

Oscillatory integral operators on manifolds and related Keakea and Nikodym problems

SONG DAI, LIUWEI GONG, SHAOMING GUO, AND RUIXIANG ZHANG

We consider Carleson-Sjölin operators on Riemannian manifolds that arise naturally from the study of Bochner-Riesz problems on manifolds. They are special cases of Hörmander-type oscillatory integral operators. We obtain improved L^p bounds of Carleson-Sjölin operators in two cases: The case where the underlying manifold has constant sectional curvature and the case where the manifold satisfies Sogge’s chaotic curvature condition [Sog99].

The two results rely on very different methods: To prove the former result, we show that on a Riemannian manifold, the distance function satisfies Bourgain’s condition in [GWZ22] if and only if the manifold has constant sectional curvature. To obtain the second result, we introduce the notion of “contact orders” to Hörmander-type oscillatory integral operators, prove that if a Hörmander-type oscillatory integral operator is of a finite contact order, then it always has better L^p bounds than “worst cases” (in spirit of Bourgain and Guth [BG11] and Guth, Hickman and Iliopoulou [GHI19]), and eventually verify that for Riemannian manifolds that satisfy Sogge’s chaotic curvature condition, their distance functions always have finite contact orders.

As byproducts, we obtain new bounds for Nikodym maximal functions on manifolds of constant sectional curvatures.

AMS 2000 SUBJECT CLASSIFICATIONS: Primary 42B25, 58J40.

1	Introduction	938
1.1	Hörmander-type oscillatory integrals	939
1.2	Curved Keakea problem	942
1.3	Carleson-Sjölin operators on manifolds	944
1.4	Nikodym problems on manifolds	945
2	Statement of main results	949

arXiv: 2310.20122

3	Proof of Theorem 2.5	955
3.1	Preliminaries in Riemannian geometry	955
3.2	Sogge's chaotic curvature condition	960
4	Proof of Theorem 2.4	977
4.1	Preliminaries in polynomial partitionings	978
4.2	Polynomial partitioning algorithms	982
4.3	Polynomial Wolff axioms	986
5	Proofs of Theorem 2.1 and Theorem 2.2	997
5.1	Preliminaries in Riemannian geometry II	997
5.2	Relation between Carleson-Sjölin on manifolds and Hörmander's problems	1000
5.3	Relation between Carleson-Sjölin on manifolds and Nikodym on manifolds	1001
5.4	Proof of Theorem 2.2: Part a)	1001
5.5	Proof of Theorem 2.2: Part b)	1005
5.6	Proof of Theorem 2.2: Part c)	1007
A	More connections between curved Kakeya problems and Nikodym problems on manifolds	1008
	References	1012

1. Introduction

We study several problems in harmonic analysis and their connections, including Hörmander-type oscillatory integrals, Carleson-Sjölin operators on manifolds, curved Kakeya problems and Nikodym problems on manifolds. In the introduction, we will introduce these four problems, and give a brief review of known results. Experts can skip the slightly long introduction and go to Section 2 directly, where new results are stated. Notations are listed at the end of the introduction.

1.1. Hörmander-type oscillatory integrals

Consider Hörmander-type oscillatory integral operators

$$(1.1) \quad T_N^{(\phi)} f(x, t) := \int_{\mathbb{R}^{n-1}} e^{iN\phi(x, t; y)} a(x, t; y) f(y) dy.$$

Here $x \in \mathbb{R}^{n-1}$, $t \in \mathbb{R}$, $y \in \mathbb{R}^{n-1}$ and $N \in \mathbb{R}$ is a large real number. To simplify notation, we often write $\mathbf{x} = (x, t)$. Moreover, $a(x, t; y)$ is a smooth function supported in a bounded open neighborhood of the origin. On the support of $a(x, t; y)$, let us assume that ϕ is a smooth function and that

(H1) $\text{rank} \nabla_x \nabla_y \phi(x, t; y) = n - 1$;

(H2) if we define

$$(1.2) \quad G_0(\mathbf{x}; y) := \partial_{y_1} \nabla_{\mathbf{x}} \phi(\mathbf{x}; y) \wedge \cdots \wedge \partial_{y_{n-1}} \nabla_{\mathbf{x}} \phi(\mathbf{x}; y),$$

then

$$(1.3) \quad \det \nabla_y^2 \langle \nabla_{\mathbf{x}} \phi(\mathbf{x}; y), G_0(\mathbf{x}; y_0) \rangle \Big|_{y=y_0} \neq 0.$$

The function $\phi(x, t; y)$ will be referred to as the phase function of the operator $T_N^{(\phi)}$, and $a(x, t; y)$ will be referred to as its amplitude function. By saying that the phase function $\phi(x, t; y)$ satisfies Hörmander's non-degeneracy condition, we mean that it satisfies (H1) and (H2) above.

We are interested in proving estimates of the form

$$(1.4) \quad \|T_N^{(\phi)} f\|_{L^p(\mathbb{R}^n)} \lesssim_{\phi, a, p, \epsilon} N^{-\frac{n}{p} + \epsilon} \|f\|_{L^p(\mathbb{R}^{n-1})},$$

for every $\epsilon > 0$ and every $N \geq 1$, and for a range of exponents p that is as large as possible.

For estimates of the form (1.4), the simplest and perhaps the most interesting phase function is

$$(1.5) \quad \phi(x, t; y) = x \cdot y + t|y|^2.$$

Fourier restriction conjecture. The estimate (1.4) holds for all

$$(1.6) \quad p \geq \frac{2n}{n-1},$$

with the phase function $\phi(x, t; y)$ given by (1.5).

Hörmander [Hor73] asked whether for phase functions ϕ satisfying (H1) and (H2), the estimate (1.4) could still hold for the same range of p as in (1.6). Let us briefly review known results on (1.4).

To simplify our discussion, we will always work in a small neighborhood of the origin, and therefore we will pick a sufficiently small $\epsilon_\phi > 0$ depending on ϕ , and assume that $a(x, t; y)$ is supported in $\mathbb{B}_{\epsilon_\phi}^{n-1} \times \mathbb{B}_{\epsilon_\phi}^1 \times \mathbb{B}_{\epsilon_\phi}^{n-1}$.¹ We often without loss of generality assume that $\phi(\mathbf{x}; y)$ is in its normal form at the origin, that is,

$$(1.7) \quad \phi(\mathbf{x}; y) = x \cdot y + t\langle y, Ay \rangle + O(|t||y|^3 + |\mathbf{x}|^2|y|^2),$$

where A is an $(n-1) \times (n-1)$ non-degenerate matrix. Normal forms (1.7) were introduced by Hörmander [Hor73] and Bourgain [Bou91] to simplify calculations. It is elementary to see ([Bou91, page 323]) that, after simple transformations, all phase functions $\phi(x, t; y)$ satisfying Hörmander's non-degeneracy condition can be written in normal form.

Theorem 1.1 (Hickman and Iliopoulou, [HI22]). *Let ϕ be a phase function of the form (1.7). Let s_0 be the signature of the matrix A . Then (1.4) holds for all*

$$(1.8) \quad p \geq \begin{cases} 2 \cdot \frac{s_0+2(n+1)}{s_0+2(n-1)} & \text{if } n \text{ is odd,} \\ 2 \cdot \frac{s_0+2n+3}{s_0+2n-1} & \text{if } n \text{ is even.} \end{cases}$$

Hickman and Iliopoulou [HI22], by generalizing the examples constructed earlier by Bourgain [Bou91], Wisewell [Wis05], Minicozzi and Sogge [MS97] and Bourgain and Guth [BG11], also showed that the range of p given by (1.8) is sharp. More precisely, there exists a phase function of the form (1.7) with s_0 being the signature of A , and the estimate (1.4) fails for p outside the ranges given in (1.8).

Several special cases of Theorem 1.1 are particularly interesting and were proven earlier.

Theorem 1.2 (Stein [Ste84], Bourgain and Guth [BG11]). *Let ϕ be a phase function of the form (1.7). Assume that A is of smallest possible signature, that is, $\text{sgn}(A) = 1$ when n is even and $\text{sgn}(A) = 0$ when n is odd. Then (1.4) holds for all*

$$(1.9) \quad p \geq \begin{cases} 2 \cdot \frac{n+1}{n-1} & \text{if } n \text{ is odd,} \\ 2 \cdot \frac{n+2}{n} & \text{if } n \text{ is even.} \end{cases}$$

¹Here and below, we refer to the notation section at the end of the introduction.

Theorem 1.3 (Lee [Lee06]; Guth, Hickman, Iliopoulou, [GHI19]). *Let ϕ be a phase function of the form (1.7). Assume that A is positive definite. Then (1.4) holds for all*

$$(1.10) \quad p \geq \begin{cases} 2 \cdot \frac{3n+1}{3n-3} & \text{if } n \text{ is odd,} \\ 2 \cdot \frac{3n+2}{3n-2} & \text{if } n \text{ is even.} \end{cases}$$

Recently, Guo, Wang and Zhang [GWZ22] imposed extra assumptions on the phase function ϕ and proved (1.4) for some p that goes beyond the sharp range given by (1.10).

Definition 1.4 (Bourgain's condition, [Bou91], [GWZ22]). *Let ϕ be a phase function satisfying Hörmander's non-degeneracy conditions. We say that it satisfies Bourgain's condition at $(\mathbf{x}_0; y_0)$ if*

$$(1.11) \quad \left((G_0 \cdot \nabla_{\mathbf{x}})^2 \nabla_y^2 \phi \right) (\mathbf{x}_0; y_0) \text{ is a multiple of } \left((G_0 \cdot \nabla_{\mathbf{x}}) \nabla_y^2 \phi \right) (\mathbf{x}_0; y_0).$$

The constant here is allowed to depend on \mathbf{x}_0 and y_0 .

Theorem 1.5 (Guo, Wang and Zhang [GWZ22]). *Let ϕ be a phase function of the form (1.7) with A positive definite. Moreover, assume that ϕ satisfies Bourgain's condition for every $(\mathbf{x}_0; y_0)$. Then (1.4) holds for all*

$$(1.12) \quad p > p_{\text{GWZ}}(n) := 2 + \frac{2.5921}{n} + O(n^{-2}).$$

Bourgain's condition is very natural when studying Hörmander-type oscillatory integrals. On the one hand, Bourgain [Bou91] proved that if the phase function ϕ fails Bourgain's condition at some $(\mathbf{x}_0; y_0)$, then (1.4) can not hold for all

$$(1.13) \quad p \geq \frac{2n}{n-1},$$

the range of the Fourier restriction conjecture (see (1.6)). More precisely, there exists $p > 2n/(n-1)$ depending only on the dimension n such that (1.4) fails at this p .

On the other hand, it is conjectured in [GWZ22] that if ϕ satisfies Bourgain's condition at every $(\mathbf{x}_0; y_0)$, then (1.4) holds for all $p \geq 2n/(n-1)$.

1.2. Curved Takeya problem

Associated to Hörmander-type oscillatory integrals, one can define Takeya sets. Given a phase function $\phi(\mathbf{x}; y)$ satisfying (H1) and (H2), we pick $\epsilon_\phi > 0$ to be a sufficiently small constant depending on ϕ .

Definition 1.6 (Curved tubes). *For $y \in \mathbb{B}_{\epsilon_\phi}^{n-1}$, $\mathbf{x} \in \mathbb{B}_{\epsilon_\phi}^n$ and $0 < \delta < \epsilon_\phi$, define*

$$(1.14) \quad \begin{aligned} \Gamma_y^{(\phi)}(\mathbf{x}) &:= \left\{ \mathbf{x}' \in \mathbb{R}^n \cap \mathbb{B}_{2\epsilon_\phi}^n : \nabla_y \phi(\mathbf{x}'; y) = \nabla_y \phi(\mathbf{x}; y) \right\}, \\ T_y^{\delta, (\phi)}(\mathbf{x}) &:= \left\{ \mathbf{x}' \in \mathbb{R}^n \cap \mathbb{B}_{2\epsilon_\phi}^n : |\nabla_y \phi(\mathbf{x}'; y) - \nabla_y \phi(\mathbf{x}; y)| < \delta \right\}. \end{aligned}$$

If \mathbf{x} is of the form $(\omega, 0)$, that is, the last coordinate is 0, then we often abbreviate $\Gamma_y^{(\phi)}(\mathbf{x})$ and $T_y^{\delta, (\phi)}(\mathbf{x})$ to $\Gamma_y^{(\phi)}(\omega)$ and $T_y^{\delta, (\phi)}(\omega)$. If it is clear from the context which ϕ is involved, we will abbreviate $\Gamma_y^{(\phi)}(\mathbf{x}), T_y^{\delta, (\phi)}(\mathbf{x})$ to $\Gamma_y(\mathbf{x}), T_y^\delta(\mathbf{x})$, respectively. We will call $T_y^\delta(\omega)$ the δ -tube associated to the phase function $\phi(\mathbf{x}; y)$ with frequency y and initial location ω ; $\Gamma_y(\omega)$ will be called the central curve of $T_y^\delta(\omega)$.

Definition 1.7 (Curved Takeya sets). *A set $E \subset \mathbb{R}^n$ with $\mathcal{L}^n(E) = 0$ is a curved Takeya set (associated to ϕ) if for all $y \in \mathbb{B}_{\epsilon_\phi}^{n-1}$ there exists an $\omega \in \mathbb{B}_{\epsilon_\phi}^{n-1}$ such that $\Gamma_y(\omega) \subset E$.*

Definition 1.8 (Curved Takeya maximal function, [Bou91]). *Given a phase function $\phi(\mathbf{x}; y)$ satisfying (H1) and (H2). For $y \in \mathbb{B}_{\epsilon_\phi}^{n-1}$ and $0 < \delta < \epsilon_\phi$, we define*

$$(1.15) \quad \mathcal{K}_\delta^{(\phi)} f(y) := \sup_{\omega \in \mathbb{B}_{\epsilon_\phi}^{n-1}} \frac{1}{\mathcal{L}^n(T_y^{\delta, (\phi)}(\omega))} \int_{T_y^{\delta, (\phi)}(\omega)} |f|.$$

If it is clear from the context which ϕ is involved, then we often abbreviate $\mathcal{K}_\delta^{(\phi)} f$ to $\mathcal{K}_\delta f$.

The problems of studying the Hausdorff dimensions of curved Takeya sets will be referred to as *curved Takeya problems*. Among all the phase functions ϕ , the one that is perhaps the most interesting is given by (1.5), that is,

$$(1.16) \quad \phi(x, t; y) = x \cdot y + t|y|^2.$$

In this case, the central curve $\Gamma_y(\omega)$ becomes

$$(1.17) \quad \{(x, t) : x + 2ty = \omega\},$$

which is a straight line. Takeya sets associated to (1.16) will be referred to as the traditional Takeya sets, or the traditional straight line Takeya sets. Moreover, for this special phase function, we have

Takeya conjecture. Let ϕ be given by (1.16). Then every Takeya set associated to ϕ must have full Hausdorff dimension, that is, it must have Hausdorff dimension n .

Maximal Takeya conjecture. Let ϕ be given by (1.16). Take $\epsilon_\phi = 1$. Then the associated Takeya maximal operator \mathcal{K}_δ satisfies

$$(1.18) \quad \|\mathcal{K}_\delta f\|_{L^n(\mathbb{B}^{n-1})} \lesssim_{n,\epsilon} \delta^{-\epsilon} \|f\|_{L^n(\mathbb{R}^n)},$$

for every $\epsilon > 0$ and every $\delta \in (0, 1)$.

It is well-known that

$$(1.19) \quad \begin{aligned} & \text{Fourier restriction conjecture} \\ \implies & \text{Maximal Takeya conjecture} \\ \implies & \text{Takeya conjecture.} \end{aligned}$$

Similar to the above implications, for Hörmander-type oscillatory integrals and curved Takeya problems, we also have

Theorem 1.9 (Wisewell [Wis05]). *Given a phase function $\phi(\mathbf{x}; y)$ satisfying (H1) and (H2). Let $\epsilon_\phi > 0$ be a sufficiently small constant depending on ϕ .*

(1) *Suppose that*

$$(1.20) \quad \|T_N^{(\phi)} f\|_{L^p(\mathbb{B}_{\epsilon_\phi}^n)} \lesssim_{\phi,p,n,a} N^{-\frac{n}{p}} \|f\|_{L^p(\mathbb{R}^{n-1})},$$

for some $p > 1$, every $N \geq 1$ and every a supported in $\mathbb{B}_{\epsilon_\phi}^{n-1} \times \mathbb{B}_{\epsilon_\phi}^1 \times \mathbb{B}_{\epsilon_\phi}^{n-1}$, then the curved Takeya maximal function is of restricted weak type (q, q) with norm at most $\delta^{-2(\frac{n}{q}-1)}$ where $q := (p/2)'$. In particular, as $p \rightarrow \frac{2n}{n-1}$, we see that $q \rightarrow n$.

(2) *If (1.20) holds for all $p > \frac{2n}{n-1}$, the largest possible range, then every curved Takeya set associated to ϕ must have Hausdorff dimension n .*

Curved Takeya problems are studied intensively in Wisewell's thesis [Wis03] and her paper [Wis05]. We refer interested readers to these two works for more results she obtained.

1.3. Carleson-Sjölin operators on manifolds

Let us work with a smooth Riemannian metric $\{g_{ij}(\mathbf{x})\}_{1 \leq i,j \leq n}$ defined on a small open neighborhood of $0 \in \mathbb{R}^n$. The Riemannian manifold is denoted by \mathcal{M} . We will only study curvature properties of \mathcal{M} near the origin. Let $\epsilon_{\mathcal{M}} > 0$ be a small constant depending on \mathcal{M} . Let $a(\mathbf{x}; \mathbf{y}) : \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}$ be a compactly supported smooth function supported on $\mathbb{B}_{\epsilon_{\mathcal{M}}}^n \times \mathbb{B}_{\epsilon_{\mathcal{M}}}^n$, and supported away from the diagonal. Define

$$(1.21) \quad T_N^{(\mathcal{M})} f(\mathbf{x}) := \int_{\mathcal{M}} e^{iN \text{dist}(\mathbf{x}, \mathbf{y})} a(\mathbf{x}; \mathbf{y}) f(\mathbf{y}) d\mathbf{y},$$

where dist refers to the distance function on \mathcal{M} , and call it a Carleson-Sjölin operator on the manifold \mathcal{M} . Our goal is to prove

$$(1.22) \quad \|T_N^{(\mathcal{M})} f\|_{L^p(\mathcal{M})} \lesssim_{\mathcal{M}, p, a, \epsilon} N^{-\frac{n}{p} + \epsilon} \|f\|_{L^p(\mathcal{M})},$$

for every $\epsilon > 0$, $N \geq 1$, and for a range of p that is as large as possible.

If one takes the metric g to the identity matrix at every point, then we have

Bochner-Riesz conjecture. The estimate (1.22) holds for all

$$(1.23) \quad p \geq \frac{2n}{n-1},$$

if \mathcal{M} is taken to be the Euclidean space.

Moreover, Tao [Tao99] proved that

$$(1.24) \quad \text{Bochner-Riesz conjecture} \implies \text{Fourier restriction conjecture}.$$

The study of Carleson-Sjölin operators on general Riemannian manifolds also has a long history, and the operator (1.21) already appeared in Minicozzi and Sogge [MS97] (see also Sogge [Sog17, page 290]). To prove bounds of the form (1.22), we will follow the Carleson-Sjölin reduction (see Carleson-Sjölin [CS72]). Let \mathcal{M}' be a hyperplane of \mathbb{R}^n intersecting $\mathbb{B}_{\epsilon_{\mathcal{M}}}^n$. Let $a(\mathbf{x})$ be a smooth function supported on $\mathbb{B}_{\epsilon_{\mathcal{M}}}^n$ satisfying

$$(1.25) \quad \text{dist}(\text{supp}(a), \mathcal{M}') > 0.$$

To prove (1.22), it suffices to prove

$$(1.26) \quad \|R_N^{(\mathcal{M}, \mathcal{M}')} f\|_{L^p(\mathcal{M})} \lesssim_{\mathcal{M}, p, a, \epsilon} N^{-\frac{n}{p} + \epsilon} \|f\|_{L^p(\mathcal{M}')},$$

for all f supported on $\mathcal{M}' \cap \mathbb{B}_{\epsilon_{\mathcal{M}}}^n$, where

$$(1.27) \quad R_N^{(\mathcal{M}, \mathcal{M}')} f(\mathbf{x}) := \int_{\mathcal{M}'} e^{iN \text{dist}(\mathbf{x}, \mathbf{y})} a(\mathbf{x}) f(\mathbf{y}) d\mathcal{H}^{n-1}(\mathbf{y}),$$

and the implicit constant in (1.26) depends also on the distance in (1.25) but not on each individual \mathcal{M}' . We will call $R_N^{(\mathcal{M}, \mathcal{M}')}$ a reduced Carleson-Sjölin operator on the manifold \mathcal{M} .

We will see below that the range of p for which (1.26) holds often determines curvature properties of the manifold \mathcal{M} near the origin.

1.4. Nikodym problems on manifolds

Take the manifold \mathcal{M} as in Subsection 1.3. Recall that $\epsilon_{\mathcal{M}} > 0$ is a small real number that is allowed to be sufficiently small depending on \mathcal{M} , and that we use dist to denote its distance function. For $\mathbf{x} \in \mathbb{B}_{\epsilon_{\mathcal{M}}/2}^n$, we use $\gamma_{\mathbf{x}}$ to denote the portion of a geodesic passing through \mathbf{x} that lies in $\mathbb{B}_{\epsilon_{\mathcal{M}}}^n$. Moreover, for $\lambda \in (0, 1)$, denote

$$(1.28) \quad \gamma_{\mathbf{x}, \lambda\text{-trun}} := \{\mathbf{x}' \in \gamma_{\mathbf{x}} : \text{dist}(\mathbf{x}, \mathbf{x}') \geq 1 - \lambda\}.$$

Definition 1.10 (Nikodym set, Sogge [Sog99]). *Let $\lambda \in (0, 1)$. A set $E \subset \mathbb{R}^n$ is said to be a λ -Nikodym set if*

$$(1.29) \quad \mathcal{L}^n(\{\mathbf{x} \in \mathbb{B}_{\epsilon_{\mathcal{M}}/2}^n : \text{There exists } \gamma_{\mathbf{x}} \text{ such that } \gamma_{\mathbf{x}, \lambda\text{-trun}} \subset E\}) > 0.$$

*A set E is said to be Nikodym if it is λ -Nikodym for every $\lambda < 1$.*²

For $\delta > 0$, let $\gamma_{\mathbf{x}}^{\delta}$ be the δ -neighborhood of $\gamma_{\mathbf{x}}$. Similarly, we define $\gamma_{\mathbf{x}, \lambda\text{-trun}}^{\delta}$.

Definition 1.11 (Nikodym maximal function, Sogge [Sog99]). *Let $\delta \in (0, 1)$ and $\lambda \in (0, 1)$. For a function f defined on \mathcal{M} and $\mathbf{x} \in \mathbb{B}_{\epsilon_{\mathcal{M}}/2}^n$, define*

$$(1.30) \quad f_{\delta}^*(\mathbf{x}) := \sup_{\gamma_{\mathbf{x}}} \delta^{-(n-1)} \int_{\gamma_{\mathbf{x}}^{\delta}} |f|,$$

²The definition of Nikodym sets has a slightly different formulation from that in Sogge [Sog99], but they are essentially the same.

and call it the Nikodym maximal function. Moreover, define

$$(1.31) \quad \mathcal{N}_{\delta,\lambda} f(\mathbf{x}) := \sup_{\gamma_{\mathbf{x}}} \delta^{-(n-1)} \int_{\gamma_{\mathbf{x},\lambda-\text{trun}}^{\delta}} |f|,$$

and call it the λ -Nikodym maximal function.

Theorem 1.12 (Sogge [Sog99], Xi [Xi17]). *Assume that the manifold \mathcal{M} with dimension $n \geq 3$ has constant sectional curvature. Then for sufficiently small $\epsilon_{\mathcal{M}} > 0$ depending on \mathcal{M} , it holds that*

$$(1.32) \quad \|f_{\delta}^*\|_{L^q(\mathbb{B}_{\epsilon_{\mathcal{M}/2}^n})} \lesssim_{q,p,\mathcal{M},\epsilon} \delta^{1-\frac{n}{p}-\epsilon} \|f\|_{L^p(\mathcal{M})},$$

for all $\delta \in (0, 1)$, $\epsilon > 0$ and all

$$(1.33) \quad 1 \leq p \leq \frac{n+2}{2}, \quad q = (n-1)p'.$$

Consequently, every Nikodym set in \mathcal{M} must have Hausdorff and Minkowski dimensions at least $\frac{n+2}{2}$.

In Sogge [Sog99], the author, after investigating bounds for Nikodym maximal operators for three dimensional manifolds of constant curvatures, also considered manifolds whose sectional curvatures are not constant. To state Sogge's result, let us first recall several concepts from Riemannian geometry.

Let \mathcal{M} be a three-dimensional manifold as in Subsection 1.3 with Riemannian metric $g = \{g_{ij}\}_{1 \leq i,j \leq 3}$. Denote by Ric the Ricci tensor on \mathcal{M} , which is a $(0, 2)$ -tensor.³ Denote

$$(1.34) \quad \bar{\text{Ric}} : T\mathcal{M} \rightarrow T\mathcal{M},$$

which satisfies

$$(1.35) \quad g(\bar{\text{Ric}}(X_1), X_2) := \text{Ric}(X_1, X_2), \quad \forall X_1, X_2 \in T\mathcal{M}.$$

Definition 1.13 (Chaotic curvature, Sogge [Sog99]).⁴ *Let γ be a geodesic parametrized by arclength with $\gamma(0) = 0 \in \mathcal{M}$. Take a unit vector $X(0) \in$*

³We refer to Subsection 3.1 for these standard definitions from Riemannian geometry.

⁴The definition of chaotic curvature here is formulated slightly differently from Sogge's, see [Sog99, Definition 3.1].

$T_{\gamma(0)}\mathcal{M}$ with $\dot{\gamma}(0) \perp X(0)$. Let $X(t) \in T_{\gamma(t)}\mathcal{M}$ be the parallel transport of $X(0)$ along γ . Denote

$$(1.36) \quad Y(t) := \bar{\text{Ric}}(X(t)),$$

and let $Y^\perp(t)$ be the projection of $Y(t)$ to the orthonormal complement of the space spanned by $X(t)$ and $\dot{\gamma}(t)$. We say that the manifold \mathcal{M} satisfies the chaotic curvature at the origin, if

$$(1.37) \quad |Y^\perp(0)| + |\nabla_{\dot{\gamma}} Y^\perp(0)| \neq 0,$$

for all geodesics γ passing through the origin, and all $X(t)$ given above.

Theorem 1.14 (Sogge [Sog99]). *Let \mathcal{M} be a three-dimensional manifold as in Subsection 1.3. Assume that \mathcal{M} satisfies the chaotic curvature condition at the origin, then every Nikodym set on \mathcal{M} must have Minkowski dimension $\geq 7/3$.*

Later, Sogge, Xi and Xu [SXX18] proved that Theorem 1.14 is stable under perturbations. Let us be slightly more precise. Let \mathcal{M} be a manifold satisfying Sogge’s chaotic curvature condition. If we give a “small” perturbation to the metric of \mathcal{M} , then the new metric also satisfies Sogge’s chaotic curvature condition.

Notations. We list notations that are used in the introduction and in the rest of the paper.

1. For $\epsilon > 0$ and $\mathbf{x} \in \mathbb{R}^n$, we let $\mathbb{B}_\epsilon^n(\mathbf{x})$ denote the ball of radius ϵ in \mathbb{R}^n centered at \mathbf{x} . If $\epsilon = 1$, we often abbreviate $\mathbb{B}_1^n(\mathbf{x})$ to $\mathbb{B}^n(\mathbf{x})$; if $\mathbf{x} = 0$, then we often abbreviate $\mathbb{B}_\epsilon^n(\mathbf{x})$ to \mathbb{B}_ϵ^n .
2. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, their last components often play a distinct role compared with the first $n - 1$ one, and therefore we often write $\mathbf{x} = (x, t), \mathbf{y} = (y, \tau)$, with $x, y \in \mathbb{R}^{n-1}$.
3. For a vector $\mathbf{v} \in \mathbb{R}^n$, we use $|\mathbf{v}|$ to denote its standard Euclidean length. For a manifold \mathcal{M} with metric tensor g , we use $|\mathbf{v}|$ to denote its length $\sqrt{g_p(\mathbf{v}, \mathbf{v})}$ for $p \in \mathcal{M}, \mathbf{v} \in T_p\mathcal{M}$.
4. We will use \widehat{f} or $(f)^\wedge$ to denote the Fourier transform of f .
5. For a set E , we will use $\mathbb{1}_E$ to denote its indicator function.
6. For a set $E \subset \mathbb{R}^n$ and $\delta > 0$, we use $\mathcal{N}_\delta(E)$ to denote the δ -neighborhood of E .

7. For two non-negative real numbers a, b and a parameter p , we use $a \lesssim_p b$ to mean that there exists a constant C_p depending only on p such that $a \leq C_p b$. For instance, let $T : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ be an operator. We use

$$(1.38) \quad \|Tf\|_{L^p(\mathbb{R}^n)} \lesssim_p \|f\|_{L^p(\mathbb{R}^n)}$$

to mean that there exists $C_p \in \mathbb{R}$ depending only on the Lebesgue exponent p such that

$$(1.39) \quad \|Tf\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)},$$

for all functions f . If it is clear from the context which parameters p are involved, then we often abbreviate $a \lesssim_p b$ to $a \lesssim b$. Similarly, we define $a \gtrsim_p b$. Moreover, we use $a \simeq_p b$ to mean $a \lesssim_p b$ and $a \gtrsim_p b$.

8. For a set $E \subset \mathbb{R}^n$, we use $\mathcal{L}^n(E)$ to refer to its Lebesgue measure.
 9. For $p \in [1, \infty]$, we use p' to denote its Hölder dual, that is, $1/p + 1/p' = 1$.
 10. All manifolds in the current paper are assumed to be smooth.
 11. For a rectangle $\square \subset \mathbb{R}^n$ and $r > 0$, we use $r\square$ to denote the rectangle with the same center as R , but dilated by r with respect to the center of R .
 12. We try to make sure that the same notations are not repeatedly used within a same section, unless otherwise specified. However, if a same notation appears in different sections, it may refer to different things.

Acknowledgements. S. G. would like to thank Sigurd Angenent, Yanyan Li and Ruobing Zhang for discussions on relevant Riemannian geometry materials in the paper, and thank Yakun Xi for pointing out the reference [SXX18]. Part of the work was done during Guo's multiple visits to Rutgers University; he would like to thank Yanyan Li for the invitations and the hospitality. S. D. is partly supported by NSF of China (No. 11971244 and No. 12071338); S. G. is partly supported by the Nankai Zhide Foundation and partly by NSF-2044828; R. Z. is supported by NSF DMS-2207281 (transferred from DMS-1856541), NSF DMS-2143989 and the Sloan Research Fellowship. The authors would like to thank the referee for carefully reading the paper and for various helpful suggestions which significantly improved the exposition of the paper.

2. Statement of main results

The first result is about the connections among the operators introduced above, and is well known. Roughly speaking, it says that

$$(2.1) \quad \begin{aligned} & \text{Hörmander-type oscillatory integrals} \\ \implies & \text{reduced Carleson-Sjölin on manifolds} \\ \implies & \lambda\text{-Nikodym maximal functions.} \end{aligned}$$

Let us be more precise. Let \mathcal{M} be a Riemannian manifold as in Subsection 1.3 of dimension $n \geq 3$, and let $\epsilon_{\mathcal{M}} > 0$ be a small constant that is allowed to depend on \mathcal{M} . Recall reduced Carleson-Sjölin operators defined in (1.26).

Theorem 2.1. *a) Reduced Carleson-Sjölin operators satisfy Hörmander's non-degeneracy conditions.
b) Assume that*

$$(2.2) \quad \|R_N^{(\mathcal{M}, \mathcal{M}')} f\|_{L^p(\mathcal{M})} \lesssim_{\mathcal{M}, \mathcal{M}', p, a} N^{-\frac{n}{p}} \|f\|_{L^p(\mathcal{M}')} ,$$

for some $p > 1$, every $N \geq 1$, every hyperplane \mathcal{M}' intersecting $\mathbb{B}_{\epsilon_{\mathcal{M}}}^n$ and every smooth amplitude function satisfying the separation condition (1.25). Then the λ -Nikodym maximal function $\mathcal{N}_{\delta, \lambda}$ is of restricted weak type (q, q) with norm

$$(2.3) \quad \lesssim_{\lambda, \mathcal{M}, p} \delta^{-2(\frac{n}{q}-1)} ,$$

where $q := (p/2)'$, for every $\lambda < 1$.

The formulation of item b) in Theorem 2.1 is taken from Wisewell's Theorem 1.9 in Subsection 1.2. Indeed, the proof of item b) is also essentially the same as that of Theorem 1.9.

We should also mention that curved Keakeya maximal operators are also closely related to λ -Nikodym maximal operators. Indeed in many interesting cases, they are essentially the same objects. Later we will use this relation, for instance in the proof of Theorem 2.2. However, due to purely technical reasons, we do not have a clean way to state such relations.

In the appendix, we will explain a key difference between curved Keakeya maximal operators and λ -Nikodym maximal operators.

Before stating the next result, let us recall the result in [GWZ22], as stated in Theorem 1.5. In [GWZ22], the authors considered Hörmander-type

oscillatory integrals, and showed that if the phase function $\phi(\mathbf{x}; y)$ satisfies Bourgain's condition everywhere, then all the current techniques that have been developed so far in the study of the Fourier restriction conjecture can also be applied to Hörmander-type oscillatory integrals.

Indeed, the same principle applies also to the study of curved Kakeya problems. Let $n \geq 3$. For phase functions ϕ satisfying the same assumptions as in Theorem 1.5, the associated curved Kakeya sets satisfy the same dimension bounds as what Hickman, Rogers and Zhang [HRZ22] and Zahl [Zah21] obtained for the traditional straight line Kakeya sets. Let us be more precise. Denote

$$(2.4) \quad q_{\text{HRZ}}(n) := 1 + \min_{2 \leq k \leq n} \max \left\{ \frac{2n}{(n-1)n + (k-1)k}, \frac{1}{n-k+1} \right\},$$

which is the exponent that appeared in [HRZ22, Theorem 1.2]. Moreover, denote

$$(2.5) \quad p_{\text{HRZ}}(n) := (q_{\text{HRZ}}(n))'.$$

Then one can follow the same argument as in [HRZ22], use the (strong) polynomial Wolff axioms for ϕ obtained in [GWZ22, Theorem 1.2, Theorem 6.2], a standard equivalence argument (see for instance [Mat15, Proposition 22.6]) and obtain

$$(2.6) \quad \|\mathcal{K}_\delta^{(\phi)} f\|_{L^p(\mathbb{B}_{\epsilon_\phi}^{n-1})} \lesssim_{\phi, \epsilon, p} \delta^{-\epsilon} \delta^{-(n-1-\frac{n}{p'})} \|f\|_{L^p(\mathbb{R}^n)},$$

for every

$$(2.7) \quad 1 < p \leq p_{\text{HRZ}}(n),$$

every $\epsilon > 0, \delta \in (0, 1)$. In particular, for every p in the above range, the exponent of δ on the right hand side is sharp. Moreover, it seems reasonable to conjecture that (2.6) holds for all $1 < p \leq n$, that is, the same range of p as in the maximal Kakeya conjecture.

As a corollary of (2.6) (see [Mat15, Theorem 22.9]), we obtain that every curved Kakeya set associated to ϕ must have Hausdorff dimension at least

$$(2.8) \quad n - \left(n - 1 - \frac{n}{q_{\text{HRZ}}(n)} \right) p_{\text{HRZ}}(n) =: d_{\text{HRZ}}(n).$$

If we let $n \rightarrow \infty$, then (see [HRZ22, Subsection 9.2])

$$(2.9) \quad d_{\text{HRZ}}(n) = (2 - \sqrt{2})n + O(1).$$

It is worth mentioning that the asymptotic (2.9) obtained by Hickman, Rogers and Zhang [HRZ22] for the dimensions of the traditional Kakeya sets (for straight lines) is the same as that of Katz and Tao [KT02]. Moreover, these two results [HRZ22] and [KT02] together give the currently best known results for the dimensions of the traditional Kakeya sets in high dimensions.

Let us state our next result, which involves bounds for reduced Carleson-Sjölin operators defined in (1.26), and for λ -Nikodym maximal operators in Definition 1.11. Let \mathcal{M} be a Riemannian manifold as in Subsection 1.3 of dimension $n \geq 3$. let $\epsilon_{\mathcal{M}} > 0$ be a small constant that is allowed to depend on \mathcal{M} .

Theorem 2.2. (a) Assume that \mathcal{M} is of constant sectional curvature. Recall the definition of $p_{\text{GWZ}}(n)$ in (1.12). Then

$$(2.10) \quad \left\| R_N^{(\mathcal{M}, \mathcal{M}')} f \right\|_{L^p(\mathcal{M})} \lesssim_{\mathcal{M}, \mathcal{M}', a, p, \epsilon} N^{-\frac{n}{p} + \epsilon} \|f\|_{L^p(\mathcal{M}')} ,$$

for all $p \geq p_{\text{GWZ}}(n)$, $\epsilon > 0$, $N \geq 1$, all hyperplanes \mathcal{M}' intersecting $\mathbb{B}_{\epsilon_{\mathcal{M}}}^n$, and all functions $a(\mathbf{x})$ satisfying the separation condition (1.25).

(b) Assume \mathcal{M} is analytic and its sectional curvature is not constant. Then we can find $p > \frac{2n}{n-1}$, a small positive $\epsilon_{\mathcal{M}} > 0$, a hyperplane \mathcal{M}' intersecting $\mathbb{B}_{\epsilon_{\mathcal{M}}}^n$, a smooth function $a(\mathbf{x})$ satisfying the separation condition (1.25), such that (2.10) fails.

(c) Assume that \mathcal{M} is of constant sectional curvature. Then

$$(2.11) \quad \left\| \mathcal{N}_{\delta, \lambda} f \right\|_{L^p(\mathbb{B}_{\epsilon_{\mathcal{M}}/2}^n)} \lesssim_{p, \epsilon, \lambda, \mathcal{M}} \delta^{-\epsilon} \delta^{-(n-1-\frac{n}{p'})} \|f\|_{L^p(\mathcal{M})} ,$$

for every $\lambda < 1$, $\epsilon > 0$, $\delta \in (0, 1)$ and every

$$(2.12) \quad 1 < p \leq p_{\text{HRZ}}(n) .$$

Consequently, every Nikodym set must have Minkowski dimension $\geq d_{\text{HRZ}}(n)$.

Recall Carleson-Sjölin operators defined in (1.21):

$$(2.13) \quad T_N^{(\mathcal{M})} f(\mathbf{x}) := \int_{\mathcal{M}} e^{iN \text{dist}(\mathbf{x}, \mathbf{y})} a(\mathbf{x}; \mathbf{y}) f(\mathbf{y}) d\mathbf{y} ,$$

where $a(\mathbf{x}; \mathbf{y}) : \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}$ is a compactly supported smooth function supported on $\mathbb{B}_{\epsilon_{\mathcal{M}}}^n \times \mathbb{B}_{\epsilon_{\mathcal{M}}}^n$, and supported away from the diagonal. By item a)

of Theorem 2.2 and Fubini's theorem (Carleson-Sjölin reduction as explained in Subsection 1.3), we obtain

Corollary 2.3. *Let \mathcal{M} be a Riemannian manifold of constant sectional curvature. Let $n \geq 3$ be the dimension of \mathcal{M} . Then*

$$(2.14) \quad \|T_N^{(\mathcal{M})} f\|_{L^p(\mathcal{M})} \lesssim_{\mathcal{M}, p, a, \epsilon} N^{-\frac{n}{p} + \epsilon} \|f\|_{L^p(\mathcal{M})},$$

for all

$$(2.15) \quad p \geq p_{\text{GWZ}}(n),$$

all $\epsilon > 0$ and $N \geq 1$.

In particular, if we take \mathcal{M} to be the standard Euclidean space, then the bound (2.14) with the range (2.15) is precisely what Guo, Wang and Zhang [GWZ22] obtained for the (Euclidean) Bochner-Riesz operator. Moreover, this bound is the currently best known bound for the Bochner-Riesz conjecture (stated in Subsection 1.3). Our bound (2.14) generalizes that of Guo, Wang and Zhang [GWZ22] for the Euclidean distance functions to distance functions on manifolds of constant curvatures.

Item (b) in Theorem 2.2 says that if \mathcal{M} does not have constant sectional curvature, then the reduced Carleson-Sjölin operator will not satisfy as good bounds as those on manifolds of constant sectional curvature. By adapting the argument in Sogge [Sog17, page 290], one can also show that for manifolds whose sectional curvatures are not constant, the estimate (2.14) also fails for some $p > 2n/(n-1)$.

Item (c) in Theorem 2.2 is an improvement over the result of Xi [Xi17] as stated in Theorem 1.12. Recall that Sogge [Sog99] proved that Nikodym sets on three dimensional manifolds of constant curvatures must have Minkowski dimension $\geq 5/2$. Xi [Xi17] generalized this result to higher dimensions, and proved that Nikodym sets on n dimensional manifolds of constant curvatures must have Minkowski dimension $\geq (n+2)/2$.

It is reasonable to conjecture that for manifolds of constant sectional curvatures, the bound (2.10) holds for the largest possible range

$$(2.16) \quad p \geq \frac{2n}{n-1},$$

the same as the range (1.6) in the Fourier restriction conjecture. It is also reasonable to conjecture that every Nikodym set on manifolds of constant

sectional curvatures must have a full Hausdorff dimension. Theorem 2.2 provides some partial evidence for such conjectures.

So far we have studied the curved Kakeya problem for ϕ satisfying Bourgain's condition, bounds for (reduced) Carleson-Sjölin operators on manifolds of constant sectional curvatures, and Nikodym problems on manifolds of constant sectional curvatures. The settings in these problems are perhaps the “best” possible in the sense that we conjecture all these problems would eventually have the same answers to their Euclidean counterparts.

Moreover, Bourgain [Bou91], Wisewell [Wis05], Minicozzi and Sogge [MS97], Bourgain and Guth [BG11], Sogge, Xi and Xu [SXX18], Guth, Hickman and Iliopoulou [GHI19] and Hickman and Iliopoulou [HI22] have constructed “worst” possible examples in these problems.

Our next goal is to study “intermediate” examples. We will only consider the case $n = 3$.

Let $\phi(x, t; y)$ be a phase function satisfying Hörmander's non-degeneracy conditions. Let $\epsilon_\phi > 0$ be a small constant depending on ϕ . Fix $(x_0, t_0; y_0) \in \mathbb{B}_{\epsilon_\phi}^{n-1} \times \mathbb{B}_{\epsilon_\phi}^1 \times \mathbb{B}_{\epsilon_\phi}^{n-1}$. Let $X_0(t)$ be the unique solution to

$$(2.17) \quad \nabla_y \phi(X_0(t) + x_0, t + t_0; y_0) = \nabla_y \phi(x_0, t_0; y_0).$$

The existence and uniqueness of the solution are guaranteed by Hörmander's non-degeneracy conditions, and that ϵ_ϕ is chosen sufficiently small. Denote

$$(2.18) \quad \phi_0(x, t; y) := \phi(x + x_0, t + t_0; y + y_0) - \phi(x_0, t_0; y + y_0).$$

Moreover, denote

$$(2.19) \quad D_{ij}(t) := \partial_{y_i} \partial_{y_j} \phi_0(X_0(t), t; 0), \quad 1 \leq i, j \leq 2,$$

and

$$(2.20) \quad \mathfrak{D}(t) := \det \begin{bmatrix} D_{11}(t) & D_{12}(t) \\ D_{21}(t) & D_{22}(t) \end{bmatrix}.$$

Take an integer $k \geq 4$. We say that the phase function ϕ is of a contact order $\leq k$ at the point (x_0, t_0) if the matrix

$$(2.21) \quad \begin{bmatrix} \mathfrak{D}'(0) & \mathfrak{D}''(0) & \dots & \mathfrak{D}^{(k)}(0) \\ D'_{11}(0) & D''_{11}(0) & \dots & D_{11}^{(k)}(0) \\ D'_{12}(0) & D''_{12}(0) & \dots & D_{12}^{(k)}(0) \\ D'_{22}(0) & D''_{22}(0) & \dots & D_{22}^{(k)}(0) \end{bmatrix}$$

has rank 4. The matrix (2.21) already implicitly appeared in Bourgain's work [Bou91], see equation (6.160) in [Bou91, page 364].

Theorem 2.4. *Let $k \in \mathbb{N}$ and $k \geq 4$. Take $n = 3$ and $\phi(x, t; y)$ a smooth phase function of the normal form (1.7), that is,*

$$(2.22) \quad \phi(\mathbf{x}; y) = x \cdot y + t \langle y, Ay \rangle + O(|t||y|^3 + |\mathbf{x}|^2|y|^2).$$

Assume that A is positive definite and that the contact order of ϕ at the origin $x = 0, t = 0, y = 0$ is $\leq k$. Then there exists $\epsilon_\phi > 0$ depending on ϕ , and

$$(2.23) \quad \epsilon_k := \frac{1}{9k - 6},$$

such that

$$(2.24) \quad \|T_N^{(\phi)} f\|_{L^p(\mathbb{R}^3)} \lesssim_{\phi, a, p, \epsilon} N^{-\frac{3}{p} + \epsilon} \|f\|_{L^p(\mathbb{R}^2)}$$

holds for all

$$(2.25) \quad p \geq \frac{10}{3} - \epsilon_k,$$

all $\epsilon > 0, N \geq 1$, and all smooth amplitude functions a supported in $\mathbb{B}_{\epsilon_\phi}^2 \times \mathbb{B}_{\epsilon_\phi}^1 \times \mathbb{B}_{\epsilon_\phi}^2$.

If we take $k = 2$ (which is not allowed by Theorem 2.4), then the range (2.25) becomes $p \geq 3.25$, the range of Guth [Gut16]. This is not a coincidence, and in the proof of Theorem 2.4 we will generalize the polynomial Wolff axioms in [Gut16] to Hörmander-type oscillatory integrals.

If we take $k = 4$, the smallest value of k that is allowed by Theorem 2.4, then the range (2.25) becomes $p \geq 3.3$, the range of Bourgain and Guth [BG11, Theorem 2]. This seems very likely just a coincidence.

The notion of contact orders also in some sense already appeared in Sogge's work [Sog99].

Theorem 2.5. *Let \mathcal{M} be a smooth manifold as in Subsection 1.3 of dimension three. Assume that \mathcal{M} satisfies Sogge's chaotic curvature condition at the origin. Then there exists $\epsilon_{\mathcal{M}} > 0$ depending only on \mathcal{M} , such that if we let*

$$(2.26) \quad \phi(x, t; y) := \text{dist}((x, t), (y, y_3)),$$

where $y_3 \in (0, \epsilon_{\mathcal{M}})$, then ϕ has a contact order ≤ 4 at $(x, t; y) = (0, 0; 0)$.

As an immediate corollary of Theorem 2.5 and Theorem 2.4, we obtain

Corollary 2.6. *Let \mathcal{M} be a three dimensional manifold as in Theorem 2.5. Then the Carleson-Sjölin operator $T_N^{(\mathcal{M})}$ on \mathcal{M} satisfies*

$$(2.27) \quad \|T_N^{(\mathcal{M})} f\|_{L^p(\mathcal{M})} \lesssim_{\mathcal{M}, p, a, \epsilon} N^{-\frac{3}{p} + \epsilon} \|f\|_{L^p(\mathcal{M})},$$

for all $p \geq 3.3$, $N \geq 1$ and $\epsilon > 0$.

Let us briefly explain the geometric intuition behind chaotic curvatures. Interested readers should read [Sog99, Section 3], especially Proposition 3.2 there, in order to have a better understanding of Sogge's chaotic curvature.

Let \mathcal{M} be a three dimensional manifold. Assume that \mathcal{M} satisfies Sogge's chaotic curvature condition. Take a coordinate patch, and express it in Fermi coordinates (see Definition 3.1 below). Take a coordinate plane in the Fermi coordinate. Then \mathcal{M} satisfying chaotic curvature conditions means that no geodesic in \mathcal{M} can have a contact order ≥ 5 with the given coordinate plane. In other words, if we have a geodesic that is first order tangent, second order tangent and third order tangent to the coordinate plane, then it cannot be fourth order tangent to it.

In the end, let us mention that Sogge's chaotic curvature is only formulated in dimension three, and Theorem 2.5 also holds only in dimension three. It is a very interesting question to generalize the notion of chaotic curvatures to higher dimensions.

3. Proof of Theorem 2.5

3.1. Preliminaries in Riemannian geometry

All the materials in this subsection can be found in standard Riemannian geometry textbooks, for instance Peterson [Pet16]. Let \mathcal{M} be a smooth Riemannian manifold of dimension n . Let g be the metric tensor on \mathcal{M} . Given a smooth local coordinate, let $\{e_i\}_{i=1}^n$ be the standard basis of the tangent space. Denote the metric expression as

$$(3.1) \quad g_{ij} := g(e_i, e_j).$$

We use $(g^{kl})_{1 \leq k, l \leq n}$ to denote the inverse of g . Denote

$$(3.2) \quad g_{ij,k} := \nabla_{e_k} g_{ij}, \quad g_{ij,kl} := \nabla_{e_k} \nabla_{e_l} g_{ij},$$

where ∇_{e_k} means directional derivative here. Similarly, we define $g_{ij,klm}$. The covariant derivative of a function f defined on \mathcal{M} is defined to be

$$(3.3) \quad \bar{\nabla} f := \sum_{ij} g^{ij} \cdot (\nabla_{e_j} f) e_i.$$

Here we add a bar on top of ∇ just to distinguish $\bar{\nabla} f$ from ∇f ; we want $\bar{\nabla} f$ to be a vector field $((1, 0)$ -tensor), while ∇f is a one-form $((0, 1)$ -tensor). The covariant derivative of a vector field Y $((1, 0)$ -tensor) is defined to be the $(1, 1)$ -tensor ∇Y given by

$$(3.4) \quad (\nabla Y)(X) := \nabla_X Y,$$

where X is a vector field and in $\nabla_X Y$ we use the standard Levi-Civita connection, which is the unique connection ∇ that is compatible with g and torsion free: For vector fields X, Y, Z ,

$$(3.5) \quad Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z);$$

$$(3.6) \quad \nabla_X Y - \nabla_Y X = [X, Y],$$

where $[X, Y]$ denotes the Lie bracket of X, Y . More definitions can be seen e.g. [Lee, Chapter 5].

Although here we need X, Y to be vector fields to obtain $\nabla_X Y$, $\nabla_X Y$ at point p only depends on the value of X at point p and the value of Y along a curve whose tangent vector at p coincide with X at p . So, along a geodesic γ , the definition $\nabla_{\dot{\gamma}} Y$ also make sense.

We then define the Hessian of a function f to be a $(0, 2)$ -tensor given by

$$(3.7) \quad (\text{Hessian } f)(X, Y) := g(\nabla_X \bar{\nabla} f, Y),$$

where X, Y are vector fields.

Christoffel symbols Γ_{ij}^k are defined by⁵

$$(3.8) \quad \nabla_{e_i} e_j = \sum_k \Gamma_{ij}^k e_k.$$

In local coordinates they can be computed by

$$(3.9) \quad \Gamma_{ij}^k = \frac{1}{2} \sum_m g^{km} (g_{mi,j} + g_{mj,i} - g_{ij,m}).$$

⁵In the literature and in textbooks, the notation Γ_{ij}^k is also often used.

Let $R : T\mathcal{M} \times T\mathcal{M} \times T\mathcal{M} \rightarrow T\mathcal{M}$ be the Riemannian curvature tensor, defined by

$$(3.10) \quad R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

Define $R_{ijk}{}^l$ by

$$(3.11) \quad R(e_i, e_j)e_k = \sum_l R_{ijk}{}^l e_l.$$

If we denote

$$(3.12) \quad R_{ijkl} = g(R(e_i, e_j)e_k, e_l),$$

then we have

$$(3.13) \quad R_{ijkl} = \sum_p R_{ijk}{}^p g_{pl}.$$

There are symmetry properties for the Riemannian curvature tensor.

$$(3.14) \quad \begin{aligned} R_{ijkl} &= -R_{jikl} = -R_{ijlk} = R_{klij}, \\ R_{ijkl} + R_{iklj} + R_{iljk} &= 0. \end{aligned}$$

In local coordinates, we have

$$(3.15) \quad R_{lki}{}^q = -\left(\partial_k \Gamma_{li}^q - \partial_l \Gamma_{ki}^q - \sum_p \Gamma_{lp}^q \Gamma_{ki}^p + \sum_p \Gamma_{kp}^q \Gamma_{li}^p\right).$$

The Ricci tensor is defined by

$$(3.16) \quad \text{Ric}(Y, Z) := \text{tr}(X \mapsto R(X, Y)Z),$$

where the right hand side means the trace of the map $X \mapsto R(X, Y)Z$. That is, under any orthonormal basis $\{E_i\}_{i=1}^n$,

$$(3.17) \quad \text{Ric}(Y, Z) = \sum_{i=1}^n g(R(E_i, Y)Z, E_i).$$

Definition 3.1 (Fermi coordinates). *Take an n dimensional manifold \mathcal{M} . Let $\gamma : [0, L] \rightarrow \mathcal{M}$ be an arclength parametrized geodesic. At $s = 0$, pick $n - 1$ tangent vectors $E_1(0), \dots, E_{n-1}(0) \in T_{\gamma(0)}\mathcal{M}$ that form an orthonormal*

basis together with $\dot{\gamma}(0)$. Let $E_1(s), \dots, E_{n-1}(s)$ be the parallel transports of $E_1(0), \dots, E_{n-1}(0)$ along $\gamma(s)$, respectively. One then assigns Fermi coordinates (x_1, \dots, x_{n-1}, t) to a point if

1. this point lies on the geodesic passing through the point $\gamma(t)$ with the unit tangent vector at this point being

$$(3.18) \quad \frac{1}{|(x_1, \dots, x_{n-1})|} \left(x_1 E_1(t) + \dots + x_{n-1} E_{n-1}(t) \right),$$

where $|(x_1, \dots, x_{n-1})|$ means Euclidean norm;

2. the distance from this point to $\gamma(t)$ is $|(x_1, \dots, x_{n-1})|$.

In Fermi coordinates, metrics satisfy

- 1.

$$(3.19) \quad \begin{bmatrix} g_{11}(\mathbf{x}) & \cdots & g_{1n-1}(\mathbf{x}) & g_{1n}(\mathbf{x}) \\ \vdots & & \vdots & \vdots \\ g_{n-11}(\mathbf{x}) & \cdots & g_{n-1n-1}(\mathbf{x}) & g_{2n}(\mathbf{x}) \\ g_{n1}(\mathbf{x}) & \cdots & g_{nn-1}(\mathbf{x}) & g_{nn}(\mathbf{x}) \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ 0 \end{bmatrix},$$

with $\mathbf{x} = (x_1, \dots, x_{n-1}, t)$,

- 2.

$$(3.20) \quad g_{ij}(\mathbf{x}) = \delta_{ij}, \quad \text{whenever } x_1 = \dots = x_{n-1} = 0.$$

In other words, the matrix $(g_{i,j})_{1 \leq i,j \leq n}$ is always the identity matrix along the geodesic γ .

- 3.

$$(3.21) \quad \frac{\partial}{\partial x_i} g_{jk}(\mathbf{x}) = 0, \quad i = 1, \dots, n-1 \quad \forall j, k, \quad \text{whenever } x_1 = \dots = x_{n-1} = 0.$$

Indeed, (3.21) can also be written as

$$(3.22) \quad \frac{\partial}{\partial x_i} g_{jk}(\mathbf{x}) = 0, \quad \forall i, j, k, \quad \text{whenever } x_1 = \dots = x_{n-1} = 0.$$

In the three dimensional case $n = 3$, the above three properties (3.19)–(3.21) can be found in [Sog99, page 5]. In higher dimensions, similar estimates hold as

well: See [Lee, Theorem 6.38], [Lee, Proposition 5.26] and [Kli95, Proposition 1.12.1].

In the end of this subsection, we state one lemma that will be used multiple times in the rest of the paper.

Let $\epsilon_{\mathcal{M}} > 0$ be a small constant depending on \mathcal{M} . Fix $\epsilon \in (0, \epsilon_{\mathcal{M}})$. Denote

$$(3.23) \quad \phi_{\epsilon}(x, t; y) := \text{dist}((x, t), (y, \epsilon)).$$

We always consider (x, t) in a small neighborhood of the origin, and (y, ϵ) in a small neighborhood of $(0, \epsilon)$.

Lemma 3.2. *Let $n \geq 2$. Fix a point (y_0, ϵ) . Let γ be a geodesic passing through $(y_0, \epsilon) \in \mathcal{M}$. Then*

$$(3.24) \quad (\partial_{y_1} \phi_{\epsilon}(\mathbf{x}; y_0), \dots, \partial_{y_{n-1}} \phi_{\epsilon}(\mathbf{x}; y_0))$$

stays constant when $\mathbf{x} = (x, t)$ moves along γ .

Proof of Lemma 3.2. We parameterize γ by an arclength parameter s such that $\gamma(0) = \mathbf{x}$ and $\gamma(L) = \mathbf{y}_0 := (y_0, \epsilon)$, where $L = \text{dist}(\mathbf{x}, \mathbf{y}_0)$. Pick a normal coordinate $z = (z_1, z_2, \dots, z_n)$ centered at \mathbf{x} and $\gamma'(0) = \partial_{z_1}$ at the origin point. Then for any $V = \sum_i V_i \partial_{z_i} \in T_{\mathbf{x}}\mathcal{M}$, the geodesic γ_V starting at \mathbf{x} with initial velocity V is represented in normal coordinate by the radial line segment

$$\gamma_V(s) = \exp_{\mathbf{x}}((sV_1, sV_2, \dots, sV_n)).$$

In our case, there is $\gamma(s) = \exp_{\mathbf{x}}((s, 0, \dots, 0))$. In particular,

$$(3.25) \quad \mathbf{y}_0 = \exp_{\mathbf{x}}((L, 0, \dots, 0)),$$

and the tangent vector

$$(3.26) \quad \gamma'(L) = d\exp_{\mathbf{x}}|_{z=(L,0,\dots,0)}\partial_{z_1}.$$

The Riemannian distance function can be expressed explicitly in normal coordinate (see Corollary 6.12 in [Lee, page 162], which is a corollary of the Gauss Lemma), and we have

$$\text{dist}(\mathbf{x}, \exp_{\mathbf{x}}(z)) = \sqrt{z_1^2 + z_2^2 + \dots + z_n^2}.$$

Then we can compute that

$$(3.27) \quad \bar{\nabla}_{\mathbf{y}} \text{dist}(\mathbf{x}, \mathbf{y}_0) = d \exp_{\mathbf{x}}|_{z=(L,0,\dots,0)} \left(\sum_i \frac{z_i \partial_{z_i}}{|z|} \right) = d \exp_{\mathbf{x}}|_{z=(L,0,\dots,0)} \partial_{z_1},$$

where $\bar{\nabla}_{\mathbf{y}}$ is the covariant gradient taken in the $\mathbf{y} = (y, \tau)$ variable. Combining equations (3.26) and (3.27), we can get

$$(3.28) \quad \bar{\nabla}_{\mathbf{y}} \text{dist}(\mathbf{x}, \mathbf{y}_0) = \gamma'(L).$$

From this we see that $\bar{\nabla}_{\mathbf{y}} \text{dist}(\mathbf{x}, \mathbf{y}_0)$ stays constant when \mathbf{x} moves along γ . In the end, by the chain rule we observe that

$$\partial_{y_i} \phi_\epsilon(\mathbf{x}; y_0) = \sum_{k,j} (\bar{\nabla}_{\mathbf{y}} \text{dist}(\mathbf{x}, \mathbf{y}_0))_k (g(\mathbf{y}_0))_{kj} \left(\frac{\partial \mathbf{y}}{\partial y} \right)_{ji} \Big|_{y=y_0},$$

which verifies that, once \mathbf{y}_0 is fixed, $\partial_{y_i} \phi_\epsilon(\mathbf{x}; y_0)$ stays constant when \mathbf{x} moves along γ . \square

3.2. Sogge's chaotic curvature condition

In this part, we will prove Theorem 2.5.

First of all, it is elementary to see that contact orders are invariant under changes of coordinates. We therefore without loss of generality assume that we are in a Fermi coordinate. Let $\epsilon > 0$ be a small number to be chosen; its smallness will depend on the manifold \mathcal{M} . Assume that our Fermi coordinate is based on the geodesic

$$(3.29) \quad \gamma(s) = (0, s), \quad \forall s \in [0, \epsilon].$$

We use e_1, e_2, e_3 to denote the coordinate vectors. Define vector fields on γ by⁶

$$(3.30) \quad E_3(s) := \frac{\partial}{\partial \tau} \in T_{\gamma(s)} \mathcal{M}, \quad E_i(s) := \frac{\partial}{\partial y_i} \in T_{\gamma(s)} \mathcal{M}, \quad i = 1, 2.$$

⁶Here we parametrize our manifold by using (y, τ) .

In other words, $E_i(s) = e_i(\gamma(s))$, $i = 1, 2, 3$, and therefore $\{E_i(s)\}_{i=1}^3$ forms an orthonormal basis for $T_{\gamma(s)}\mathcal{M}$. Denote

$$(3.31) \quad \phi_\epsilon(x, t; y) := \text{dist}((x, t), (y, \epsilon))$$

and

$$(3.32) \quad \Phi(x, t; y, \tau) := \text{dist}((x, t), (y, \tau)).$$

Moreover, denote

$$(3.33) \quad \phi_{0,\epsilon}(x, t; y) := \phi_\epsilon(x, t; y) - \phi_\epsilon(0, 0; y),$$

and

$$(3.34) \quad \Phi_0(x, t; y, \tau) := \Phi(x, t; y, \tau) - \Phi(0, 0; y, \tau).$$

Here (x, t) and (y, τ) are points on \mathcal{M} . Our goal is prove that if \mathcal{M} satisfies Sogge's chaotic curvature condition at the origin, then the contact order of ϕ_ϵ is ≤ 4 at $(x, t) = (0, 0)$, $y = 0$ if ϵ is chosen to be sufficiently small.

Before computing contact orders, let us first compute what Sogge's chaotic curvature says in the Fermi coordinate. Recall Definition 1.13. We pick the geodesic in Definition 1.13 to be (3.29). The vector field $X(s)$ in Definition 1.13 is the parallel transport of a unit vector $X(0) \perp \dot{\gamma}(0)$. Therefore, we can write

$$(3.35) \quad X(s) = c_1 E_1(s) + c_2 E_2(s), \quad c_1^2 + c_2^2 = 1.$$

The vector field $Y(s)$ in Definition 1.13 is therefore

$$(3.36) \quad Y(s) = \bar{\text{Ric}}(X(s)),$$

and

$$(3.37) \quad \begin{aligned} Y^\perp(s) &= g\left(Y(s), c_2 E_1(s) - c_1 E_2(s)\right)(c_2 E_1(s) - c_1 E_2(s)) \\ &= (c_1, c_2) \begin{bmatrix} \text{Ric}_{11}(s) & \text{Ric}_{12}(s) \\ \text{Ric}_{21}(s) & \text{Ric}_{22}(s) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} (c_2 E_1(s) - c_1 E_2(s)) \end{aligned}$$

where

$$(3.38) \quad \text{Ric}_{ij}(s) := g(\bar{\text{Ric}}(E_i(s)), E_j(s)) = \text{Ric}(E_i(s), E_j(s)).$$

Moreover,

$$(3.39) \quad \begin{aligned} & \nabla_{\dot{\gamma}(s)} Y^\perp(s) \\ &= (c_1, c_2) \begin{bmatrix} \partial_s \text{Ric}_{11}(s) & \partial_s \text{Ric}_{12}(s) \\ \partial_s \text{Ric}_{21}(s) & \partial_s \text{Ric}_{22}(s) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} (c_2 E_1(s) - c_1 E_2(s)), \end{aligned}$$

where we used the fact that $E_1(s), E_2(s)$ are parallel transports along $\gamma(s)$, which says

$$(3.40) \quad \nabla_{\dot{\gamma}(s)} E_j(s) = 0, \quad j = 1, 2.$$

As we assume Sogge's chaotic curvature condition, we therefore can conclude that

$$(3.41) \quad \begin{aligned} & |Y^\perp(0)| + |\nabla_{\dot{\gamma}} Y^\perp(0)| \\ &= \left| (c_1, c_2) \begin{bmatrix} \text{Ric}_{11}(0) & \text{Ric}_{12}(0) \\ \text{Ric}_{21}(0) & \text{Ric}_{22}(0) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \right| \\ &+ \left| (c_1, c_2) \begin{bmatrix} \partial_s \text{Ric}_{11}(0) & \partial_s \text{Ric}_{12}(0) \\ \partial_s \text{Ric}_{21}(0) & \partial_s \text{Ric}_{22}(0) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \right| \neq 0, \end{aligned}$$

for all c_1, c_2 satisfying $c_1^2 + c_2^2 = 1$.

We compute $\text{Ric}_{ij}(0)$ and $\partial_s \text{Ric}_{ij}(0)$. By the definition of Ricci tensors in (3.16), and that we are working with an orthonormal basis, we have

$$(3.42) \quad \begin{bmatrix} \text{Ric}_{11}(s) & \text{Ric}_{12}(s) \\ \text{Ric}_{21}(s) & \text{Ric}_{22}(s) \end{bmatrix} = \begin{bmatrix} R_{2112}(s) + R_{3113}(s) & R_{3123}(s) \\ R_{3123}(s) & R_{1221}(s) + R_{3223}(s) \end{bmatrix},$$

where

$$(3.43) \quad R_{ijkl}(s) := g(R(E_i(s), E_j(s))E_k(s), E_l(s)),$$

and we used basic symmetries of the Riemannian curvature tensor. We continue to compute the right hand side of (3.42).

Claim 3.3. *For $1 \leq i, j \leq 2$, we have*

$$(3.44) \quad R_{i33j}(\mathbf{x}) = -\frac{1}{2}g_{33,ij}(\mathbf{x}),$$

whenever $x_1 = x_2 = 0$.

Proof of Claim 3.3. By the formula (3.15) and the property (3.21) of Fermi coordinates, we have

$$\begin{aligned}
 R_{i33j} &= R_{i33}{}^j = \partial_i \Gamma_{33}^j \\
 (3.45) \quad &= \frac{1}{2} \partial_i g^{jk} (g_{3k,3} + g_{3k,3} - g_{33,k}) + \frac{1}{2} g^{jk} (g_{3k,3i} + g_{3k,3i} - g_{33,ki}) \\
 &= \frac{1}{2} (2g_{3j,3i} - g_{33,ji}).
 \end{aligned}$$

To prove the claim, it remains to show that

$$(3.46) \quad g_{3j,3i}(\mathbf{x}) = 0, \quad 1 \leq i, j \leq 2,$$

whenever $x_1 = x_2 = 0$.

To show this, we will differentiate both sides of (3.19), that is,

$$(3.47) \quad \begin{bmatrix} g_{11}(\mathbf{x}) & g_{12}(\mathbf{x}) & g_{13}(\mathbf{x}) \\ g_{21}(\mathbf{x}) & g_{22}(\mathbf{x}) & g_{23}(\mathbf{x}) \\ g_{31}(\mathbf{x}) & g_{32}(\mathbf{x}) & g_{33}(\mathbf{x}) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}.$$

Take $\partial_{x_1}^2$ on both sides, we obtain

$$(3.48) \quad \frac{\partial^2 g}{\partial x_1^2} \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} + 2 \frac{\partial g}{\partial x_1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0.$$

By taking ∂_3 , that is, taking ∂_t , we further obtain

$$(3.49) \quad \frac{\partial^3 g}{\partial t \partial x_1^2} \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} + 2 \frac{\partial^2 g}{\partial t \partial x_1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0,$$

from which we conclude that

$$(3.50) \quad g_{13,13}(\mathbf{x}) = 0,$$

whenever $x_1 = x_2 = 0$. Similarly, we can prove that

$$(3.51) \quad g_{23,23}(\mathbf{x}) = 0,$$

whenever $x_1 = x_2 = 0$. To prove the rest of (3.46), we take $\partial_1 \partial_2 \partial_3$ on both sides of (3.47), and we obtain

$$(3.52) \quad g_{23,13}(\mathbf{x}) + g_{13,23}(\mathbf{x}) = 0,$$

whenever $x_1 = x_2 = 0$. Recall the calculations in (3.45). By basic symmetries of Riemannian curvature tensors, we have $R_{i33j} = R_{j33i}$, which further implies that

$$(3.53) \quad g_{3j,3i} = g_{3i,3j}.$$

This, combined with (3.52), will finish all the cases of (3.46). \square

We return to computing the right hand side of (3.42). There is one term $R_{2112}(s)$ that is not covered by Claim 3.3. It is natural to try to follow the same strategy of Claim 3.3 and compute it. Whenever $x_1 = x_2 = 0$, we have

$$(3.54) \quad \begin{aligned} -R_{2112} &= \partial_1 \Gamma_{21}^2 - \partial_2 \Gamma_{11}^2 \\ &= \frac{1}{2} \partial_1 \left(\sum_l g^{2l} (g_{2l,1} + g_{1l,2} - g_{21,l}) \right) - \frac{1}{2} \partial_2 \left(g^{2l} (g_{1l,1} + g_{1l,1} - g_{11,l}) \right) \\ &= \frac{1}{2} \sum_l g^{2l} (g_{2l,11} + g_{1l,21} - g_{21,l1}) - \frac{1}{2} g^{2l} (g_{1l,11} + g_{1l,11} - g_{11,l1}) \\ &= \frac{1}{2} (g_{22,11} - g_{12,12} - g_{12,12} + g_{11,22}). \end{aligned}$$

By taking derivatives on both sides of (3.47), one can further simplify (3.54) to

$$(3.55) \quad R_{2112} = -\frac{3}{2} g_{11,22}.$$

This finishes computing (3.42), and we have

$$(3.56) \quad \begin{aligned} &\begin{bmatrix} \text{Ric}_{11}(s) & \text{Ric}_{12}(s) \\ \text{Ric}_{21}(s) & \text{Ric}_{22}(s) \end{bmatrix} \\ &= - \begin{bmatrix} \frac{3}{2} g_{11,22}(s) + \frac{1}{2} g_{33,11}(s) & \frac{1}{2} g_{33,21}(s) \\ \frac{1}{2} g_{33,21}(s) & \frac{3}{2} g_{11,22}(s) + \frac{1}{2} g_{33,22}(s) \end{bmatrix}, \end{aligned}$$

where

$$(3.57) \quad g_{ij,kl}(s) := g_{ij,kl}(\gamma(s)).$$

Let us go back to (3.41) and try to express it using $g_{ij,kl}$ and their derivatives. Direct calculations show that (3.41) is equivalent to saying that

$$(3.58) \quad \begin{aligned} & |(c_2^2 - c_1^2)g_{33,21}(0) + c_1c_2(g_{33,11}(0) - g_{33,22}(0))| \\ & + |(c_2^2 - c_1^2)g_{33,213}(0) + c_1c_2(g_{33,113}(0) - g_{33,223}(0))| \neq 0 \end{aligned}$$

for all $c_1^2 + c_2^2 = 1$. Let us point out here that the term $R_{2112} = -\frac{3}{2}g_{11,22}$ does not appear in (3.58) as it gets cancelled out in the middle of the computation. This is what Sogge meant in his paper [Sog99] by saying that the chaotic curvature conditions involves “off-diagonal” parts of the Ricci tensors.

So far we have finished computing the chaotic curvature. Now we start computing the contact order of ϕ_ϵ .

Claim 3.4. *We have*

$$(3.59) \quad \frac{\partial}{\partial y_i} \frac{\partial}{\partial y_j} \phi_{0,\epsilon} \Big|_{\substack{(x,t)=\gamma(s), \\ (y,\tau)=(0,\epsilon)}} = \left(\text{Hessian } \Phi_0 \right) \Big|_{\substack{(x,t)=\gamma(s), \\ (y,\tau)=(0,\epsilon)}} \left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} \right)$$

for every $s \in [0, \epsilon)$, $1 \leq i, j \leq 2$. Here Hessian is the covariant Hessian in the (y, τ) variables (see (3.7)). Moreover,

$$(3.60) \quad \left(\text{Hessian } \Phi_0 \right) \Big|_{\substack{(x,t)=\gamma(s), \\ (y,\tau)=(0,\epsilon)}} \left(Y, \frac{\partial}{\partial \tau} \right) = 0, \quad \forall Y \in T_{(0,\epsilon)}\mathcal{M},$$

for every $s \in [0, \epsilon)$, $1 \leq i \leq 2$.

Proof of Claim 3.4. By definition,

$$(3.61) \quad \begin{aligned} \left(\text{Hessian } \Phi_0 \right) \left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} \right) &= g \left(\nabla_{\frac{\partial}{\partial y_i}} (\bar{\nabla} \Phi_0), \frac{\partial}{\partial y_j} \right) \\ &= \nabla_{\frac{\partial}{\partial y_i}} \left(\nabla_{\frac{\partial}{\partial y_j}} \Phi_0 \right) - g \left(\bar{\nabla} \Phi_0, \nabla_{\frac{\partial}{\partial y_i}} \frac{\partial}{\partial y_j} \right). \end{aligned}$$

By (3.28), we obtain

$$(3.62) \quad \nabla \Phi_0 \Big|_{\substack{(x,t)=\gamma(s), \\ (y,\tau)=(0,\epsilon)}} = 0$$

for every s . This finishes the proof of (3.59). The other identity (3.60) can be proven similarly. \square

Remark 3.5. *The proof of Claim 3.4 does not rely on Fermi coordinate, and Claim 3.4 holds true for distance functions in general coordinates. This will be used later in Lemma 5.4, in particular, in (5.29).*

Let us write

$$(3.63) \quad \begin{aligned} & \left(\text{Hessian } \Phi_0 \right) \Big|_{\substack{(x,t)=\gamma(s), \\ (y,\tau)=(0,\epsilon)}} \left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} \right) \\ &= \text{Hessian } \Phi_0 \Big|_{\gamma(s)} \left(E_i(\epsilon), E_j(\epsilon) \right) =: W_{ij}(s, \epsilon). \end{aligned}$$

In the middle term, we have left out the valuation $(y, \tau) = (0, \epsilon)$, as this information can be read from the fact that $E_i(\epsilon) \in T_{\gamma(\epsilon)}\mathcal{M}$ and $\gamma(\epsilon) = (0, \epsilon)$. Denote

$$(3.64) \quad W(s, \epsilon) := \begin{bmatrix} W_{11}(s, \epsilon) & W_{12}(s, \epsilon) \\ W_{21}(s, \epsilon) & W_{22}(s, \epsilon) \end{bmatrix}$$

and

$$(3.65) \quad \mathfrak{W}(s, \epsilon) := \det W(s, \epsilon).$$

Our goal is now to compute W_{ij} for $1 \leq i, j \leq 2$, given on the right hand side of (3.63), and show that the matrix

$$(3.66) \quad \begin{bmatrix} \partial_s \mathfrak{W}(0, \epsilon) & \partial_s^2 \mathfrak{W}(0, \epsilon) & \partial_s^3 \mathfrak{W}(0, \epsilon) & \partial_s^4 \mathfrak{W}(0, \epsilon) \\ \partial_s W_{11}(0, \epsilon) & \partial_s^2 W_{11}(0, \epsilon) & \partial_s^3 W_{11}(0, \epsilon) & \partial_s^4 W_{11}(0, \epsilon) \\ \partial_s W_{12}(0, \epsilon) & \partial_s^2 W_{12}(0, \epsilon) & \partial_s^3 W_{12}(0, \epsilon) & \partial_s^4 W_{12}(0, \epsilon) \\ \partial_s W_{22}(0, \epsilon) & \partial_s^2 W_{22}(0, \epsilon) & \partial_s^3 W_{22}(0, \epsilon) & \partial_s^4 W_{22}(0, \epsilon) \end{bmatrix}$$

has rank 4, if $\epsilon > 0$ is chosen sufficiently small.

Now let us discuss manifolds in general dimension n . Consider a geodesic $\gamma(\cdot)$ parameterized by arc length. Let $E_n(\cdot)$ be the tangent vector $\dot{\gamma}(\cdot)$. Use parallel transport to complete the orthonormal basis $\{E_1(\cdot), \dots, E_n(\cdot)\}$. Next, consider $n - 1$ families of vector fields $X_j(s, \cdot)$, $j = 1, \dots, n - 1$ satisfying the Jacobi equation and the following boundary conditions:

$$(3.67) \quad \begin{aligned} & \nabla_{\dot{\gamma}(s')}^2 X_j(s, s') + R(X_j(s, s'), \dot{\gamma}(s')) \dot{\gamma}(s') = 0, \\ & X_j(s, s) = 0, \quad X_j(s, \epsilon) = E_j(\epsilon). \end{aligned}$$

Here R stands for the Riemannian curvature tensor, and

$$(3.68) \quad \nabla_{\dot{\gamma}(s')}^2 Y := \nabla_{\dot{\gamma}(s')}(\nabla_{\dot{\gamma}(s')} Y)$$

for a vector field Y . More background of the Jacobi equation can be seen, e.g. [Lee, Chapter 10].

Claim 3.6. *For $j = 1, \dots, n-1$ and fixed s , it holds that*

$$(3.69) \quad X_j(s, s') \perp \dot{\gamma}(s')$$

for all $s' \in [s, \epsilon]$.

Proof of Claim 3.6. We need to prove that

$$(3.70) \quad g\left(X_j(s, s'), \dot{\gamma}(s')\right) = 0, \quad \forall s' \in [s, \epsilon].$$

Note that by the initial conditions in (3.67), we have

$$(3.71) \quad g\left(X_j(s, s), \dot{\gamma}(s)\right) = 0, \quad g\left(X_j(s, \epsilon), \dot{\gamma}(\epsilon)\right) = 0.$$

By the compatibility with metric (3.5), we have

$$(3.72) \quad \begin{aligned} \frac{\partial}{\partial s'} g\left(X_j(s, s'), \dot{\gamma}(s')\right) \\ = g\left(\nabla_{\dot{\gamma}(s')} X_j(s, s'), \dot{\gamma}(s')\right) + g\left(X_j(s, s'), \nabla_{\dot{\gamma}(s')} \dot{\gamma}(s')\right). \end{aligned}$$

As γ is a geodesic, we have

$$(3.73) \quad \nabla_{\dot{\gamma}(s')} \dot{\gamma}(s') = 0.$$

In order to use the information from Jacobi fields in (3.67), we take another derivative on both sides of (3.72), and obtain

$$(3.74) \quad \begin{aligned} \frac{\partial^2}{\partial (s')^2} g\left(X_j(s, s'), \dot{\gamma}(s')\right) &= g\left(\nabla_{\dot{\gamma}(s')}^2 X_j(s, s'), \dot{\gamma}(s')\right) \\ &= -g\left(R(X_j(s, s'), \dot{\gamma}(s')) \dot{\gamma}(s'), \dot{\gamma}(s')\right) = 0. \end{aligned}$$

In the last step, we used basic symmetries of the Riemannian curvature tensors. The claim follows from the initial conditions in the Jacobi equation. \square

To proceed, we need Remark 4.11 in Sakai [Sak96, page 110].

Lemma 3.7 (Sakai [Sak96]). *Take two distinct points $(x, t), (y, \tau) \in \mathcal{M}$ and let d_0 be their distance. Let $\gamma_0 : [0, d_0] \rightarrow \mathcal{M}$ be the arc-length parametrized*

geodesic connecting these two points. Take $n - 1$ vectors $V_1, \dots, V_{n-1} \in T_{(y, \tau)} \mathcal{M}$. Let $Z_j(s)$ be the Jacobi field along γ_0 satisfying the boundary conditions

$$(3.75) \quad Z_j(0) = 0, \quad Z_j(d_0) = V_j, \quad j = 1, \dots, n - 1.$$

Moreover, let

$$(3.76) \quad Z_j^\perp(s) := Z_j(s) - g(Z_j(s), \dot{\gamma}_0(s))\dot{\gamma}_0(s), \quad j = 1, \dots, n - 1.$$

Then for $i \neq j$,

$$(3.77) \quad \left(\text{Hessian } \Phi((x, t), (y, \tau)) \right) (V_i, V_j) = g \left(\nabla_{\dot{\gamma}_0(d_0)} Z_i^\perp(d_0), Z_j^\perp(d_0) \right).$$

Here Hessian is the covariant Hessian in the (y, τ) variables.

By Claim 3.6 and Lemma 3.7, we obtain

$$(3.78) \quad \text{Hessian } \Phi|_{\gamma(s)} (E_i(\epsilon), E_j(\epsilon)) = g \left(\nabla_{\dot{\gamma}(s')} X_i(s, s'), X_j(s, s') \right) \Big|_{s'=\epsilon}.$$

Note that on the left hand side of (3.78), we have Φ but not Φ_0 .

In $n = 2$ case, we introduce a matrix

$$(3.79) \quad A(s, s') = \begin{bmatrix} a_{11}(s, s') & a_{12}(s, s') \\ a_{21}(s, s') & a_{22}(s, s') \end{bmatrix}$$

by writing

$$(3.80) \quad X_j(s, s') = a_{j1}(s, s')E_1(s') + a_{j2}(s, s')E_2(s'), \quad j = 1, 2.$$

Let us rewrite (3.67) using the new notation. The first equation in (3.67) becomes

$$(3.81) \quad \partial_{s'}^2 a_{j1}(s, s')E_1(s') + \partial_{s'}^2 a_{j2}(s, s')E_2(s') \\ + R(a_{j1}(s, s')E_1(s') + a_{j2}(s, s')E_2(s'), \dot{\gamma}(s'))\dot{\gamma}(s') = 0,$$

which is equivalent to the following two equations

$$(3.82) \quad \partial_{s'}^2 a_{j1}(s, s') + a_{j1}(s, s')g \left(R(E_1(s'), \dot{\gamma}(s'))\dot{\gamma}(s'), E_1(s') \right) \\ + a_{j2}(s, s')g \left(R(E_2(s'), \dot{\gamma}(s'))\dot{\gamma}(s'), E_1(s') \right) = 0,$$

and

$$(3.83) \quad \partial_{s'}^2 a_{j2}(s, s') + a_{j1}(s, s')g\left(R(E_1(s'), \dot{\gamma}(s'))\dot{\gamma}(s'), E_2(s')\right) \\ + a_{j2}(s, s')g\left(R(E_2(s'), \dot{\gamma}(s'))\dot{\gamma}(s'), E_2(s')\right) = 0.$$

Denote

$$(3.84) \quad R(s') := \begin{bmatrix} R_{1331}(s') & R_{1332}(s') \\ R_{2331}(s') & R_{2332}(s') \end{bmatrix}$$

where

$$(3.85) \quad R_{ijkl}(s') := g(R(E_i(s'), E_j(s'))E_k(s'), E_l(s')).$$

Then (3.67) can be written as⁷

$$(3.86) \quad \begin{aligned} \partial_{s'}^2 A(s, s') + A(s, s')R(s') &= 0, \\ A(s, s) &= 0, \quad A(s, \epsilon) = I_{2 \times 2}. \end{aligned}$$

Here 0 stands for 2×2 zero matrix, $I_{2 \times 2}$ is the 2×2 identity matrix. Moreover, (3.78) is equal to

$$(3.87) \quad \begin{aligned} &g\left(\nabla_{\dot{\gamma}(s')}(a_{i1}(s, s')E_1(s') + a_{i2}(s, s')E_2(s'), \right. \\ &\quad \left. a_{j1}(s, s')E_1(s') + a_{j2}(s, s')E_2(s'))\right)\Big|_{s'=\epsilon} \\ &= \partial_{s'} a_{i1}(s, s')a_{j1}(s, s') + \partial_{s'} a_{i2}(s, s')a_{j2}(s, s')\Big|_{s'=\epsilon} = \partial_{s'} a_{ij}\Big|_{s'=\epsilon}, \end{aligned}$$

where in the last step we used the initial condition at $s' = \epsilon$.

Recall that our goal was to compute (3.63) and prove that (3.66) has rank four. By (3.87), we obtain

$$(3.88) \quad W_{ij}(s, \epsilon) = \text{Hessian } \Phi_0|_{\gamma(s)}\left(E_i(\epsilon), E_j(\epsilon)\right) = \partial_{s'} a_{ij}(s, \epsilon) - \partial_{s'} a_{ij}(0, \epsilon).$$

When computing (3.63) and proving that (3.66) has rank four, we will consider the Taylor expansion of (3.88) at $s = 0$:

$$(3.89) \quad W(s, \epsilon) = \frac{s}{1!} \partial_s \partial_{s'} A(0, \epsilon) + \frac{s^2}{2!} \partial_s^2 \partial_{s'} A(0, \epsilon) + \dots$$

⁷Here derivatives are taken component-wise.

In the following computation, it will be convenient to introduce some notation. We will write

$$(3.90) \quad R'(s') := \frac{\partial}{\partial s'} R(s'),$$

where partial derivatives are taken component-wise. Moreover, we will write

$$(3.91) \quad R_0 := R(0), \quad R'_0 := R'(0).$$

Claim 3.8. *We have*

$$(3.92) \quad \begin{aligned} \partial_s \partial_{s'} A(0, \epsilon) &= \frac{1}{\epsilon^2} \left(I_{2 \times 2} + \frac{R_0}{3} \epsilon^2 + \frac{R'_0}{6} \epsilon^3 + O(\epsilon^4) \right), \\ \partial_s^2 \partial_{s'} A(0, \epsilon) &= \frac{2!}{\epsilon^3} \left(I_{2 \times 2} + \frac{R'_0}{12} \epsilon^3 + O(\epsilon^4) \right), \\ \partial_s^3 \partial_{s'} A(0, \epsilon) &= \frac{3!}{\epsilon^4} \left(I_{2 \times 2} + O(\epsilon^4) \right), \\ \partial_s^4 \partial_{s'} A(0, \epsilon) &= \frac{4!}{\epsilon^5} \left(I_{2 \times 2} + O(\epsilon^4) \right). \end{aligned}$$

The implicit constants in $O(\epsilon^4)$ depend only on the manifold.

The proof of Claim 3.8 involves heavy calculations, and will therefore be postponed to the end of this subsection.

We stop the Taylor expansion in (3.89) at the fifth order, and write

$$(3.93) \quad \begin{aligned} W(s, \epsilon) &= \frac{s}{\epsilon^2} \left(I_{2 \times 2} + \frac{R_0}{3} \epsilon^2 + \frac{R'_0}{6} \epsilon^3 + O(\epsilon^4) \right) + \frac{s^2}{\epsilon^3} \left(I_{2 \times 2} + \frac{R'_0}{12} \epsilon^3 + O(\epsilon^4) \right) \\ &\quad + \frac{s^3}{\epsilon^4} \left(I_{2 \times 2} + O(\epsilon^4) \right) + \frac{s^4}{\epsilon^5} \left(I_{2 \times 2} + O(\epsilon^4) \right) + O_\epsilon(s^5). \end{aligned}$$

Here $O_\epsilon(s^5)$ means that the implicit constant there is allowed to depend on ϵ . This dependence is harmless because when computing contact orders (in the s variable), we will always fix ϵ and consider $s \rightarrow 0$. Recall that we need to prove (3.66) has rank four. To simplify notation, we will make the change of variables $s \mapsto \epsilon s$, and consider the matrix

$$(3.94) \quad \begin{aligned} \epsilon W(\epsilon s, \epsilon) &= s \left(I_{2 \times 2} + \frac{R_0}{3} \epsilon^2 + \frac{R'_0}{6} \epsilon^3 + O(\epsilon^4) \right) + s^2 \left(I_{2 \times 2} + \frac{R'_0}{12} \epsilon^3 + O(\epsilon^4) \right) \\ &\quad + s^3 \left(I_{2 \times 2} + O(\epsilon^4) \right) + s^4 \left(I_{2 \times 2} + O(\epsilon^4) \right) + O_\epsilon(s^5). \end{aligned}$$

By Claim 3.3,

$$(3.95) \quad R(s) = - \begin{bmatrix} \frac{g_{33,11}(s)}{2} & \frac{g_{33,12}(s)}{2} \\ \frac{g_{33,21}(s)}{2} & \frac{g_{33,22}(s)}{2} \end{bmatrix},$$

and therefore

$$(3.96) \quad R'(s) = - \begin{bmatrix} \frac{g_{33,113}(s)}{2} & \frac{g_{33,123}(s)}{2} \\ \frac{g_{33,213}(s)}{2} & \frac{g_{33,223}(s)}{2} \end{bmatrix},$$

where similarly to (3.57), we use the notation

$$(3.97) \quad g_{ij,klm}(s) := g_{ij,klm}(\gamma(s)).$$

Moreover, we denote

$$(3.98) \quad \mathfrak{g}_{ij,kl} := g_{ij,kl}(0), \quad \mathfrak{g}_{ij,klm} := g_{ij,klm}(0),$$

which will significantly simplify the notation. Write

$$(3.99) \quad \epsilon W(\epsilon s, \epsilon) = \begin{bmatrix} (\star)_{11} & (\star)_{12} \\ (\star)_{21} & (\star)_{22} \end{bmatrix} + O_\epsilon(s^5),$$

where

$$(3.100) \quad \begin{aligned} (\star)_{11} &:= s \left(1 - \frac{1}{6} \mathfrak{g}_{33,11} \epsilon^2 - \frac{1}{12} \mathfrak{g}_{33,113} \epsilon^3 + O(\epsilon^4) \right) \\ &\quad + s^2 \left(1 - \frac{1}{24} \mathfrak{g}_{33,113} \epsilon^3 + O(\epsilon^4) \right) + s^3(1 + O(\epsilon^4)) + s^4(1 + O(\epsilon^4)), \\ (\star)_{22} &:= s \left(1 - \frac{1}{6} \mathfrak{g}_{33,22} \epsilon^2 - \frac{1}{12} \mathfrak{g}_{33,223} \epsilon^3 + O(\epsilon^4) \right) \\ &\quad + s^2 \left(1 - \frac{1}{24} \mathfrak{g}_{33,223} \epsilon^3 + O(\epsilon^4) \right) + s^3(1 + O(\epsilon^4)) + s^4(1 + O(\epsilon^4)), \end{aligned}$$

and

$$(3.102) \quad (\star)_{12} = (\star)_{21} := s \left(-\frac{1}{6} \mathfrak{g}_{33,12} \epsilon^2 - \frac{1}{12} \mathfrak{g}_{33,123} \epsilon^3 + O(\epsilon^4) \right)$$

$$+ s^2 \left(-\frac{1}{24} \mathfrak{g}_{33,123} \epsilon^3 + O(\epsilon^4) \right) + s^3 O(\epsilon^4) + s^4 O(\epsilon^4).$$

We compute $\det(\epsilon W(\epsilon s, \epsilon))$ and obtain

$$(3.103) \quad (\star)_{11}(\star)_{22} - (\star)_{12}(\star)_{21} = (\star)_1 s^2 + (\star)_2 s^3 + (\star)_3 s^4 + O_\epsilon(s^5),$$

where

$$(3.104) \quad \begin{aligned} (\star)_1 &:= 1 - \frac{1}{6} \epsilon^2 (\mathfrak{g}_{33,11} + \mathfrak{g}_{33,22}) - \frac{1}{12} \epsilon^3 (\mathfrak{g}_{33,113} + \mathfrak{g}_{33,223}) + O(\epsilon^4), \\ (\star)_2 &:= 2 - \frac{1}{6} \epsilon^2 (\mathfrak{g}_{33,11} + \mathfrak{g}_{33,22}) - \frac{1}{8} \epsilon^2 (\mathfrak{g}_{33,113} + \mathfrak{g}_{33,223}) + O(\epsilon^4), \\ (\star)_3 &:= 3 - \frac{1}{6} \epsilon^2 (\mathfrak{g}_{33,11} + \mathfrak{g}_{33,22}) - \frac{1}{8} \epsilon^2 (\mathfrak{g}_{33,113} + \mathfrak{g}_{33,223}) + O(\epsilon^4). \end{aligned}$$

To prove that (3.66) has rank four, it suffices to show that the matrix

$$(3.105) \quad \begin{bmatrix} (\star\star)_{11} & 1 - \frac{1}{24} \mathfrak{g}_{33,113} \epsilon^3 + O(\epsilon^4) & 1 + O(\epsilon^4) & 1 + O(\epsilon^4) \\ (\star\star)_{21} & 1 - \frac{1}{24} \mathfrak{g}_{33,223} \epsilon^3 + O(\epsilon^4) & 1 + O(\epsilon^4) & 1 + O(\epsilon^4) \\ (\star\star)_{31} & -\frac{1}{24} \mathfrak{g}_{33,123} \epsilon^3 + O(\epsilon^4) & O(\epsilon^4) & O(\epsilon^4) \\ 0 & (\star)_1 & (\star)_2 & (\star)_3 \end{bmatrix}$$

with

$$(3.106) \quad \begin{aligned} (\star\star)_{11} &:= 1 - \frac{1}{6} \mathfrak{g}_{33,11} \epsilon^2 - \frac{1}{12} \mathfrak{g}_{33,113} \epsilon^3 + O(\epsilon^4), \\ (\star\star)_{21} &:= 1 - \frac{1}{6} \mathfrak{g}_{33,22} \epsilon^2 - \frac{1}{12} \mathfrak{g}_{33,223} \epsilon^3 + O(\epsilon^4), \\ (\star\star)_{31} &:= -\frac{1}{6} \mathfrak{g}_{33,12} \epsilon^2 - \frac{1}{12} \mathfrak{g}_{33,123} \epsilon^3 + O(\epsilon^4), \end{aligned}$$

is non-degenerate whenever $\epsilon > 0$ is picked to be small enough. We subtract the second row from the first row, and obtain

$$(3.107) \quad \begin{bmatrix} (\star\star\star)_{11}, & -\frac{1}{24} (\mathfrak{g}_{33,113} - \mathfrak{g}_{33,223}) \epsilon^3 + O(\epsilon^4) & O(\epsilon^4) & O(\epsilon^4) \\ (\star\star\star)_{21} & 1 - \frac{1}{24} \mathfrak{g}_{33,223} \epsilon^3 + O(\epsilon^4) & 1 + O(\epsilon^4) & 1 + O(\epsilon^4) \\ (\star\star\star)_{31} & -\frac{1}{24} \mathfrak{g}_{33,123} \epsilon^3 + O(\epsilon^4) & O(\epsilon^4) & O(\epsilon^4) \\ 0 & (\star)_1 & (\star)_2 & (\star)_3 \end{bmatrix},$$

where

$$\begin{aligned}
 (\star\star\star)_{11} &:= -\frac{1}{6}(\mathfrak{g}_{33,11} - \mathfrak{g}_{33,22})\epsilon^2 - \frac{1}{12}(\mathfrak{g}_{33,113} - \mathfrak{g}_{33,223})\epsilon^3 + O(\epsilon^4), \\
 (3.108) \quad (\star\star\star)_{21} &:= 1 - \frac{1}{6}\mathfrak{g}_{33,22}\epsilon^2 - \frac{1}{12}\mathfrak{g}_{33,223}\epsilon^3 + O(\epsilon^4), \\
 (\star\star\star)_{31} &:= -\frac{1}{6}\mathfrak{g}_{33,12}\epsilon^2 - \frac{1}{12}\mathfrak{g}_{33,123}\epsilon^3 + O(\epsilon^4).
 \end{aligned}$$

It is not difficult to see that the determinant is $O(\epsilon^5)$. We compute the coefficient of ϵ^5 , and obtain

$$(3.109) \quad -\frac{1}{6 \ast 24} \left((\mathfrak{g}_{33,11} - \mathfrak{g}_{33,22})\mathfrak{g}_{33,123} - (\mathfrak{g}_{33,113} - \mathfrak{g}_{33,223})\mathfrak{g}_{33,12} \right).$$

Recall from (3.58) that Sogge's chaotic curvature implies that

$$\begin{aligned}
 (3.110) \quad & |(c_2^2 - c_1^2)\mathfrak{g}_{33,21} + c_1c_2(\mathfrak{g}_{33,11} - \mathfrak{g}_{33,22})| \\
 & + |(c_2^2 - c_1^2)\mathfrak{g}_{33,213} + c_1c_2(\mathfrak{g}_{33,113} - \mathfrak{g}_{33,223})| \neq 0
 \end{aligned}$$

for all $c_1^2 + c_2^2 = 1$. Let us write (3.110) slightly differently, but equivalently, as

$$\begin{aligned}
 (3.111) \quad & |c_1\mathfrak{g}_{33,21} + c_2(\mathfrak{g}_{33,11} - \mathfrak{g}_{33,22})| \\
 & + |c_1\mathfrak{g}_{33,213} + c_2(\mathfrak{g}_{33,113} - \mathfrak{g}_{33,223})| \neq 0
 \end{aligned}$$

for all $c_1^2 + c_2^2 = 1$. To see that (3.109) $\neq 0$, we just pick

$$(3.112) \quad c_1 = c(\mathfrak{g}_{33,113} - \mathfrak{g}_{33,223}), \quad c_2 = -c(\mathfrak{g}_{33,123}),$$

in (3.111), for some appropriately chosen constant c ; the only thing we need to make sure is that $\mathfrak{g}_{33,113} - \mathfrak{g}_{33,223}$ and $\mathfrak{g}_{33,123}$ do not vanish simultaneously. However, it is elementary to see that if they do vanish simultaneously, then (3.111) can never hold. This finishes the proof that (3.109) $\neq 0$, thus the item (a) of Theorem 2.5, modulo the proof of Claim 3.8.

Proof of Claim 3.8. To compute (3.89), let us first recall how the matrix A is defined in (3.86). To study this linear systems of equations, we will introduce two auxiliary linear systems of equations:

$$\begin{aligned}
 (3.113) \quad & B_1''(s') + B_1(s')R(s') = 0_{2 \times 2}, \\
 & B_1(0) = I_{2 \times 2}, \quad B_1'(0) = 0_{2 \times 2},
 \end{aligned}$$

and

$$(3.114) \quad \begin{aligned} B_2''(s') + B_2(s')R(s') &= 0_{2 \times 2}, \\ B_2(0) &= 0_{2 \times 2}, \quad B_2'(0) = I_{2 \times 2}, \end{aligned}$$

where $B_1(s')$ and $B_2(s')$ are two 2×2 matrices with entries being functions depending on s' . As all these systems of equations are linear, we know we can find smooth matrix-valued functions $C_1(s)$ and $C_2(s)$ such that

$$(3.115) \quad A(s, s') = C_1(s)B_1(s') + C_2(s)B_2(s').$$

Before computing partial derivatives of A , let us collect some useful data. By taking derivatives on (3.113) and (3.114), we obtain

$$(3.116) \quad \begin{aligned} B_1(0) &= I, \quad B_1'(0) = 0, \quad B_1''(0) = -R_0, \quad B_1'''(0) = -R_0'; \\ B_2(0) &= 0, \quad B_2'(0) = I, \quad B_2''(0) = 0, \quad B_2'''(0) = -R_0, \quad B_2^{(4)}(0) = -2R_0', \end{aligned}$$

where we abbreviate $I_{2 \times 2}$ to I . By Taylor's expansion, we have

$$(3.117) \quad \begin{aligned} B_1(\epsilon) &= I - \frac{R_0}{2!}\epsilon^2 - \frac{R_0'}{3!}\epsilon^3 + O(\epsilon^4), \\ B_2(\epsilon) &= \epsilon \left(I - \frac{R_0}{3!}\epsilon^2 - 2\frac{R_0'}{4!}\epsilon^3 + O(\epsilon^4) \right). \end{aligned}$$

By taking inverses, we obtain

$$(3.118) \quad B_2^{-1}(\epsilon) = \frac{1}{\epsilon} \left(I + \frac{R_0}{3!}\epsilon^2 + \frac{2R_0'}{4!}\epsilon^3 + O(\epsilon^4) \right).$$

Direct computation shows that

$$(3.119) \quad \begin{aligned} B_2^{-1}(\epsilon)B_1(\epsilon) &= \frac{1}{\epsilon} \left(I + \frac{R_0}{3!}\epsilon^2 + \frac{2R_0'}{4!}\epsilon^3 + O(\epsilon^4) \right) \left(I - \frac{R_0}{2!}\epsilon^2 - \frac{R_0'}{3!}\epsilon^3 + O(\epsilon^4) \right) \\ &= \frac{1}{\epsilon} \left(I - \frac{R_0}{3}\epsilon^2 - \frac{R_0'}{12}\epsilon^3 + O(\epsilon^4) \right). \end{aligned}$$

Recall the initial condition in (3.86), that is, the condition that

$$(3.120) \quad A(s, s) = 0, \quad A(s, \epsilon) = I_{2 \times 2}.$$

By taking derivatives in s in (3.120), we obtain that

$$\begin{aligned}
 (3.121) \quad & \partial_s A(s, s) + \partial_{s'} A(s, s) = 0, \quad \partial_s A(s, \epsilon) = 0, \\
 & \partial_s^2 A(s, s) + 2\partial_s \partial_{s'} A(s, s) + \partial_{s'}^2 A(s, s) = 0, \quad \partial_s^2 A(s, \epsilon) = 0, \\
 & \partial_s^3 A(s, s) + 3\partial_s^2 \partial_{s'} A(s, s) + 3\partial_s \partial_{s'}^2 A(s, s) + \partial_{s'}^3 A(s, s) = 0, \quad \partial_s^3 A(s, \epsilon) = 0.
 \end{aligned}$$

Taking $s = 0$ in (3.121), we obtain

$$\begin{aligned}
 (3.122) \quad & C_1(0) = 0, \quad C_2(0) = B_2^{-1}(\epsilon), \\
 & C_1'(0) = -B_2^{-1}(\epsilon), \quad C_2'(0) = B_2^{-1}(\epsilon)B_1(\epsilon)B_2^{-1}(\epsilon), \\
 & C_1''(0) = -2C_2'(0), \quad C_2''(0) = -C_1''(0)B_1(\epsilon)B_2^{-1}(\epsilon), \\
 & C_1'''(0) = -3C_2''(0) - 2B_2^{-1}(\epsilon)R_0, \quad C_2'''(0) = -C_1'''(0)B_1(\epsilon)B_2^{-1}(\epsilon).
 \end{aligned}$$

We compute $\partial_s \partial_{s'} A(0, \epsilon)$. Using the relation (3.115), we obtain

$$\begin{aligned}
 (3.123) \quad & \partial_s \partial_{s'} A(0, \epsilon) = C_1'(0)B_1'(\epsilon) + C_2'(0)B_2'(\epsilon) \\
 & = -B_2^{-1}(\epsilon)B_1'(\epsilon) + B_2^{-1}(\epsilon)B_1(\epsilon)B_2^{-1}(\epsilon)B_2'(\epsilon).
 \end{aligned}$$

This is further equal to

$$\begin{aligned}
 (3.124) \quad & \left(\frac{1}{\epsilon} I + \frac{R_0}{3!} \epsilon + \frac{2R_0'}{4!} \epsilon^2 + O(\epsilon^3) \right) \left(R_0 \epsilon + \frac{R_0'}{2!} \epsilon^2 + O(\epsilon^3) \right) \\
 & + \frac{1}{\epsilon^2} \left(I + \frac{R_0}{3!} \epsilon^2 + \frac{2R_0'}{4!} \epsilon^3 + O(\epsilon^4) \right) \left(I - \frac{R_0}{2!} \epsilon^2 - \frac{R_0'}{3!} \epsilon^3 + O(\epsilon^4) \right) \\
 & \left(I + \frac{R_0}{3!} \epsilon^2 + \frac{2R_0'}{4!} \epsilon^3 + O(\epsilon^4) \right) \left(I - \frac{R_0}{2!} \epsilon^2 - \frac{2R_0'}{3!} \epsilon^3 + O(\epsilon^4) \right) \\
 & = \left(R_0 + \frac{R_0'}{2} \epsilon + O(\epsilon^2) \right) + \frac{1}{\epsilon^2} \left(I - \frac{2}{3} R_0 \epsilon^2 - \frac{2}{3!} R_0' \epsilon^3 + O(\epsilon^4) \right) \\
 & = \frac{1}{\epsilon^2} \left(I + \frac{R_0}{3} \epsilon^2 + \frac{R_0'}{6} \epsilon^3 + O(\epsilon^4) \right).
 \end{aligned}$$

This finishes the calculation for the first identity in the claim.

We compute $\partial_s^2 \partial_{s'} A(0, \epsilon)$. First, observe that

$$(3.125) \quad \partial_s^2 \partial_{s'} A(0, \epsilon) = 2B_2^{-1}(\epsilon)B_1(\epsilon)\partial_s \partial_{s'} A(0, \epsilon).$$

We use the first identity in the claim, and see that the last display is equal

to

$$(3.126) \quad \frac{2}{\epsilon^3} \left(I + \frac{R_0}{3!} \epsilon^2 + \frac{2R'_0}{4!} \epsilon^3 + O(\epsilon^4) \right) \\ \times \left(I - \frac{R_0}{2!} \epsilon^2 - \frac{R'_0}{3!} \epsilon^3 + O(\epsilon^4) \right) \left(I + \frac{R_0}{3} \epsilon^2 + \frac{R'_0}{6} \epsilon^3 + O(\epsilon^4) \right),$$

which is further equal to

$$(3.127) \quad \frac{2}{\epsilon^3} \left(I + \frac{2R'_0}{4!} \epsilon^3 + O(\epsilon^4) \right) = \frac{2}{\epsilon^3} I + \frac{R'_0}{3!} + O(\epsilon).$$

This finishes the second identity in the claim.

Next, we compute $\partial_s^3 \partial_{s'} A(0, \epsilon)$. By (3.115), we obtain

$$(3.128) \quad \partial_s^3 \partial_{s'} A(0, \epsilon) = C_1'''(0) \cdot B_1'(\epsilon) + C_2'''(0) \cdot B_2'(\epsilon).$$

By (3.122), this is equal to

$$(3.129) \quad \begin{aligned} & C_1'''(0) B_1'(\epsilon) - C_1'''(0) B_1(\epsilon) B_2^{-1}(\epsilon) B_2'(\epsilon) \\ &= C_1'''(0) (B_1'(\epsilon) - B_1(\epsilon) B_2^{-1}(\epsilon) B_2'(\epsilon)) \\ &= - (3C_2'''(0) + 2B_2^{-1}(\epsilon) R(0)) (B_1'(\epsilon) - B_1(\epsilon) B_2^{-1}(\epsilon) B_2'(\epsilon)) \\ &= - (6B_2^{-1}(\epsilon) B_1(\epsilon) B_2^{-1}(\epsilon) B_1(\epsilon) B_2^{-1}(\epsilon) + 2B_2^{-1}(\epsilon) R_0) \\ &\quad \times (B_1'(\epsilon) - B_1(\epsilon) B_2^{-1}(\epsilon) B_2'(\epsilon)). \end{aligned}$$

We compute the two factors in the last term separately. The first fact is equal to

$$(3.130) \quad \begin{aligned} & \frac{6}{\epsilon^3} \left(I - \frac{R_0}{3} \epsilon^2 - \frac{R'_0}{12} \epsilon^3 + O(\epsilon^4) \right) \left(I - \frac{R_0}{3} \epsilon^2 - \frac{R'_0}{12} \epsilon^3 + O(\epsilon^4) \right) \\ & \times \left(I + \frac{R_0}{3!} \epsilon^2 + \frac{2R'_0}{4!} \epsilon^3 + O(\epsilon^4) \right) \\ & + \frac{2}{\epsilon} \left(I + \frac{R_0}{3!} \epsilon^2 + \frac{2R'_0}{4!} \epsilon^3 + O(\epsilon^4) \right) \cdot R_0 \\ &= \frac{6}{\epsilon^3} \left(I - \frac{R_0}{2} \epsilon^2 - \frac{R'_0}{12} \epsilon^3 + O(\epsilon^4) \right) + \frac{2}{\epsilon} \cdot R_0 + O(\epsilon) \\ &= \frac{6}{\epsilon^3} \left(I - \frac{R_0}{6} \epsilon^2 - \frac{R'_0}{12} \epsilon^3 + O(\epsilon^4) \right). \end{aligned}$$

The second factor is equal to

$$\begin{aligned}
 & -R_0 \cdot \epsilon - \frac{R'_0}{2} \cdot \epsilon^2 + O(\epsilon^3) \\
 & - \frac{1}{\epsilon} \left(I - \frac{R_0}{2} \epsilon^2 - \frac{R'_0}{6} \epsilon^3 + O(\epsilon^4) \right) \left(I + \frac{R_0}{6} \epsilon^2 + \frac{2R'_0}{4!} \epsilon^3 + O(\epsilon^4) \right) \\
 (3.131) \quad & \times \left(I - \frac{R_0}{2} \epsilon^2 - \frac{R'_0}{3} \cdot \epsilon^3 + O(\epsilon^4) \right) \\
 & = -R_0 \epsilon - \frac{R'_0}{2} \cdot \epsilon^2 + O(\epsilon^3) - \frac{1}{\epsilon} \left(I - \frac{5}{6} R_0 \epsilon^2 - \frac{5}{12} \cdot R'_0 \epsilon^3 + O(\epsilon^4) \right) \\
 & = -\frac{1}{\epsilon} \left(I + \frac{1}{6} R_0 \epsilon^2 + \frac{1}{12} R'_0 \epsilon^3 + O(\epsilon^4) \right).
 \end{aligned}$$

Multiplying these two factors, we obtain the desired equation for $\partial_s^3 \partial_{s'} A(0, \epsilon)$.

The last equation involving $\partial_s^4 \partial_{s'} A(0, \epsilon)$ can be proven similarly. We leave out the computations. \square

4. Proof of Theorem 2.4

Let $\epsilon_\phi > 0$ be a sufficiently small constant depending on ϕ . Let $a(x, t; y)$ be supported in $\mathbb{B}_{\epsilon_\phi}^{n-1} \times \mathbb{B}_{\epsilon_\phi}^1 \times \mathbb{B}_{\epsilon_\phi}^{n-1}$. For $N \geq 1$, $\mathbf{x} = (x, t) \in \mathbb{R}^3$, $y \in \mathbb{R}^2$, denote

$$(4.1) \quad \phi^N(\mathbf{x}; y) := N\phi(\mathbf{x}/N; y), \quad a^N(\mathbf{x}; y) := a(\mathbf{x}/N; y).$$

Define an operator

$$(4.2) \quad T^N f(\mathbf{x}) := \int e^{i\phi^N(\mathbf{x}; y)} a^N(\mathbf{x}; y) dy.$$

Note that $T^N f$ is just a rescaled version of $T_N^{(\phi)} f$. The goal of this section is to prove the following theorem.

Theorem 4.1. *Under the same assumptions as in Theorem 2.4, we have*

$$(4.3) \quad \|T^N f\|_{L^p(\mathbb{R}^3)} \lesssim_{\phi, a, p, \epsilon} N^\epsilon \|f\|_{L^p(\mathbb{R}^2)},$$

for all

$$(4.4) \quad p \geq \frac{10}{3} - \epsilon_k,$$

all $\epsilon > 0$ and $N \geq 1$.

To prove Theorem 4.1, we follow the idea of Guth [Gut18] and Bourgain and Guth [BG11], and reduce it to “broad” estimates.

Let $K \geq 1$. We divide $\mathbb{B}_{\epsilon_\phi}^2$ into dyadic squares τ of side length K^{-1} . Denote $f_\tau := f \cdot \mathbb{1}_\tau$. Fix a ball $\mathbb{B}_{K^2}^3(\mathbf{x}_0)$. Define

$$(4.5) \quad \mu_{T^N f}(\mathbb{B}_{K^2}^3(\mathbf{x}_0)) := \min_{\tau_1, \dots, \tau_{A_0}} \left(\max_{\tau \neq \tau_{A_0'}, 1 \leq A_0' \leq A_0} \|T^N f_\tau\|_{L^p(\mathbb{B}_{K^2}^3(\mathbf{x}_0))}^p \right),$$

where A_0 is a large parameter whose choice will become clear later. For $U \subset \mathbb{R}^3$, define

$$(4.6) \quad \|T^N f\|_{\text{BL}_{A_0}^p(U)} := \left(\sum_{\mathbb{B}_{K^2}^3(\mathbf{x}_0)} \frac{|\mathbb{B}_{K^2}^3(\mathbf{x}_0) \cap U|}{|\mathbb{B}_{K^2}^3(\mathbf{x}_0)|} \mu_{T^N f}(\mathbb{B}_{K^2}^3(\mathbf{x}_0)) \right)^{1/p},$$

where the sum runs over a finitely overlapping collection of balls $\mathbb{B}_{K^2}^3(\mathbf{x}_0)$ that covers \mathbb{R}^3 . This is called the broad part of $T^N f$.

Theorem 4.2. *For every $\epsilon > 0$, there exists A_0 such that*

$$(4.7) \quad \|T^N f\|_{\text{BL}_{A_0}^p(\mathbb{R}^3)} \lesssim_{K, \epsilon} N^\epsilon \|f\|_{L^2}^{2/p} \|f\|_{L^\infty}^{1-2/p},$$

for every p satisfying (4.4), every $K \geq 1, \epsilon > 0$, and $N \geq 1$. Moreover, the implicit constant depends polynomially on K .

Reducing Theorem 4.1 to Theorem 4.2 can be done via standard arguments in the literature (for instance Guth, Hickman and Iliopoulou [GHI19, Proposition 11.1]).

The rest of this section is devoted to the proof of Theorem 4.2.

4.1. Preliminaries in polynomial partitionings

We follow the notation from [GWZ22]. Take

$$(4.8) \quad 1 \leq r \leq R = N.$$

The only reason of introducing the parameter R is for the forthcoming notation to be consistent with that in [GWZ22]. Take a collection Θ_r of dyadic cubes of side length $\frac{9}{11}r^{-1/2}$ covering the ball \mathbb{B}^2 , the unit ball in \mathbb{R}^2 . We take a smooth partition of unity $(\psi_\theta)_{\theta \in \Theta_r}$ with $\text{supp } \psi_\theta \subset \frac{11}{10}\theta$ for the ball \mathbb{B}^2 such that

$$|\partial_\xi^\alpha \psi_\theta(\xi)| \lesssim_\alpha r^{|\alpha|/2},$$

for all $\xi \in \mathbb{R}^2$ and all multi-indices $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$ with $\|\alpha\| := |\alpha_1| + |\alpha_2|$. We denote by ξ_θ the center of θ . Given a function h , we perform a Fourier series decomposition to the function $h\psi_\theta$ on the region $\frac{11}{9}\theta$ and obtain

$$h(\xi)\psi_\theta(\xi) \cdot \mathbb{1}_{\frac{11}{10}\theta}(\xi) = \left(\frac{r^{1/2}}{2\pi}\right)^2 \sum_{v \in r^{1/2}\mathbb{Z}^2} (h\psi_\theta)^\wedge(v) e^{2\pi i v \cdot \xi} \mathbb{1}_{\frac{11}{10}\theta}(\xi).$$

Let $\tilde{\psi}_\theta$ be a non-negative smooth cutoff function supported on $\frac{11}{9}\theta$ and equal to 1 on $\frac{11}{10}\theta$. We can therefore write

$$h(\xi)\psi_\theta(\xi) \cdot \tilde{\psi}_\theta(\xi) = \left(\frac{r^{1/2}}{2\pi}\right)^2 \sum_{v \in r^{1/2}\mathbb{Z}^2} (h\psi_\theta)^\wedge(v) e^{2\pi i v \cdot \xi} \tilde{\psi}_\theta(\xi).$$

If we also define

$$h_{\theta,v}(\xi) := \left(\frac{r^{1/2}}{2\pi}\right)^2 (h\psi_\theta)^\wedge(v) e^{2\pi i v \cdot \xi} \tilde{\psi}_\theta(\xi),$$

then we have

$$(4.9) \quad h = \sum_{(\theta,v) \in \Theta_r \times r^{1/2}\mathbb{Z}^2} h_{\theta,v}.$$

For $\omega \in \mathbb{B}_{\epsilon_\phi}^2$ and $\xi \in \mathbb{B}_{\epsilon_\phi}^2$, we define $\Phi = \Phi(\omega, t; \xi)$ by

$$(4.10) \quad \nabla_\xi \phi\left(\Phi(\omega, t; \xi), t; \xi\right) = \omega.$$

Let us remark here that Φ is only locally defined; the existence and uniqueness of Φ can be guaranteed by picking ϵ_ϕ to be small enough. Let

$$(4.11) \quad \delta := \epsilon^C, \quad C \text{ large universal constant},$$

where ϵ is as in (4.3). We define curved $r^{1/2+\delta}$ -tubes as

$$(4.12) \quad T_{\theta,v} := \left\{ (x, t) : \left| \frac{x}{N} - \Phi\left(\frac{v}{N}, \frac{t}{N}; \xi_\theta\right) \right| \leq \frac{r^{1/2+\delta}}{N}, \quad t \in [0, r] \right\}.$$

This finishes the wave packet decomposition for the ball $\mathbb{B}_r^3 \subset \mathbb{R}^3$. We use $\mathbb{T}[\mathbb{B}_r^3]$ to denote the collection of the wave packets $T_{\theta,v}$.

Similarly, for $\mathbf{x}_0 \in \mathbb{B}_N^3$, we can define wave packet decompositions for the ball $\mathbb{B}_r^3(\mathbf{x}_0)$, and use $\mathbb{T}[\mathbb{B}_r^3(\mathbf{x}_0)]$ to denote the collection of the resulting wave packets. Let us be more precise. Take $\mathbf{x}_0 \in \mathbb{B}_N^3$ and consider the ball $\mathbb{B}_r^3(\mathbf{x}_0)$. For $h : \mathbb{B}_{\epsilon_\phi}^2 \rightarrow \mathbb{C}$ integrable, define

$$(4.13) \quad \tilde{h}(\xi) := e^{2\pi i \phi(\mathbf{x}_0; \xi)} h(\xi)$$

so that

$$(4.14) \quad T^N h(\mathbf{x}) = \tilde{T}^N \tilde{h}(\tilde{\mathbf{x}}), \quad \tilde{\mathbf{x}} := \mathbf{x} - \mathbf{x}_0,$$

where \tilde{T}^N is the Hörmander-type operator with the phase function

$$(4.15) \quad \tilde{\phi}(\mathbf{x}; \xi) := \phi(\mathbf{x} + \frac{\mathbf{x}_0}{N}; \xi) - \phi(\frac{\mathbf{x}_0}{N}; \xi).$$

If $\mathbf{x} \in \mathbb{B}_r^3(\mathbf{x}_0)$, then $\tilde{\mathbf{x}} \in \mathbb{B}_r^3$, and we can apply the wave packet decomposition above to $\tilde{T}^N \tilde{h}(\tilde{\mathbf{x}})$. We have

$$(4.16) \quad T^N h(\mathbf{x}) = \sum_{T \in \mathbb{T}[\mathbb{B}_r^3(\mathbf{x}_0)]} T^N h_T(\mathbf{x}),$$

whenever $|\mathbf{x} - \mathbf{x}_0| \lesssim r$.

Next, we introduce a few key notions that will appear in the forthcoming polynomial partitioning algorithms.

Definition 4.3 (Cells). *Let $P : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a non-zero polynomial. Denote*

$$(4.17) \quad Z(P) := \{z \in \mathbb{R}^3 : P(z) = 0\}.$$

We let $\text{cell}(P)$ denote the collection of all the connected components of $\mathbb{R}^3 \setminus Z(P)$. Each element in $\text{cell}(P)$ will be referred to as a cell of P .

Definition 4.4 (Transverse complete intersection). *Take the dimension $n = 3$ and $0 \leq m \leq 2$. Let $P_1, \dots, P_{n-m} : \mathbb{R}^n \rightarrow \mathbb{R}$ be polynomials. We consider the common zero set*

$$(4.18) \quad Z(P_1, \dots, P_{n-m}) := \{x \in \mathbb{R}^n : P_1(x) = \dots = P_{n-m}(x) = 0\}.$$

Suppose that for all $z \in Z(P_1, \dots, P_{n-m})$, one has

$$(4.19) \quad \bigwedge_{j=1}^{n-m} \nabla P_j(z) \neq 0.$$

Then a connected branch of this set, or a union of connected branches of this set, is called an m -dimensional transverse complete intersection. Given a set Z of the form (4.18), the degree of Z is defined by

$$\min \left(\prod_{i=1}^{n-m} \deg(P_i) \right),$$

where the minimum is taken over all possible representations of

$$Z = Z(P_1, \dots, P_{n-m}).$$

Lemma 4.5 (Polynomial partitioning lemma, Guth [Gut18], Hickman and Rogers [HR19]). Fix $r \gg 1, d \in \mathbb{N}$ and suppose $F \in L^1(\mathbb{R}^3)$ is non-negative and supported on $\mathbb{B}_r^3(\mathbf{x}_0) \cap \mathcal{N}_{r^{1/2+\delta_0}}(Z)$ for some \mathbf{x}_0 and $0 < \delta_0 \ll 1$, where Z is an m dimensional transverse complete intersection of degree at most d . At least one of the following cases holds:

Cellular case. There exists a polynomial $P : \mathbb{R}^3 \rightarrow \mathbb{R}$ of degree $O(d)$ with the following properties:

- (1) We can find a sub-collection of cells $\text{cell}'(P) \subset \text{cell}(P)$ with $\#\text{cell}'(P) \simeq d^m$ and each $O' \in \text{cell}'(P)$ has diameter at most r/d .
- (2) If we define

$$\mathcal{O} := \{O' \setminus \mathcal{N}_{r^{1/2+\delta_0}}(Z(P)) : O' \in \text{cell}'(P)\},$$

then

$$(4.20) \quad \int_O F \simeq d^{-m} \int_{\mathbb{R}^n} F,$$

for all $O \in \mathcal{O}$.

Algebraic case. There exists an $(m-1)$ -dimensional transverse complete intersection Y of degree at most $O(d)$ such that

$$(4.21) \quad \int_{\mathbb{B}_r^3(\mathbf{x}_0) \cap \mathcal{N}_{r^{1/2+\delta_0}}(Z)} F \lesssim \int_{\mathbb{B}_r^3(\mathbf{x}_0) \cap \mathcal{N}_{r^{1/2+\delta_0}}(Y)} F.$$

Before we proceed, let us explain the role that the degree parameter d will play in the forthcoming argument. Recall that our goal is to prove Theorem 4.2. Once p and ϵ are fixed, the degree parameter d will also be fixed, and will be picked to be sufficiently large depending on p and ϵ . For instance,

we will apply Lemma 4.5 repeatedly. Each time when we apply this lemma, we will lose an absolute constant C in (4.20) or (4.21). Note that by item (1) in Lemma 4.5, the radius of a cell will shrink by at least a factor of $1/d$. If we start with a ball of radius N , we know that Lemma 4.5 will be applied for at most $\log_d N$ many times, and in this process, the implicit constants will accumulate to $C^{\log_d N}$. If we pick d to be sufficiently large, then $C^{\log_d N}$ is much smaller compared to N^ϵ .

4.2. Polynomial partitioning algorithms

Our goal is to prove Theorem 4.2, that is, we will prove

$$(4.22) \quad \|T^N f\|_{\text{BL}_{A_0}^p(\mathbb{R}^3)} \lesssim_{K,\epsilon} N^\epsilon \|f\|_{L^2}^{2/p} \|f\|_{L^\infty}^{1-2/p},$$

for all

$$(4.23) \quad p \geq \frac{10}{3} - \epsilon_k,$$

all $K \geq 1$, $\epsilon > 0$ and $N \geq 1$.

We will recycle the polynomial partitioning algorithm in [GWZ22], which is a variant of that in Hickman and Rogers [HR19]. Let us be slightly more precise. We will repeat precisely the polynomial partitioning algorithm in Subsections 5.2-5.4 in [GWZ22], where Lemma 4.5 was repeatedly applied. In this algorithm, we will fix small parameters $\delta_j, j = 0, 1, 2, 3$, satisfying

$$(4.24) \quad \delta \ll \delta_3 \ll \delta_2 \ll \delta_1 \ll \delta_0 \ll \epsilon.$$

For instance, one can take $\delta_j = \delta_{j-1}^{10}$ for $j = 3, 2, 1$ and $\delta_0 = \epsilon^{10}, \delta = \delta_3^{10}$. Here δ is the same as that in (4.11). We partition \mathbb{B}_R^3 into a finitely overlapping collection of balls $\{B_\iota\}_\iota$, each of which is of radius $R^{1-\delta}$. Let us recall the output of this algorithm in [GWZ22, page 48].

Output 1. We obtain a sequence of nodes

$$(4.25) \quad \mathbf{n}_0^*, \mathbf{n}_1^*, \dots, \mathbf{n}_{\ell_0}^*.$$

Each node \mathbf{n}_ℓ^* with $0 \leq \ell \leq \ell_0 \in \mathbb{N}$ is a collection of open sets in \mathbb{R}^3 , and is assigned several parameters: A dimension parameter $\dim(\mathbf{n}_\ell^*)$ and a radius

parameter $\rho(\mathfrak{n}_\ell^*)$. As we are in \mathbb{R}^3 , we have $\dim(\mathfrak{n}_\ell^*)$ takes values in $\{2, 3\}$, and is non-increasing in ℓ . Let $\ell_- \in \{0, 1, \dots, \ell_0\}$ be such that

$$(4.26) \quad \dim(\mathfrak{n}_\ell^*) = 3, \quad \text{if } 0 \leq \ell < \ell_-,$$

and $\dim(\mathfrak{n}_\ell^*) = 2$ otherwise. It may happen that $\ell_- = \ell_0$. However, this case is easy to handle, and we therefore assume that we always have $\ell_- < \ell_0$.

Output 2. The node $\mathfrak{n}_{\ell_-}^*$ is particularly important. Denote

$$(4.27) \quad \mathfrak{S}_3 := \mathfrak{n}_0^*, \quad \mathfrak{S}_2 := \mathfrak{n}_{\ell_-}^*,$$

and

$$(4.28) \quad r_3 := \rho(\mathfrak{S}_3) = R, \quad r_2 := \rho(\mathfrak{S}_2), \quad r_1 := 1.$$

The node \mathfrak{n}_0^* consists of only one element, \mathbb{B}_R^3 . Elements in \mathfrak{S}_2 are of the form

$$(4.29) \quad B_{r_2} \cap \mathcal{N}_{r_2^{1/2+\delta_2}}(S_2),$$

where $B_{r_2} \subset \mathbb{B}_R^3$ is a ball of radius r_2 and S_2 is an algebraic variety of dimension two.⁸ To simplify notation, we will use S_2 to refer to (4.29), whenever it is clear from the context that we are talking about \mathfrak{S}_2 .

Output 3. Each open set $O \in \mathfrak{n}_{\ell_0}^*$ has diameter at most R^{δ_0} . This is the stopping condition of the algorithm (see [GWZ22, page 41]), which says that the algorithm terminates whenever we reach a scale $\leq R^{\delta_0}$. Each $O \in \mathfrak{n}_{\ell_0}^*$ is associated with a function $f_{\iota, O}$, which is built with a collection of tubes from $\mathbb{T}[B_{\rho(\mathfrak{n}_{\ell_0}^*)}]$, that is,

$$(4.30) \quad f_{\iota, O} = \sum_{T \in \mathbb{T}'[B_{\rho(\mathfrak{n}_{\ell_0}^*)}]} f_T,$$

where $\mathbb{T}'[B_{\rho(\mathfrak{n}_{\ell_0}^*)}]$ is a sub-collection of tubes in $\mathbb{T}[B_{\rho(\mathfrak{n}_{\ell_0}^*)}]$, and $B_{\rho(\mathfrak{n}_{\ell_0}^*)} \subset \mathbb{R}^3$ is the ball of radius $\rho(\mathfrak{n}_{\ell_0}^*)$ that contains O .

Output 4. For each ι and each $S_2 \in \mathfrak{S}_2$ with $S_2 \cap B_\iota \neq \emptyset$, there is an associated function f_{ι, S_2}^* . This function is built with a collection of tubes from $\mathbb{T}[B_{r_2}]$, where B_{r_2} is as in (4.29). Most importantly, all the tubes in this collection are contained in S_2 .

⁸This is why in the previous item we let $\dim(\mathfrak{n}_{\ell_-}^*) = 2$.

Output 5. In the end, we have parameters D_3, D_2, D_1 that are integer powers of d satisfying $D_3 = 1$, and

$$(4.31) \quad D_1 \leq r_2, \quad r_1 D_1 D_2 D_3 \leq R, \quad r_2 D_2 D_3 \leq R.$$

See Lemma 5.10 in [GWZ22, page 50].⁹

Next, let us state what estimates the outputs satisfy. Denote $p_3 = p$, $\alpha_3 = \beta_3 = 1$, $\beta_2 = \alpha_2$, and $\alpha_2 \in [0, 1]$ is to be determined. Let p_2 be such that

$$(4.32) \quad \frac{1}{p_3} = \frac{1 - \alpha_2}{2} + \frac{\alpha_2}{p_2}.$$

Then the above outputs satisfy the following properties.

Property 1. We can find $C_p > 0$ depending only on p and $A_2 \in \mathbb{N}$ with $A_2 \leq A_0$ depending only on ϵ , such that

$$(4.33) \quad \|T^N f\|_{\text{BL}_{A_0}^p(\mathbb{B}_R^3)} \lesssim R^{C_p \delta_0} (r_2 D_2)^{\frac{1}{2}(1-\beta_2)} \|f\|_2^{1-\beta_2} \left(\sum_{S_2 \in \mathfrak{S}_2} \sum_{\iota} \|T^N f_{\iota, S_2}^*\|_{\text{BL}_{A_2}^{p_2}(B_{r_2})}^{p_2} \right)^{\frac{\beta_2}{p_2}},$$

where B_{r_2} is the ball of radius r_2 that contains S_2 , given as in (4.29). We remark here that C_p will also appear below, and its precise values will not be important and may change from line to line.

Property 2. We also have

$$(4.34) \quad \sum_{S_2 \in \mathfrak{S}_2} \|f_{\iota, S_2}^*\|_2^2 \lesssim R^{C_p \delta_0} D_2 \|f\|_2^2.$$

Property 3. In the end, we have

$$(4.35) \quad \max_{S_2 \in \mathfrak{S}_2} \|f_{\iota, S_2}^*\|_2^2 \lesssim R^{C_p \delta_0} D_2^{-2} \|f\|_2^2$$

and

$$(4.36) \quad \max_{S_2 \in \mathfrak{S}_2} \max_{\ell(\theta)=\rho^{-1/2}} \|f_{\iota, S_2}^*\|_{L_{\text{avg}}^2(\theta)}^2 \lesssim R^{C_p \delta_0} \max_{\ell(\theta)=\rho^{-1/2}} \|f\|_{L_{\text{avg}}^2(\theta)}^2$$

⁹The bound $D_1 \leq r_2$ is not stated explicitly in Lemma 5.10 in [GWZ22, page 50], but it can be proven easily via the same argument.

for all $1 \leq \rho \leq r_2$, where the max runs over all squares $\theta \subset \mathbb{R}^2$ of side length $\ell(\theta) = \rho^{-1/2}$, and

$$(4.37) \quad \|h\|_{L^2_{\text{avg}}(\theta)}^2 := \frac{1}{\mathcal{L}^2(\theta)} \int_{\theta} |h|^2$$

for a function $h : \mathbb{R}^2 \rightarrow \mathbb{R}$.

These three properties are taken from [GWZ22, page 51], where we set $n = 3, n' = 2$. We do not need Property 4 there because it is relevant only for estimates in \mathbb{R}^n with $n \geq 4$.

In Property 1, we connect the scale $r_3 = \rho(\mathfrak{n}_0^*)$ with the scale $r_2 = \rho(\mathfrak{n}_{\ell_-}^*)$. Now we state a few estimates that connect the scale $r_3 = \rho(\mathfrak{n}_0^*)$ with the smallest scale $\rho(\mathfrak{n}_{\ell_0}^*)$. More precisely, we have

(4.38)

$$\|T^N f\|_{\text{BL}_{A_0}^p(\mathbb{B}_R^3)} \lesssim R^{C_p \delta_0} (r_2 D_2)^{\frac{1}{2}(1-\beta_2)} D_1^{\frac{\beta_2}{p_2}} D_2^{\frac{\beta_2}{p_2}} \|f\|_2^{\frac{2}{p_3}} \max_{O \in \mathfrak{n}_{\ell_0}^*} \|f_{\iota, O}\|_2^{1-\frac{2}{p_3}},$$

where ι refers to the unique $B_{\iota} \subset \mathbb{B}_R^3$ of radius $R^{1-\delta}$ containing O . This estimate is precisely from equations (9.1)–(9.2) in [GWZ22, page 86]. Moreover,

$$(4.39) \quad \max_{O \in \mathfrak{n}_{\ell_0}^*} \|f_{\iota, O}\|_2^2 \lesssim R^{C_p \delta_0} r_2^{-\frac{1}{2}} D_1^{-1} \max_{S_2 \in \mathfrak{S}_2} \|f_{\iota, S_2}^*\|_2^2,$$

which is equation (9.3) in [GWZ22, page 87], with $n = 3, n' = 2$. This finishes recalling the outputs of the algorithm in [GWZ22] and the properties of the outputs.

Before we proceed, let us first see how to prove (4.22) for the smaller range $p \geq 10/3$. We combine (4.38), (4.39), (4.35) and obtain

$$(4.40) \quad \begin{aligned} & \|T^N f\|_{\text{BL}_{A_0}^p(\mathbb{B}_R^3)} \\ & \lesssim R^{C_p \delta_0} (r_2 D_2)^{\frac{1}{2}(1-\beta_2)} D_1^{\frac{\beta_2}{p_2}} D_2^{\frac{\beta_2}{p_2}} r_2^{-\frac{1}{2}(\frac{1}{2}-\frac{1}{p})} D_1^{-(\frac{1}{2}-\frac{1}{p})} D_2^{-2(\frac{1}{2}-\frac{1}{p})} \|f\|_2 \\ & \lesssim R^{C_p \delta_0} (r_2 D_2)^{\frac{1}{2}(1-\beta_2)} D_1^{\frac{\beta_2}{2}-2(\frac{1}{2}-\frac{1}{p})} r_2^{-\frac{1}{2}(\frac{1}{2}-\frac{1}{p})} D_2^{\frac{\beta_2}{2}-3(\frac{1}{2}-\frac{1}{p})} \|f\|_2, \end{aligned}$$

where in the second inequality we used (4.32). Recall the relations in (4.31), and that β_2 is a free parameter to choose. We impose the constraint that

$$(4.41) \quad \frac{\beta_2}{2} - 2\left(\frac{1}{2} - \frac{1}{p}\right) \geq 0,$$

and obtain that the last term in (4.40) can be further bounded by

$$(4.42) \quad \begin{aligned} & R^{C_p \delta_0} (r_2 D_2)^{\frac{1}{2}(1-\beta_2)} r_2^{\frac{\beta_2}{2} - 2(\frac{1}{2} - \frac{1}{p})} r_2^{-\frac{1}{2}(\frac{1}{2} - \frac{1}{p})} D_2^{\frac{\beta_2}{2} - 3(\frac{1}{2} - \frac{1}{p})} \|f\|_2 \\ & \lesssim R^{C_p \delta_0} r_2^{\frac{1}{2} - \frac{5}{2}(\frac{1}{2} - \frac{1}{p})} D_2^{\frac{1}{2} - 3(\frac{1}{2} - \frac{1}{p})} \|f\|_2. \end{aligned}$$

Note that the exponent of D_2 is always negative as $p > 3$. Moreover, by letting the exponent of r_2 be negative, we see the exponent $p \geq 10/3$.

4.3. Polynomial Wolff axioms

Now let us focus on the improved range (4.23). This will rely on polynomial Wolff axioms. Recall the definition of the function Φ in (4.10). For $\kappa \in (0, 1)$, we use θ to denote a dyadic square on \mathbb{R}^2 of side length κ . Moreover, we use ξ_θ to denote the center of θ . For $v \in \mathbb{R}^2$ with $|v| \leq 1$, define

$$(4.43) \quad T_{\xi_\theta, v, \Phi}(\kappa, 1) := \{(x, t) \in \mathbb{R}^2 \times \mathbb{R} : |x - \Phi(v, t; \xi_\theta)| \leq \kappa, |t| \leq 1\}.$$

In $T_{\xi_\theta, v, \Phi}(\kappa, 1)$, we use 1 to indicate that it is a (curved) tube of length one; it is a rescaled version of the tubes defined in (4.12). The main reason of rescaling the curved tubes is for our notation to be consistent with that in [GWZ22, Section 3]; our $T_{\xi_\theta, v, \Phi}(\kappa, 1)$ is precisely $T_{\xi_\theta, v, \Phi}(\delta, 1)$ in equation (3.2) in [GWZ22], and we are not using δ because it was used previously.

For a collection \mathbb{T} of tubes $\{T_{\xi_\theta, v, \Phi}(\kappa, 1)\}$, we say that the tubes in \mathbb{T} point in different direction if no any two tubes share the same ξ_θ .

Theorem 4.6. *Let $S \subset \mathbb{B}^3$ be a semi-algebraic set of complexity $\leq E$. Let $\mathbb{T} = \{T_{\xi_\theta, v, \Phi}(\kappa, 1)\}$ be a collection of tubes pointing in different directions. Then*

$$(4.44) \quad \#\{T \in \mathbb{T} : T \subset S\} \lesssim_{\phi, E, \epsilon'} \mathcal{L}^3(S) \kappa^{-2-\epsilon'},$$

for every $\epsilon' > 0$.

Proof of Theorem 4.6. We will apply [GWZ22, Theorem 3.1]. To apply this theorem, it suffices to show that there exists $\epsilon_\phi > 0$ such that

$$(4.45) \quad \int_{|t| \leq \epsilon_\phi} |\det(\nabla_v \Phi(v, t; \xi) \cdot M + \nabla_\xi \Phi(v, t; \xi))| dt \gtrsim_\phi 1,$$

for all $|v| \leq \epsilon_\phi$, $|\xi| \leq \epsilon_\phi$ and 2×2 matrices M . In particular, the implicit constant is not allowed to depend on v, ξ, M . This is the step where we will use the assumption that ϕ is of contact order $\leq k$ at the origin.

To prove (4.45), let us first prove that the notion of contact orders is stable under perturbations. To be more precise, we claim that the phase function ϕ is of contact order $\leq k$ at $(x_0, t_0; \xi_0)$ whenever $|x_0| \leq \epsilon_\phi$, $|t_0| \leq \epsilon_\phi$ and $|\xi_0| \leq \epsilon_\phi$. Denote

$$(4.46) \quad \phi_0(x, t; \xi) := \phi(x + x_0, t + t_0; \xi + \xi_0) - \phi(x_0, t_0; \xi + \xi_0).$$

Moreover, denote

$$(4.47) \quad D_{ij}(t; x_0, t_0; \xi_0) := \partial_{\xi_i} \partial_{\xi_j} \phi_0(X_0(t), t; 0), \quad 1 \leq i, j \leq 2,$$

and

$$(4.48) \quad \mathfrak{D}(t; x_0, t_0; \xi_0) := \det \begin{bmatrix} D_{11}(t; x_0, t_0; \xi_0) & D_{12}(t; x_0, t_0; \xi_0) \\ D_{21}(t; x_0, t_0; \xi_0) & D_{22}(t; x_0, t_0; \xi_0) \end{bmatrix}.$$

Recall that our assumption is that ϕ is of contact order $\leq k$ at $x_0 = 0, t_0 = 0, \xi_0 = 0$, that is, the matrix

$$(4.49) \quad \begin{bmatrix} \mathfrak{D}'(0; 0, 0; 0) & \mathfrak{D}''(0; 0, 0; 0) & \dots & \mathfrak{D}^{(k)}(0; 0, 0; 0) \\ D'_{11}(0; 0, 0; 0) & D''_{11}(0; 0, 0; 0) & \dots & D_{11}^{(k)}(0; 0, 0; 0) \\ D'_{12}(0; 0, 0; 0) & D''_{12}(0; 0, 0; 0) & \dots & D_{12}^{(k)}(0; 0, 0; 0) \\ D'_{22}(0; 0, 0; 0) & D''_{22}(0; 0, 0; 0) & \dots & D_{22}^{(k)}(0; 0, 0; 0) \end{bmatrix}$$

has rank 4, and therefore we can find a minor of order 4×4 whose rank is 4. The claim follows from continuity.

Now let us prove (4.45). Recall the definition of Φ from (4.10) that

$$(4.50) \quad \nabla_\xi \phi(\Phi(v, t; \xi), t; \xi) = v.$$

Taking derivatives in v on both sides, we obtain

$$(4.51) \quad \nabla_x \nabla_\xi \phi \left(\Phi(v, t; \xi), t; \xi \right) \nabla_v \Phi(v, t; \xi) = I_{2 \times 2}.$$

Taking derivatives in ξ on both sides of (4.50), we obtain

$$(4.52) \quad \nabla_x \nabla_\xi \phi \left(\Phi(v, t; \xi), t; \xi \right) \nabla_\xi \Phi(v, t; \xi) + \nabla_\xi^2 \phi \left(\Phi(v, t; \xi), t; \xi \right) = 0.$$

Therefore (4.45) amounts to proving that

$$(4.53) \quad \int_{|t| \leq \epsilon_\phi} \left| \det \left(M + \nabla_\xi^2 \phi \left(\Phi(v, t; \xi), t; \xi \right) \right) \right| dt \gtrsim_\phi 1,$$

uniformly in v, ξ and M . Write

$$(4.54) \quad \begin{aligned} & \det \left(M + \nabla_\xi^2 \phi \left(\Phi(v, t; \xi), t; \xi \right) \right) \\ &= \det \left(\widetilde{M} + \nabla_\xi^2 \phi \left(\Phi(v, t; \xi), t; \xi \right) - \nabla_\xi^2 \phi \left(\Phi(v, 0; \xi), 0; \xi \right) \right), \end{aligned}$$

for some new 2×2 matrix

$$(4.55) \quad \widetilde{M} = \begin{bmatrix} \widetilde{m}_{11} & \widetilde{m}_{12} \\ \widetilde{m}_{21} & \widetilde{m}_{22} \end{bmatrix}.$$

We take $t_0 = 0$ in (4.47), x_0 in (4.47) satisfying

$$(4.56) \quad \nabla_\xi \phi(x_0, 0; \xi) = v,$$

and obtain

$$(4.57) \quad \begin{aligned} (4.54) &= \mathfrak{D}(t; x_0, 0; \xi) + \widetilde{m}_{11} D_{22}(t; x_0, 0; \xi) + \widetilde{m}_{22} D_{11}(t; x_0, 0; \xi) \\ &\quad - (\widetilde{m}_{12} + \widetilde{m}_{21}) D_{12}(t; x_0, 0; \xi) + \det(\widetilde{M}). \end{aligned}$$

Let us write

$$(4.58) \quad \det \left(M + \nabla_\xi^2 \phi \left(\Phi(v, t; \xi), t; \xi \right) \right) = c_0 + \sum_{k'=1}^k c_{k'} t^{k'} + \text{higher order terms},$$

where $c_{k'}$ are constants that depend on the choice of v, ξ and M . From (4.49) and continuity (on the parameters x_0 and ξ), we know that

$$(4.59) \quad \sum_{k'=1}^k |c_{k'}| \gtrsim_{\phi} 1 + \|(\tilde{m}_{11}, \tilde{m}_{22}, \tilde{m}_{12} + \tilde{m}_{21})\|_1,$$

where

$$(4.60) \quad \|(m_1, m_2, m_3)\|_1 := |m_1| + |m_2| + |m_3|,$$

for $m_1, m_2, m_3 \in \mathbb{R}$. Moreover, note that the right hand side of (4.57) depends linearly on $\tilde{m}_{11}, \tilde{m}_{22}$ and $\tilde{m}_{12} + \tilde{m}_{21}$, and therefore the higher order terms on the right hand side of (4.58) are

$$(4.61) \quad (1 + \|(m_1, m_2, m_3)\|_1) O_{\phi, k}(t^{k+1}),$$

where the implicit constant in $O_{\phi, k}$ depends only on ϕ and k . By dividing the coefficients in (4.58) by a constant, we see that the desired bound (4.53) follows from

Claim 4.7. *Let $k \geq 1$ be an integer. Let $W : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function of the form*

$$(4.62) \quad c_0 + \sum_{k'=1}^k c_{k'} t^{k'} + O_{\phi, k}(t^{k+1})$$

satisfying

$$(4.63) \quad \sum_{k'=1}^k |c_{k'}| \gtrsim_{\phi, k} 1,$$

where the implicit constant in (4.62) depends on ϕ and k . Then

$$(4.64) \quad \int_{|t| \leq \epsilon_{\phi}} |W(t)| dt \gtrsim_{\phi, k} 1.$$

Proof of Claim 4.7. The proof is almost immediate. From (4.63) we can conclude that there exist $\epsilon_{\phi, k}$ and $k'' \in \{0, 1, \dots, k\}$ such that

$$(4.65) \quad |W^{(k'')}(t)| \gtrsim_{\phi, k} 1,$$

for all $|t| \leq \epsilon_{\phi,k}$, which implies that $|W(t)|$ can not stay close to 0 for too long time, and further implies the desired lower bound (4.64). \square

Claim 4.7 implies the lower bound (4.53), and we have therefore finished the proof of Theorem 4.6. \square

Let us state a corollary of Theorem 4.6. Recall the outputs of the polynomial partitioning algorithm. In particular, recall that $r_2 = \rho(\mathfrak{S}_2)$, that elements in \mathfrak{S}_2 are of the form

$$(4.66) \quad B_{r_2} \cap \mathcal{N}_{r_2^{1/2+\delta_2}}(S_2),$$

where $B_{r_2} \subset \mathbb{B}_R^3$ is a ball of radius r_2 and S_2 is an algebraic variety of dimension two, and that the function f_{ι,S_2}^* is built with a collection of tubes from $\mathbb{T}[B_{r_2}]$ that are contained in (4.66).

Corollary 4.8. *Assume that $r_2 = R$. Then for every S_2 , it holds that*

$$(4.67) \quad \|f_{\iota,S_2}^*\|_2^2 \lesssim_{\phi,\epsilon} r_2^{-1/2+\delta_0} \|f\|_\infty^2.$$

Here the relation for $\delta_2, \delta_0, \epsilon$ are given in (4.24), and the implicit constant depends on ϵ .

Proof of Corollary 4.8. The argument is a minor variant of the one introduced by Guth [Gut16]. Write $h := f_{\iota,S_2}^*$. Let us write h using the wave packet decomposition in Subsection 4.1,

$$(4.68) \quad h = \sum_{\theta} h_{\theta}, \quad h_{\theta} := \sum_v h_{\theta,v},$$

where θ denotes a frequency square of side length $r_2^{-1/2}$. By the orthogonality estimate in (4.36),

$$(4.69) \quad \|h\|_2^2 \lesssim \sum_{\theta} \|h_{\theta}\|_2^2 \lesssim r_2^{-1} \sum_{\theta} \|f\|_{L_{\text{avg}}^2(\theta)}^2,$$

where the factor r_2^{-1} comes from taking averaged integral over θ . By Theorem 4.6, the number of θ that contributes to the above sum is $\lesssim_{\epsilon} r_2^{1/2+\delta_0}$. This, combined with (4.69), implies the desired bound. \square

Before we proceed to polynomial Wolff axioms at general scales (general r_2 instead of $r_2 = R$), let us first see why (4.67) is more favorable compared

with previously used estimates in (4.40). In (4.40), we combined (4.39), (4.35), and obtained

$$(4.70) \quad \max_{O \in \mathfrak{n}_{\ell_0}^*} \|f_{\iota, O}\|_2^2 \lesssim R^{C_p \delta_0} r_2^{-\frac{1}{2}} D_1^{-1} D_2^{-2} \|f\|_2^2.$$

Now we replace (4.35) by (4.67), and obtain

$$(4.71) \quad \begin{aligned} \max_{O \in \mathfrak{n}_{\ell_0}^*} \|f_{\iota, O}\|_2^2 &\lesssim R^{C_p \delta_0} r_2^{-\frac{1}{2}} D_1^{-1} \max_{S_2 \in \mathfrak{S}_2} \|f_{\iota, S_2}^*\|_2^2 \\ &\lesssim R^{C_p \delta_0} r_2^{-\frac{1}{2}} D_1^{-1} r_2^{-\frac{1}{2}} \|f\|_\infty^2. \end{aligned}$$

Recall the relation $r_2 D_2 \leq R$ in (4.31). In other words, if $r_2 \simeq R$, then $D_2 \simeq 1$. In this case, the bound (4.71) is much better than the bound (4.70); the trade-off is that $\|f\|_2$ in (4.70) is replaced by $\|f\|_\infty$, which we can afford.

Next, let us try to understand rescaled versions of the above polynomial Wolff axioms. More precisely, we will prove an analog of Corollary 4.8 for general $r_2 \leq R$.

Proposition 4.9. *For every $S_2 \in \mathfrak{S}_2$ and $r_2 = \rho(\mathfrak{S}_2)$, it holds that*

$$(4.72) \quad \|f_{\iota, S_2}^*\|_2^2 \lesssim_{\phi, \epsilon} \min \left\{ \left(\frac{N}{r_2} \right)^{k-2} r_2^{-1/2}, \left(r_2^{-1/2} + \frac{r_2}{N} \right) \right\} (r_2)^{\delta_0} \|f\|_\infty^2.$$

Here δ_0 and ϵ are the same as in Corollary 4.8.

Proof of Proposition 4.9. To simplify notation, let us without loss of generality assume that S_2 is contained in the ball $\mathbb{B}_{r_2}^3 \subset \mathbb{B}_R^3$, that is, the ball of radius r_2 centered at the origin. In this case, wave packets are defined in (4.12), that is

$$(4.73) \quad T_{\theta, v} := \left\{ (x, t) : \left| \frac{x}{N} - \Phi \left(\frac{v}{N}, \frac{t}{N}; \xi_\theta \right) \right| \leq \frac{r_2^{1/2+\delta}}{N}, t \in [0, r] \right\}.$$

Note that $T_{\theta, v}$ is a curved tube of length r_2 . Let us rescale it by a factor of r_2 , and consider everything at the unit scale:

$$(4.74) \quad T_{\theta, v}^\circ := \left\{ (x, t) : \left| \frac{r_2 x}{N} - \Phi \left(\frac{r_2 v}{N}, \frac{r_2 t}{N}; \xi_\theta \right) \right| \leq \frac{r_2^{1/2+\delta}}{N}, t \in [0, 1] \right\}.$$

Rewrite the defining equation for $T_{\theta, v}^\circ$ in (4.74) as

$$(4.75) \quad \left| x - N_2 \Phi \left(\frac{v}{N_2}, \frac{t}{N_2}; \xi_\theta \right) \right| \leq \frac{r_2^{1/2+\delta}}{r_2},$$

where $N_2 := N/r_2$. Denote

$$(4.76) \quad \Phi_{N_2}(v, t; \xi) := N_2 \Phi\left(\frac{v}{N_2}, \frac{t}{N_2}; \xi\right).$$

Recall that Φ satisfies equation (4.50), that is,

$$(4.77) \quad \nabla_\xi \phi\left(\Phi(v, t; \xi), t; \xi\right) = v,$$

which be written as

$$(4.78) \quad \frac{v}{N_2} = \nabla_\xi \phi\left(\frac{1}{N_2} \Phi_{N_2}(v, t; \xi), \frac{t}{N_2}; \xi\right) = \nabla_\xi \phi_{N_2}(\Phi_{N_2}(v, t; \xi), t; \xi),$$

with

$$(4.79) \quad \phi_{N_2}(x, t; \xi) := N_2 \phi\left(\frac{x}{N_2}, \frac{t}{N_2}; \xi\right).$$

Denote

$$(4.80) \quad T_{\xi_\theta, v, \Phi_{N_2}}(\kappa, 1) := \{(x, t) \in \mathbb{R}^2 \times \mathbb{R} : |x - \Phi_{N_2}(v, t; \xi_\theta)| \leq \kappa, |t| \leq 1\}.$$

Similarly to Theorem 4.6, we have

Claim 4.10. *Let $S \subset \mathbb{B}^3$ be a semi-algebraic set of complexity $\leq E$. Let $\mathbb{T} = \{T_{\xi_\theta, v, \Phi_{N_2}}(\kappa, 1)\}$ be a collection of tubes pointing in different directions. Then*

$$(4.81) \quad \#\{T \in \mathbb{T} : T \subset S\} \lesssim_{\phi, E, \epsilon} \min\{\mathcal{L}^3(S)(N_2)^{k-2}, \mathcal{L}^3(S_{N_2^{-1}})\} \kappa^{-2-\epsilon'},$$

for every $\epsilon' > 0$, where k is the contact order of ϕ and $S_{N_2^{-1}}$ denotes the N_2^{-1} neighborhood of S .

Proof of Claim 4.10. The proof of the upper bound

$$(4.82) \quad \#\{T \in \mathbb{T} : T \subset S\} \lesssim_{\phi, E, \epsilon} \mathcal{L}^3(S)(N_2)^{k-2} \kappa^{-2-\epsilon'}$$

is essentially the same as that of Theorem 4.6. The key difference is that (4.45) is no longer true anymore, and instead we have

$$(4.83) \quad \int_{|t| \leq \epsilon_\phi} |\det(\nabla_v \Phi_{N_2}(v, t; \xi) \cdot M + \nabla_\xi \Phi_{N_2}(v, t; \xi))| dt \gtrsim_\phi (N_2)^{-k+2},$$

uniformly in v, ξ and 2×2 matrices M . With (4.83) in hand, one can repeat the proof of Theorem 3.1 in [GWZ22], and obtain Claim 4.10.

To prove (4.82), it remains to prove (4.83). Let us first write it using ϕ_{N_2} . By (4.78) and (4.79), it is equivalent to prove that

$$(4.84) \quad \int_{|t| \leq \epsilon_\phi} \left| \det \left(M + \nabla_\xi^2 \phi_{N_2} \left(\Phi_{N_2}(v, t; \xi), t; \xi \right) \right) \right| dt \gtrsim_\phi (N_2)^{-k+2},$$

uniformly in v, ξ and 2×2 matrices M . By using the stability phenomenon that we observed in (4.49), we can without loss of generality assume that we are working with $v = 0, \xi = 0$. Under this simplification, we have

$$(4.85) \quad \Phi_{N_2}(0, t; 0) = 0, \quad \forall t,$$

as ϕ is of a normal form, and therefore (4.84) becomes

$$(4.86) \quad \int_{|t| \leq \epsilon_\phi} \left| \det \left(M + \nabla_\xi^2 \phi_{N_2}(0, t; 0) \right) \right| dt \gtrsim_\phi (N_2)^{-k+2},$$

which follows from Claim 4.7.

Let us turn to the other upper bound. We without loss of generality assume that

$$(4.87) \quad \phi(\mathbf{x}; \xi) = \langle x, \xi \rangle + \frac{1}{2} t |\xi|^2 + \psi(\mathbf{x}; \xi), \quad \mathbf{x} = (x, t),$$

where

$$(4.88) \quad \psi(\mathbf{x}; \xi) = O(|t| |\xi|^3 + |\mathbf{x}|^2 |\xi|^2).$$

Under this form, we have

$$(4.89) \quad \phi_{N_2}(\mathbf{x}; \xi) = \langle x, \xi \rangle + \frac{1}{2} t |\xi|^2 + N_2 \psi \left(\frac{x}{N_2}, \frac{t}{N_2}; \xi \right).$$

Let us write

$$(4.90) \quad \psi(\mathbf{x}; \xi) = t Q_3(\xi) + \psi_2(\mathbf{x}; \xi),$$

with

$$(4.91) \quad Q_3(\xi) = O(|\xi|^3), \quad \psi_2(\mathbf{x}; \xi) = O(|\mathbf{x}|^2 |\xi|^2).$$

We have

$$(4.92) \quad \phi_{N_2}(\mathbf{x}; \xi) = \langle x, \xi \rangle + \frac{1}{2}t|\xi|^2 + tQ_3(\xi) + N_2\psi_2\left(\frac{x}{N_2}, \frac{t}{N_2}; \xi\right).$$

The tube $T_{\xi_\theta, v, \Phi_{N_2}}(\kappa, 1)$ is given by the κ neighborhood of the curve

$$(4.93) \quad \{(x, t) : \nabla_\xi \phi_{N_2}(x, t; \xi_\theta) = v\}.$$

Define

$$(4.94) \quad \tilde{\phi}_{N_2}(\mathbf{x}; \xi) = \langle x, \xi \rangle + \frac{1}{2}t|\xi|^2 + tQ_3(\xi).$$

Define the tube $\tilde{T}_{\xi_\theta, v, \Phi_{N_2}}(\kappa, 1)$ to be the κ neighborhood of the curve

$$(4.95) \quad \{(x, t) : \nabla_\xi \tilde{\phi}_{N_2}(x, t; \xi_\theta) = v\}.$$

Take a point (x, t) from the curve (4.93) and a point (x', t) from the curve (4.95). By adding a zero, we obtain

$$(4.96)$$

$$\nabla_\xi \phi_{N_2}(x, t; \xi_\theta) - \nabla_\xi \phi_{N_2}(x', t; \xi_\theta) + \nabla_\xi \phi_{N_2}(x', t; \xi_\theta) - \nabla_\xi \tilde{\phi}_{N_2}(x', t; \xi_\theta) = 0.$$

By Hörmander's non-degeneracy condition, we know that

$$(4.97) \quad |\nabla_\xi \phi_{N_2}(x, t; \xi_\theta) - \nabla_\xi \phi_{N_2}(x', t; \xi_\theta)| \simeq |x - x'|.$$

Moreover,

$$(4.98) \quad |\nabla_\xi \phi_{N_2}(x', t; \xi_\theta) - \nabla_\xi \tilde{\phi}_{N_2}(x', t; \xi_\theta)| \lesssim N_2^{-1}.$$

As a consequence, we know that the tube $T_{\xi_\theta, v, \Phi_{N_2}}(\kappa, 1)$ is in the N_2^{-1} neighborhood of the tube $\tilde{T}_{\xi_\theta, v, \Phi_{N_2}}(\kappa, 1)$.

Claim 4.11. *Let $S \subset \mathbb{B}^3$ be a semi-algebraic set of complexity $\leq E$. Let $\mathbb{T} = \{\tilde{T}_{\xi_\theta, v, \Phi_{N_2}}(\kappa, 1)\}$ be a collection of tubes pointing in different directions. Then*

$$(4.99) \quad \#\{\tilde{T} \in \mathbb{T} : \tilde{T} \subset S\} \lesssim_{\phi, E, \epsilon} \mathcal{L}^3(S) \kappa^{-2-\epsilon'},$$

for every $\epsilon' > 0$.

Proof of Claim 4.11. The proof of Claim 4.11 is the same as that of Theorem 4.6. Let us give a very brief sketch here. The Hessian of $\tilde{\phi}_{N_2}$ in the ξ variables is

$$(4.100) \quad \begin{bmatrix} t + t\partial_{\xi_1}\partial_{\xi_1}Q_3(\xi) & t\partial_{\xi_1}\partial_{\xi_2}Q_3(\xi) \\ t\partial_{\xi_2}\partial_{\xi_1}Q_3(\xi) & t + t\partial_{\xi_2}\partial_{\xi_2}Q_3(\xi) \end{bmatrix}.$$

From this, one can check easily that an analogue of (4.53) holds, that is,

$$(4.101) \quad \int_{|t| \leq \epsilon_\phi} \left| \det \left(M + \nabla_\xi^2 \tilde{\phi}_{N_2}(0, t; \xi) \right) \right| dt \gtrsim_\phi 1,$$

uniformly in M and ξ . Here in the argument of $\tilde{\phi}_{N_2}$ we simply set $x = 0$ because the Hessian of $\tilde{\phi}_{N_2}$ in ξ is constant in x anyway. This finishes the proof of Claim 4.11. \square

As a consequence of (4.98) and Claim 4.11, we obtain that

$$(4.102) \quad \#\{T \in \mathbb{T} : T \subset S\} \lesssim_{\phi, E, \epsilon} \mathcal{L}^3(S_{N_2^{-1}}) \kappa^{-2-\epsilon'},$$

for every $\epsilon' > 0$, where $S_{N_2^{-1}}$ denotes the N_2^{-1} neighborhood of S . This finishes the proof of Claim 4.10. \square

Once we prove Claim 4.10, the desired estimate in the proposition is immediate, and the argument is exactly the same as in the proof of Corollary 4.8. We apply Claim 4.10 with $\kappa = r_2^{-1/2}$ and $\mathcal{L}^3(S) \simeq \kappa$, and see that wave packets that are contained in S_2 point in at most

$$(4.103) \quad \min \left\{ (N_2)^{k-2} r_2^{-1/2}, \left(r_2^{-1/2} + \frac{r_2}{N} \right) \right\} (r_2)^{1+\delta_0}$$

many different directions; here we applied [YC04, Corollary 5.7] to bound the Lebesgue measure term on right hand side of (4.102). This finishes the proof of the proposition. \square

Now we have all the tools to finish the proof of Theorem 4.2. The starting point is again to apply (4.38) and (4.39), which we write down again:

$$(4.104) \quad \begin{aligned} \|T^N f\|_{\text{BL}_{A_0}^p(\mathbb{B}_R^3)} &\lesssim R^{C_p \delta_0} (r_2 D_2)^{\frac{1}{2}(1-\beta_2)} D_1^{\frac{\beta_2}{p_2}} D_2^{\frac{\beta_2}{p_2}} \|f\|_2^{\frac{2}{p_3}} \max_{O \in \mathbf{n}_{\ell_0}^*} \|f_{\iota, O}\|_2^{1-\frac{2}{p_3}} \\ &\lesssim R^{C_p \delta_0} (r_2 D_2)^{\frac{1}{2}(1-\beta_2)} D_1^{\frac{\beta_2}{p_2}} D_2^{\frac{\beta_2}{p_2}} \|f\|_2^{\frac{2}{p_3}} r_2^{-\frac{1}{2}(\frac{1}{2}-\frac{1}{p_3})} D_1^{-(\frac{1}{2}-\frac{1}{p_3})} \max_{S_2 \in \mathfrak{S}_2} \|f_{\iota, S_2}^*\|_2^{1-\frac{2}{p_3}}. \end{aligned}$$

To control the right hand side, recall that in (4.35), we already proved

$$(4.105) \quad \max_{S_2 \in \mathfrak{S}_2} \|f_{\iota, S_2}^*\|_2^2 \lesssim R^{C_p \delta_0} D_2^{-2} \|f\|_2^2 \lesssim R^{C_p \delta_0} D_2^{-2} \|f\|_\infty^2,$$

where in the last step we used Hölder's inequality. Moreover, Proposition 4.9 says that

$$(4.106) \quad \|f_{\iota, S_2}^*\|_2^2 \lesssim \min \left\{ \left(\frac{N}{r_2} \right)^{k-2} r_2^{-1/2}, \left(r_2^{-1/2} + \frac{r_2}{N} \right) \right\} r_2^{\delta_0} \|f\|_\infty^2,$$

uniformly in S_2 . Before we continue, let us simplify the right hand side of (4.106). We discuss two cases

$$(4.107) \quad r_2 \leq N^{2/3}, \text{ or } r_2 \geq N^{2/3}.$$

In the former case, we have

$$(4.108) \quad \|f_{\iota, S_2}^*\|_2^2 \lesssim r_2^{-1/2} r_2^{\delta_0} \|f\|_\infty^2.$$

In the latter case, we have

$$(4.109) \quad \|f_{\iota, S_2}^*\|_2^2 \lesssim \min \left\{ \left(\frac{N}{r_2} \right)^{k-2} r_2^{-1/2}, \frac{r_2}{N} \right\} r_2^{\delta_0} \|f\|_\infty^2 \lesssim r_2^{-\frac{1}{2(k-1)} + \delta_0} \|f\|_\infty^2.$$

Therefore, we always have

$$(4.110) \quad \|f_{\iota, S_2}^*\|_2^2 \lesssim r_2^{-\frac{1}{2(k-1)} + \delta_0} \|f\|_\infty^2.$$

Let $\gamma \in [0, 1]$ to be determined. We take a geometric average of (4.105) and (4.110), and obtain

$$(4.111) \quad \max_{S_2 \in \mathfrak{S}_2} \|f_{\iota, S_2}^*\|_2^2 \lesssim R^{C_p \delta_0} r_2^{-\frac{1}{2(k-1)}(1-\gamma)} D_2^{-2\gamma} \|f\|_\infty^2.$$

We combine (4.104) and (4.111), and obtain

$$(4.112) \quad \|T^N f\|_{\text{BL}_{A_0}^p(\mathbb{B}_R^3)} \lesssim R^{C_p \delta_0} (r_2 D_2)^{\frac{1}{2}(1-\beta_2)} D_1^{\frac{\beta_2}{p_2}} D_2^{\frac{\beta_2}{p_2}} r_2^{-\frac{1}{2}(\frac{1}{2}-\frac{1}{p})} D_1^{-(\frac{1}{2}-\frac{1}{p})} \\ \times r_2^{-\frac{1}{2(k-1)}(1-\gamma)(\frac{1}{2}-\frac{1}{p})} D_2^{-2\gamma(\frac{1}{2}-\frac{1}{p})} \|f\|_2^{\frac{2}{p}} \|f\|_\infty^{1-\frac{2}{p}}.$$

Here we unify the notation $N = R, p_3 = p$. We simplify coefficient on the the right hand side of (4.112), and write it as

(4.113)

$$\begin{aligned} & R^{C_p \delta_0} (r_2 D_2)^{\frac{1}{2}(1-\beta_2)} D_1^{\frac{\beta_2}{p^2}} D_2^{\frac{\beta_2}{p^2}} r_2^{-\frac{1}{2}(\frac{1}{2}-\frac{1}{p})} D_1^{-(\frac{1}{2}-\frac{1}{p})} \\ & \quad \times r_2^{-\frac{1}{2(k-1)}(1-\gamma)(\frac{1}{2}-\frac{1}{p})} D_2^{-2\gamma(\frac{1}{2}-\frac{1}{p})} \\ & \lesssim R^{C_p \delta_0} (r_2 D_2)^{\frac{1}{2}(1-\beta_2)} D_1^{\frac{\beta_2}{2}-2(\frac{1}{2}-\frac{1}{p})} D_2^{\frac{\beta_2}{2}-(1+2\gamma)(\frac{1}{2}-\frac{1}{p})} r_2^{-(\frac{k}{2(k-1)}-\frac{\gamma}{2(k-1)})(\frac{1}{2}-\frac{1}{p})} \\ & \lesssim R^{C_p \delta_0} (r_2)^{\frac{1}{2}(1-\beta_2)} D_1^{\frac{\beta_2}{2}-2(\frac{1}{2}-\frac{1}{p})} D_2^{\frac{1}{2}-(1+2\gamma)(\frac{1}{2}-\frac{1}{p})} r_2^{-(\frac{k}{2(k-1)}-\frac{\gamma}{2(k-1)})(\frac{1}{2}-\frac{1}{p})}. \end{aligned}$$

Recall that $\beta_2 \in [0, 1]$ is a free parameter we can choose. We give it the constraint that

$$(4.114) \quad \frac{\beta_2}{2} - 2\left(\frac{1}{2} - \frac{1}{p}\right) \geq 0.$$

Under this constraint, we apply (4.31) and obtain

$$(4.115) \quad (4.113) \leq R^{C_p \delta_0} r_2^{\frac{1}{2}-2(\frac{1}{2}-\frac{1}{p})} D_2^{\frac{1}{2}-(1+2\gamma)(\frac{1}{2}-\frac{1}{p})} r_2^{-(\frac{k}{2(k-1)}-\frac{\gamma}{2(k-1)})(\frac{1}{2}-\frac{1}{p})}.$$

By letting the exponents of r_2 and D_2 be zero, we obtain

$$(4.116) \quad p = 3 + \frac{k-1}{3k-2}, \quad \gamma = \frac{3k-2}{4k-3}.$$

This finishes the proof of Theorem 4.2.

5. Proofs of Theorem 2.1 and Theorem 2.2

In this section, we will prove Theorem 2.1 and Theorem 2.2.

5.1. Preliminaries in Riemannian geometry II

Let $\epsilon_{\mathcal{M}} > 0$ be a small constant depending on \mathcal{M} . Fix $\epsilon \in (0, \epsilon_{\mathcal{M}})$. In Section 3.2, we studied the distance function

$$(5.1) \quad \phi_{\epsilon}(x, t; y) := \text{dist}((x, t), (y, \epsilon)).$$

We always consider (x, t) in a small neighborhood of the origin, and (y, ϵ) in a small neighborhood of $(0, \epsilon)$. As a preparation for the proofs of Theorem 2.1 and Theorem 2.2, we will collect more properties of this function.

For given $\mathbf{x}_0 = (x_0, t_0)$ and y_0 , we let $\tilde{\gamma} : [0, \tilde{L}] \rightarrow \mathcal{M}$ be the arc-length parametrized geodesic satisfying $\tilde{\gamma}(0) = (x_0, t_0)$ and $\tilde{\gamma}(\tilde{L}) = (y_0, \epsilon)$. Define

$$(5.2) \quad \vec{V}(x_0, t_0; y_0) := \frac{d}{ds} \tilde{\gamma}(0).$$

Lemma 5.1. *It holds that*

$$(5.3) \quad \vec{V}(x_0, t_0; y_0) \perp (\partial_{x_1} \partial_{y_i} \phi_\epsilon(x_0, t_0; y_0), \dots, \partial_{x_{n-1}} \partial_{y_i} \phi_\epsilon(x_0, t_0; y_0), \partial_t \partial_{y_i} \phi_\epsilon(x_0, t_0; y_0))$$

for every $i = 1, \dots, n-1$. Here \perp means perpendicular under the Euclidean inner product.

Proof of Lemma 5.1. This is a corollary of Lemma 3.2, from which we know that $\partial_{y_i} \phi_\epsilon(x, t; y_0)$ stays constant when (x, t) moves along $\tilde{\gamma}$. \square

Lemma 5.2. *For smooth functions $f : \mathcal{M} \rightarrow \mathbb{C}$, it holds that*

$$(5.4) \quad (\vec{V} \cdot \nabla_{\mathbf{x}}) f \Big|_{\substack{\mathbf{x}=(x_0, t_0) \\ y=y_0}} = \frac{d}{ds} (f(\tilde{\gamma}(s))) \Big|_{s=0}$$

and

$$(5.5) \quad (\vec{V} \cdot \nabla_{\mathbf{x}})^2 f \Big|_{\substack{\mathbf{x}=(x_0, t_0) \\ y=y_0}} = \frac{d^2}{ds^2} (f(\tilde{\gamma}(s))) \Big|_{s=0},$$

where if we write

$$(5.6) \quad \vec{V}(x, t; y) = (V_1(x, t; y), \dots, V_n(x, t; y)),$$

then

$$(5.7) \quad \vec{V} \cdot \nabla_{\mathbf{x}} := V_1(x, t; y) \partial_{x_1} + \dots + V_{n-1}(x, t; y) \partial_{x_{n-1}} + V_n(x, t; y) \partial_t,$$

and

$$(5.8) \quad (\vec{V} \cdot \nabla_{\mathbf{x}})^2 := \sum_{i,j} V_i V_j \partial_i \partial_j + \sum_{i,j} V_j (\partial_j V_i) \partial_i.$$

In the last equation, we used the convention that

$$(5.9) \quad \partial_i = \partial_{x_i}, \quad i = 1, \dots, n-1; \quad \partial_n = \partial_t,$$

the same as previously used, say in (3.30).

Proof of Lemma 5.2. We only prove (5.5). The right hand side of (5.5) equals

$$(5.10) \quad \frac{d}{ds} \left(\frac{d}{ds} (f(\tilde{\gamma}(s))) \right) \Big|_{s=0} = \frac{d}{ds} \left(\sum_i \partial_i f \frac{d\tilde{\gamma}_i(s)}{ds} \right) \Big|_{s=0},$$

where

$$(5.11) \quad \tilde{\gamma}(s) = (\tilde{\gamma}_1(s), \dots, \tilde{\gamma}_n(s)).$$

By (5.2), we have

$$(5.12) \quad (5.10) = \frac{d}{ds} \left(\sum_i (\partial_i f)(\tilde{\gamma}(s)) V_i(\tilde{\gamma}(s); y_0) \right) \Big|_{s=0},$$

which, by the chain rule, equals the left side of (5.5). This finishes the proof of the claim. \square

Before we proceed, let us state a corollary of Lemma 5.2. Recall from Definition 1.4 that the phase function $\phi_\epsilon(x, t; y)$ is said to satisfy Bourgain's condition at $(x_0, t_0; y_0)$ if

$$(5.13) \quad \left((G_0 \cdot \nabla_{\mathbf{x}})^2 \nabla_y^2 \phi_\epsilon \right) (\mathbf{x}_0; y_0) \text{ is a multiple of } ((G_0 \cdot \nabla_{\mathbf{x}}) \nabla_y^2 \phi_\epsilon) (\mathbf{x}_0; y_0),$$

where

$$(5.14) \quad G_0(\mathbf{x}; y) := \partial_{y_1} \nabla_{\mathbf{x}} \phi_\epsilon(\mathbf{x}; y) \wedge \dots \wedge \partial_{y_{n-1}} \nabla_{\mathbf{x}} \phi_\epsilon(\mathbf{x}; y),$$

and ∇_y^2 is the standard Euclidean Hessian. By Lemma 2.3 in [GWZ22], Bourgain's condition is invariant under multiplying G_0 by a non-zero scalar function which is allowed to depend on x_0, y_0 . This, combined with Lemma 5.2, gives the following equivalent form of Bourgain's condition for the specific phases under consideration.

Corollary 5.3. *Bourgain's condition holds for the phase ϕ_ϵ at the point $(x_0, t_0; y_0)$ if and only if*

$$(5.15) \quad \frac{d^2}{ds^2} (\partial_{y_i} \partial_{y_j} \phi_\epsilon(\tilde{\gamma}(s); y_0)) \Big|_{s=0} = C(\mathbf{x}_0; y_0) \frac{d}{ds} (\partial_{y_i} \partial_{y_j} \phi_\epsilon(\tilde{\gamma}(s); y_0)) \Big|_{s=0},$$

where $C(\mathbf{x}_0; y_0) \in \mathbb{R}$ is allowed to depend on \mathbf{x}_0 and y_0 , but not on $1 \leq i, j \leq n-1$.

Proof of Corollary 5.3. According to Lemma 5.1, we know that the vector field \vec{V} satisfies the definition (5.14) in Bourgain's condition. So, we can use (5.13) and both (5.4) and (5.5) to get the conclusion (5.15). \square

5.2. Relation between Carleson-Sjölin on manifolds and Hörmander's problems

In this subsection, we will prove item a) of Theorem 2.1. This is well-known. Moreover, we also know that Carleson-Sjölin on manifolds is elliptic, in the sense that it satisfies (H1) (H2), and if we write the relevant phase function in the normal form as in (1.7), then the matrix A is always positive definite.

The proof is very short, and therefore we will include the proof here. Recall that a reduced Carleson-Sjölin operator is given by

$$(5.16) \quad R_N^{(\mathcal{M}, \mathcal{M}')} f(\mathbf{x}) = \int_{\mathcal{M}'} e^{iN \text{dist}(\mathbf{x}, \mathbf{y})} a(\mathbf{x}) f(\mathbf{y}) d\mathcal{H}^{n-1}(\mathbf{y}),$$

where \mathcal{M}' is a submanifold of \mathcal{M} . Let us without loss of generality assume that \mathcal{M}' is given by $\{(y, \epsilon) : y \in \mathbb{R}^{n-1}\}$ for some $\epsilon \neq 0$, and that the smooth amplitude function $a(\mathbf{x})$ is supported in a small neighborhood of the origin. Our oscillatory integral becomes

$$(5.17) \quad \int_{\mathbb{R}^{n-1}} e^{iN \phi_\epsilon(x, t; y)} a(\mathbf{x}) f(y) dy,$$

where $\mathbf{x} = (x, t)$ and

$$(5.18) \quad \phi_\epsilon(x, t; y) := \text{dist}((x, t), (y, \epsilon)).$$

We need to show that

$$(5.19) \quad \text{rank} \nabla_x \nabla_y \phi_\epsilon(x_0, t_0; y_0) = n - 1,$$

and that

$$(5.20) \quad \det \nabla_y^2 \langle \nabla_{\mathbf{x}} \phi_\epsilon(\mathbf{x}_0; y), G_0(\mathbf{x}; y_0) \rangle \Big|_{y=y_0} \neq 0,$$

for all (x_0, t_0) near $(0, 0)$ and y_0 near 0, where G_0 is defined in (5.14). This follows immediately from continuity and the fact that (5.19) and (5.20) hold for the Euclidean distance function.

We remark here that by applying the tools from Riemannian geometry we introduced above, one can avoid the continuity argument and make the choice of ϵ more explicit. To keep our presentation short, we will not pursue this direction.

5.3. Relation between Carleson-Sjölin on manifolds and Nikodym on manifolds

The goal of this subsection is to prove item b) of Theorem 2.1. As mentioned below Theorem 2.1, the proof is essentially the same as the proof of Theorem 1.9; the only extra input is Lemma 3.2.

Let us be more precise. We continue to use the notation

$$(5.21) \quad \phi_\epsilon(x, t; y) := \text{dist}((x, t), (y, \epsilon)).$$

For the given phase function ϕ_ϵ , recall from Definition 1.6 that curved tubes in the curved Keakeya problem associated to ϕ_ϵ are given by

$$(5.22) \quad T_y^{\delta, (\phi_\epsilon)}(\mathbf{x}) := \left\{ \mathbf{x}' \in \mathbb{B}_{2\epsilon_{\phi_\epsilon}}^n : |\nabla_y \phi_\epsilon(\mathbf{x}'; y) - \nabla_y \phi_\epsilon(\mathbf{x}; y)| < \delta \right\}.$$

However, Lemma 3.2, together with Hörmander's non-degeneracy conditions (H1) and (H2) at the very beginning of the paper, says precisely that this is the δ -neighborhood of a geodesic passing through (y, ϵ) . The Nikodym maximal function $\mathcal{N}_{\delta, \lambda} f(y, \epsilon)$ is essentially the curved Keakeya maximal function $\mathcal{K}_\delta^{(\phi_\epsilon)} f(y)$. Here we need to assume that $\lambda < 1$ just to avoid the singularities of the distance function along the diagonals. We refer the rest of the details to Wisewell's thesis [Wis03, page 26].

Before finishing this subsection, let us make a remark that the Nikodym maximal function bound we can deduce here is stronger than what we need for $\mathcal{N}_{\delta, \lambda}$. More precisely, the Nikodym maximal function bound concerns the L^p norm on $\mathbb{B}_{\epsilon_{\mathcal{M}/2}}^n$, which is an integral over an n -dimensional object. However, the bound we can deduce from the argument above concerns the L^p norm on each hyperplane \mathcal{M}' , which is of course much stronger because of Fubini's theorem. This also explains one key difference between Nikodym maximal operators and curved Keakeya maximal operators.

5.4. Proof of Theorem 2.2: Part a)

Let us without loss of generality assume that \mathbf{x} is near $0 \in \mathbb{R}^n$ and \mathbf{y} is near $(0, \epsilon)$ where $0 \in \mathbb{R}^{n-1}$ and $\epsilon > 0$ is a fixed small number. What we need to

prove is

$$(5.23) \quad \left\| \int_{\mathbb{R}^{n-1}} e^{iN\phi_\epsilon(x,t;y)} a(\mathbf{x}; y) g(y) dy \right\|_{L^p(\mathbb{R}^n)} \lesssim_{\mathcal{M}, a, \epsilon'} N^{-\frac{n}{p} + \epsilon'} \|g\|_{L^p(\mathbb{R}^{n-1})},$$

for every $\epsilon' > 0$, where

$$(5.24) \quad \phi_\epsilon(x, t; y) := \text{dist}((x, t), (y, \epsilon)),$$

and $a(\mathbf{x}; y)$ is a smooth bump function with $\mathbf{x} = (x, t)$ supported near $0 \in \mathbb{R}^n$ and y supported near $0 \in \mathbb{R}^{n-1}$. To prove (5.23), by [GWZ22, Theorem 1.3], it suffices to prove

Lemma 5.4. *The phase function $\phi_\epsilon(x, t; y)$ satisfies Bourgain's condition everywhere if \mathcal{M} has constant sectional curvature.*

Proof of Lemma 5.4. Take two points $\mathbf{x}_0 = (x_0, t_0)$ and (y_0, ϵ) on the manifold, and we would like to check Bourgain's condition. By Corollary 5.3, it suffices to prove that

$$(5.25) \quad \frac{d^2}{ds^2} (\partial_{y_i} \partial_{y_j} \phi_\epsilon(\tilde{\gamma}(s); y_0))|_{s=0} = C(\mathbf{x}_0; y_0) \frac{d}{ds} (\partial_{y_i} \partial_{y_j} \phi_\epsilon(\tilde{\gamma}(s); y_0))|_{s=0},$$

where $\tilde{\gamma}$ is the arc-length parametrized geodesic connecting (x_0, t_0) and (y_0, ϵ) , and $C(\mathbf{x}_0; y_0) \in \mathbb{R}$ is allowed to depend on \mathbf{x}_0 and y_0 , but not on $1 \leq i, j \leq n-1$.

Denote

$$(5.26) \quad \Phi(x, t; y, \tau) := \text{dist}((x, t), (y, \tau)).$$

We would like to connect $\partial_{y_i} \partial_{y_j} \phi_\epsilon$ with the covariant Hessian of Φ . Without loss of generality, assume that $(x_0, t_0) = (0, 0)$ and $y_0 = 0$. Note that in [GWZ22, Corollary 2.2], it is proved that Bourgain's condition is independent of the choice of coordinates. Therefore, we can for the sake of simplicity assume that we are in the same setting as in Subsection 3.2, that is, we are in the Fermi coordinate based on the geodesic

$$(5.27) \quad \gamma(s) = (0, s), \quad \forall s \in [0, \epsilon],$$

and if we define

$$(5.28) \quad E_i(s) := \frac{\partial}{\partial y_i} \in T_{\gamma(s)} \mathcal{M}, \quad i = 1, \dots, n-1, \quad E_n(s) := \frac{\partial}{\partial \tau} \in T_{\gamma(s)} \mathcal{M},$$

then $\{E_i(s)\}_{i=1}^n$ forms an orthonormal basis for $T_{\gamma(s)}\mathcal{M}$.

Recall the notation from (3.31)–(3.34). We apply Claim 3.4 and obtain

$$(5.29) \quad \frac{\partial}{\partial y_i} \frac{\partial}{\partial y_j} \phi_{0,\epsilon} \Big|_{\substack{(x,t)=\gamma(s), \\ y=0}} = \left(\text{Hessian } \Phi_0 \right) \Big|_{\substack{(x,t)=\gamma(s), \\ (y,\tau)=(0,\epsilon)}} \left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} \right)$$

for every $s \in [0, \epsilon)$, $1 \leq i, j \leq n-1$. Here Hessian is the covariant Hessian in the (y, τ) variables. In (5.29), we connected $\partial_{y_i} \partial_{y_j} \phi_{0,\epsilon}$ with the covariant Hessian of Φ_0 , but not $\partial_{y_i} \partial_{y_j} \phi_\epsilon$ with the covariant Hessian of Φ . However, by taking derivatives $\nabla_{\dot{\gamma}}$ on both sides of (5.29), we immediately obtain

$$(5.30) \quad \frac{\partial^\iota}{\partial s^\iota} \frac{\partial}{\partial y_i} \frac{\partial}{\partial y_j} \phi_\epsilon(\gamma(s); 0) = \frac{\partial^\iota}{\partial s^\iota} \left(\text{Hessian } \Phi(\gamma(s); (0, \epsilon)) \right) \left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} \right)$$

for every $\iota \in \mathbb{N}$, $\iota \geq 1$ and every $1 \leq i, j \leq n-1$, $s \in [0, \epsilon)$. Therefore, to check Bourgain's condition, it is equivalent to check that

$$(5.31) \quad \begin{aligned} & \frac{\partial^2}{\partial s^2} \left(\text{Hessian } \Phi(\gamma(s); (0, \epsilon)) \right) \left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} \right) \Big|_{s=0} \\ &= C \frac{\partial}{\partial s} \left(\text{Hessian } \Phi(\gamma(s); (0, \epsilon)) \right) \left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} \right) \Big|_{s=0}, \end{aligned}$$

for every $1 \leq i, j \leq n-1$. Recall the second equation in Claim 3.4 that

$$(5.32) \quad \left(\text{Hessian } \Phi_0 \right) \Big|_{\substack{(x,t)=\gamma(s), \\ (y,\tau)=(0,\epsilon)}} \left(Y, \frac{\partial}{\partial \tau} \right) = 0, \quad \forall Y \in T_{(0,\epsilon)}\mathcal{M},$$

for every $s \in [0, \epsilon)$, $1 \leq i \leq n-1$. If we identify $\partial/\partial y_n$ with $\partial/\partial \tau$, then Bourgain's condition is equivalent to saying that (5.31) holds for $1 \leq i, j \leq n$. Moreover, because $\{E_i(s)\}_{i=1}^n$ is a parallel orthonormal frame, we can say that Bourgain's condition is equivalent to

$$(5.33) \quad \nabla_{\dot{\gamma}}^2 \text{Hessian } \Phi \Big|_{\substack{(x,t)=(0,0) \\ (y,\tau)=(0,\epsilon)}} = C \nabla_{\dot{\gamma}} \text{Hessian } \Phi \Big|_{\substack{(x,t)=(0,0) \\ (y,\tau)=(0,\epsilon)}}$$

for some $C \in \mathbb{R}$.

To prove this, the first few steps are the same as in those in Subsection 3.2. Define Jacobi fields (previously defined in (3.67))

$$(5.34) \quad \begin{aligned} & \nabla_{\dot{\gamma}(s')}^2 X_j(s, s') + R(X_j(s, s'), \dot{\gamma}(s')) \dot{\gamma}(s') = 0, \\ & X_j(s, s) = 0, \quad X_j(s, \epsilon) = E_j(\epsilon). \end{aligned}$$

By Claim 3.6 and Lemma 3.7, we obtain (see (3.78))

$$(5.35) \quad \text{Hessian } \Phi|_{\gamma(s)} \left(E_i(\epsilon), E_j(\epsilon) \right) = g \left(\nabla_{\dot{\gamma}(s')} X_i(s, s'), X_j(s, s') \right) \Big|_{s'=\epsilon}.$$

Define $a_{ji}(s, s')$ by

$$(5.36) \quad X_j(s, s') = \sum_{i=1}^{n-1} a_{ji}(s, s') E_i(s'), \quad j = 1, \dots, n-1.$$

Denote

$$(5.37) \quad A(s, s') = [a_{ij}(s, s')]_{1 \leq i, j \leq n-1}.$$

Then the Jacobi fields (5.34) can be written as

$$(5.38) \quad \begin{aligned} \partial_{s'}^2 A(s, s') + A(s, s') R(s') &= 0, \\ A(s, s) &= 0, \quad A(s, \epsilon) = I_{(n-1) \times (n-1)}, \end{aligned}$$

where

$$(5.39) \quad R(s') := [R_{innj}(s')]_{1 \leq i, j \leq n-1}$$

and

$$(5.40) \quad R_{ijkl}(s') := g(R(E_i(s'), E_j(s')) E_k(s'), E_l(s')).$$

Moreover, as calculated in (3.87), we have

$$(5.41) \quad (5.35) = \partial_{s'} a_{ij} \Big|_{s'=\epsilon}.$$

Recall that Bourgain's condition is equivalent to (5.31), and now it is further equivalent to

$$(5.42) \quad \partial_s^2 \partial_{s'} A \Big|_{s=0, s'=\epsilon} = C \partial_s \partial_{s'} A \Big|_{s=0, s'=\epsilon},$$

for some $C \in \mathbb{R}$.

Let $\kappa \in \mathbb{R}$ be the sectional curvature of \mathcal{M} . As the sectional curvature of \mathcal{M} is constant, we have (see for instance [Pet16, page 84])

$$(5.43) \quad R(w, v)v = \kappa(w - g(w, v)v),$$

where $g(v, v) = 1$. Therefore $R(s') = \kappa I_{(n-1) \times (n-1)}$. We solve (5.38) explicitly, and obtain

$$(5.44) \quad A(s, s') = A_0^{-1}(\epsilon - s)A_0(s' - s),$$

where

$$(5.45) \quad A_0(r) := \begin{cases} \frac{\sin \sqrt{\kappa} r}{\sqrt{\kappa}} I_{(n-1) \times (n-1)} & \text{if } \kappa > 0, \\ r I_{(n-1) \times (n-1)} & \text{if } \kappa = 0, \\ \frac{\sinh \sqrt{-\kappa} r}{\sqrt{-\kappa}} I_{(n-1) \times (n-1)} & \text{if } \kappa < 0. \end{cases}$$

Compute

$$(5.46) \quad \partial_{s'} A|_{s'=\epsilon} = \begin{cases} \frac{\sqrt{\kappa} \cos(\sqrt{\kappa}(\epsilon-s))}{\sin(\sqrt{\kappa}(\epsilon-s))} I_{(n-1) \times (n-1)} & \text{if } \kappa > 0, \\ \frac{1}{\epsilon-s} I_{(n-1) \times (n-1)} & \text{if } \kappa = 0, \\ \frac{\sqrt{-\kappa} \cosh(\sqrt{-\kappa}(\epsilon-s))}{\sinh(\sqrt{-\kappa}(\epsilon-s))} I_{(n-1) \times (n-1)} & \text{if } \kappa < 0. \end{cases}$$

From this, one can see that (5.42) holds. This finishes the proof of the lemma. \square

5.5. Proof of Theorem 2.2: Part b)

By [GWZ22, Theorem 1.1], it suffices to prove that Bourgain's condition fails at least at one point, if the sectional curvature of \mathcal{M} is not constant. We will argue by contradiction, assume that Bourgain's condition holds at every point, and then derive that the sectional curvature must be constant.

Before we start the proof, let us mention that in the subsection we will need Claim 3.8 for manifolds of a general dimension n ; however in the proof of this claim, the dimension parameter actually does not appear explicitly, and the same argument works for all dimensions n .

Recall the Jacobi fields (5.38), and that Bourgain's condition is equivalent to (see (5.42))

$$(5.47) \quad \partial_s^2 \partial_{s'} A|_{s=0, s'=\epsilon} = C \partial_s \partial_{s'} A|_{s=0, s'=\epsilon},$$

for some $C \in \mathbb{R}$. To study (5.38), we follow the strategy in the proof of Claim 3.8, and introduce the following two systems of equations (see (3.113))

and (3.114))

$$(5.48) \quad \begin{aligned} B_1''(s') + B_1(s')R(s') &= 0_{(n-1) \times (n-1)}, \\ B_1(0) &= I_{(n-1) \times (n-1)}, \quad B_1'(0) = 0_{(n-1) \times (n-1)}, \end{aligned}$$

and

$$(5.49) \quad \begin{aligned} B_2''(s') + B_2(s')R(s') &= 0_{(n-1) \times (n-1)}, \\ B_2(0) &= 0_{(n-1) \times (n-1)}, \quad B_2'(0) = I_{(n-1) \times (n-1)}. \end{aligned}$$

Below we abbreviate $0_{(n-1) \times (n-1)}$ to 0, and $I_{(n-1) \times (n-1)}$ to I . Recall that in (3.125), we obtained that

$$(5.50) \quad \partial_s^2 \partial_{s'} A(0, \epsilon) = 2B_2^{-1}(\epsilon)B_1(\epsilon)\partial_s \partial_{s'} A(0, \epsilon).$$

As we assume that Bourgain's condition holds everywhere, we can find a scalar function $C(\epsilon)$ such that

$$(5.51) \quad B_1(\epsilon) = C(\epsilon)B_2(\epsilon).$$

By taking the second order derivative in (5.51), we obtain

$$(5.52) \quad B_1''(\epsilon) = C''(\epsilon)B_2(\epsilon) + 2C'(\epsilon)B_2'(\epsilon) + C(\epsilon)B_2''(\epsilon).$$

This, combined with the first equation in (5.48), implies that

$$(5.53) \quad C''(\epsilon)B_2(\epsilon) + 2C'(\epsilon)B_2'(\epsilon) = 0.$$

By taking a further derivative in ϵ , we obtain

$$(5.54) \quad C'''(\epsilon)B_2(\epsilon) + 3C''(\epsilon)B_2'(\epsilon) + 2C'(\epsilon)B_2''(\epsilon) = 0.$$

By (5.53) and the first equation in (5.49), we obtain

$$(5.55) \quad C'(\epsilon)C'''(\epsilon)B_2(\epsilon) - \frac{3}{2}(C''(\epsilon))^2B_2(\epsilon) - 2(C'(\epsilon))^2R(\epsilon)B_2(\epsilon) = 0.$$

Because of the initial condition in (5.49), we see that $B_2(\epsilon)$ is always invertible, whenever $\epsilon > 0$ is taken to be small enough. Consequently, (5.55) is equivalent to

$$(5.56) \quad C'(\epsilon)C'''(\epsilon)I - \frac{3}{2}(C''(\epsilon))^2I - 2(C'(\epsilon))^2R(\epsilon) = 0.$$

Recall in (3.119), we obtained

$$(5.57) \quad B_2^{-1}(\epsilon)B_1(\epsilon) = \frac{1}{\epsilon} \left(I - \frac{R_0}{3}\epsilon^2 - \frac{R'_0}{12}\epsilon^3 + O(\epsilon^4) \right),$$

where $R_0 := R(0)$, $R'_0 := R'(0)$ and the implicit constant in $O(\epsilon)$ depends only on the manifold. This, combined with the definition of $C(\epsilon)$ in (5.51), implies that $C'(\epsilon) \neq 0$ whenever $\epsilon \neq 0$ is taken small enough. Therefore, (5.55) can be further written as

$$(5.58) \quad R(\epsilon) = \left(\frac{C'''(\epsilon)}{2C'(\epsilon)} - \frac{3(C'''(\epsilon))^2}{4(C'(\epsilon))^2} \right) I =: \kappa(\epsilon)I.$$

For two given tangent vectors $E_i(\epsilon), E_n(\epsilon)$ with $1 \leq i \leq n-1$, the sectional curvature at $\gamma(\epsilon)$ associated to these two tangent vectors is

$$(5.59) \quad R(E_i(\epsilon), E_n(\epsilon), E_n(\epsilon), E_i(\epsilon)) = \kappa(\epsilon),$$

which is independent of $1 \leq i \leq n-1$.

Recall that $\{E_i(\epsilon)\}_{i=1}^n$ forms an orthonormal basis for $T_{\gamma(\epsilon)}\mathcal{M}$. So far we have proven that the sectional curvature at $\gamma(\epsilon)$ associated to the pair of vectors $E_i(\epsilon), E_n(\epsilon)$ with $1 \leq i \leq n-1$ is independent of i . Let us write it as $\kappa(\gamma(\epsilon), n)$. The index n plays a special role here because the geodesic γ is chosen such that $\dot{\gamma}(\epsilon) = E_n(\epsilon)$. Now we consider all possible geodesics passing through the point $\gamma(\epsilon)$. Similarly, we will obtain that

$$(5.60) \quad R(E_i(\epsilon), E_{n'}(\epsilon), E_{n'}(\epsilon), E_i(\epsilon)) = \kappa(\gamma(\epsilon), n'),$$

where $\kappa(\gamma(\epsilon), n') \in \mathbb{R}$ is some constant that is independent of $i \neq n'$. By basic symmetries of Riemannian tensors, we can conclude that

$$(5.61) \quad R(E_i(\epsilon), E_j(\epsilon), E_j(\epsilon), E_i(\epsilon))$$

is the same for all choices of $i \neq j$. By Schur's lemma, this constant must also be independent of the choice of the point $\gamma(\epsilon)$, that is, \mathcal{M} must have constant sectional curvature.

5.6. Proof of Theorem 2.2: Part c)

To prove the λ -Nikodym maximal function bound in part c) of Theorem 2.2, one just need to repeat the argument in Hickman, Roger and Zhang [HRZ22],

similarly to how one proves the bound (2.6) for curved Keakeya maximal functions associated to phase functions satisfying Bourgain's condition. Lemma 5.4 guarantees that the relevant phase functions satisfy Bourgain's condition, which further guarantees that we have (strong) polynomial Wolff axioms as required by [HRZ22], and Subsection 5.3 explains the connection between Nikodym maximal functions and curved Keakeya maximal functions.

The Minkowski dimension bound in part c) of Theorem 2.2 follows from standard argument connecting Nikodym maximal function bounds and Minkowski dimensions of Nikodym sets, see for instance [Sog99, Corollary 2.2].

Appendix A. More connections between curved Keakeya problems and Nikodym problems on manifolds

In this section, we will show that not every curved Keakeya problem can be viewed as a Nikodym problem on manifolds. Let us be more precise. We will find a phase function $\phi(x, t; \xi) : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfying Hörmander's non-degeneracy condition, and show that no matter how we pick Riemannian metric tensor $\{g_{ij}(x, t)\}_{1 \leq i, j \leq 3}$ on $\mathbb{R}^2 \times \mathbb{R}$, the curves

$$(A.1) \quad \{(x, t) : \nabla_\xi \phi(x, t; \xi) = \omega\},$$

where ξ, ω are parameters, are never geodesics.

Let us take a phase function $\phi(x, t; \xi)$ satisfying Hörmander's nondegeneracy condition that will be picked later. Let us assume that we can find a Riemannian metric tensor $\{g_{ij}(x, t)\}_{1 \leq i, j \leq 3}$ such that

$$(A.2) \quad \{(x, t) \in \mathbb{B}_{2\epsilon_\phi}^3 : \nabla_\xi \phi(x, t; \xi) = \omega\}$$

is a geodesic for every $\xi \in \mathbb{B}_{\epsilon_\phi}^2$ and $\omega \in \mathbb{B}_{\epsilon_\phi}^2$. Here $\epsilon_\phi > 0$ is a small real number that depends on ϕ .

Fix (x_0, t_0) . We write a geodesic passing through this point as $(X_\xi(t), t)$ where

$$(A.3) \quad (\nabla_\xi \phi)(X_\xi(t), t; \xi) = (\nabla_\xi \phi)(x_0, t_0; \xi), \quad X_\xi(t_0) = x_0.$$

We compute $\frac{\partial}{\partial t} X_\xi(t)$, and obtain the tangent vector at the starting point. To do so, we take the derivative of (A.3) in t , and obtain

$$(A.4) \quad \nabla_x \nabla_\xi \phi(X_\xi(t), t; \xi) \cdot \frac{\partial}{\partial t} X_\xi(t) + \partial_t \nabla_\xi \phi(X_\xi(t), t; \xi) = 0.$$

Therefore,

$$(A.5) \quad \frac{\partial X_\xi(t)}{\partial t} = (\nabla_x \nabla_\xi \phi)^{-1} \cdot \partial_t \nabla_\xi \phi,$$

and the tangent vector of the geodesic (A.3) is parallel to

$$(A.6) \quad \begin{bmatrix} (\nabla_x \nabla_\xi \phi)^{-1} \cdot \partial_t \nabla_\xi \phi \\ 1 \end{bmatrix}.$$

We take a further derivative in t on both sides of (A.4), and obtain

$$(A.7) \quad \begin{aligned} \nabla_x \nabla_\xi \phi \cdot \frac{\partial^2}{\partial t^2} X_\xi(t) + \left(\frac{\partial}{\partial t} X_\xi(t) \right)^T \cdot \nabla_x^2 \nabla_\xi \phi \cdot \frac{\partial}{\partial t} X_\xi(t) \\ + 2 \partial_t \nabla_x \nabla_\xi \phi \cdot \frac{\partial}{\partial t} X_\xi(t) + \partial_t^2 \nabla_\xi \phi = 0. \end{aligned}$$

By moving terms, we obtain

$$(A.8) \quad \begin{aligned} \frac{\partial^2}{\partial t^2} X_\xi(t) = 2(\nabla_x \nabla_\xi \phi)^{-1} \partial_t \nabla_x \nabla_\xi \phi \cdot (\nabla_x \nabla_\xi \phi)^{-1} \partial_t \nabla_\xi \phi + (\nabla_x \nabla_\xi \phi)^{-1} \partial_t^2 \nabla_\xi \phi \\ + (\nabla_x \nabla_\xi \phi)^{-1} \cdot \left(\left(\frac{\partial}{\partial t} X_\xi(t) \right)^T \cdot \nabla_x^2 \partial_{\xi_i} \phi \cdot \frac{\partial}{\partial t} X_\xi(t) \right)^T_{1 \leq i \leq n-1}. \end{aligned}$$

To simplify notations, we will write $x = (x_1, x_2)$ and treat t as the third spatial variable; we will use $\Gamma_{\alpha\beta}^\mu$, $1 \leq \alpha, \beta, \mu \leq 3$ for Christoffel symbols. If we parametrize geodesics by using the third spatial variable t , that is, if

$$(A.9) \quad (x^1(t), x^2(t), x^3(t))$$

is a geodesic with $x^3(t) = t$, then

$$(A.10) \quad \frac{d^2 x^\mu}{dt^2} = - \sum_{\alpha, \beta} \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} + \sum_{\alpha, \beta} \Gamma_{\alpha\beta}^3 \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} \frac{dx^\mu}{dt},$$

where $\mu = 1, 2, 3$. We therefore have

$$(A.11) \quad \begin{bmatrix} \frac{\partial^2 X_\xi(t)}{\partial t^2} \\ 0 \end{bmatrix} + \begin{bmatrix} \left(\left(\frac{\partial X_\xi(t)}{\partial t} \right)^T, 1 \right) \cdot [\Gamma_{bc}^1]_{1 \leq b, c \leq 3} \cdot \left(\left(\frac{\partial X_\xi(t)}{\partial t} \right)^T, 1 \right)^T \\ \left(\left(\frac{\partial X_\xi(t)}{\partial t} \right)^T, 1 \right) \cdot [\Gamma_{bc}^2]_{1 \leq b, c \leq 3} \cdot \left(\left(\frac{\partial X_\xi(t)}{\partial t} \right)^T, 1 \right)^T \\ 0 \end{bmatrix}$$

$$- \left(\left(\frac{\partial X_\xi(t)}{\partial t} \right)^T, 1 \right) \cdot [\Gamma_{bc}^3]_{1 \leq b, c \leq 3} \cdot \left(\left(\frac{\partial X_\xi(t)}{\partial t} \right)^T, 1 \right)^T \begin{bmatrix} \frac{\partial X_\xi(t)}{\partial t} \\ 0 \end{bmatrix} = 0, \quad \forall t.$$

Recall how to pass from metrics to Christoffel symbols:

$$(A.12) \quad \partial_i g_{jk} = \sum_l \Gamma_{ij}^l g_{lk} + \sum_l \Gamma_{ik}^l g_{jl},$$

or

$$(A.13) \quad \Gamma_{kl}^i = \frac{1}{2} \sum_m g^{im} \left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right).$$

The first identity (A.12) can be written as

$$(A.14) \quad \partial_i \mathbf{g} = \begin{bmatrix} \Gamma_{i1}^1 & \Gamma_{i1}^2 & \Gamma_{i1}^3 \\ \Gamma_{i2}^1 & \Gamma_{i2}^2 & \Gamma_{i2}^3 \\ \Gamma_{i3}^1 & \Gamma_{i3}^2 & \Gamma_{i3}^3 \end{bmatrix} \cdot \mathbf{g} + \mathbf{g} \cdot \begin{bmatrix} \Gamma_{i1}^1 & \Gamma_{i2}^1 & \Gamma_{i3}^1 \\ \Gamma_{i1}^2 & \Gamma_{i2}^2 & \Gamma_{i3}^2 \\ \Gamma_{i1}^3 & \Gamma_{i2}^3 & \Gamma_{i3}^3 \end{bmatrix},$$

where

$$(A.15) \quad \mathbf{g} := (g_{ij})_{1 \leq i, j \leq 3}.$$

We now start constructing the phase function ϕ . We start with the phase function

$$(A.16) \quad \phi_0(x, t; \xi) = \langle x, \xi \rangle + t\xi_1\xi_2 + \frac{1}{2}t^2\xi_1^2,$$

which was constructed by Bourgain [Bou91]. Let us take a point (z_1, z_2, τ) and compute the Christoffel symbols at this point.

We first compute all the geodesics passing through this point. Suppose

$$(A.17) \quad \begin{aligned} x_1 + t\xi_2 + t^2\xi_1 &= w_1, \\ x_2 + t\xi_1 &= w_2. \end{aligned}$$

Then we see that

$$(A.18) \quad \begin{aligned} X_1(t) &= z_1 + \tau\xi_2 + \tau^2\xi_1 - t\xi_2 - t^2\xi_1, \\ X_2(t) &= z_2 + \tau\xi_1 - t\xi_1. \end{aligned}$$

All geodesics passing through (z_1, z_2, τ) can be written as

$$(A.19) \quad (X_1(t), X_2(t), t).$$

By the equation (A.10), we obtain

$$(A.20) \quad \begin{aligned} -2\xi_1 = & -(-\xi_2 - 2\tau\xi_1, -\xi_1, 1)(\Gamma_{\alpha\beta}^1)_{1 \leq \alpha, \beta \leq 3}(-\xi_2 - 2\tau\xi_1, -\xi_1, 1)^T \\ & + (-\xi_2 - 2\tau\xi_1)(-\xi_2 - 2\tau\xi_1, -\xi_1, 1)(\Gamma_{\alpha\beta}^3)_{1 \leq \alpha, \beta \leq 3}(-\xi_2 - 2\tau\xi_1, -\xi_1, 1)^T, \end{aligned}$$

and

$$(A.21) \quad \begin{aligned} 0 = & -(-\xi_2 - 2\tau\xi_1, -\xi_1, 1)(\Gamma_{\alpha\beta}^2)_{1 \leq \alpha, \beta \leq 3}(-\xi_2 - 2\tau\xi_1, -\xi_1, 1)^T \\ & + (-\xi_1)(-\xi_2 - 2\tau\xi_1, -\xi_1, 1)(\Gamma_{\alpha\beta}^3)_{1 \leq \alpha, \beta \leq 3}(-\xi_2 - 2\tau\xi_1, -\xi_1, 1)^T. \end{aligned}$$

Christoffel symbols are not unique. If we take

$$(A.22) \quad \begin{aligned} (\Gamma_{\alpha\beta}^1)_{1 \leq \alpha, \beta \leq 3} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \\ (\Gamma_{\alpha\beta}^2)_{1 \leq \alpha, \beta \leq 3} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (\Gamma_{\alpha\beta}^3)_{1 \leq \alpha, \beta \leq 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \end{aligned}$$

then (A.20) and (A.21) are satisfied. Eventually, one can check directly that under the metric

$$(A.23) \quad \mathbf{g}(x_1, x_2, t) = \begin{bmatrix} 1 & -t & -x_2 \\ -t & t^2 + 1 & x_2 t \\ -x_2 & x_2 t & x_2^2 + 1 \end{bmatrix},$$

the curve (A.19) is a geodesic for all (z_1, z_2, τ) and ξ .

Next, let us modify Bourgain's example and show that metrics may not always exist. Take

$$(A.24) \quad \phi(x, t; \xi) = \langle x, \xi \rangle + t\xi_1\xi_2 + t^2P(\xi_1),$$

where $P(\xi_1) = O(|\xi_1|^2)$ is to be chosen. Consider all the curves determined by ϕ passing through the origin. They can be written as

$$(A.25) \quad X_1(t) = -t\xi_2 - t^2P'(\xi_1),$$

$$(A.26) \quad X_2(t) = -t\xi_1.$$

Recall that if metric tensor exists, then we always have (A.10). We check this equation at the origin. Note that $\frac{dx^\alpha}{dt}$ stays the same as in Bourgain's example. However, the left hand side of (A.10) changes dramatically. More precisely,

$$(A.27) \quad \frac{dx^1}{dt}\Big|_{t=0} = -\xi_2, \quad \frac{dx^2}{dt}\Big|_{t=0} = -\xi_1, \quad \frac{dx^3}{dt}\Big|_{t=0} = 1.$$

Therefore, the right hand side of (A.10) at the origin is (at most) cubic in ξ . However, the left hand side of (A.10) is

$$(A.28) \quad (-2P'(\xi_1), 0, 0)^T.$$

If for instance we take $P(\xi_1) = \xi_1^5$, then there does not exist any metric tensor to make (A.10) hold for every ξ .

References

- [Bou91] Bourgain, J. *L^p -estimates for oscillatory integrals in several variables*. Geom. Funct. Anal. 1 (1991), no. 4, 321–374.
- [BG11] Bourgain, J. and Guth, L. *Bounds on oscillatory integral operators based on multilinear estimates*. Geom. Funct. Anal. 21 (2011), no. 6, 1239–1295.
- [CS72] Carleson, L. and Sjölin, P. *Oscillatory integrals and multiplier problem for the disc*. Studia Mathematica 3 (1972), no. 44, 287–299.
- [GWZ22] Guo, S., Wang, H. and Zhang, R. *A dichotomy for Hörmander-type oscillatory integral operators*. [arXiv:2210.05851](https://arxiv.org/abs/2210.05851).
- [Gut16] Guth, L. *A restriction estimate using polynomial partitioning*. J. Amer. Math. Soc. 29 (2016), no. 2, 371–413.
- [Gut18] Guth, L. *Restriction estimates using polynomial partitioning II*. Acta Mathematica 221 (2018), no. 1, 81–142.
- [GHI19] Guth, L., Hickman, J. and Iliopoulou, M. *Sharp estimates for oscillatory integral operators via polynomial partitioning*. Acta Math. 223 (2019), no. 2, 251–376.
- [HI22] Hickman, J. and Iliopoulou, M. *Sharp L^p estimates for oscillatory integral operators of arbitrary signature*. Math. Z. 301 (2022), no. 1, 1143–1189.
- [HR19] Hickman, J. and Rogers, K. *Improved Fourier restriction estimates in higher dimensions*. Camb. J. Math. 7 (2019), no. 3, 219–282.

- [HRZ22] Hickman, J., Rogers, K. and Zhang, R. *Improved bounds for the Kakeya maximal conjecture in higher dimensions*. Amer. J. Math. 144 (2022), no. 6, 1511–1560.
- [Hor73] Hörmander, L. *Oscillatory integrals and multipliers on FL^p* . Ark. Mat. 11 (1973), 1–11.
- [KT02] Katz, N. and Tao, T. *New bounds for Kakeya problems*. Dedicated to the memory of Thomas H. Wolff. J. Anal. Math. 87 (2002), 231–263.
- [Kli95] Klingenberg, Wilhelm. *Riemannian Geometry*, 2nd rev. ed. W. de Gruyter, Berlin; New York, 1995. ISBN 3-11-014593-6.
- [Lee] Lee, J. *Introduction to Riemannian Manifolds*, Second edition. Graduate Texts in Mathematics, 176. Springer, Cham, 2018. xiii+437 pp. ISBN: 978-3-319-91754-2; 978-3-319-91755-9.
- [Lee06] Lee, S. *Linear and bilinear estimates for oscillatory integral operators related to restriction to hypersurfaces*. J. Funct. Anal. 241 (2006), no. 1, 56–98.
- [Mat15] Mattila, P. *Fourier Analysis and Hausdorff Dimension*. Cambridge Studies in Advanced Mathematics, 150. Cambridge University Press, Cambridge, 2015. xiv+440 pp. ISBN: 978-1-107-10735-9
- [MS97] Minicozzi, W. and Sogge, C. *Negative results for Nikodym maximal functions and related oscillatory integrals in curved space*. Math. Res. Lett. 4 (1997), no. 2–3, 221–237.
- [Pet16] Petersen, P. *Riemannian Geometry*, Third edition. Graduate Texts in Mathematics, 171. Springer, Cham, 2016. xviii+499 pp. ISBN: 978-3-319-26652-7; 978-3-319-26654-1
- [Sak96] Sakai, T. *Riemannian Geometry*. Translated from the 1992 Japanese original by the author. Translations of Mathematical Monographs, 149. American Mathematical Society, Providence, RI, 1996. xiv+358 pp. ISBN: 0-8218-0284-4
- [Sog99] Sogge, C. *Concerning Nikodym-type sets in 3-dimensional curved spaces*. J. Amer. Math. Soc. 12 (1999), no. 1, 1–31.
- [SXX18] Sogge, C., Xi, Y. and Xu, H. *On instability of the Nikodym maximal function bounds over Riemannian manifolds*. J. Geom. Anal. 28 (2018), no. 3, 2886–2901.

- [Sog17] Sogge, C. *Fourier Integrals in Classical Analysis*. Second edition. Cambridge Tracts in Mathematics, 210. Cambridge University Press, Cambridge, 2017. xiv+334 pp. ISBN: 978-1-107-12007-5
- [Ste84] Stein, E. *Oscillatory integrals in Fourier analysis*. In: Beijing Lectures in Harmonic Analysis (Beijing, 1984), 307–355, Ann. of Math. Stud., 112, Princeton Univ. Press, Princeton, NJ, 1986.
- [Tao99] Tao, T. *The Bochner-Riesz conjecture implies the restriction conjecture*. Duke Math. J. 96 (1999), 363–375.
- [Wis03] Wisewell, L. *Oscillatory Integrals and Curved Minkowski Sets*. PhD thesis, University of Edinburgh, 2003.
- [Wis05] Wisewell, L. *Minkowski sets of curves*. Geom. Funct. Anal. 15 (2005), no. 6, 1319–1362.
- [Xi17] Xi, Y. *On Minkowski-Nikodym type maximal inequalities*. Trans. Amer. Math. Soc. 369 (2017), no. 9, 6351–6372.
- [YC04] Yomdin, Y., Comte, G. *Tame Geometry with Application in Smooth Analysis*. Lecture Notes in Mathematics, vol. 1834. Springer, Berlin, 2004.
- [Zah21] Zahl, J. *New Minkowski estimates using Gromov’s algebraic lemma*. Adv. Math. 380 (2021), 107596.

SONG DAI
CENTER FOR APPLIED MATHEMATICS AND KL-AAGDM
TIANJIN UNIVERSITY
TIANJIN 300072
CHINA
E-mail address: song.dai@tju.edu.cn

LIUWEI GONG
THE CHINESE UNIVERSITY OF HONG KONG
HONG KONG
CHINA
RUTGERS UNIVERSITY
NEW JERSEY
USA
E-mail address: lg689@rutgers.edu

SHAOMING GUO
CHERN INSTITUTE OF MATHEMATICS AND LPMC
NANKAI UNIVERSITY
TIANJIN
CHINA
UNIVERSITY OF WISCONSIN-MADISON
MADISON, WI
USA
E-mail address: shaomingguo2018@gmail.com

RUIXIANG ZHANG
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA, BERKELEY
BERKELEY, CA 94720
USA
E-mail address: ruixiang@berkeley.edu

RECEIVED JANUARY 7, 2024