-y^1|+|x^2-y^2|)^{1+\theta}}\frac{1}{|x^2-y^2|}1_{I\dot{+}n_1}(y)1_I(x)\,\mathrm{d}x\,\mathrm{d}y$$

Simply split into $|x^2-y^2|\le|x^1-y^1|$ and $|x^2-y^2|>|x^1-y^1|$ when integrating over y^2, then after the trivial integration over x^2, we end up with 3.12 as well.

3.3. **Identical.** In this case, we extend

$$\langle T(h^0_I,h^0_I),h_I\rangle=\sum_{J_i\in\mathrm{ch}(I)}\langle T(h^0_I1_{J_1},h^0_I1_{J_2}),h_I1_{J_3}\rangle.$$

If $J_i\ne J_j$ for some $i\ne j$, then it can be handled similar as the adjacent case. Otherwise by the weak boundedness property we have

$$\Big|\sum_{J\in\mathrm{ch}(I)}\langle T(h^0_I1_J,h^0_I1_J),h_I1_J\rangle\Big|\lesssim\sum_{J\in\mathrm{ch}(I)}\frac{|J|}{|I|^{\frac{3}{2}}}=\frac{|I|^{\frac{3}{2}}}{|K|^2}.$$

This completes the proof of theorem 3.9. □

4. Proof of the main results

We begin with the proof of Theorem 1.6.

Proof of Theorem 1.6. By Theorem 3.9 we have already proved the 'if' part of the $T1$ theorem. For the 'only if' part, it suffices to prove the $L^\infty\times L^\infty\to\mathrm{BMO}$ boundedness of T. We will show that there exists some constant C_Q such that

$$\sup_Q\Big(\frac{1}{|Q|}\int_Q|T(\varphi_1,\varphi_2)-C_Q|^r\Big)^{\frac{1}{r}}\lesssim\|\varphi_1\|_{L^\infty}\|\varphi_2\|_{L^\infty}. \tag{4.1}$$

Indeed, we split

$$\begin{aligned}T(\varphi_1,\varphi_2)&=T(\varphi_11_{3Q},\varphi_21_{3Q})+T(\varphi_11_{3Q},\varphi_21_{(3Q)^c})+T(\varphi_11_{(3Q)^c},\varphi_21_{3Q})\\&\quad+T(\varphi_11_{(3Q)^c},\varphi_21_{(3Q)^c}).\end{aligned}$$

First of all, we use the $L^p \times L^q \to L^r$ boundedness of T,

$$\Big(\frac{1}{|Q|}\int_Q |T(\varphi_1 1_{3Q}, \varphi_2 1_{3Q})|^r\Big)^{\frac{1}{r}} \lesssim \Big(\frac{1}{|Q|}\Big)^{\frac{1}{r}} \|\varphi_1\|_{L^\infty}|Q|^{\frac{1}{p}}\|\varphi_2\|_{L^\infty}|Q|^{\frac{1}{q}} = \|\varphi_1\|_{L^\infty}\|\varphi_2\|_{L^\infty}.$$

Secondly, note that if $Q = I \times J$,

$$(3Q)^c = [3I \times (3J)^c] \cup [(3I)^c \times 3J] \cup [(3I)^c \times (3J)^c].$$

Then for any fixed $x \in Q$ we have

$$\begin{aligned}
&|T(\varphi_1 1_{3Q}, \varphi_2 1_{3I\times(3J)^c})(x)| \\
&\le \|\varphi_1\|_{L^\infty}\|\varphi_2\|_{L^\infty}\int_{3I\times(3J)^c}\int_{3Q}\frac{D_\theta(x,y,z)}{\prod_{i=1}^{2}(|x^i-y^i|+|x^i-z^i|)^2}\,\mathrm{d}y\,\mathrm{d}z \\
&\lesssim \|\varphi_1\|_{L^\infty}\|\varphi_2\|_{L^\infty}\int_{3I\times 3I}\frac{\mathrm{d}y^1\,\mathrm{d}z^1}{(|x^1-y^1|+|x^1-z^1|)^{2-\frac{\theta}{4}}}\int_{3J\times(3J)^c}\frac{\mathrm{d}y^2\,\mathrm{d}z^2}{(|x^2-y^2|+|x^2-z^2|)^{2+\frac{\theta}{4}}} \\
&\lesssim \|\varphi_1\|_{L^\infty}\|\varphi_2\|_{L^\infty}.
\end{aligned}$$

Similarly

$$|T(\varphi_1 1_{3Q}, \varphi_2 1_{(3I)^c\times 3J})(x)| \lesssim \|\varphi_1\|_{L^\infty}\|\varphi_2\|_{L^\infty}.$$

Lastly, we also have

$$\begin{aligned}
&|T(\varphi_1 1_{3Q}, \varphi_2 1_{(3I)^c\times(3J)^c})(x) - T(\varphi_1 1_{3Q}, \varphi_2 1_{(3I)^c\times(3J)^c})(c_Q)| \\
&\lesssim \|\varphi_1\|_{L^\infty}\|\varphi_2\|_{L^\infty}\int_{3Q}\int_{(3I)^c\times(3J)^c}\frac{D_{\min\{\theta,\alpha/2\}}(x,y,z)}{\prod_{i=1}^{2}(|x^i-y^i|+|x^i-z^i|)^2} \\
&\qquad\qquad \times\Big[\frac{\ell(Q)^\alpha}{(|x^1-y^1|+|x^1-z^1|)^\alpha}+\frac{\ell(Q)^\alpha}{(|x^2-y^2|+|x^2-z^2|)^\alpha}\Big]\,\mathrm{d}z\,\mathrm{d}y \\
&\lesssim \|\varphi_1\|_{L^\infty}\|\varphi_2\|_{L^\infty}\int_{3Q}\int_{(3I)^c\times(3J)^c} \\
&\quad\Big[\frac{\ell(Q)^\alpha}{(|x^1-y^1|+|x^1-z^1|)^{2+\alpha-\min\{\theta,\frac{\alpha}{2}\}}(|x^2-y^2|+|x^2-z^2|)^{2+\min\{\theta,\frac{\alpha}{2}\}}} \\
&\quad+\frac{\ell(Q)^\alpha}{(|x^1-y^1|+|x^1-z^1|)^{2+\min\{\theta,\frac{\alpha}{2}\}}(|x^2-y^2|+|x^2-z^2|)^{2+\alpha-\min\{\theta,\frac{\alpha}{2}\}}}\Big]\,\mathrm{d}z\,\mathrm{d}y \\
&\lesssim \|\varphi_1\|_{L^\infty}\|\varphi_2\|_{L^\infty}.
\end{aligned}$$

The term involved with $T(\varphi_1 1_{(3Q)^c}, \varphi_2 1_{3Q})$ is completely similar. Finally, we will bound the term involving

$$\begin{aligned}
T(\varphi_1 1_{(3Q)^c}, \varphi_2 1_{(3Q)^c}) &= T(\varphi_1 1_{3I\times(3J)^c}, \varphi_2 1_{3I\times(3J)^c}) + T(\varphi_1 1_{3I\times(3J)^c}, \varphi_2 1_{(3I)^c\times 3J}) \\
&\quad + T(\varphi_1 1_{3I\times(3J)^c}, \varphi_2 1_{(3I)^c\times(3J)^c}) + \cdots,
\end{aligned}$$

where we have 9 terms in the right hand side, each of them can be estimated in a similar way as above. Therefore, we have proved (4.1) with suitable choice of C_Q. This proves the claim.

Next we prove $T : L^1 \times L^1 \to L^{\frac{1}{2},\infty}$ boundedly if $\theta \in (1, 2]$. Without loss of generality, we may assume $\|f_1\|_{L^1} = \|f_2\|_{L^1} = 1$, we shall prove

$$|\{x \in \mathbb{R}^2 : |T(f_1, f_2)(x)| > \lambda\}| \lesssim \lambda^{-1/2}.$$

Applying Calderón-Zygmund decomposition to f_1 and f_2 at height $\lambda^{1/2}$, we get two families of disjoint cubes $\{Q_1^j\}, \{Q_2^k\}$, respectively. Meanwhile we have $f_i = g_i + b_i$ with $\|g_i\|_{L^\infty} \le \lambda^{1/2}$, $\|g_i\|_{L^1} \le 1$ and

$$b_1 = \sum_j b_1^j, \quad \int b_1^j = 0, \quad \|b_1^j\|_{L^1} \lesssim \lambda^{1/2}|Q_1^j|, \quad \sum_j |Q_1^j| \lesssim \lambda^{-1/2};$$
$$b_2 = \sum_k b_2^k, \quad \int b_2^k = 0, \quad \|b_2^k\|_{L^1} \lesssim \lambda^{1/2}|Q_2^k|, \quad \sum_k |Q_2^k| \lesssim \lambda^{-1/2}.$$

Then we have

$$\begin{aligned}|\{x \in \mathbb{R}^2 : |T(f_1, f_2)(x)| > \lambda\}| \le{}& \sum_j |3Q_1^j| + \sum_k |3Q_2^k| + |\{x \in \mathbb{R}^2 : |T(g_1, g_2)(x)| > \lambda/4\}| \\ &+ |\{x \in \mathbb{R}^2 \setminus \bigcup_k 3Q_2^k : |T(g_1, b_2)(x)| > \lambda/4\}| \\ &+ |\{x \in \mathbb{R}^2 \setminus \bigcup_j 3Q_1^j : |T(b_1, g_2)(x)| > \lambda/4\}| \\ &+ |\{x \in \mathbb{R}^2 \setminus \bigcup_{j,k} \big(3Q_1^j \cup 3Q_2^k\big) : |T(b_1, b_2)(x)| > \lambda/4\}|.\end{aligned}$$

We already know that

$$\sum_j |3Q_1^j| + \sum_k |3Q_2^k| \lesssim \lambda^{-1/2}.$$

By the $L^p \times L^q \to L^r$ boundedness, it is also easy to see that

$$\begin{aligned}|\{x \in \mathbb{R}^2 : |T(g_1, g_2)(x)| > \lambda/4\}| &\lesssim \lambda^{-r}\|T(g_1, g_2)\|_{L^r}^r \lesssim \lambda^{-r}\|g_1\|_{L^p}^r\|g_2\|_{L^q}^r \\ &\le \lambda^{-r}\lambda^{(p-1)r/(2p)}\lambda^{(q-1)r/(2q)} = \lambda^{-1/2}\end{aligned}$$

Next we bound

$$\begin{aligned}&|\{x \in \mathbb{R}^2 \setminus \bigcup_k 3Q_2^k : |T(g_1, b_2)(x)| > \lambda/4\}| \\ &\le \frac{4}{\lambda}\sum_k \int_{\mathbb{R}^2\setminus 3Q_2^k} |T(g_1, b_2^k)(x)|\,\mathrm{d}x \\ &= \frac{4}{\lambda}\sum_k \Big[\int_{3I_2^k \times (3J_2^k)^c} + \int_{(3I_2^k)^c \times (3J_2^k)} + \int_{(3I_2^k)^c \times (3J_2^k)^c}\Big] =: I_1 + I_2 + I_3,\end{aligned}$$

where we have denoted $Q_2^k := I_2^k \times J_2^k$. First of all,

$$I_1 \le \frac{4}{\lambda}\sum_k \int_{3I_2^k \times (3J_2^k)^c} \iint |K(x, y, z)g_1(y)b_2^k(z)|\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x$$

$$\leq 4\lambda^{-1/2}\sum_k \int |b_2^k(z)| \int_{3I_2^k\times(3J_2^k)^c}\int \frac{D_\theta(x,y,z)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z}{\prod_{i=1}^2(|x^i-y^i|+|x^i-z^i|)^2}$$
$$\lesssim \lambda^{-1/2}\sum_k \int |b_2^k(z)| \int_{3I_2^k\times(3J_2^k)^c}\int \frac{\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z}{(|x^1-y^1|+|x^1-z^1|)^{2-\frac{\theta}{4}}(|x^2-y^2|+|x^2-z^2|)^{2+\frac{\theta}{4}}},$$

where we have used $D_\theta \leq D_{\theta/4}$. Note that for fixed $z\in I_2^k\times J_2^k$, we have

$$\int_{3I_2^k}\int \frac{\mathrm{d}y^1\,\mathrm{d}x^1}{(|x^1-y^1|+|x^1-z^1|)^{2-\frac{\theta}{4}}} \lesssim \int_{3I_2^k}\frac{\mathrm{d}x^1}{|x^1-z^1|^{1-\frac{\theta}{4}}} \lesssim \ell(Q_2^k)^{\frac{\theta}{4}}$$

and

$$\int_{(3J_2^k)^c}\int \frac{\mathrm{d}y^2\,\mathrm{d}x^2}{(|x^2-y^2|+|x^2-z^2|)^{2+\frac{\theta}{4}}} \lesssim \int_{(3J_2^k)^c}\frac{\mathrm{d}x^2}{|x^2-z^2|^{1+\frac{\theta}{4}}} \lesssim \ell(Q_2^k)^{-\frac{\theta}{4}}.$$

Thus

$$I_1 \lesssim \lambda^{-1/2}\sum_k\int |b_2^k(z)|\,\mathrm{d}z \lesssim \lambda^{-1/2}.$$

The estimate of I_2 is completely similar. For I_3, we have

$$\begin{aligned}
I_3 &\leq 4\lambda^{-1/2}\sum_k\int_{(3I_2^k)^c\times(3J_2^k)^c}\iint |K(x,y,z)-K(x,y,c_{Q_2^k})||b_2^k(z)|\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x\\
&\lesssim \lambda^{-1/2}\sum_k\int |b_2^k(z)|\,\mathrm{d}z\int_{(3I_2^k)^c\times(3J_2^k)^c}\int \frac{D_{\min\{\theta,\alpha/2\}}(x,y,z)}{\prod_{i=1}^2(|x^i-y^i|+|x^i-z^i|)^2}\\
&\qquad\qquad \times\Big[\frac{\ell(Q_2^k)^\alpha}{(|x^1-y^1|+|x^1-z^1|)^\alpha}+\frac{\ell(Q_2^k)^\alpha}{(|x^2-y^2|+|x^2-z^2|)^\alpha}\Big]\,\mathrm{d}y\,\mathrm{d}x\\
&\lesssim \lambda^{-1/2}.
\end{aligned}$$

The estimate of $|\{x\in\mathbb{R}^2\setminus\bigcup_j 3Q_1^j : |T(b_1,g_2)(x)|>\lambda/4\}|$ is similar. It remains to control

$$\begin{aligned}
&|\{x\in\mathbb{R}^2\setminus\bigcup_{j,k}\big(3Q_1^j\cup 3Q_2^k\big) : |T(b_1,b_2)(x)|>\lambda/4\}|\\
&\leq \frac{4}{\lambda}\sum_{j,k}\int_{\mathbb{R}^2\setminus(3Q_1^j\cup 3Q_2^k)}\Big|\int K(x,y,z)b_1^j(y)b_2^k(z)\,\mathrm{d}y\,\mathrm{d}z\Big|\,\mathrm{d}x.
\end{aligned}$$

By symmetry we may assume $\ell(Q_1^j)\leq \ell(Q_2^k)$. Note that

$$\mathbb{R}^2\setminus(3Q_1^j\cup 3Q_2^k) = (3Q_1^j)^c\cap(3Q_2^k)^c$$

and

$$\begin{aligned}
(3Q_1^j)^c &= \big[3I_1^j\times(3J_1^j)^c\big]\cup\big[(3I_1^j)^c\times 3J_1^j\big]\cup\big[(3I_1^j)^c\times(3J_1^j)^c\big]\\
(3Q_2^k)^c &= \big[3I_2^k\times(3J_2^k)^c\big]\cup\big[(3I_2^k)^c\times 3J_2^k\big]\cup\big[(3I_2^k)^c\times(3J_2^k)^c\big].
\end{aligned}$$

We first consider the combination $\big[3I_1^j\times(3J_1^j)^c\big]\cap\big[3I_2^k\times(3J_2^k)^c\big]$. We have

$$\frac{4}{\lambda}\sum_{j,k}\int_{\big[3I_1^j\times(3J_1^j)^c\big]\cap\big[3I_2^k\times(3J_2^k)^c\big]}\Big|\iint K(x,y,z)b_1^j(y)b_2^k(z)\,\mathrm{d}y\,\mathrm{d}z\Big|\,\mathrm{d}x$$

$$
\begin{aligned}
&\lesssim \frac{1}{\lambda}\sum_{j,k}\int_{\left[3I_1^j\times(3J_1^j)^c\right]\cap\left[3I_2^k\times(3J_2^k)^c\right]}\iint \frac{|b_1^j(y)b_2^k(z)|\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x}{(|x^1-y^1|)^{2-\theta}(|x^2-c_{J_1^j}|+|x^2-c_{J_2^k}|)^{2+\theta}}\\
&\lesssim \frac{1}{\lambda}\sum_{j,k}\ell(Q_1^j)^{(\theta-1)}\int_{(3J_1^j)^c\cap(3J_2^k)^c}\iint \frac{|b_1^j(y)b_2^k(z)|\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x^2}{(|x^2-c_{J_1^j}|+|x^2-c_{J_2^k}|)^{2+\theta}}\\
&\lesssim \sum_{j,k}\ell(Q_1^j)^{(\theta-1)}|Q_1^j||Q_2^k|\int_{(3J_1^j)^c\cap(3J_2^k)^c}\frac{\mathrm{d}x^2}{(|x^2-c_{J_1^j}|+|x^2-c_{J_2^k}|+|c_{I_1^j}-c_{I_2^k}|)^{2+\theta}}\\
&\sim \sum_{j}\ell(Q_1^j)^{(\theta+1)}\sum_k\int_{Q_2^k}\int_{(3J_1^j)^c\cap(3J_2^k)^c}\frac{\mathrm{d}x^2\,\mathrm{d}z}{(|x^2-c_{J_1^j}|+|x^2-z^2|+|c_{I_1^j}-z^1|)^{2+\theta}}\\
&\le \sum_{j}\ell(Q_1^j)^{(\theta+1)}\int_{(3J_1^j)^c}\int\frac{\mathrm{d}z\,\mathrm{d}x^2}{(|x^2-c_{J_1^j}|+|x^2-z^2|+|c_{I_1^j}-z^1|)^{2+\theta}}\\
&\lesssim \sum_j|Q_1^j|\lesssim\lambda^{-1/2}.
\end{aligned}
$$

Next we consider the combination $\left[3I_1^j\times(3J_1^j)^c\right]\cap\left[(3I_2^k)^c\times 3J_2^k\right]$. We have

$$
\begin{aligned}
&\frac{4}{\lambda}\sum_{j,k}\int_{\left[3I_1^j\times(3J_1^j)^c\right]\cap\left[(3I_2^k)^c\times3J_2^k\right]}\Big|\iint K(x,y,z)b_1^j(y)b_2^k(z)\,\mathrm{d}y\,\mathrm{d}z\Big|\,\mathrm{d}x\\
&\le \frac{4}{\lambda}\sum_{j,k}\int_{\left[3I_1^j\times(3J_1^j)^c\right]\cap\left[(3I_2^k)^c\times3J_2^k\right]}\iint\frac{|b_1^j(y)b_2^k(z)|}{\prod_{i=1}^2(|x^i-y^i|+|x^i-z^i|)^2}D_\theta(x,y,z)\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x\\
&\sim \frac{1}{\lambda}\sum_{j,k}\int_{\left[3I_1^j\times(3J_1^j)^c\right]\cap\left[(3I_2^k)^c\times3J_2^k\right]}\iint\frac{|b_1^j(y)b_2^k(z)|}{\prod_{i=1}^2(|x^i-c^i_{Q_1^j}|+|x^i-z^i|)^2}D_\theta(x,c_{Q_1^j},z)\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x\\
&\le \frac{1}{\lambda}\sum_{j,k}\int_{\left[3I_1^j\times(3J_1^j)^c\right]\cap\left[(3I_2^k)^c\times3J_2^k\right]}\int\frac{|b_2^k(z)|\lambda^{1/2}|Q_1^j|}{\prod_{i=1}^2(|x^i-c^i_{Q_1^j}|+|x^i-z^i|)^2}D_\theta(x,c_{Q_1^j},z)\,\mathrm{d}z\,\mathrm{d}x\\
&\le \lambda^{-1/2}\sum_k\int_{(3I_2^k)^c\times3J_2^k}\iint\frac{|b_2^k(z)|D_\theta(x,y,z)}{\prod_{i=1}^2(|x^i-y^i|+|x^i-z^i|)^2}\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x,
\end{aligned}
$$

which can be estimated similarly as I_2. Likewise, the combination $\left[3I_1^j\times(3J_1^j)^c\right]\cap\left[(3I_2^k)^c\times(3J_2^k)^c\right]$ can be reduced to something similar as I_3. By symmetry we have considered all combinations, we are done.

□

Next we prove the first part of Theorem 1.5. The second part will be a consequence of Theorem 1.9.

4.1. **Counterexample.** Let $\theta \in (0,2]$, $t_1, t_2 > 0$, and ϕ be a non-negative bump function. Define

$$K(x,y) = K_{t_1,t_2,\theta_2}(x,y) = \left(\frac{t_1}{t_2} + \frac{t_2}{t_1}\right)^{-\theta} \prod_{i=1}^{2} \frac{1}{t_i^2} \phi(\frac{x^i}{t_i})\phi(\frac{y^i}{t_i}).$$

Let us first check that $K(x-y, x-z)$ is a bilinear exotic Calderón-Zygmund operator uniformly on the parameters t_1, t_2. Indeed, suppose by symmetry that $t_1 \le t_2$, then

$$\begin{aligned}
&\big(|x^1|+|y^1|\big)^{2+\alpha_1+\beta_1}\big(|x^2|+|y^2|\big)^{2+\alpha_2+\beta_2}\big|\partial^\alpha\partial^\beta K(x,y)\big| \\
&\lesssim \Big(\frac{t_1}{t_2}\Big)^{\theta} \prod_{i=1}^{2} \Big(\frac{|x^i|+|y^i|}{t_i}\Big)^{2+\alpha_i+\beta_i} \Big(1+\frac{|x^i|}{t_i}\Big)^{-N}\Big(1+\frac{|y^i|}{t_i}\Big)^{-N} \\
&\le \Big(\frac{t_1}{t_2}\Big)^{\theta} \prod_{i=1}^{2} \Big(\frac{|x^i|+|y^i|}{t_i}\Big)^{2+\alpha_i+\beta_i} \Big(1+\frac{|x^i|+|y^i|}{t_i}\Big)^{-N} \\
&\lesssim \Big(\frac{t_1}{t_2}\Big)^{2\theta} \frac{(|x^1|+|y^1|)^{2+\alpha_1+\beta_1+\theta} t_1^{N-2-\alpha_1-\beta_1-\theta}}{(|x^1|+|y^1|+t_1)^N} \\
&\qquad\qquad \times \frac{(|x^2|+|y^2|)^{2+\alpha_2+\beta_2-\theta} t_2^{N-2-\alpha_2-\beta_2+\theta}}{(|x^2|+|y^2|+t_2)^N} \Big(\frac{|x^2|+|y^2|}{|x^1|+|y^1|}\Big)^{\theta} \\
&\le \Big(\frac{|x^2|+|y^2|}{|x^1|+|y^1|}\Big)^{\theta}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\big(|x^1|+|y^1|\big)^{2+\alpha_1+\beta_1}\big(|x^2|+|y^2|\big)^{2+\alpha_2+\beta_2}\big|\partial^\alpha\partial^\beta K(x,y)\big| \\
&\lesssim \min\Big\{\Big(\frac{|x^2|+|y^2|}{|x^1|+|y^1|}\Big)^{\theta}, \Big(\frac{|x^1|+|y^1|}{|x^2|+|y^2|}\Big)^{\theta}\Big\} \sim \Big(\frac{|x^1|+|y^1|}{|x^2|+|y^2|} + \frac{|x^2|+|y^2|}{|x^1|+|y^1|}\Big)^{-\theta}.
\end{aligned}$$

From $\alpha, \beta \in \{(0,0),(0,1),(1,0)\}$ with $|\alpha|+|\beta| \le 1$, we get the desired kernel estimates. For the boundedness, notice that

$$|T(f_1,f_2)(x)| = \Big|\iint K(x-y,x-z) f_1(y) f_2(z)\,\mathrm{d}y\,\mathrm{d}z\Big| \lesssim (M_1M_2f_1)(x)(M_1M_2f_2)(x).$$

Now we fix t_1, t_2, θ momentarily and let R be a rectangle with sidelengths t_1, t_2, then for $f_1, f_2 \ge 0$,

$$\begin{aligned}
T(f_1,f_2)(x) &= \iint K(x-y,x-z) f_1(y) f_2(z)\,\mathrm{d}y\,\mathrm{d}z \\
&= \iint \left(\frac{t_1}{t_2}+\frac{t_2}{t_1}\right)^{-\theta} \prod_{i=1}^{2} \frac{1}{t_i^2}\phi(\frac{x^i-y^i}{t_i})\phi(\frac{x^i-z^i}{t_i}) f_1(y) f_2(z)\,\mathrm{d}y\,\mathrm{d}z \\
&\gtrsim \operatorname{ecc}(R)^{-\theta} 1_R \langle f_1\rangle_R \langle f_2\rangle_R,
\end{aligned}$$

where

$$\operatorname{ecc}(R) := \max\Big\{\frac{t_1}{t_2}, \frac{t_2}{t_1}\Big\}$$

is the eccentricity of R. Suppose now for $p, q \in (1,\infty)$ with $1/r = 1/p + 1/q$ we have

$$\|T(f_1,f_2)w\|_{L^r} \le C([(w_1,w_2)]_{A_{(p,q)}}) \|f_1w_1\|_{L^p} \|f_2w_2\|_{L^q}$$

for all $(w_1, w_2) \in A_{(p,q)}$ and $f_1 \in L^p(w_1^p)$, $f_2 \in L^q(w_2^q)$. Then

$$\mathrm{ecc}(R)^{-\theta} w^r(R)^{\frac{1}{r}} \langle f_1 \rangle_R \langle f_2 \rangle_R \lesssim C([(w_1, w_2)]_{A_{(p,q)}}) \|f_1 w_1\|_{L^p} \|f_2 w_2\|_{L^q}.$$

Let $f_1 = 1_R \sigma_1 = 1_R w_1^{-p'}$, $f_2 = 1_R \sigma_2 = 1_R w_2^{-q'}$, then

$$\langle w^r \rangle_R^{\frac{1}{r}} \langle \sigma_1 \rangle_R^{\frac{1}{p'}} \langle \sigma_2 \rangle_R^{\frac{1}{q'}} \lesssim C([(w_1, w_2)]_{A_{(p,q)}}) \,\mathrm{ecc}(R)^{\theta}.$$

Now let $\sigma_1(x) = |x|^{\alpha_1}$ with $p'/(2r) < \alpha_1 < p'/r$, $\sigma_2(x) = |x|^{\alpha_2}$ with $q'/(2r) < \alpha_2 < q'/r$. Then since

$$p'/r \le 2p' < 2(2p'-1), \qquad q'/r \le 2q' < 2(2q'-1),$$

we have $\sigma_1 \in A_{2p'}, \sigma_2 \in A_{2q'}$. Now that

$$w^r = (w_1 w_2)^r = |x|^{-\alpha_1 r/p' - \alpha_2 r/q'}$$

and $-\alpha_1 r/p' - \alpha_2 r/q' \in (-2, -1)$, we see that $w^r \in A_1 \subset A_{2r}$. Therefore we get $(w_1, w_2) \in A_{(p,q)}$. Now consider rectangles of the form $R = (0, \varepsilon) \times (\varepsilon, 1)$ with $\mathrm{ecc}(R) \sim 1/\varepsilon$. On R, we have $|x| = |(x^1, x^2)| \sim x^2$, hence

$$\langle \sigma_i \rangle_R \sim \varepsilon^{-1} \int_0^{\varepsilon} \int_{\varepsilon}^1 (x^2)^{\alpha_i} \,\mathrm{d}x^2 \,\mathrm{d}x^1 \sim 1,$$

$$\langle w^r \rangle_R \sim \varepsilon^{-1} \int_0^{\varepsilon} \int_{\varepsilon}^1 (x^2)^{-\alpha_1 \frac{r}{p'} - \alpha_2 \frac{r}{q'}} \,\mathrm{d}x^2 \,\mathrm{d}x^1 \sim \varepsilon^{1 - \alpha_1 \frac{r}{p'} - \alpha_2 \frac{r}{q'}} \sim \mathrm{ecc}(R)^{\alpha_1 \frac{r}{p'} + \alpha_2 \frac{r}{q'} - 1}.$$

Combining the analysis above, we obtain

$$\mathrm{ecc}(R)^{\alpha_1 \frac{1}{p'} + \alpha_2 \frac{1}{q'} - \frac{1}{r}} \lesssim \mathrm{ecc}(R)^{\theta}.$$

Since we can let $\mathrm{ecc}(R) \to \infty$, we must have $\frac{\alpha_1}{p'} + \frac{\alpha_2}{q'} - \frac{1}{r} \le \theta$. Let $\alpha_1 \to \frac{p'}{r}, \alpha_2 \to \frac{q'}{r}$, then $1/r \le \theta$, then let $p, q \to 1$ so that $1/r \to 2$, we get $\theta \ge 2$.

Thus weighted boundedness cannot hold in general for bilinear exotic CZOs if $\theta < 2$. Next, we prove some sparse estimates, from which we get the desired weighted estimates when $\theta = 2$. Define the bilinear sharp grand maximal function

$$\begin{aligned}\mathcal{M}^{\#}_{T,3}(f_1, f_2)(x) = \sup_{Q \ni x} \operatorname*{ess\,sup}_{\xi, \eta \in Q} \big| \big(T(f_1, f_2)(\xi) - T(f_1 1_{3Q}, f_2 1_{3Q})(\xi)\big) \\ - \big(T(f_1, f_2)(\eta) - T(f_1 1_{3Q}, f_2 1_{3Q})(\eta)\big)\big|.\end{aligned}$$

4.2. **Lemma.** *Let T be a bilinear exotic CZO with $\theta = 2$. Then*

$$\mathcal{M}^{\#}_{T,3}(f_1, f_2)(x) \lesssim M_* f_1(x) M_* f_2(x),$$

where the right-hand side is the strong maximal function, with supremum over all axes-parallel rectangles containing x.

Proof. Fix a cube $Q = I \times J$ and some $x, \xi, \eta \in Q$. Note that

$$\begin{aligned}&T(f_1, f_2)(\xi) - T(f_1 1_{3Q}, f_2 1_{3Q})(\xi) \\ &= T(f_1 1_{3Q}, f_2 1_{(3Q)^c})(\xi) + T(f_1 1_{(3Q)^c}, f_2 1_{3Q})(\xi) + T(f_1 1_{(3Q)^c}, f_2 1_{(3Q)^c})(\xi).\end{aligned}$$

As usual, we split

$$(3Q)^c = [3I \times (3J)^c] \cup [(3I)^c \times 3J] \cup [(3I)^c \times (3J)^c].$$

First of all, we have

$$\begin{aligned}|T(f_1 1_{3Q}, f_2 1_{3I\times(3J)^c})(\xi)| &\lesssim \iint \frac{f_1 1_{3Q}(y) f_2 1_{3I\times(3J)^c}(z)\,\mathrm{d}y\,\mathrm{d}z}{(|\xi^2-y^2|+|\xi^2-z^2|)^4}\\
&\sim \iint \frac{f_1 1_{3Q}(y) f_2 1_{3I\times(3J)^c}(z)\,\mathrm{d}y\,\mathrm{d}z}{|x^2-z^2|^4}\\
&\lesssim M_* f_1(x)\int_{3I\times(3J)^c}\frac{|Q| f_2(z)\,\mathrm{d}z}{|x^2-z^2|^4}\\
&\lesssim M_* f_1(x)\sum_{\ell\ge 1}\int_{3I}\int_{|x^2-z^2|\sim 3^\ell \ell(Q)}\frac{|Q| f_2(z)\,\mathrm{d}z}{|x^2-z^2|^4}\\
&\lesssim M_* f_1(x) M_* f_2(x).\end{aligned}$$

Similarly,

$$|T(f_1 1_{3Q}, f_2 1_{(3I)^c\times 3J})(\xi)| \lesssim M_* f_1(x) M_* f_2(x).$$

Finally, similar as before

$$\begin{aligned}&|T(f_1 1_{3Q}, f_2 1_{(3I)^c\times(3J)^c})(\xi) - T(f_1 1_{3Q}, f_2 1_{(3I)^c\times(3J)^c})(\eta)|\\
&\lesssim \int_{3Q}\int_{(3I)^c\times(3J)^c}|f_1(y)f_2(z)|\\
&\quad\Big[\frac{\ell(Q)^\alpha}{(|\xi^1-y^1|+|\xi^1-z^1|)^{2+\alpha-\min\{\theta,\frac{\alpha}{2}\}}(|\xi^2-y^2|+|\xi^2-z^2|)^{2+\min\{\theta,\frac{\alpha}{2}\}}}\\
&\quad + \frac{\ell(Q)^\alpha}{(|\xi^1-y^1|+|\xi^1-z^1|)^{2+\min\{\theta,\frac{\alpha}{2}\}}(|\xi^2-y^2|+|\xi^2-z^2|)^{2+\alpha-\min\{\theta,\frac{\alpha}{2}\}}}\Big]\,\mathrm{d}z\,\mathrm{d}y\\
&\le \int_{3Q}\int_{(3I)^c\times(3J)^c}|f_1(y)f_2(z)|\Big[\frac{\ell(Q)^\alpha}{|x^1-z^1|^{2+\alpha-\min\{\theta,\frac{\alpha}{2}\}}|x^2-z^2|^{2+\min\{\theta,\frac{\alpha}{2}\}}}\\
&\quad + \frac{\ell(Q)^\alpha}{|x^1-z^1|^{2+\min\{\theta,\frac{\alpha}{2}\}}|x^2-z^2|^{2+\alpha-\min\{\theta,\frac{\alpha}{2}\}}}\Big]\,\mathrm{d}z\,\mathrm{d}y\\
&\lesssim M_* f_1(x)\int_{(3I)^c\times(3J)^c}|f_2(z)|\Big[\frac{\ell(Q)^{2+\alpha}}{|x^1-z^1|^{2+\alpha-\min\{\theta,\frac{\alpha}{2}\}}|x^2-z^2|^{2+\min\{\theta,\frac{\alpha}{2}\}}}\\
&\quad + \frac{\ell(Q)^{2+\alpha}}{|x^1-z^1|^{2+\min\{\theta,\frac{\alpha}{2}\}}|x^2-z^2|^{2+\alpha-\min\{\theta,\frac{\alpha}{2}\}}}\Big]\,\mathrm{d}z\,\mathrm{d}y\\
&\lesssim M_* f_1(x) M_* f_2(x).\end{aligned}$$

Other combinations can be handled in a similar way, we are done.

□

We also need the following result, which is a variant of Theorem 3.4 in [13].

4.3. **Theorem.** [13, Theorem 3.4] *Let $1 \le q, r < \infty$ and $s = \max(q, r)$. Let f_1, f_2 be compactly supported functions from L^s. Assume that T is bounded from $L^q \times L^q \to L^{q/2,\infty}$ and $\mathcal{M}^{\#}_{T,3}$ is bounded from $L^r \times L^r \to L^{r/2,\infty}$. Then there exists a $\frac{1}{2\cdot 3^n}$-sparse family $\mathcal{S}$ such that*

$$|T(f_1, f_2)(x)| \le C \sum_{Q\in\mathcal{S}} \langle |f_1|\rangle_{s,Q}\langle |f_2|\rangle_{s,Q} 1_Q(x)$$

for a.e. $x \in \mathbb{R}^n$, *where* $C = c_n\big(\|T\|_{L^q\times L^q\to L^{q/2,\infty}} + \|\mathcal{M}^{\#}_{T,3}\|_{L^r\times L^r\to L^{r/2,\infty}}\big)$.

Observe that by Theorem 1.6, T is bounded from $L^1 \times L^1 \to L^{1/2,\infty}$, and by Lemma 4.2 and Hölder's inequality for the weak spaces we know that $\mathcal{M}^{\#}_{T,3}$ is bounded from $L^r \times L^r \to L^{r/2,\infty}$ with $1 < r < 2$ and

$$\|\mathcal{M}^{\#}_{T,3}(f_1,f_2)\|_{L^{r/2,\infty}} \lesssim \|M_*f_1M_*f_2\|_{L^{r/2,\infty}} \lesssim \|M_*f_1\|_{L^{r,\infty}}\|M_*f_2\|_{L^{r,\infty}} \lesssim (r')^2.$$

Hence

$$|T(f_1,f_2)(x)| \lesssim (s')^2\sum_{Q\in\mathcal{S}}\langle|f_1|\rangle_{s,Q}\langle|f_2|\rangle_{s,Q}1_Q(x).$$

Then the desired qualitative weighted estimates are well-known, see e.g. [3] and [15]. Here we provide the details for the quantitative weighted estimates.

Proof of Theorem 1.9. Suppose that $(w_1,w_2) \in A_{(3,3)}$, then by the reverse Hölder property of $\sigma_i = w_i^{-\frac{3}{2}}$ $(i=1,2)$ we take

$$s = 1 + \frac{1}{c_n\max\{[\sigma_1]_{A_\infty},[\sigma_2]_{A_\infty},[w^{\frac{3}{2}}]_{A_\infty}\}},$$

where $w = w_1w_2$, then

$$\sup_Q\Big(\frac{1}{|Q|}\int_Q w^{\frac{3s}{3-s}}\Big)^{\frac{3-s}{3s}}\prod_{i=1}^2\Big(\frac{1}{|Q|}\int_Q w_i^{-\frac{3s}{3-s}}\Big)^{\frac{3-s}{3s}} \sim [\vec{w}]_{A_{(3,3)}}.$$

Denote $\widetilde{\sigma}_i = w_i^{-\frac{3s}{3-s}}$ for $i=1,2$, for $h \geq 0$ we have

$$\begin{aligned}
&\sum_{Q\in\mathcal{S}}\langle|f_1|\rangle_{s,Q}\langle|f_2|\rangle_{s,Q}\int_Q h\\
&= \sum_{Q\in\mathcal{S}}\Big(\frac{\widetilde{\sigma}_1(Q)}{|Q|}\Big)^{\frac1s}\Big(\frac{1}{\widetilde{\sigma}_1(Q)}\int_Q|f_1|^s\Big)^{\frac1s}\Big(\frac{\widetilde{\sigma}_2(Q)}{|Q|}\Big)^{\frac1s}\Big(\frac{1}{\widetilde{\sigma}_2(Q)}\int_Q|f_2|^s\Big)^{\frac1s}\int_Q h\\
&\lesssim [\vec{w}]^{\frac{3}{3-s}}_{A_{(3,3)}}\sum_{Q\in\mathcal{S}}\Big(\frac{1}{\widetilde{\sigma}_1(Q)}\int_Q|f_1|^s\Big)^{\frac1s}\Big(\frac{1}{\widetilde{\sigma}_2(Q)}\int_Q|f_2|^s\Big)^{\frac1s}\Big(\frac{1}{w^{\frac{3s}{3-s}}(Q)}\int_Q h^s\Big)^{\frac1s}|E_Q|\\
&\lesssim [\vec{w}]^{\frac32}_{A_{(3,3)}}\int M^{\mathcal{D}}_{s,\widetilde{\sigma}_1}(f_1\widetilde{\sigma}_1^{-\frac1s})M^{\mathcal{D}}_{s,\widetilde{\sigma}_2}(f_2\widetilde{\sigma}_2^{-\frac1s})M^{\mathcal{D}}_{s,w^{\frac{3s}{3-s}}}(hw^{-\frac{3}{3-s}})w^{\frac{s}{3-s}}w_1^{-\frac{s}{3-s}}w_2^{-\frac{s}{3-s}}\\
&\leq [\vec{w}]^{\frac32}_{A_{(3,3)}}\|M^{\mathcal{D}}_{s,\widetilde{\sigma}_1}(f_1\widetilde{\sigma}_1^{-\frac1s})\|_{L^3(\widetilde{\sigma}_1)}\|M^{\mathcal{D}}_{s,\widetilde{\sigma}_2}(f_2\widetilde{\sigma}_2^{-\frac1s})\|_{L^3(\widetilde{\sigma}_2)}\|M^{\mathcal{D}}_{s,w^{\frac{3s}{3-s}}}(hw^{-\frac{3}{3-s}})\|_{L^3(w^{\frac{3s}{3-s}})}\\
&\lesssim [\vec{w}]^{\frac32}_{A_{(3,3)}}\|f_1w_1\|_{L^3}\|f_2w_2\|_{L^3}\|hw^{-1}\|_{L^3}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|T(f_1,f_2)w\|_{L^{\frac32}} &\lesssim (s')^2[\vec{w}]^{\frac32}_{A_{(3,3)}}\|f_1w_1\|_{L^3}\|f_2w_2\|_{L^3}\\
&\lesssim [\vec{w}]^{\frac32\cdot3}_{A_{(3,3)}}\|f_1w_1\|_{L^3}\|f_2w_2\|_{L^3}.
\end{aligned}$$

Quantitative extrapolation from [19, Theorem 2.2] gives that

$$\|T(f,g)w\|_{L^p} \lesssim [\vec{w}]^{3\max\{p,p_1',p_2'\}}_{A_{\vec{p}}}\|fw_1\|_{L^{p_1}}\|gw_2\|_{L^{p_2}}.$$

□

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