On Permutations with Bounded Drop Size

Joanna N. Chen[1] and William Y.C. Chen[2]

[1]College of Science
Tianjin University of Technology
Tianjin 300384, P.R. China

[2]Center for Applied Mathematics
Tianjin University
Tianjin 300072, P.R. China

[1]joannachen@tjut.edu.cn, [2]chenyc@tju.edu.cn

Abstract

The *maximum drop size* of a permutation π of $[n] = \{1, 2, \ldots, n\}$ is defined to be the maximum value of $i - \pi(i)$. Chung, Claesson, Dukes and Graham found polynomials $P_k(x)$ that can be used to determine the number of permutations of $[n]$ with d descents and maximum drop size at most k. Furthermore, Chung and Graham gave combinatorial interpretations of the coefficients of $Q_k(x) = x^k P_k(x)$ and $R_{n,k}(x) = Q_k(x)(1 + x + \cdots + x^k)^{n-k}$, and raised the question of finding a bijective proof of the symmetry property of $R_{n,k}(x)$. In this paper, we construct a map φ_k on the set of permutations with maximum drop size at most k. We show that φ_k is an involution and it induces a bijection in answer to the question of Chung and Graham. The second result of this paper is a proof of a unimodality conjecture of Hyatt concerning the type B analogue of the polynomials $P_k(x)$.

Keywords: descent polynomial, unimodal polynomial, maximum drop size

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1 Introduction

This paper is concerned with the study of permutations of $[n] = \{1, 2, \ldots, n\}$ having d descents and maximum drop size at most k. Let this number be denoted by $E^k(n, d)$. Chung, Claesson, Dukes and Graham [3] found polynomials $P_k(x)$ that can be used to determine the number $E^k(n, d)$. They proved that the polynomials $P_k(x)$ are unimodal. Furthermore, Chung and Graham obtained combinatorial interpretations for the polynomials $Q_k(x) = x^k P_k(x)$ and $R_{n,k}(x) = Q_k(x)(1 + x + \cdots + x^k)^{n-k}$, and asked for a combinatorial interpretation of the symmetry property of $R_{n,k}(x)$. The first result of this paper is to present a bijection in answer to the question of Chung and Graham. The second result of this paper is a proof of a conjecture of Hyatt [7] on the unimodality of the type B analogue of the polynomials $P_k(x)$.

Let us give an overview of notation and terminology. Let S_n denote the set of permutations of $[n]$. For a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ in S_n, we say that a number $1 \leq i \leq n-1$ is a *descent* of π if $\pi_i > \pi_{i+1}$. The *descent set* of $\pi \in S_n$, denoted by $\mathrm{Des}(\pi)$, is defined by

$$\mathrm{Des}(\pi) = \{i \in [n-1] : \pi_i > \pi_{i+1}\}.$$

Let $\mathrm{des}(\pi)$ denote the number of descents of $\pi \in S_n$. An *excedance* of π is an index i such that $\pi_i > i$ and a *drop* of π is an index i such that $i > \pi_i$. It is well-known that the number of excedances and the number of descents are equidistributed over S_n. It is clear that the number of excedances and the number of drops have the same distribution over S_n. If i is a drop of a permutation $\pi \in S_n$, then we define the drop size to be $i - \pi_i$. The *maximum drop size* of π is

$$\mathrm{maxdrop}(\pi) = \max\{i - \pi_i \colon 1 \leq i \leq n\}.$$

For example, let $\pi = 43562187$. The set of excedances of π is given by $\{1, 2, 3, 4, 7\}$, the set of drops of π is given by $\{5, 6, 8\}$, $\mathrm{des}(\pi) = 4$, and $\mathrm{maxdrop}(\pi) = 5$.

Diaconis and Graham [5] studied the permutation statistic "Spearman's disarray", which is related to the drop size. This statistic, called "total displacement" by Knuth [8], is defined as

$$\sum_{i=1}^{n} |\pi_i - i| = 2\sum_{\pi_i > i}(\pi_i - i) = 2\sum_{i > \pi_i}(i - \pi_i).$$

Petersen and Tenner [9] introduced a permutation statistic called the depth in terms of factorizations of the elements into products of reflections. It turns out that the depth of a permutation is half of its total displacement.

Chung, Claesson, Dukes and Graham [3] obtained a polynomial $P_k(x)$ that can be used to determine the number $E^k(n,d)$ of permutations of $[n]$ with d descents and maximum drop size at most k. Let $\mathcal{A}_{n,k}$ denote the set of permutations of $[n]$ with maximum drop size at most k. The *k-maxdrop-restricted descent polynomial* is defined by

$$A_{n,k}(y) = \sum_{\pi \in \mathcal{A}_{n,k}} y^{\mathrm{des}(\pi)} = \sum_{d \geq 0} E^k(n,d) y^d.$$

Clearly, for $k \geq n$, we have $\mathcal{A}_{n,k} = S_n$ and $A_{n,k}(y)$ becomes the Eulerian polynomial

$$A_n(y) = \sum_{\pi \in S_n} y^{\mathrm{des}(\pi)}.$$

Notice that here we have adopted the definition of the Eulerian polynomial as used by Chung et al. [3], which differs from the definition given in Stanley [10] by a factor of y. Chung, Claesson, Dukes and Graham [3] obtained the following recurrence relation for $A_{n,k}(y)$.

Theorem 1.1 (Chung, Claesson, Dukes and Graham, [3]) *For* $n, k \geq 0$,

$$A_{n+k+1,k}(y) = \sum_{i=1}^{k+1} \binom{k+1}{i} (y-1)^{i-1} A_{n+k+1-i,k}(y),$$

where $A_{i,k}(y) = A_i(y)$ *for* $0 \leq i \leq k$.

Using the recurrence relation for $A_{n,k}(y)$ in Theorem 1.1, Chung, Claesson, Dukes and Graham introduced the polynomials

$$P_k(x) = \sum_{l=0}^{k} A_{k-l}(x^{k+1})(x^{k+1}-1)^l \sum_{i=l}^{k} \binom{i}{l} x^{-i}, \tag{1.1}$$

and derived the following expression for $A_{n,k}(y)$ which can be used to determine the number $E^k(n,d)$.

Theorem 1.2 (Chung, Claesson, Dukes and Graham,[3]) *For* $n, k \geq 0$,

$$A_{n,k}(y) = \sum_{d} \beta_k((k+1)d)y^d, \tag{1.2}$$

where

$$\sum_{j} \beta_k(j)x^j = P_k(x) \left(\frac{1-x^{k+1}}{1-x} \right)^{n-k}. \tag{1.3}$$

By the definition of $A_{n,k}(y)$, one sees from the above theorem that $E^k(n,d)$ equals the coefficient of $x^{(k+1)d}$ in

$$P_k(x)(1+x+x^2+\cdots+x^k)^{n-k}.$$

We say a sequence $(s_1, s_2, \ldots, s_n)$ is *unimodal* if there exists an integer $1 \leq t \leq n$ such that $s_1 \leq s_2 \leq \cdots \leq s_t$ and $s_t \geq s_{t+1} \geq \cdots \geq s_n$. A polynomial is said to be *unimodal* if the sequence of its coefficients is unimodal. Chung, Claesson, Dukes and Graham [3] proved that the polynomial $P_k(x)$ is unimodal for all k.

Furthermore, Chung and Graham [4] found combinatorial interpretations of the coefficients of the polynomials $Q_k(x) = x^k P_k(x)$ and $R_{n,k}(x) = Q_k(x)(1+x+\cdots+x^k)^{n-k}$. They used the notation $\left\langle {n \atop i} \right\rangle^j$ for the number of permutations $\pi \in S_n$ such that $\mathrm{des}(\pi) = i$ and $\pi_n = j$ and the notation $\left\langle {n \atop i} \right\rangle^j_{[k]}$ for the number of permutations $\pi \in \mathcal{A}_{n,k}$ such that $\mathrm{des}(\pi) = i$ and $\pi_n = j$. In this paper, we write $E(n,i;j)$ for $\left\langle {n \atop i} \right\rangle^j$ and $E^k(n,i;j)$ for $\left\langle {n \atop i} \right\rangle^j_{[k]}$.

Theorem 1.3 (Chung and Graham, [4]) *For $n \geq 0$,*

$$Q_n(x) = \sum_{0 \leq i,j \leq n} E(n+1, i; j+1) x^{(n+1)i+j}.$$

Theorem 1.4 (Chung and Graham, [4]) *For $n \geq k \geq 0$,*

$$R_{n,k}(x) = \sum_{0 \leq i \leq n} \sum_{0 \leq j \leq k} E^k(n+1, i; n+1-k+j) x^{(k+1)i+j}.$$

Chung and Graham [4] showed that the polynomials $Q_n(x)$ and $R_{n,k}(x)$ are symmetric. They constructed a bijection for the symmetry of $Q_n(x)$, and they raised the question of finding a bijective proof of the symmetry of $R_{n,k}(x)$. More precisely, the symmetry property of $R_{n,k}(x)$ can be described as follows. Assume that

$$R_{n,k}(x) = \sum_{r=0}^{(n+2)k} c_{n,k,r} x^r.$$

The symmetry of $R_{n,k}(x)$ states that for $0 \leq r \leq (n+2)k$ and $0 \leq r' \leq (n+2)k$ such that $r + r' = (n+2)k$, we have $c_{n,k,r} = c_{n,k,r'}$. For example, for $n = 4$ and $k = 2$, we have

$$R_{4,2}(x) = x^2 + 3x^3 + 7x^4 + 10x^5 + 12x^6 + 10x^7 + 7x^8 + 3x^9 + x^{10}.$$

For $0 \leq r \leq (n+2)k$, one can uniquely express r as $r = (k+1)i + j$, where $0 \leq i \leq n$ and $0 \leq j \leq k$. Thus Theorem 1.4 can be written as

$$c_{n,k,r} = E^k(n+1, i; n+1-k+j).$$

Consequently, the symmetry of $R_{n,k}(x)$ takes the following form.

Theorem 1.5 (Chung and Graham, [3]) *For $n \geq k \geq 0$, the polynomials $R_{n,k}(x)$ are symmetric. In other words, for $r = (k+1)i + j$ and $r' = (k+1)i' + j'$ such that $r + r' = (n+2)k$, where $0 \leq i, i' \leq n, 0 \leq j, j' \leq k$, we have*

$$E^k(n+1, i; n+1-k+j) = E^k(n+1, i'; n+1-k+j').$$

As an example, let $n = 4$, $k = 2$, $r = 4$ and $r' = 8$. Writing $r = 3 \cdot 1 + 1$ and $r' = 3 \cdot 2 + 2$, by Theorem 1.4, we find that $c_{4,2,4} = E^2(5, 1; 4) = 7$ and $c_{4,2,8} = E^2(5, 2; 5) = 7$. Permutations enumerated by $E^2(5, 1; 4)$ and $E^2(5, 2; 5)$ are given in Table 1.1.

In Section 2, we construct a map φ_k on Γ^k by a recursive procedure, where Γ^k is the set of permutations with maximum drop size at most k. Then, we prove that φ_k induces a bijection for Theorem 1.5.

$\pi \in \mathcal{A}_{5,2}$ with $\mathrm{des}(\pi) = 1$ and $\pi_5 = 4$	$\pi \in \mathcal{A}_{5,2}$ with $\mathrm{des}(\pi) = 2$ and $\pi_5 = 5$
1 2 3 5 4	3 2 1 4 5
1 2 5 3 4	4 2 1 3 5
1 3 5 2 4	2 1 4 3 5
1 5 2 3 4	3 1 4 2 5
2 5 1 3 4	1 4 3 2 5
3 5 1 2 4	4 3 1 2 5
5 1 2 3 4	4 1 3 2 5

Table 1.1: Permutations enumerated by $E^2(5, 1; 4)$ and $E^2(5, 2; 5)$.

In Section 3, we consider the unimodality of the type B analogue of the polynomials $P_k(x)$. As pointed out by Chung et al. [3], the maxdrop statistic is related to the bubble sorting algorithm. Let $\mathcal{B}_n$ denote the type B Coxeter group of rank n, that is, the group of signed permutations on $[n]$. Hyatt [7] found a natural way to extend the bubble sorting algorithm to signed permutations. Moreover, he introduced the notion of the maximum drop size of a signed permutation.

Recall that a signed permutation $\pi = \pi_1\pi_2\cdots\pi_n$ can be viewed as a permutation of $[n]$ for which each element may be associated with a minus sign. We shall use the bar notation $\bar{i}$ to signify an element i with a minus sign. The *descent set* of a signed permutation π is defined to be

$$\mathrm{Des}_{\mathrm{B}}(\pi) = \{i \in [0, n-1] : \pi_i > \pi_{i+1}\},$$

where we assume that $\pi_0 = 0$, see Brenti [1]. Let π be a signed permutation in $\mathcal{B}_n$. The number of descents of π is denoted by $\mathrm{des}_{\mathrm{B}}(\pi)$. Hyatt [7] defined the *maximum drop size* of π as given by

$$\mathrm{maxdrop}_{\mathrm{B}}(\pi) = \max\Big\{\max\{i - \pi_i : \pi_i > 0\}, \max\{i : \pi_i < 0\}\Big\}.$$

For example, let $\pi = \bar{4}3\bar{5}62\bar{1}87$. Then we have $\mathrm{des}_{\mathrm{B}}(\pi) = 5$ and $\mathrm{maxdrop}_{\mathrm{B}}(\pi) = 6$.

Let $\mathcal{B}_{n,k}$ denote the set of signed permutations of $[n]$ with maximum drop size at most k, and let $E_B^k(n, d)$ denote the number of signed permutations in $\mathcal{B}_{n,k}$ with d descents.

The *type B k-maxdrop-restricted descent polynomial* is defined by

$$B_{n,k}(y) = \sum_{\pi \in \mathcal{B}_{n,k}} y^{\mathrm{des}_{\mathrm{B}}(\pi)} = \sum_{d \geq 0} E_B^k(n, d) y^d.$$

When $k \geq n$, $\mathcal{B}_{n,k} = B_n$ and $B_{n,k}(y)$ becomes the type B Eulerian polynomial $B_n(y)$, which is defined by

$$B_n(y) = \sum_{\pi \in \mathcal{B}_n} y^{\mathrm{des}_{\mathrm{B}}(\pi)}.$$

Hyatt [7] showed that $B_{n,k}(y)$ satisfied the following recurrence relation.

Theorem 1.6 (Hyatt, [7]) *For $n, k \geq 0$,*

$$B_{n+k+1,k}(y) = \sum_{i=1}^{k+1} \binom{k+1}{i} (y-1)^{i-1} B_{n+k+1-i,k}(y),$$

where $B_{i,k}(y) = B_i(y)$ for $0 \leq i \leq k$.

Using the above recurrence relation for $B_{n,k}(y)$, Hyatt obtained the following type B analogue of the polynomials $P_k(x)$,

$$T_k(x) = \sum_{l=0}^{k} B_{k-l}(x^{k+1})(x^{k+1}-1)^l \sum_{i=l}^{k} \binom{i}{l} x^{-i}, \tag{1.4}$$

which determines the number $E_B^k(n, d)$.

Theorem 1.7 (Hyatt, [7]) *For $n, k \geq 0$,*

$$B_{n,k}(y) = \sum_{d} \gamma_k((k+1)d) y^d, \tag{1.5}$$

where

$$\sum_{j} \gamma_k(j) x^j = T_k(x) \left(\frac{1-x^{k+1}}{1-x} \right)^{n-k}. \tag{1.6}$$

The above theorem implies that $E_B^k(n, d)$ equals the coefficient of $x^{(k+1)d}$ in

$$T_k(x)(1 + x + x^2 + \cdots + x^k)^{n-k}.$$

The following conjecture was posed by Hyatt [7].

Conjecture 1.8 (Hyatt, [7]) *The polynomial $T_k(x)$ is unimodal for $k \geq 0$.*

The second result of this paper is a proof of the above conjecture, which will be given in Section 3.

2 Combinatorial proof of the symmetry of $R_{n,k}(x)$

In this section, we give a combinatorial proof of Theorem 1.5. For $k \geq 0$, let Γ^k be the set of permutations with maximum drop size at most k. We construct a map φ_k on Γ^k by a recursive procedure. We shall prove that φ_k is an involution on Γ^k and it induces a bijection for Theorem 1.5.

To describe the map φ_k, we begin with some notation. Given $\pi \in S_n$ and $1 \leq i \leq n+1$, let $\pi \leftarrow i$ denote the permutation $\mu = \mu_1\mu_2\cdots\mu_{n+1}$ in S_{n+1} that is obtained from π by adding i at the end of π and increasing the elements $i, i+1, \ldots, n$ by 1. For example, $3421 \leftarrow 3 = 45213$.

For $n \geq 1$, let $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation in Γ^k. The permutation $\varphi_k(\pi)$ is recursively constructed as follows. If $n = 1$, define $\varphi_k(1) = 1$. We now assume that $n \geq 2$. Let $i = \mathrm{des}(\pi)$ and $j = \pi_n - n + k$. Assume that π' is the permutation of $[n-1]$ that is order isomorphic to $\pi_1\pi_2\cdots\pi_{n-1}$. In other words, write $\pi = \pi' \leftarrow \pi_n$. In order to recursively construct $\varphi_k(\pi)$, it is necessary to verify that $\mathrm{maxdrop}(\pi') \leq k$, that is, $t - \pi'_t \leq k$ for $1 \leq t \leq n-1$. We consider two cases. If $\pi'_t = \pi_t$, then $t - \pi'_t = t - \pi_t \leq k$. If $\pi'_t = \pi_t - 1$, by the definition of π', we get $\pi_t > \pi_n$. Thus $t - \pi'_t = t + 1 - \pi_t \leq n - \pi_n \leq k$. So π' is a permutation of length $n-1$ in Γ^k. This enables us to define

$$\varphi_k(\pi) = \varphi_k(\pi') \leftarrow (n - k + j'),$$

where j' is uniquely determined by n, k, i and j, as given below

$$i' = \left\lfloor \frac{(n+1)k - (k+1)i - j}{k+1} \right\rfloor, \tag{2.1}$$

$$j' = (n+1)k - (k+1)i - j - (k+1)i'. \tag{2.2}$$

For example, let $\pi = 12354$. It can be checked that $\pi \in \Gamma^1$. So we also have $\pi \in \Gamma^2$. To demonstrate that the map φ_k is indeed dependent on k, let us compute $\varphi_2(\pi)$ and $\varphi_1(\pi)$. To compute $\varphi_2(\pi)$, we have $i = \mathrm{des}(\pi) = 1$ and $j = \pi_5 - 5 + 2 = 1$. By relations (2.1) and (2.2), we get $i' = 2$ and $j' = 2$. Write $\pi = \pi' \leftarrow \pi_5 = 1234 \leftarrow 4$. By the definition of the map φ_2, we get $\varphi_2(\pi) = \varphi_2(\pi') \leftarrow 5$. We now turn to $\varphi_2(\pi')$. Repeating the above process, we obtain that $\pi'' = 123$, $\pi''' = 12$ and $\pi'''' = 1$. It follows that $\varphi_2(\pi'''') = 1$, $\varphi_2(\pi''') = 21$, $\varphi_2(\pi'') = 321$ and $\varphi_2(\pi') = 3214$. So we find that $\varphi_2(\pi) = 32145$. Similarly, we obtain that $\varphi_1(\pi) = 21534$. It can be seen that $\varphi_2(\pi) \neq \varphi_1(\pi)$.

The following theorem states that for $k \geq 0$, φ_k is an involution, that is, for any $\pi \in \Gamma^k$, we have $\varphi_k^2(\pi) = \pi$.

Theorem 2.1 *For $k \geq 0$, the map φ_k is an involution on Γ^k.*

To prove the above theorem, we need the following property of the map φ_k. Let $\Gamma^k(n, i; j)$ denote the set of permutations on $[n]$ enumerated by $E^k(n, i; n-k+j)$, that is, the set of permutations on $[n]$ with maximum drop size at most k such that the descent number equals i and the last element equals $n - k + j$.

Theorem 2.2 *For $n \geq 1$, $n \geq k \geq 0$, $0 \leq i \leq n-1$, $0 \leq j \leq k$ and a permutation π in $\Gamma^k(n, i; j)$, we have $\varphi_k(\pi) \in \Gamma^k(n, i'; j')$, where i' and j' are given by relations (2.1) and (2.2).*

Proof. We proceed by induction on n. For $n = 1$, we have $1 \in \Gamma^k(1, 0; k)$. By (2.1) and (2.2), we deduce that $i' = 0$ and $j' = k$. Clearly, $\varphi_k(1) \in \Gamma^k(1, 0; k)$ for any $k \geq 0$. This proves the case for $n = 1$. Assume that the theorem holds for $n - 1$, where $n \geq 2$. We aim to show that it is valid for n.

Write $\pi = \pi_1\pi_2 \cdots \pi_n$ and assume that $\sigma = \sigma_1\sigma_2 \cdots \sigma_{n-1}$ is the permutation of $[n - 1]$ that is order isomorphic to $\pi_1\pi_2 \cdots \pi_{n-1}$, that is, $\pi = \sigma \leftarrow \pi_n$. Denote $\varphi_k(\pi)$ by $\beta = \beta_1\beta_2 \cdots \beta_n$. By the recursive construction of φ_k, we have

$$\beta = \varphi_k(\sigma) \leftarrow (n - k + j'), \tag{2.3}$$

where j' is given by (2.1) and (2.2).

To show that $\beta \in \Gamma^k(n, i'; j')$, denote $\varphi_k(\sigma)$ by $\alpha = \alpha_1\alpha_2 \cdots \alpha_{n-1}$. Let

$$\begin{aligned} s &= \operatorname{des}(\sigma), && (2.4)\\ t &= \sigma_{n-1} - n + 1 + k, && (2.5)\\ s' &= \left\lfloor \frac{nk - s(k+1) - t}{k+1} \right\rfloor, && (2.6)\\ t' &= nk - s(k+1) - t - s'(k+1). && (2.7) \end{aligned}$$

In the above notation, we have $\sigma \in \Gamma^k(n - 1, s; t)$. By the induction hypothesis, $\alpha \in \Gamma^k(n - 1, s'; t')$. This implies that $\operatorname{maxdrop}(\alpha) \leq k$. It can be seen from (2.3) that $\beta_n = n - k + j'$ and $\beta_i \geq \alpha_i$ for $1 \leq i \leq n - 1$, so that $\operatorname{maxdrop}(\beta) \leq \max\{\operatorname{maxdrop}(\alpha), k - j'\}$. It follows that $\operatorname{maxdrop}(\beta) \leq k$.

It remains to verify that $\operatorname{des}(\beta) = i'$. In view of (2.3), it suffices to check that $i' = s' + 1$ when $\alpha_{n-1} \geq \beta_n$ and $i' = s'$ when $\alpha_{n-1} < \beta_n$. Since $\beta_n = n - k + j'$ and $\alpha_{n-1} = n - 1 - k + t'$, we need to show that $i' = s' + 1$ when $j' - t' \leq -1$ and $i' = s'$ when $j' - t' > -1$. To this end, we need the following four relations (2.8)-(2.11).

By the definition t, we have $0 \leq t \leq k$. Since $0 \leq j \leq k$, we find that

$$-k \leq j - t \leq k. \tag{2.8}$$

Similarly,

$$-k \leq j' - t' \leq k. \tag{2.9}$$

By (2.2) and (2.7), we see that

$$\begin{aligned} i(k+1) + j + i'(k+1) + j' &= (n+1)k, && (2.10)\\ s(k+1) + t + s'(k+1) + t' &= nk. && (2.11) \end{aligned}$$

Since $i = \operatorname{des}(\pi)$, $s = \operatorname{des}(\sigma)$ and $\pi = \sigma \leftarrow \pi_n$, we have $i = s$ or $i = s + 1$. So there are two cases.

Case 1: $i = s$, so $\pi_{n-1} < \pi_n$, and so $j - t > -1$. By (2.10) and (2.11),

$$(i' - s')(k + 1) = k - (j - t) - (j' - t').$$

If $j' - t' \le -1$, by (2.9), we see that $k \ge 1$. By (2.8) and the assumption $j - t > -1$, we deduce that $-1 < j - t \le k$. By (2.9) and the assumption $j' - t' \le -1$, we find that $-k \le j' - t' \le -1$. It follows that $(i' - s')(k + 1) \in [1, 2k]$, where $k \ge 1$. Hence we arrive at the assertion that $i' = s' + 1$.

If $j' - t' > -1$, by (2.9), we find that $-1 < j' - t' \le k$. By (2.8) and the assumption $j - t > -1$, we get $-1 < j - t \le k$. Thus, $(i' - s')(k + 1) \in [-k, k]$. So we deduce that $i' = s'$.

Case 2: $i = s + 1$, so $\pi_{n-1} > \pi_n$, and so $j - t \le -1$. By (2.8) and the assumption $j - t \le -1$, we deduce that $k \ge 1$. It follows from (2.10) and (2.11) that

$$(i' - s')(k + 1) = -1 - (j - t) - (j' - t'). \tag{2.12}$$

If $j' - t' \le -1$, we claim that $k \ge 2$. Assume to the contrary that $k = 1$. By (2.8) and (2.9), we obtain that $j' - t' = -1$ and $j - t = -1$. By (2.12), we deduce that $2(i' - s') = 1$, a contradiction. This proves that $k \ge 2$. Using (2.8) and the assumption $j - t \le -1$, we find that $-k \le j - t \le -1$. Similarly, we have $-k \le j' - t' \le -1$. It follows that $(i' - s')(k + 1) \in [1, 2k - 1]$, where $k \ge 2$. So we reach the conclusion that $i' = s' + 1$.

If $j' - t' > -1$, by (2.9), we deduce that $-1 < j' - t' \le k$. By (2.8) and the assumption $j - t \le -1$, we find that $-k \le j - t \le -1$. It follows that $(i' - s')(k + 1) \in [-k, k - 1]$, where $k \ge 1$. This implies that $i' = s'$.

Up to now, we have shown that $i' = s' + 1$ when $j' - t' \le -1$ and $i' = s'$ when $j' - t' > -1$. This yields that $\mathrm{des}(\beta) = i'$, and hence the proof is complete. ∎

We are now ready to finish the proof of Theorem 2.1.

Proof of Theorem 2.1. Let $\pi = \pi_1 \pi_2 \cdots \pi_n$ be a permutation in Γ^k, we aim to show that $\varphi_k^2(\pi) = \pi$. We proceed by induction on n. When $n = 1$, it is obvious that $\varphi_k^2(1) = 1$. So the theorem is valid for $n = 1$. Assume that the theorem holds for $n - 1$, where $n \ge 2$, that is, for any permutation $\sigma = \sigma_1 \sigma_2 \cdots \sigma_{n-1}$, we have $\varphi_k^2(\sigma) = \sigma$. Denote $\varphi_k^2(\pi)$ by $\gamma = \gamma_1 \gamma_2 \cdots \gamma_n$.

To prove that $\gamma = \pi$, write $\pi = \sigma \leftarrow \pi_n$, where $\sigma = \sigma_1 \sigma_2 \cdots \sigma_{n-1}$. Let $i = \mathrm{des}(\pi)$ and $j = \pi_n - n + k$; that is, π is a permutation in $\Gamma^k(n, i, j)$. By Theorem 2.2, we know that $\varphi_k(\pi) = \varphi_k(\sigma \leftarrow (n - k + j)) \in \Gamma^k(n, i'; j')$, where i' and j' are given by (2.1) and (2.2). By the construction of φ_k, we have

$$\varphi_k(\pi) = \varphi_k\big(\sigma \leftarrow (n - k + j)\big) = \varphi_k(\sigma) \leftarrow (n - k + j'). \tag{2.13}$$

Let i'' and j'' be the integers obtained from i' and j' by using (2.1) and (2.2). A direct computation indicates that $i'' = i$ and $j'' = j$. Applying (2.13) twice yields that

$$\gamma = \varphi_k^2(\pi) = \varphi_k^2(\sigma) \leftarrow (n - k + j).$$

But the induction hypothesis says that $\varphi_k^2(\sigma) = \sigma$, so we get

$$\gamma = \sigma \leftarrow (n-k+j) = \pi.$$

This completes the proof. ∎

To conclude this section, we notice that when restricted to $\Gamma^k(n, i; j)$ the map φ_k serves as a combinatorial interpretation of Theorem 1.5 with $n+1$ replaced by n. For $n \geq 1$, $n \geq k \geq 0$, $r = (k+1)i + j$ and $r' = (k+1)i' + j'$ such that $r + r' = (n+1)k$, $0 \leq i, i' \leq n-1$ and $0 \leq j, j' \leq k$, it is easy to see that the integers i' and j' are uniquely determined by n, k, i, j, as given by relations (2.1) and (2.2). Combining Theorems 2.1 and 2.2, we are led to the following bijection.

Theorem 2.3 *For $n \geq 1$, $n \geq k \geq 0$, $r = (k+1)i + j$ and $r' = (k+1)i' + j'$ such that $r + r' = (n+1)k$, $0 \leq i, i' \leq n-1$ and $0 \leq j, j' \leq k$, φ_k induces a bijection from $\Gamma^k(n, i; j)$ to $\Gamma^k(n, i'; j')$.*

3 The unimodality of $T_k(x)$

In this section, we prove a conjecture of Hyatt [7] on the unimodality of a type B analogue of the polynomials $P_k(x)$. Let $\mathcal{B}_n$ be the set of signed permutations on $[n]$. For $\pi \in \mathcal{B}_n$, Hyatt defined the maximum drop size of π as follows. We say π has a drop at position i if $i > \pi(i)$. If π has a drop at position i, the drop size at this position is defined to be $\min\{i - \pi(i), i\}$. The *type B maximum drop size* of π, denoted $\mathrm{maxdrop_B}(\pi)$, is the maximum value of all drop sizes of π; that is,

$$\mathrm{maxdrop_B}(\pi) = \max\Big\{ \max\{i - \pi_i : \pi_i > 0\}, \max\{i : \pi_i < 0\} \Big\}.$$

Based on the type B descent number and the maximum drop size of a signed permutation, for $k \geq 0$, Hyatt introduced a type B analogue of the polynomial $P_k(x)$, denoted $T_k(x)$. Recall that the type B Eulerian polynomials are associated with the type B descent number of a signed permutation, which are given by

$$B_n(y) = \sum_{\pi \in \mathcal{B}_n} y^{\mathrm{des_B}(\pi)}.$$

The polynomials $T_k(x)$ are defined by

$$T_k(x) = \sum_{l=0}^{k} B_{k-l}(x^{k+1})(x^{k+1} - 1)^l \sum_{i=l}^{k} \binom{i}{l} x^{-i}.$$

Let $E_B^k(n, d)$ be the number of signed permutations on $[n]$ with d type B descents and type B maximum drop size at most k. For $k \geq 0$, Hyatt showed that $E_B^k(n, d)$ equals the coefficient of $x^{(k+1)d}$ in $T_k(x)(1+x+x^2+\cdots+x^k)^{n-k}$, and he conjectured that $T_k(x)$ is unimodal.

To prove this conjecture, we define the polynomials $H_k(x)$ as given by

$$\begin{aligned} H_k(x) = & \sum_{l=0}^{k} B_{k-l}(x^{2k+2})(x^{2k+2}-1)^l \sum_{s=l}^{k} \binom{s}{l} x^{2k+1-s} \\ & + \sum_{l=0}^{k} B_{k-l}(x^{-2k-2})(x^{-2k-2}-1)^l \sum_{s=l}^{k} \binom{s}{l} x^{2(k+1)^2+s}. \end{aligned} \tag{3.1}$$

As will be shown that the sequence of coefficients of $T_k(x)$ is a subsequence of those of $H_k(x)$. Thus the unimodality of $T_k(x)$ follows from the unimodality of $H_k(x)$.

Let $\widetilde{T}_k(x) = x^k T_k(x)$, that is,

$$\widetilde{T}_k(x) = \sum_{l=0}^{k} B_{k-l}(x^{k+1})(x^{k+1}-1)^l \sum_{i=l}^{k} \binom{i}{l} x^{k-i}. \tag{3.2}$$

k	$\widetilde{T}_k(x)$
0	1
1	$x + 2x^2 + x^3$
2	$x^2 + 4x^3 + 6x^4 + 6x^5 + 4x^6 + 2x^7 + x^8$
3	$x^3 + 8x^4 + 12x^5 + 18x^6 + 23x^7 + 32x^8 + 32x^9 + 28x^{10} + 23x^{11}$
	$+ 8x^{12} + 4x^{13} + 2x^{14} + x^{15}$

Table 3.2: The polynomials $\widetilde{T}_k(x)$ for $0 \le k \le 3$.

For $0 \le k \le 3$, the polynomials $\widetilde{T}_k(x)$ are given in Table 3.2. Analogous to the array representation of $Q_k(x)$ given by Chung and Graham [4], we define an array representation of $\widetilde{T}_k(x)$. For $0 \le i \le k+1$ and $0 \le j \le k$, the (i,j)-entry $t_k(i,j)$ is set to be the coefficient of $x^{(k+1)i+j}$ of $\widetilde{T}_k(x)$, that is,

$$\widetilde{T}_k(x) = \sum_{i=0}^{k+1} \sum_{j=0}^{k} t_k(i,j) x^{(k+1)i+j}. \tag{3.3}$$

Similarly, we can arrange the coefficients of $H_k(x)$ in a $(k+2) \times 2(k+1)$ array h_k so that

$$H_k(x) = \sum_{i=0}^{k+1} \sum_{j=0}^{2k+1} h_k(i,j) x^{2(k+1)i+j}.$$

In fact, for any $k \ge 0$, h_k can be obtained from t_k as described in the following lemma.

Lemma 3.1 *For $k \geq 0$, h_k can be obtained by rotating t_k 180 degrees (in either direction), and adjoining the rotated array to the left side of t_k.*

For example, the array h_2 can be obtained from the array t_2 by the following operations. First, rotate the array t_2 180 degrees. Then adjoin this rotated array to the left side of t_2. Table 3.3 gives the array t_2 and Table 3.4 illustrates the corresponding array h_2.

0	0	1
4	6	6
4	2	1
0	0	0

Table 3.3: The array t_2

0	0	0	0	0	1
1	2	4	4	6	6
6	6	4	4	2	1
1	0	0	0	0	0

Table 3.4: The array h_2

To prove Lemma 3.1, we need the following property.

Lemma 3.2 *For $k \geq 0$, define*

$$F_k(x) = \sum_{l=0}^{k} B_{k-l}(x^{k+2})(x^{k+2}-1)^l \sum_{i=l}^{k} \binom{i}{l} x^{k+1-i}. \tag{3.4}$$

Arrange the coefficients of $F_k(x)$ in a $(k+2) \times (k+2)$ array f_k so that

$$F_k(x) = \sum_{i=0}^{k+1} \sum_{j=0}^{k+1} f_k(i,j) x^{(k+2)i+j}.$$

Then the array f_k can be obtained from t_k by adjoining a column of zeros to the left of t_k.

Proof. To prove that f_k can be obtained from t_k by inserting a column of zeros in front of t_k, we proceed to verify that $f_k(i,0) = 0$ for $0 \leq i \leq k+1$ and $f_k(i,j+1) = t_k(i,j)$ for $0 \leq i \leq k+1$ and $0 \leq j \leq k$.

For convenience, for $0 \leq l \leq k$, let

$$\begin{aligned} U_l(t) &= B_{k-l}(t)(t-1)^l, \\ V_l(t) &= \sum_{i=l}^{k} \binom{i}{l} t^{k-i}. \end{aligned}$$

Notice that $U_l(t)$ is a polynomial in t of degree k and $V_l(t)$ is a polynomial in t of degree at most k.

From the expression (3.4) of $F_k(x)$, we see that

$$F_k(x) = \sum_{l=0}^{k} x U_l(x^{k+2}) V_l(x).$$

Since $U_l(x^{k+2})$ can be seen as a polynomial in x^{k+2} and the degree of $V_l(x)$ is at most k, we deduce that the coefficient of $x^{(k+2)i}$ in $F_k(x)$ equals zero for $0 \le i \le k+1$. Hence $f_k(i,0) = 0$ for $0 \le i \le k+1$.

Next we prove that $t_k(i,j) = f_k(i,j+1)$ for $0 \le i \le k+1$ and $0 \le j \le k$. We shall adopt the common notation $[x^l]\,p(x)$ for the coefficient of x^l in a polynomial $p(x)$. It suffices to show that

$$[x^{(k+1)i+j}]\,\widetilde{T}_k(x) = [x^{(k+2)i+j+1}]\,F_k(x). \tag{3.5}$$

From the expression (3.2) of $\widetilde{T}_k(x)$, it follows that

$$\widetilde{T}_k(x) = \sum_{l=0}^{k} U_l(x^{k+1})V_l(x).$$

Recalling that $V_l(x)$ is a polynomial in x of degree at most k, for $0 \le i \le k+1$ and $0 \le j \le k$, it is easily checked that

$$\begin{aligned} [x^{(k+1)i+j}]\,\widetilde{T}_k(x) &= \sum_{l=0}^{k} \Big([x^{(k+1)i}]\,U_l(x^{k+1})\Big)\Big([x^j]\,V_l(x)\Big) \\ &= \sum_{l=0}^{k} \Big([t^i]\,U_l(t)\Big)\Big([x^j]\,V_l(x)\Big). \end{aligned} \tag{3.6}$$

Similarly, we have

$$\begin{aligned} [x^{(k+2)i+j+1}]\,F_k(x) &= \sum_{l=0}^{k} \Big([x^{(k+2)i}]\,U_l(x^{k+2})\Big)\Big([x^{j+1}]\,xV_l(x)\Big) \\ &= \sum_{l=0}^{k} \Big([x^{(k+2)i}]\,U_l(x^{k+2})\Big)\Big([x^{j}]\,V_l(x)\Big) \\ &= \sum_{l=0}^{k} \Big([t^i]\,U_l(t)\Big)\Big([x^j]\,V_l(x)\Big). \end{aligned} \tag{3.7}$$

Hence (3.5) follows from (3.6) and (3.7). So we arrive at the conclusion that $f_k(i,j+1) = t_k(i,j)$ for $0 \le i \le k+1$ and $0 \le j \le k$. This completes the proof. ∎

We are now ready to give a proof of Lemma 3.1.

Proof of Lemma 3.1. Write $H_k(x)$ as

$$H_k(x) = H_k'(x) + H_k''(x),$$

where

$$H_k'(x) = \sum_{l=0}^{k} B_{k-l}(x^{2k+2})(x^{2k+2}-1)^l \sum_{s=l}^{k} \binom{s}{l} x^{2k+1-s}, \tag{3.8}$$

$$H_k''(x) \ = \ \sum_{l=0}^{k} B_{k-l}(x^{-2k-2})(x^{-2k-2}-1)^l \sum_{s=l}^{k} \binom{s}{l} x^{2(k+1)^2+s}. \qquad (3.9)$$

Assume $H_k'(x)$ has an array representation h_k' such that

$$H_k'(x) = \sum_{i=0}^{k+1} \sum_{j=0}^{2k+1} h_k'(i,j) x^{2(k+1)i+j},$$

and $H_k''(x)$ has an array representation h_k'' such that

$$H_k''(x) = \sum_{i=0}^{k+1} \sum_{j=0}^{2k+1} h_k''(i,j) x^{2(k+1)i+j}.$$

Clearly, we have $h_k = h_k' + h_k''$. Using Lemma 3.2 repeatedly, we deduce that h_k' can be obtained form t_k by adjoining $k+1$ columns of zeros to the left side of t_k. Table 3.5 gives an example of h_k' for $k = 2$.

From the expression (3.8) of $H_k'(x)$ and the expression (3.9) of $H_k''(x)$, we see that

$$H_k''(x) = H_k'(x^{-1}) x^{2(k+1)(k+2)-1}.$$

Hence, in the array representation, we deduce that h_k'' can be obtained from h_k' by rotating h_k' 180 degrees. For example, the array h_2'' in Table 3.6 is constructed from the array h_2' in Table 3.5.

0	0	0	0	0	1
0	0	0	4	6	6
0	0	0	4	2	1
0	0	0	0	0	0

Table 3.5: The array h_2'

0	0	0	0	0	0
1	2	4	0	0	0
6	6	4	0	0	0
1	0	0	0	0	0

Table 3.6: The array h_2''

By the fact that $h_k = h_k' + h_k''$ and the constructions of h_k' and h_k'', we see that the first $k+1$ columns of h_k can be obtained from t_k by a rotation of 180 degrees and t_k remains to be the last $k+1$ columns of h_k. This completes the proof. ∎

As a consequence of Lemma 3.1, we have the following property.

Corollary 3.3 *For $k \geq 0$, the polynomial $H_k(x)$ is symmetric.*

In the array representation, the symmetry of $H_k(x)$ means that for $0 \leq i \leq k+1$ and $0 \leq j \leq 2k+1$,

$$h_k(i,j) = h_k(k+1-i, 2k+1-j). \qquad (3.10)$$

It is clear from Lemma 3.1 that the coefficients of $T_k(x)$ form a subsequence of those of $H_k(x)$. We shall prove that for $k \geq 0$, $H_k(x)$ is unimodal.

Theorem 3.4 *The polynomial $H_k(x)$ is unimodal for all $k \geq 0$.*

To prove Theorem 3.4, we introduce the polynomials $G_k(x)$ which will be used to derive a recurrence relation of the coefficients of $H_k(x)$.

Based on the definition (3.1) of $H_k(x)$, we define

$$\begin{aligned} G_k(x) = \ & \frac{1}{x}\sum_{l=0}^{k} B_{k-l}(x^{2k+4})(x^{2k+4}-1)^l \sum_{s=l}^{k} \binom{s}{l} x^{2k+3-s} \\ & + \sum_{l=0}^{k} B_{k-l}(x^{-2k-4})(x^{-2k-4}-1)^l \sum_{s=l}^{k} \binom{s}{l} x^{2(k+1)(k+2)+s}. \end{aligned} \tag{3.11}$$

Let g_k be an array representation of $G_k(x)$ such that

$$G_k(x) = \sum_{i=0}^{k+1} \sum_{j=0}^{2k+3} g_k(i,j) x^{2(k+2)i+j}.$$

We claim that the array g_k can be obtained from h_k by adding a column of zeros after the $(k+1)$-st column and adding a column of zeros after the $2(k+1)$-st column of h_k. The verification of this fact is similar to that of Lemma 3.1, hence the details are ommitted. Table 3.7 gives the array g_2.

0	0	0	0	0	0	1	0
1	2	4	0	4	6	6	0
6	6	4	0	4	2	1	0
1	0	0	0	0	0	0	0

Table 3.7: The array g_2

Lemma 3.5 *For $k \geq 0$, we have*

$$H_{k+1}(x) = G_k(x) \cdot (x + x^2 + \cdots + x^{2k+4}) \tag{3.12}$$

Proof. We aim to show that

$$(1-x) \cdot H_{k+1}(x) = xG_k(x) \cdot (1 - x^{2k+4}), \tag{3.13}$$

which is equivalent to (3.12). By the definition of $H_k(x)$ in (3.1), we see that $(1-x) \cdot H_{k+1}(x)$ equals

$$(1-x)\sum_{l=0}^{k+1} B_{k+1-l}(x^{2k+4})(x^{2k+4}-1)^l \sum_{s=l}^{k+1} \binom{s}{l} x^{2k+3-s}$$

$$
\begin{aligned}
&+(1-x)\sum_{l=0}^{k+1}B_{k+1-l}(x^{-2k-4})(x^{-2k-4}-1)^l\sum_{s=l}^{k+1}\binom{s}{l}x^{2(k+2)^2+s}\\
&=(1-x)\sum_{l=1}^{k+1}B_{k+1-l}(x^{2k+4})(x^{2k+4}-1)^l\sum_{s=l}^{k+1}\binom{s}{l}x^{2k+3-s}\\
&\quad+(1-x)\sum_{l=1}^{k+1}B_{k+1-l}(x^{-2k-4})(x^{-2k-4}-1)^l\sum_{s=l}^{k+1}\binom{s}{l}x^{2(k+2)^2+s}\\
&\quad+(1-x)B_{k+1}(x^{2k+4})\sum_{s=0}^{k+1}x^{2k+3-s}+(1-x)B_{k+1}(x^{-2k-4})\sum_{s=0}^{k+1}x^{2(k+2)^2+s}\\
&=-\sum_{l=0}^{k}B_{k-l}(x^{2k+4})(x^{2k+4}-1)^{l+1}\sum_{s=l}^{k}\binom{s}{l}x^{2k+3-s}\\
&\quad+\sum_{l=0}^{k}B_{k-l}(x^{-2k-4})(x^{-2k-4}-1)^{l+1}\sum_{s=l}^{k}\binom{s}{l}x^{2(k+2)^2+s+1}\\
&\quad+\sum_{l=0}^{k}B_{k-l}(x^{2k+4})(x^{2k+4}-1)^{l+1}\binom{k+1}{l+1}x^{k+2}\\
&\quad-\sum_{l=0}^{k}B_{k-l}(x^{-2k-4})(x^{-2k-4}-1)^{l+1}\binom{k+1}{l+1}x^{(k+2)(2k+5)}\\
&\quad+B_{k+1}(x^{2k+4})x^{k+2}(1-x^{k+2})+B_{k+1}(x^{-2k-4})x^{2(k+2)^2}(1-x^{k+2}). \qquad (3.14)
\end{aligned}
$$

On the other hand, by the definition of $G_k(x)$ in (3.11), we find that

$$
\begin{aligned}
xG_k(x)\cdot(1-x^{2k+4}) &= -\sum_{l=0}^{k}B_{k-l}(x^{2k+4})(x^{2k+4}-1)^{l+1}\sum_{s=l}^{k}\binom{s}{l}x^{2k+3-s}\\
&\quad+\sum_{l=0}^{k}B_{k-l}(x^{-2k-4})(x^{-2k-4}-1)^{l+1}\sum_{s=l}^{k}\binom{s}{l}x^{2(k+2)^2+s+1}.
\end{aligned}
$$

Comparing the above expression for $xG_k(x)\cdot(1-x^{2k+4})$ and the the first two summations in (3.14), to prove (3.13), it suffices to show that

$$
\begin{aligned}
&B_{k+1}(x^{2k+4})x^{2k+4}-B_{k+1}(x^{-2k-4})x^{2(k+2)^2}\\
&\quad=\sum_{l=0}^{k+1}B_{k+1-l}(x^{2k+4})(x^{2k+4}-1)^l\binom{k+1}{l}x^{k+2}\\
&\qquad-\sum_{l=0}^{k+1}B_{k+1-l}(x^{-2k-4})(x^{-2k-4}-1)^l\binom{k+1}{l}x^{(k+2)(2k+5)}. \qquad (3.15)
\end{aligned}
$$

It is known that the type B Eulerian polynomial $B_n(t)$ is a symmetric polynomial of degree n, that is,

$$B_n(t) = B_n(t^{-1})t^n,$$

see Brenti [1]. Hence we have

$$B_{k+1}(x^{2k+4})x^{2k+4} - B_{k+1}(x^{-2k-4})x^{2(k+2)^2} = 0.$$

Thus (3.15) is equivalent to the following relation

$$\begin{aligned}
&\sum_{l=0}^{k+1} B_{k+1-l}(x^{2k+4})(x^{2k+4}-1)^l \binom{k+1}{l} \\
&\qquad = \sum_{l=0}^{k+1} B_{k+1-l}(x^{-2k-4})(x^{-2k-4}-1)^l \binom{k+1}{l} x^{2(k+2)^2}. \qquad (3.16)
\end{aligned}$$

Setting $t = x^{2k+4}$ and $n = k+1$, (3.16) can be rewritten as

$$\sum_{l=0}^{n} B_{n-l}(t)(t-1)^l \binom{n}{l} = \sum_{l=0}^{n} B_{n-l}(t^{-1})(t^{-1}-1)^l \binom{n}{l} t^{n+1}. \qquad (3.17)$$

To prove (3.17), we need the following formula

$$\sum_{n\geq 0} B_n(t)\frac{x^n}{n!} = \frac{(1-t)e^{x(1-t)}}{1-te^{2x(1-t)}}, \qquad (3.18)$$

which was obtained by Chow and Gessel [2]. Using (3.18), we get

$$\begin{aligned}
&\sum_{n>1}\sum_{j=0}^{n} B_{n-j}(t)(t-1)^j \binom{n}{j} \frac{x^n}{n!} \\
&\qquad = \Big(\sum_{n\geq 0} B_n(t)\frac{x^n}{n!}\Big)\Big(\sum_{n\geq 0}(t-1)^n\frac{x^n}{n!}\Big) - 1 \\
&\qquad = \frac{te^{2x(1-t)}-t}{1-te^{2x(1-t)}}. \qquad (3.19)
\end{aligned}$$

Similarly, using (3.18) we find that

$$\begin{aligned}
&\sum_{n>1}\sum_{j=0}^{n} B_{n-j}(t^{-1})(t^{-1}-1)^j \binom{n}{j} t^{n+1}\frac{x^n}{n!} \\
&\qquad = t\Big(\sum_{n\geq 0} B_n(t^{-1})\frac{x^n}{n!}\Big)\Big(\sum_{n\geq 0}(t-1)^n\frac{(tx)^n}{n!}\Big) - t \\
&\qquad = \frac{te^{2x(1-t)}-t}{1-te^{2x(1-t)}}. \qquad (3.20)
\end{aligned}$$

Combining (3.19) and (3.20), we arrive at (3.17). This completes the proof. ∎

Based on Lemma 3.5 and the relationship between the array representation of $H_k(x)$ and the array representation of $G_k(x)$, we establish the following recurrence relations for the array representation of $H_k(x)$.

Corollary 3.6 *For $0 \leq i \leq k+1$ and $0 \leq j \leq k$, we have*

$$\begin{aligned} h_k(i,j) &= h_{k-1}(i,0) + h_{k-1}(i,1) + \cdots + h_{k-1}(i,j-1) \\ &\quad + h_{k-1}(i-1,j) + h_{k-1}(i-1,j+1) + \cdots + h_{k-1}(i-1,2k-1), \end{aligned} \tag{3.21}$$

and for $0 \leq i \leq k+1$ and $k+1 \leq j \leq 2k+1$, we have

$$\begin{aligned} h_k(i,j) &= h_{k-1}(i,0) + h_{k-1}(i,1) + \cdots + h_{k-1}(i,j-2) \\ &\quad + h_{k-1}(i-1,j-1) + h_{k-1}(i-1,j) + \cdots + h_{k-1}(i-1,2k-1), \end{aligned} \tag{3.22}$$

where we assume that $h_k(i,j) = 0$ when $i < 0$.

We are now in a position to complete the proof of Theorem 3.4.

Proof of Theorem 3.4. We proceed by induction on k. For $k = 0$, by the expression (3.1) of $H_k(x)$, we get $H_0(x) = x + x^2$, which is unimodal. Assume that $H_{k-1}(x)$ is unimodal, where $k \geq 1$. We aim to prove that $H_k(x)$ is unimodal.

Assume that $k \geq 1$. Let $(a_0, a_1, \cdots, a_{2k^2+2k-1})$ denote the sequence of coefficients of $H_{k-1}(x)$. By the symmetry of $H_{k-1}(x)$ as given in Corollary 3.3, we have $a_i = a_{2k^2+2k-1-i}$. Hence, by the induction hypothesis, we have

$$a_0 \leq a_1 \leq \cdots \leq a_{k^2+k-1}. \tag{3.23}$$

Assume that $(b_0, b_1, \cdots, b_{2k^2+6k+3})$ is the sequence of coefficients of $H_k(x)$. By the symmetry of $H_k(x)$, to prove that $H_k(x)$ is unimodal, it suffices to prove that

$$b_0 \leq b_1 \leq \cdots \leq b_{k^2+3k+1}. \tag{3.24}$$

Indeed, we can restate the above inequalities in terms of the array representation h_k of $H_k(x)$. Recall that

$$H_k(x) = \sum_{i=0}^{k+1} \sum_{j=0}^{2k+1} h_k(i,j) x^{2(k+1)i+j}.$$

Clearly, $h_k(i,j) = b_{2(k+1)i+j}$ for $0 \leq i \leq k+1$ and $0 \leq j \leq 2k+1$. When k is odd, (3.24) can be restated as follows,

(i) $h_k(i,j+1) - h_k(i,j) \geq 0$ for $0 \leq i \leq \lfloor \frac{k+2}{2} \rfloor - 1$ and $0 \leq j \leq 2k$;

(ii) $h_k(i,j+1) - h_k(i,j) \geq 0$ for $i = \lfloor \frac{k+2}{2} \rfloor$ and $0 \leq j \leq k-1$;

(iii) $h_k(i,0) - h_k(i-1, 2k+1) \geq 0$ for $1 \leq i \leq \lfloor \frac{k+2}{2} \rfloor$.

Similarly, when k is even, (3.24) can be recast into the following assertions:

(iv) $h_k(i, j+1) - h_k(i,j) \geq 0$ for $0 \leq i \leq \frac{k}{2}$ and $0 \leq j \leq 2k$;

(v) $h_k(i,0) - h_k(i-1, 2k+1) \geq 0$ for $1 \leq i \leq \frac{k}{2}$.

We now proceed to prove the above assertions. It follows from (3.21) that for $0 \leq i \leq k+1$ and $0 \leq j \leq k-1$,

$$h_k(i, j+1) - h_k(i,j) = h_{k-1}(i,j) - h_{k-1}(i-1,j). \tag{3.25}$$

Using (3.22), we find that for $0 \leq i \leq k+1$ and $k+1 \leq j \leq 2k$,

$$h_k(i, j+1) - h_k(i,j) = h_{k-1}(i,j-1) - h_{k-1}(i-1,j-1). \tag{3.26}$$

Moreover, by (3.21) and (3.22), it is easy to check that for $0 \leq i \leq k+1$,

$$h_k(i,k) = h_k(i,k+1), \tag{3.27}$$

$$h_k(i,0) = h_k(i-1,2k+1). \tag{3.28}$$

We first consider the case when k is odd. To prove (i), we assume that $0 \leq i \leq \lfloor \frac{k+2}{2} \rfloor - 1$ and $0 \leq j \leq 2k$. Here are three subcases. When $0 \leq j \leq k-1$, we claim that $h_k(i,j+1) - h_k(i,j) \geq 0$. From (3.25) we see that

$$h_k(i,j+1) - h_k(i,j) = a_{2ki+j} - a_{2ki-2k+j}.$$

Since $0 \leq i \leq \lfloor \frac{k+2}{2} \rfloor - 1$ and $0 \leq j \leq k-1$, noting $2\lfloor \frac{k+2}{2} \rfloor = k+1$, we find that

$$2ki + j \leq 2k\Big(\Big\lfloor \frac{k+2}{2} \Big\rfloor - 1\Big) + k - 1 = k^2 - 1.$$

Clearly, we have $2ki + j \geq 2ki - 2k + j$. Thus we may use the induction hypothesis to deduce that $a_{2ki+j} - a_{2ki-2k+j} \geq 0$, which is equivalent to the claim.

When $k+1 \leq j \leq 2k$, we claim that $h_k(i,j+1) - h_k(i,j) \geq 0$. By (3.26), we get

$$h_k(i,j+1) - h_k(i,j) = a_{2ki+j-1} - a_{2ki-2k+j-1}.$$

Using the same argument as in the case when $0 \leq j \leq k-1$, we deduce that

$$2ki + j - 1 \leq 2k\Big(\Big\lfloor \frac{k+2}{2} \Big\rfloor - 1\Big) + 2k - 1 = k^2 + k - 1.$$

Similarly, we have $2ki + j - 1 \geq 2ki - 2k + j - 1$. Hence we may use the induction hypothesis to deduce that $a_{2ki+j-1} - a_{2ki-2k+j-1} \geq 0$, as claimed.

Recall that $h_k(i,k+1) = h_k(i,k)$ for $0 \le i \le k+1$ as given in (3.27). On the other hand, when $j=k$, assertion (i) becomes the relation $h_k(i,k+1) - h_k(i,k) \ge 0$ for $0 \le i \le \lfloor \frac{k+2}{2} \rfloor - 1$, which is valid since the equality holds. Combining the above three cases, assertion (i) is proved.

To prove (ii), we assume that $i = \lfloor \frac{k+2}{2} \rfloor$ and $0 \le j \le k-1$. We claim that $h_k(i,j+1) - h_k(i,j) \ge 0$. By (3.25) and the symmetry relation (3.10), we find that

$$\begin{aligned} h_k(i,j+1) - h_k(i,j) &= h_{k-1}(i,j) - h_{k-1}(i-1,j) \\ &= h_{k-1}(k-i, 2k-1-j) - h_{k-1}(i-1,j) \\ &= a_{2k(k-i)+2k-1-j} - a_{2k(i-1)+j}. \end{aligned}$$

Since $i = \lfloor \frac{k+2}{2} \rfloor$ and $0 \le j \le k-1$, we see that

$$2k(k-i) + 2k - 1 - j \le 2k\Big(k - \Big\lfloor \frac{k+2}{2} \Big\rfloor\Big) + 2k - 1 = k^2 + k - 1,$$

and

$$2k(k-i) + 2k - 1 - j \ge 2k(i-1) + j.$$

Hence we may use the induction hypothesis to deduce that $a_{2k(k-i)+2k-1-j} - a_{2k(i-1)+j} \ge 0$. This proves the claim, and hence assertion (ii) holds.

Note that by (3.28), we have $h_k(i,0) = h_k(i-1, 2k+1)$ for $1 \le i \le \lfloor \frac{k+2}{2} \rfloor$. This proves assertion (iii).

Next we turn to the case when k is even.

To prove (iv), we assume that $0 \le i \le \frac{k}{2}$ and $0 \le j \le 2k$. When $0 \le i \le \frac{k}{2}$ and $0 \le j \le k-1$, we claim that $h_k(i,j+1) - h_k(i,j) \ge 0$. By (3.25), we see that

$$h_k(i,j+1) - h_k(i,j) = a_{2ki+j} - a_{2ki-2k+j}.$$

By the assumptions $0 \le i \le \frac{k}{2}$ and $0 \le j \le k-1$, we see that

$$2ki + j \le k^2 + k - 1.$$

So we may use the induction hypothesis to deduce that $a_{2ki+j} - a_{2ki-2k+j} \ge 0$. This proves the claim.

When $0 \le i \le \frac{k}{2} - 1$ and $k+1 \le j \le 2k$, we claim that $h_k(i,j+1) - h_k(i,j) \ge 0$. By (3.26), we find that

$$h_k(i,j+1) - h_k(i,j) = a_{2ki+j-1} - a_{2ki-2k+j-1}.$$

By the assumptions $0 \le i \le \frac{k}{2} - 1$ and $k+1 \le j \le 2k$, we see that

$$2ki + j - 1 \le k^2 - 1.$$

Hence the induction hypothesis can be used to get $a_{2ki+j-1} - a_{2ki-2k+j-1} \geq 0$, which is equivalent to the claim.

When $i = \frac{k}{2}$ and $k+1 \leq j \leq 2k$, we claim that $h_k(i, j+1) - h_k(i, j) \geq 0$. By (3.26) and the symmetry relation (3.10), we find that

$$\begin{aligned} h_k(i, j+1) - h_k(i, j) &= h_{k-1}(i, j-1) - h_{k-1}(i-1, j-1) \\ &= h_{k-1}(k-i, 2k-j) - h_{k-1}(i-1, j-1) \\ &= a_{2k(k-i)+2k-j} - a_{2k(i-1)+j-1} \end{aligned}$$

Using the assumptions $i = \frac{k}{2}$ and $k+1 \leq j \leq 2k$, we get

$$2k(k-i) + 2k - j \leq k^2 + k - 1,$$

and

$$2k(k-i) + 2k - j \geq 2k(i-1) + j - 1.$$

By the induction hypothesis, we obtain that $a_{2k(k-i)+2k-j} - a_{2k(i-1)+j-1} \geq 0$. This proves the claim.

Using the fact $h_k(i, k) = h_k(i, k+1)$ for $0 \leq i \leq k+1$ as given in (3.27), it can be easily checked that assertion (iv) is true for $j = k$. So we proved assertion (iv) for all the cases of j. Clearly, by (3.28), we have $h_k(i, 0) = h_k(i-1, 2k+1)$ for $1 \leq i \leq \frac{k}{2}$. This confirms assertion (v), and so the proof is complete. ∎

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