Harnack Inequality for Distribution Dependent Second-order Stochastic Differential Equations*

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Abstract

By investigating the regularity of the nonlinear semigroup P_t^* associated to the distribution dependent second-order stochastic differential equations, the Harnack inequality is derived when the drift is Lipschitz continuous in the measure variable under the distance induced by the functions being $\beta(\beta > \frac{2}{3})$-Hölder continuous on the degenerate component and square root of Dini continuous on the non-degenerate one. The results extend the existing ones in which the drift is Lipschitz continuous in L^2-Wasserstein distance.

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1 Introduction

The second-order stochastic differential equations (SDEs), as a classical degenerate model, includes the kinetic Fokker-Planck equation (see [15]). In [4] the authors study the regularity of stochastic kinetic equations. [5] investigates the Bismut formula, gradient estimate and Harnack inequality by using the method of coupling by change of measure. [20] and [21] focus on the derivative formula. [18] proves the the hypercontractivity. One can refer to the references in the papers mentioned above for more related results.

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On the other hand, the McKean-Vlasov SDEs, presented in [10], can be used to characterize the nonlinear Fokker-Planck-Kolmogorov equations. Recently, there are plentiful results on the study of McKean-Vlasov SDEs, including the well-posedness, Harnack inequality, Bismut derivative formula, exponential ergodicity, estimate of heat kernel, see [1, 2, 6, 7, 8, 11, 13, 14, 17, 19, 22] and references therein for more details. For the well-posedness, the drifts can be not continuous in the measure variable under the weak topology, for instance [8, 14, 19] and so on. However, with respect to the Harnack inequality, most results concentrate on the case that the coefficients are Lipschitz continuous in L^2-Wasserstein distance.

Quite recently, the first author and his co-author have established the log-Harnack inequality and Bismut derivative formula for non-degenerate McKean-Vlasov SDEs in [9], where for the log-Harnack inequality, the drifts are only assumed to be Lipschitz continuous under the distance induced by square root of Dini continuous functions, which allows the drifts even being not Dini continuous in the L^2-Wasserstein distance.

In this paper, we adopt the similar technique in [9] to study the Harnack inequality for distribution dependent second-order SDEs, where the drift is Lipschitz continuous in the measure variable under the distance induced by the functions being $\beta(\beta > \frac{2}{3})$-Hölder continuous on the degenerate component and square root of Dini continuous on the non-degenerate one. Compared with [9], we need to calculate the gradient estimate of the semigroup with frozen distribution in non-degenerate and degenerate components respectively. Moreover, the gradient estimate on the degenerate component of the semigroup acting on a function only depending on the non-degenerate component is also derived, which is crucial in the proof of the main result, see Theorem 3.1(2) below.

Let $n \in \mathbb{N}^+$ and $\mathscr{P}(\mathbb{R}^n)$ be the set of all probability measures on $\mathbb{R}^n$ equipped with the weak topology. For $k \geq 1$, define

$$\mathscr{P}_k(\mathbb{R}^n) := \left\{\mu \in \mathscr{P}(\mathbb{R}^n) : \|\mu\|_k := \left(\int_{\mathbb{R}^n} |x|^k \mu(\mathrm{d}x)\right)^{\frac{1}{k}} < \infty\right\}.$$

$\mathscr{P}_k(\mathbb{R}^n)$ is a Polish space under the L^k-Wasserstein distance

$$\mathbb{W}_k(\mu,\nu) = \inf_{\pi \in \mathscr{C}(\mu,\nu)} \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} |x-y|^k \pi(\mathrm{d}x,\mathrm{d}y)\right)^{\frac{1}{k}}, \quad \mu,\nu \in \mathscr{P}_k(\mathbb{R}^n),$$

where $\mathscr{C}(\mu,\nu)$ is the set of all couplings of μ and ν.

For any $x \in \mathbb{R}^{2d}$, let $x^{(1)}$ denote the first d components and $x^{(2)}$ denote the last d components, i.e. $x = (x^{(1)}, x^{(2)}) \in \mathbb{R}^{2d}$ with $x^{(i)} \in \mathbb{R}^d, i = 1,2$. Throughout the paper, fix $T > 0$. Consider the following distribution dependent second-order SDEs on $\mathbb{R}^{2d}$:

$$\begin{cases} \mathrm{d}X_t^{(1)} = X_t^{(2)}\mathrm{d}t, \\ \mathrm{d}X_t^{(2)} = B_t(X_t, \mathscr{L}_{X_t})\mathrm{d}t + \sigma_t \mathrm{d}W_t, \end{cases} \tag{1.1}$$

where $W = (W_t)_{t\geq 0}$ is a d-dimensional standard Brownian motion with respect to a complete filtration probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$, and $\sigma : [0,T] \to \mathbb{R}^d \otimes \mathbb{R}^d$, $B : [0,T] \times \mathbb{R}^{2d} \times \mathscr{P}(\mathbb{R}^{2d}) \to \mathbb{R}^d$ are measurable.

Recall that for two probability measures $\mu, \nu \in \mathscr{P}(\mathbb{R}^n)$, the relative entropy and total variation distance are defined as follows:

$$\mathrm{Ent}(\nu|\mu) := \begin{cases} \int_{\mathbb{R}^n} (\log \frac{\mathrm{d}\nu}{\mathrm{d}\mu}) \mathrm{d}\nu, & \text{if } \nu \text{ is absolutely continuous with respect to } \mu, \\ \infty, & \text{otherwise;} \end{cases}$$

and

$$\|\mu - \nu\|_{var} := \sup_{|f| \le 1} |\mu(f) - \nu(f)|.$$

By Pinsker's inequality (see [12]),

$$\|\mu - \nu\|_{var}^2 \le 2\mathrm{Ent}(\nu|\mu), \quad \mu, \nu \in \mathscr{P}(\mathbb{R}^n). \tag{1.2}$$

Throughout the paper, we will use C or c as a constant, the values of which may depend on T and may change from one place to another. For a function f on $\mathbb{R}^{2d}$ and $i = 1, 2$, let $\nabla^{(i)} f(x)$ stand for the gradient with respect to $x^{(i)}$.

The paper is organized as follows: In Section 2, we state the main results, i.e. the Harnack inequality for distribution dependent second-order SDEs and the proof is provided in Section 3; In Section 4, the well-posedness for degenerate McKean-Vlasov SDEs is investigated, where the drifts are assumed to be Lipschitz continuous in the measure variable under the weighted variation distance plus the L^k-Wasserstein distance.

2 Main Results

Let

$$\mathscr{A} := \left\{ \varphi : [0, \infty) \to [0, \infty) \text{ is increasing and concave, } \varphi(0) = 0, \ \int_0^1 \frac{\varphi(r)^2}{r} \mathrm{d}r \in (0, \infty) \right\}.$$

For $\beta \in (0, 1]$, $\varphi \in \mathscr{A}$, define

$$\rho_{\beta,\varphi}(x, y) = |x^{(1)} - y^{(1)}|^\beta + \varphi(|x^{(2)} - y^{(2)}|), \quad x, y \in \mathbb{R}^{2d}.$$

Since $\varphi \in \mathscr{A}$, we conclude that $(\mathbb{R}^{2d}, \rho_{\beta,\varphi})$ is a Polish space. For a (real or vector valued) function f on $\mathbb{R}^{2d}$, let

$$[f]_{\beta,\varphi} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho_{\beta,\varphi}(x, y)}.$$

Let

$$\mathscr{P}_{\beta,\varphi}(\mathbb{R}^{2d}) := \left\{ \mu \in \mathscr{P}(\mathbb{R}^{2d}) : \int_{\mathbb{R}^{2d}} (|x^{(1)}|^\beta + \varphi(|x^{(2)}|)) \mu(\mathrm{d}x) < \infty \right\}.$$

Define the Wasserstein distance induced by $\rho_{\beta,\varphi}$:

$$\mathbb{W}_{\beta,\varphi}(\mu, \nu) := \inf_{\pi \in \mathscr{C}(\mu,\nu)} \int_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} \rho_{\beta,\varphi}(x, y) \pi(\mathrm{d}x, \mathrm{d}y), \quad \mu, \nu \in \mathscr{P}_{\beta,\varphi}(\mathbb{R}^{2d}).$$

Then $\mathbb{W}_{\beta,\varphi}$ is a complete distance on $\mathscr{P}_{\beta,\varphi}(\mathbb{R}^{2d})$. Moreover, we have the Kantorovich dual formula (see e.g. [3, Theorem 5.10]):

$$\mathbb{W}_{\beta,\varphi}(\mu,\nu) := \sup_{[f]_{\beta,\varphi}\leq 1} |\mu(f)-\nu(f)|, \quad \mu,\nu\in\mathscr{P}_{\beta,\varphi}(\mathbb{R}^{2d}).$$

Noting that for any $\mu,\nu\in\mathscr{P}_{\beta,\varphi}(\mathbb{R}^{2d})$, $\{f : f\in\mathscr{B}_b(\mathbb{R}^{2d}), [f]_{\beta,\varphi}\leq 1\}$ is dense in $\{f : [f]_{\beta,\varphi}\leq 1\}$ under $L^1(\mu+\nu)$, we have

$$\mathbb{W}_{\beta,\varphi}(\mu,\nu) := \sup_{f\in\mathscr{B}_b(\mathbb{R}^{2d}),[f]_{\beta,\varphi}\leq 1} |\mu(f)-\nu(f)|, \quad \mu,\nu\in\mathscr{P}_{\beta,\varphi}(\mathbb{R}^{2d}).$$

Furthermore, it follows from the concavity of φ and $\varphi(0)=0$ that

$$\varphi(rt)\leq r\varphi(t), \quad t>0, r\geq 1, \tag{2.1}$$

see [9, (2.1)] for more details. By (2.1) for $t=1$ and $\varphi(r)\leq\varphi(1), r\in[0,1]$, we conclude that

$$\varphi(r)\leq\varphi(1)(1+r), \quad r\geq 0. \tag{2.2}$$

It follows from (2.2) that for any $k\geq 1$,

$$\sup_{[f]_{\beta,\varphi}\leq 1}|f(x)-f(0)|\leq |x^{(1)}|^\beta+\varphi(|x^{(2)}|)\leq 2(\varphi(1)+1)(1+|x|^k), \quad x\in\mathbb{R}^{2d}. \tag{2.3}$$

Therefore $\mathscr{P}_k(\mathbb{R}^{2d})\subset\mathscr{P}_{\beta,\varphi}(\mathbb{R}^{2d})$ for $k\geq 1$ and

$$\frac{1}{2(\varphi(1)+1)}\mathbb{W}_{\beta,\varphi}(\mu,\nu)\leq \|\mu-\nu\|_{k,var} := \sup_{|f|\leq 1+|\cdot|^k}|\mu(f)-\nu(f)|, \quad \mu,\nu\in\mathscr{P}_k(\mathbb{R}^{2d}). \tag{2.4}$$

To obtain the Harnack inequality, we make the following assumptions.

(A1) σ_t is invertible and $\|\sigma_t\|+\|\sigma_t^{-1}\|$ is bounded in $t\in[0,T]$.

(A2) For any $t\in[0,T], \gamma\in\mathscr{P}_2(\mathbb{R}^{2d})$, $B_t(\cdot,\gamma)$ is differentiable. Moreover, there exist $K_B>0$, $\varphi\in\mathscr{A}$ and $\beta\in(\frac{2}{3},1]$ such that

$$\begin{aligned}&|\nabla B_t(x,\gamma)|\leq K_B,\\ &|B_t(x,\gamma)-B_t(x,\bar\gamma)|\leq K_B(\mathbb{W}_2(\gamma,\bar\gamma)+\mathbb{W}_{\beta,\varphi}(\gamma,\bar\gamma)),\\ &|B_t(0,\delta_0)|\leq K_B, \quad t\in[0,T], x\in\mathbb{R}^{2d}, \gamma,\bar\gamma\in\mathscr{P}_2(\mathbb{R}^{2d}).\end{aligned}$$

By **(A2)** and (2.4) for $k=1$, there exist constants $C_1,C_2>0$ such that

$$\begin{aligned}|B_t(x,\gamma)|&\leq C_1(1+|x|+(\mathbb{W}_2(\gamma,\delta_0)+\mathbb{W}_{\beta,\varphi}(\gamma,\delta_0)))\\ &\leq C_2(1+|x|+\|\gamma\|_2), \quad t\in[0,T], x\in\mathbb{R}^{2d}, \gamma\in\mathscr{P}_2(\mathbb{R}^{2d}).\end{aligned} \tag{2.5}$$

So, according to Theorem 4.1 below, under **(A1)-(A2)**, (1.1) is well-posed in $\mathscr{P}_2(\mathbb{R}^{2d})$. Denote the solution to (1.1) with $\mathscr{L}_{X_0} = \mu_0 \in \mathscr{P}_2(\mathbb{R}^{2d})$ by $X_t^{\mu_0}$. Let $P_t^*\mu_0 = \mathscr{L}_{X_t^{\mu_0}}$ and

$$P_t f(\mu_0) := \mathbb{E}[f(X_t^{\mu_0})] = \int_{\mathbb{R}^{2d}} f \mathrm{d}\{P_t^*\mu_0\}.$$

For simplicity, we denote $X_t^x = X_t^{\delta_x}$ and $P_t f(x) = P_t f(\delta_x)$ for $x \in \mathbb{R}^{2d}$. The next result characterizes the Harnack inequality for (1.1).

Theorem 2.1. *Assume* **(A1)-(A2)**. *Then the following assertions hold.*

(1) There exists a constant $c > 0$ such that for any positive $f \in \mathscr{B}_b(\mathbb{R}^{2d})$,

$$P_t \log f(\tilde{\gamma}) \le \log P_t f(\gamma) + \frac{c}{t^3}\mathbb{W}_2(\gamma,\tilde{\gamma})^2, \quad t \in (0,T], \gamma, \tilde{\gamma} \in \mathscr{P}_2(\mathbb{R}^{2d}). \tag{2.6}$$

Consequently, it holds

$$\|P_t^*\gamma - P_t^*\tilde{\gamma}\|_{var}^2 \le 2\mathrm{Ent}(P_t^*\gamma|P_t^*\tilde{\gamma}) \le \frac{2c}{t^3}\mathbb{W}_2(\gamma,\tilde{\gamma})^2, \quad t \in (0,T], \gamma, \tilde{\gamma} \in \mathscr{P}_2(\mathbb{R}^{2d}).$$

(2) There exists $c > 0$ such that for any $p > 1$, $t \in (0,T]$, $\gamma, \tilde{\gamma} \in \mathscr{P}_2(\mathbb{R}^{2d})$ and $f \in \mathscr{B}_b^+(\mathbb{R}^{2d})$,

$$\begin{aligned}(P_t f(\tilde{\gamma}))^p \le P_t f^p(\gamma) \exp\left\{\frac{cp}{(p-1)}\mathbb{W}_2(\gamma,\tilde{\gamma})^2\right\} \\ \times \inf_{\pi \in \mathscr{C}(\gamma,\tilde{\gamma})} \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \exp\left\{\frac{cp}{(p-1)t^3}|x-y|^2\right\}\pi(\mathrm{d}x,\mathrm{d}y).\end{aligned}$$

Remark 2.2. *(1) In Theorem 2.1, we assume*

$$\sup_{t\in[0,T], x\in\mathbb{R}^{2d}, \gamma\in\mathscr{P}_2(\mathbb{R}^{2d})} |\nabla B_t(x,\gamma)| < \infty,$$

which coincides with the assumption in [5, Theorem 4.4] when B is distribution free, see also [5, Theorem 4.5] for more general case in which ∇B can be unbounded.

(2) In [9], the first author and his co-author derive the log-Harnack inequality for singular distribution dependent SDEs with non-degenerate and multiplicative noise, where the drift is assumed to be Lipschitz continuous in the measure variable under $\mathbb{W}_2 + \mathbb{W}_\varphi$ for some $\varphi \in \mathscr{A}$. Compared with [9], in the present degenerate case, $\mathbb{W}_\varphi$ is replaced by $\mathbb{W}_{\beta,\varphi}$ for some $\beta > \frac{2}{3}$ and $\varphi \in \mathscr{A}$ due to the attendance of the degenerate component.

(3) In addition, different from [9], where the power of the time variable in the log-Harnack inequality is -1, the right hand side of (2.6) contains t^{-3} instead, which is caused by the degenerate component. To illustrate this point, we consider the simplest second-order model:

$$\begin{cases}\mathrm{d}X_t^{(1)} = X_t^{(2)}\mathrm{d}t, \\ \mathrm{d}X_t^{(2)} = \mathrm{d}W_t, \quad X_0 = x \in \mathbb{R}^{2d}.\end{cases}$$

Note that X_t is a $2d$-dimensional Gaussian process with covariance matrix equal to

$$\begin{pmatrix} \frac{t^3}{3} I_{d\times d} & \frac{t^2}{2} I_{d\times d} \\ \frac{t^2}{2} I_{d\times d} & t I_{d\times d} \end{pmatrix}.$$

This means that it is reasonable for t^{-3} instead of t^{-1} in (2.6).

3 Proof of Theorem 2.1

Before moving on, we give a brief outline of the strategy for proving Theorem 2.1. Observe that B is singular in the measure variable, i.e.

$$|B_t(x,\gamma) - B_t(x,\bar{\gamma})| \le K_B(\mathbb{W}_2(\gamma,\bar{\gamma}) + \mathbb{W}_{\beta,\varphi}(\gamma,\bar{\gamma})).$$

To derive the log-Harnack inequality (2.6), it is naturally for us to prove a regularity estimate like

$$\mathbb{W}_{\beta,\varphi}(P_t^*\gamma, P_t^*\bar{\gamma}) \le g(t)\mathbb{W}_2(\gamma,\bar{\gamma}), \quad t \in (0,T] \tag{3.1}$$

for some reasonable function $g : (0,T] \to (0,\infty)$, which is the main result of Lemma 3.5 below. To this end, the gradient estimate implied by the Bismut formula for distribution independent second-order SDEs is required. Once (3.1) is in hand, one can adopt the coupling by change of measure to derive the Harnack inequality in Theorem 2.1, see Proof of Theorem 2.1 below for more details.

3.1 Bismut formula for distribution independent second-order S-DEs

The Bismut derivative formula in the first assertion of the following theorem has been established in [5, 16, 20, 21], where the Malliavin calculus or coupling by change of measure plays an important role. The second one is new and will be helpful in the proof of Lemma 3.5 below. For reader's convenience, we will use the coupling by change of measure to complete the proof.

Theorem 3.1. *Assume* **(A1)**-**(A2)** *and $B_t(x,\mu)$ does not depend on μ. Then the following assertions hold.*

(1) Let

$$\gamma_{s,t}(h) := \left[\frac{(t-s)}{t} - \frac{3s(t-s)}{t^2}\right] h^{(2)} - \frac{6s(t-s)}{t^3} h^{(1)}, \quad 0 \le s \le t \le T. \tag{3.2}$$

Denote $\gamma'_{s,t}(h)$ the derivative of $\gamma_{s,t}(h)$ with respect to s. Then for any $f \in \mathscr{B}_b(\mathbb{R}^{2d})$, it holds

$$\nabla_h P_t f(x) = \mathbb{E}[f(X_t^x) N_t(h)], \quad x, h \in \mathbb{R}^{2d}, t \in (0,T]$$

with

$$N_t(h) = \int_0^t \left\langle \sigma_s^{-1}\left[\nabla B_s(X_s^x)\left(h^{(1)} + \int_0^s \gamma_{u,t}(h)\mathrm{d}u, \quad \gamma_{s,t}(h)\right) - \gamma'_{s,t}(h)\right], \mathrm{d}W_s\right\rangle,$$

and

$$|\gamma_{s,t}(h)| \le c(|h^{(2)}| + t^{-1}|h^{(1)}|), \quad |\gamma'_{s,t}(h)| \le c(t^{-1}|h^{(2)}| + t^{-2}|h^{(1)}|), \quad s \in [0,t]$$

for some constant $c > 0$.

(2) For any $f \in \mathscr{B}_b(\mathbb{R}^{2d})$ *with* $f(x^{(1)}, x^{(2)})$ *independent of* $x^{(1)}$,

$$\nabla_v^{(1)} P_t f(x) = \mathbb{E}[f((X_t^x)^{(2)}) \int_0^t \langle \sigma_s^{-1}\nabla_v^{(1)} B_s(X_s^x), \mathrm{d}W_s\rangle], \quad v \in \mathbb{R}^d, t \in [0,T]. \tag{3.3}$$

Consequently, for any $f \in \mathscr{B}_b(\mathbb{R}^{2d})$ *with* $f(x^{(1)}, x^{(2)})$ *independent of* $x^{(1)}$,

$$|\nabla^{(1)} P_t f(x)| \le C\left(\mathbb{E}|f((X_t^x)^{(2)})|^2\right)^{\frac{1}{2}} t^{\frac{1}{2}}, \quad t \in [0,T] \tag{3.4}$$

for some constant $C > 0$.

Proof. (1) Fix $t \in (0,T]$. For any $\varepsilon \in (0,1]$, $h \in \mathbb{R}^{2d}$, let $(Y_s(\varepsilon))_{s\in[0,t]}$ solve the equation

$$\begin{cases} \mathrm{d}(Y_s(\varepsilon))^{(1)} = (Y_s(\varepsilon))^{(2)}\mathrm{d}s, \\ \mathrm{d}(Y_s(\varepsilon))^{(2)} = B_s(X_s^x)\mathrm{d}s + \sigma_s \mathrm{d}W_s + \varepsilon\gamma'_{s,t}(h)\mathrm{d}s, \quad Y_0(\varepsilon) = x + \varepsilon h. \end{cases} \tag{3.5}$$

Then it is easy to see that

$$Y_s(\varepsilon) = X_s^x + \left(\varepsilon h^{(1)} + \varepsilon\int_0^s \gamma_{u,t}(h)\mathrm{d}u, \quad \varepsilon\gamma_{s,t}(h)\right). \quad s \in [0,t], \tag{3.6}$$

In particular, $Y_t(\varepsilon) = X_t^x$ due to (3.2). Let

$$\Phi_s^\varepsilon = B_s(Y_s(\varepsilon)) - B_s(X_s^x) - \varepsilon\gamma'_{s,t}(h), \quad s \in [0,t].$$

Then **(A2)** and (3.6) imply

$$|\Phi_s^\varepsilon| \le K_B\left[\varepsilon|h^{(1)}| + \varepsilon\int_0^s |\gamma_{u,t}(h)|\mathrm{d}u + \varepsilon|\gamma_{s,t}(h)|\right] + \varepsilon|\gamma'_{s,t}(h)|, \quad s \in [0,t]. \tag{3.7}$$

Observing **(A1)** and

$$|\gamma_{s,t}(h)| \le c(|h^{(2)}| + t^{-1}|h^{(1)}|), \quad |\gamma'_{s,t}(h)| \le c(t^{-1}|h^{(2)}| + t^{-2}|h^{(1)}|), \quad s \in [0,t] \tag{3.8}$$

for some constant $c > 0$, Girsanov's theorem implies that

$$\tilde{W}_s := W_s - \int_0^s \sigma_u^{-1}\Phi_u^\varepsilon \mathrm{d}u, \quad s \in [0,t]$$

is a d-dimensional Brownian motion on $[0,t]$ under $\mathbb{Q}_t^\varepsilon = R_t^\varepsilon \mathbb{P}$, where

$$R_t^\varepsilon = \exp\bigg[\int_0^t \langle \sigma_u^{-1}\Phi_u^\varepsilon, \mathrm{d}W_u\rangle - \frac{1}{2}\int_0^t |\sigma_u^{-1}\Phi_u^\varepsilon|^2 \mathrm{d}u\bigg].$$

Then (3.5) reduces to

$$\begin{cases} \mathrm{d}(Y_s(\varepsilon))^{(1)} = M(Y_s(\varepsilon))^{(2)}\mathrm{d}s, \\ \mathrm{d}(Y_s(\varepsilon))^{(2)} = B_s(Y_s(\varepsilon))\mathrm{d}s + \sigma_s \mathrm{d}\tilde{W}_s, \quad Y_0(\varepsilon) = x + \varepsilon h, \end{cases}$$

which yields that the law of $Y_t(\varepsilon)$ under $\mathbb{Q}_t^\varepsilon$ coincides with that of $X_t^{x+\varepsilon h}$ under $\mathbb{P}$. As a result, we get

$$P_t f(x+\varepsilon h) = \mathbb{E}^{\mathbb{Q}_t^\varepsilon} f(Y_t(\varepsilon)) = \mathbb{E}^{\mathbb{Q}_t^\varepsilon} f(X_t^x) = \mathbb{E}[R_t^\varepsilon f(X_t^x)], \quad f \in \mathscr{B}_b(\mathbb{R}^{2d}).$$

It follows from (3.7) and (3.8) that

$$\begin{aligned} \sup_{\varepsilon\in(0,1]} \mathbb{E}\left|\frac{R_t^\varepsilon - 1}{\varepsilon}\right|^2 &= \sup_{\varepsilon\in(0,1]} \frac{\mathbb{E}[(R_t^\varepsilon)^2] - 1}{\varepsilon^2} \\ &\le \sup_{\varepsilon\in(0,1]} \frac{\mathrm{esssup}_\Omega \exp\left\{\int_0^t |\sigma_u^{-1}\Phi_u^\varepsilon|^2 \mathrm{d}u\right\} - 1}{\varepsilon^2} \\ &\le \sup_{\varepsilon\in(0,1]} \mathrm{esssup}_\Omega \left\{ \exp\left\{\int_0^t |\sigma_u^{-1}\Phi_u^\varepsilon|^2 \mathrm{d}u\right\} \frac{\int_0^t |\sigma_u^{-1}\Phi_u^\varepsilon|^2 \mathrm{d}u}{\varepsilon^2} \right\} < \infty, \end{aligned}$$

which implies that $\{\frac{R_t^\varepsilon - 1}{\varepsilon}\}_{\varepsilon\in(0,1]}$ is uniformly integrable. As a result, by the dominated convergence theorem, **(A1)**-**(A2)** and (3.6), we have

$$\lim_{\varepsilon\to 0} \mathbb{E}\left|\frac{R_t^\varepsilon - 1}{\varepsilon} - N_t(h)\right| = 0, \tag{3.9}$$

which derives

$$\nabla_h P_t f(x) = \mathbb{E}[f(X_t^x) N_t(h)], \quad f \in \mathscr{B}_b(\mathbb{R}^{2d}).$$

This combined with (3.8) completes the proof.

(2) For any $x \in \mathbb{R}^{2d}, v \in \mathbb{R}^d$, let $\hat{X}_s^\varepsilon = ((X_s^x)^{(1)} + \varepsilon v, (X_s^x)^{(2)})$. Then it is clear that

$$\begin{cases} \mathrm{d}(\hat{X}_s^\varepsilon)^{(1)} = (\hat{X}_s^\varepsilon)^{(2)}\mathrm{d}s, \\ \mathrm{d}(\hat{X}_s^\varepsilon)^{(2)} = B_s(X_s^x)\mathrm{d}s + \sigma_s \mathrm{d}W_s, \quad \hat{X}_0^\varepsilon = (x^{(1)} + \varepsilon v, x^{(2)}). \end{cases}$$

Rewrite it as

$$\begin{cases} \mathrm{d}(\hat{X}_s^\varepsilon)^{(1)} = (\hat{X}_s^\varepsilon)^{(2)}\mathrm{d}s, \\ \mathrm{d}(\hat{X}_s^\varepsilon)^{(2)} = B_s(\hat{X}_s^\varepsilon)\mathrm{d}s + \sigma_s \mathrm{d}\hat{W}_s, \qquad \hat{X}_0^\varepsilon = (x^{(1)} + \varepsilon v, x^{(2)}). \end{cases} \tag{3.10}$$

with

$$\mathrm{d}\hat{W}_s = \mathrm{d}W_s - \sigma_s^{-1}[B_s(\hat{X}_s^\varepsilon) - B_s(X_s^x)]\mathrm{d}s.$$

Let

$$\eta_s^\varepsilon = \sigma_s^{-1}[B_s(\hat{X}_s^\varepsilon) - B_s(X_s^x)], \quad \hat{R}_t^\varepsilon = \exp\bigg[\int_0^t \langle \eta_u^\varepsilon, \mathrm{d}W_u\rangle - \frac{1}{2}\int_0^t |\eta_u^\varepsilon|^2 \mathrm{d}u\bigg].$$

Girsanov's theorem yields that $(\hat{W}_s)_{s\in[0,t]}$ is a d-dimensional Brownian motion under $\hat{\mathbb{Q}}_t^\varepsilon = \hat{R}_t^\varepsilon \mathbb{P}$ and hence (3.10) implies that the law of $\hat{X}_t^\varepsilon$ under $\hat{\mathbb{Q}}_t^\varepsilon$ coincides with that of $X_t^{(x^{(1)}+\varepsilon v, x^{(2)})}$ under $\mathbb{P}$, which together with $(\hat{X}_t^\varepsilon)^{(2)} = (X_t^x)^{(2)}$ yields that for any $f \in \mathscr{B}_b(\mathbb{R}^{2d})$ with $f(x^{(1)}, x^{(2)})$ independent of $x^{(1)}$,

$$P_t f(x^{(1)} + \varepsilon v, x^{(2)}) = \mathbb{E}^{\hat{\mathbb{Q}}_t^\varepsilon} f((\hat{X}_t^\varepsilon)^{(2)}) = \mathbb{E}^{\hat{\mathbb{Q}}_t^\varepsilon} f((X_t^x)^{(2)}) = \mathbb{E}[\hat{R}_t^\varepsilon f((X_t^x)^{(2)})].$$

By **(A1)**-**(A2)** and the same argument to derive (3.9), we may apply the dominated convergence theorem to derive

$$\lim_{\varepsilon\to 0} \mathbb{E}\left|\frac{\hat{R}_t^\varepsilon - 1}{\varepsilon} - \int_0^t \langle \sigma_s^{-1}\nabla_v^{(1)} B_s(X_s^x), \mathrm{d}W_s\rangle\right| = 0,$$

which derives (3.3). Finally, (3.4) follows from (3.3), Cauchy-Schwarz's inequality and **(A1)**-**(A2)**. □

3.2 Proof of Theorem 2.1

For any $\gamma \in \mathscr{P}_2(\mathbb{R}^{2d})$, consider the decoupled SDE:

$$(3.11) \qquad \begin{cases} \mathrm{d}(X_t^{x,\gamma})^{(1)} = (X_t^{x,\gamma})^{(2)}\mathrm{d}t, \\ \mathrm{d}(X_t^{x,\gamma})^{(2)} = B_t(X_t^{x,\gamma}, P_t^*\gamma)\mathrm{d}t + \sigma_t \mathrm{d}W_t, \end{cases} \qquad X_0^{x,\gamma} = x \in \mathbb{R}^{2d},$$

where as in Section 2, $P_t^*\gamma = \mathscr{L}_{X_t^\gamma}$ for X_t^γ being the solution to (1.1) with $\mathscr{L}_{X_0} = \gamma$.

In view of (2.5), **(A1)**-**(A2)** implies **(C1)**-**(C2)** below for $k = 2$. So, it follows from Theorem 4.1 below that

$$(3.12) \qquad \|P_t^*\gamma\|_2^2 \le C_1(1 + \|\gamma\|_2^2), \quad t \in [0, T],$$

for some constant $C_1 > 0$. This together with **(A1)**-**(A2)** implies that SDE (3.11) is well-posed and for any $p \ge 1$, there exists a constant $C_2 > 0$ such that

$$(3.13) \qquad \mathbb{E} \sup_{t\in[0,T]} |X_t^{x,\gamma}|^p \le C_2(1 + |x|^p + \|\gamma\|_2^p).$$

Let P_t^γ be the associated Markov semigroup, i.e.

$$P_t^\gamma f(x) := \mathbb{E}[f(X_t^{x,\gamma})], \quad t \in [0, T], x \in \mathbb{R}^{2d}, f \in \mathscr{B}_b(\mathbb{R}^{2d}).$$

Then it holds

$$P_t f(\gamma) = \int_{\mathbb{R}^{2d}} f(x)(P_t^*\gamma)(\mathrm{d}x) = \int_{\mathbb{R}^{2d}} P_t^\gamma f(x)\gamma(\mathrm{d}x), \quad f \in \mathscr{B}_b(\mathbb{R}^{2d}). \tag{3.14}$$

For any $\gamma \in \mathscr{P}_2(\mathbb{R}^{2d})$, consider the flow

$$\begin{cases} \mathrm{d}\theta_t^{(1)}(x,\gamma) = \theta_t^{(2)}(x,\gamma)\mathrm{d}t, \\ \mathrm{d}\theta_t^{(2)}(x,\gamma) = B_t(\theta_t(x,\gamma), P_t^*\gamma)\mathrm{d}t, \quad t \in [0,T], \theta_0(x,\gamma) = x \in \mathbb{R}^{2d}. \end{cases} \tag{3.15}$$

Lemma 3.2. *Assume* **(A1)**-**(A2)**. *Then for any $p \geq 1$, there exists a constant $c_p > 0$ such that*

$$\mathbb{E}|(X_t^{x,\gamma})^{(1)} - \theta_t^{(1)}(x,\gamma)|^p \leq c_p t^{\frac{3p}{2}}, \quad t \in [0,T], x \in \mathbb{R}^{2d}, \gamma \in \mathscr{P}_2(\mathbb{R}^{2d}), \tag{3.16}$$

$$\mathbb{E} \sup_{s\in[0,t]} |(X_s^{x,\gamma})^{(2)} - \theta_s^{(2)}(x,\gamma)|^p \leq c_p t^{\frac{p}{2}}, \quad t \in [0,T], x \in \mathbb{R}^{2d}, \gamma \in \mathscr{P}_2(\mathbb{R}^{2d}). \tag{3.17}$$

Proof. Observe that

$$\begin{cases} (X_t^{x,\gamma})^{(1)} - \theta_t^{(1)}(x,\gamma) = \int_0^t [(X_s^{x,\gamma})^{(2)} - \theta_s^{(2)}(x,\gamma)]\mathrm{d}s, \\ (X_t^{x,\gamma})^{(2)} - \theta_t^{(2)}(x,\gamma) = \int_0^t [B_s(X_s^{x,\gamma}, P_s^*\gamma) - B_s(\theta_s(x,\gamma), P_s^*\gamma)]\mathrm{d}s + \int_0^t \sigma_s \mathrm{d}W_s. \end{cases}$$

So, it is sufficient to prove (3.17). By BDG's inequality and **(A2)**, we find constants $c_1(p), c_2(p) > 0$ such that

$$\begin{aligned} &\mathbb{E} \sup_{s\in[0,t]} |X_s^{x,\gamma} - \theta_s(x,\gamma)|^p \\ &\leq c_1(p) \int_0^t \mathbb{E} \sup_{s\in[0,r]} |X_s^{x,\gamma} - \theta_s(x,\gamma)|^p \mathrm{d}r + c_1(p)\mathbb{E} \sup_{s\in[0,t]} \left| \int_0^s \sigma_r \mathrm{d}W_r \right|^p \\ &\leq c_2(p) \int_0^t \mathbb{E} \sup_{s\in[0,r]} |X_s^{x,\gamma} - \theta_s(x,\gamma)|^p \mathrm{d}r + c_2(p) t^{\frac{p}{2}}. \end{aligned}$$

So, we derive from (3.13) and Gronwall's inequality that

$$\mathbb{E} \sup_{s\in[0,t]} |X_s^{x,\gamma} - \theta_s(x,\gamma)|^p \leq c_2(p)\mathrm{e}^{c_2(p)T} t^{\frac{p}{2}}, \quad 0 \leq t \leq T.$$

Therefore, the proof is completed. □

Lemma 3.3. *Assume* **(A1)**-**(A2)**. *Then there exists a constant $c > 0$ such that*

$$\mathbb{W}_2(P_t^*\gamma, P_t^*\tilde{\gamma}) \leq c\mathbb{W}_2(\gamma, \tilde{\gamma}) + c\int_0^t \mathbb{W}_{\beta,\varphi}(P_s^*\gamma, P_s^*\tilde{\gamma})\mathrm{d}s, \quad t \in [0,T], \gamma, \tilde{\gamma} \in \mathscr{P}_2(\mathbb{R}^{2d}). \tag{3.18}$$

Proof. Take $\mathscr{F}_0$-measurable random variables $X_0^\gamma, X_0^{\tilde\gamma}$ such that

$$\mathscr{L}_{X_0^\gamma} = \gamma, \quad \mathscr{L}_{X_0^{\tilde\gamma}} = \tilde\gamma, \quad \mathbb{W}_2(\gamma,\tilde\gamma)^2 = \mathbb{E}|X_0^\gamma - X_0^{\tilde\gamma}|^2. \tag{3.19}$$

By **(A2)**, we find a constant $c_1 > 1$ such that

$$\begin{aligned}\mathbb{E}\Big[\sup_{s\in[0,t]}|X_s^{\tilde\gamma} - X_s^\gamma|^2\Big] &\le c_1\mathbb{E}|X_0^{\tilde\gamma} - X_0^\gamma|^2 \\ &\quad + c_1\left(\int_0^t \{\mathbb{W}_{\beta,\varphi}(P_s^*\gamma, P_s^*\tilde\gamma) + \mathbb{W}_2(P_s^*\gamma, P_s^*\tilde\gamma)\}\mathrm{d}s\right)^2 \\ &\quad + c_1\mathbb{E}\int_0^t |X_s^{\tilde\gamma} - X_s^\gamma|^2\mathrm{d}s, \quad t\in[0,T].\end{aligned}$$

So, it follows from the inequality $\mathbb{W}_2(P_s^*\gamma, P_s^*\tilde\gamma)^2 \le \mathbb{E}|X_s^{\tilde\gamma} - X_s^\gamma|^2$ and Gronwall's inequality that (3.18) holds. □

The following Hölder inequality for concave functions comes from [9, Lemma 2.4].

Lemma 3.4. *Let $\alpha : [0,\infty) \to [0,\infty)$ be concave. Then for any non-negative random variables ξ and η,*

$$\mathbb{E}[\alpha(\xi)\eta] \le \|\eta\|_{L^p(\mathbb{P})}\alpha\Big(\|\xi\|_{L^{\frac{p}{p-1}}(\mathbb{P})}\Big), \quad p\ge 1. \tag{3.20}$$

The following Lemma is crucial in the proof of the desired Harnack inequality.

Lemma 3.5. *Assume* **(A1)-(A2)**. *Then there exists a constant $c > 0$ such that*

$$\mathbb{W}_{\beta,\varphi}(P_t^*\gamma, P_t^*\tilde\gamma) \le c\mathbb{W}_2(\tilde\gamma,\gamma)\Big\{\frac{\varphi(t^{\frac12})}{\sqrt{t}} + t^{\frac{3(\beta-1)}{2}}\Big\}, \quad t\in(0,T], \gamma,\tilde\gamma\in\mathscr{P}_2(\mathbb{R}^{2d}). \tag{3.21}$$

Consequently, there exists a constant $\tilde c > 0$ such that for any $\gamma,\tilde\gamma\in\mathscr{P}_2(\mathbb{R}^{2d})$,

$$\sup_{t\in[0,T]}\mathbb{W}_2(P_t^*\gamma, P_t^*\tilde\gamma) \le \tilde c\mathbb{W}_2(\tilde\gamma,\gamma). \tag{3.22}$$

The idea of the proof is from [9, Lemma 2.5], where the noise is non-degenerate. We outline it in the following. In view of the triangle inequality, it holds

$$\begin{aligned}\mathbb{W}_{\beta,\varphi}(P_t^*\gamma, P_t^*\tilde\gamma) &\le \int_0^1 \Big|\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathbb{W}_{\beta,\varphi}(P_t^*\gamma, P_t^*\gamma^\varepsilon)\Big|\mathrm{d}\varepsilon \\ &\le \int_0^1 \limsup_{r\downarrow 0}\frac{|\mathbb{W}_{\beta,\varphi}(P_t^*\gamma, P_t^*\gamma^{\varepsilon+r}) - \mathbb{W}_{\beta,\varphi}(P_t^*\gamma, P_t^*\gamma^\varepsilon)|}{r}\mathrm{d}\varepsilon \\ &\le \int_0^1 \limsup_{r\downarrow 0}\frac{\mathbb{W}_{\beta,\varphi}(P_t^*\gamma^\varepsilon, P_t^*\gamma^{\varepsilon+r})}{r}\mathrm{d}\varepsilon,\end{aligned}$$

where $\gamma^\varepsilon := \mathscr{L}_{X_0^{\gamma^\varepsilon}}$, $\varepsilon\in[0,2]$ with $X_0^{\gamma^\varepsilon} := X_0^\gamma + \varepsilon(X_0^{\tilde\gamma} - X_0^\gamma)$ and X_0^γ and $X_0^{\tilde\gamma}$ being in (3.19). By the definition of $\mathbb{W}_{\beta,\varphi}$ and (3.14), for any $\varepsilon, r\in[0,1]$ and $t\in[0,T]$, we have

$$\begin{aligned}
\mathbb{W}_{\beta,\varphi}(P_t^*\gamma^{\varepsilon+r}, P_t^*\gamma^\varepsilon)^2 &= \sup_{f\in\mathscr{B}_b(\mathbb{R}^{2d}),[f]_{\beta,\varphi}\le 1} |P_tf(\gamma^{\varepsilon+r}) - P_tf(\gamma^\varepsilon)|^2\\
&\le 2\sup_{f\in\mathscr{B}_b(\mathbb{R}^{2d}),[f]_{\beta,\varphi}\le 1} |\gamma^{\varepsilon+r}(P_t^{\gamma^{\varepsilon+r}}f - P_t^{\gamma^\varepsilon}f)|^2\\
&\quad + 2\sup_{f\in\mathscr{B}_b(\mathbb{R}^{2d}),[f]_{\beta,\varphi}\le 1} |\gamma^{\varepsilon+r}(P_t^{\gamma^\varepsilon}f) - \gamma^\varepsilon(P_t^{\gamma^\varepsilon}f)|^2\\
&=: I_1(t) + I_2(t).
\end{aligned}\tag{3.23}$$

Therefore, to prove Lemma 3.5, it is sufficient to derive the estimates for $I_1(t)$ and $I_2(t)$, which will be provided in Lemma 3.6 and Lemma 3.7 below respectively. $I_1(t)$ involves in the $\mathbb{W}_{\beta,\varphi}$-distance for two diffusion processes with the same initial values but different drifts and it can be dealt with by Girsanov's transform. As to $I_2(t)$, it requires the gradient estimate of the decoupled SDE (3.11), which is in fact a distribution independent SDE with parameter γ in drift so that Theorem 3.1 is available. Moreover, to utilize $[f]_{\beta,\varphi<1}$, we will replace f with $f - f(\theta_t(x_0,\gamma))$ for any fixed $x_0\in\mathbb{R}^{2d}$, where the flow $\theta_t(x_0,\gamma)$ solves (3.15) with x_0 in place of x.

Lemma 3.6. *Assume* **(A1)**-**(A2)**. *Then there exists a constant $c>0$ such that*

$$I_1(t) \le c\psi(\varepsilon,r)\left(r^2\mathbb{W}_2(\gamma,\tilde\gamma)^2 + \int_0^t \mathbb{W}_{\beta,\varphi}(P_s^*\gamma^\varepsilon, P_s^*\gamma^{\varepsilon+r})^2\mathrm{d}s\right),\quad t\in[0,T],\tag{3.24}$$

where

$$\psi(\varepsilon,r) := \mathrm{e}^{cr^2\mathbb{W}_2(\gamma,\tilde\gamma)^2 + c\int_0^T \mathbb{W}_{\beta,\varphi}(P_s^*\gamma^\varepsilon, P_s^*\gamma^{\varepsilon+r})^2\mathrm{d}s}.\tag{3.25}$$

Proof. Firstly, by the definition of γ^ε, we have

$$\|\gamma^\varepsilon\|_2^2 \le 8\|\gamma\|_2^2 + 8\|\tilde\gamma\|_2^2,\quad \varepsilon\in[0,2],\tag{3.26}$$

and

$$\mathbb{W}_2(\gamma^\varepsilon,\gamma^{\varepsilon+r})^2 \le r^2\mathbb{W}_2(\gamma,\tilde\gamma)^2,\quad \varepsilon,r\in[0,1].\tag{3.27}$$

For any $\varepsilon\in[0,2]$, consider the SDE

$$\begin{cases}\mathrm{d}(X_t^{x,\gamma^\varepsilon})^{(1)} = (X_t^{x,\gamma^\varepsilon})^{(2)}\mathrm{d}t,\\ \mathrm{d}(X_t^{x,\gamma^\varepsilon})^{(2)} = B_t(X_t^{x,\gamma^\varepsilon}, P_t^*\gamma^\varepsilon)\mathrm{d}t + \sigma_t\mathrm{d}W_t,\quad X_0^{x,\gamma^\varepsilon} = x\in\mathbb{R}^{2d}, t\in[0,T].\end{cases}\tag{3.28}$$

For any $r,\varepsilon\in[0,1]$, define

$$\eta_t^{\varepsilon,r} = \sigma_t^{-1}[B_t(X_t^{x,\gamma^\varepsilon}, P_t^*\gamma^{\varepsilon+r}) - B_t(X_t^{x,\gamma^\varepsilon}, P_t^*\gamma^\varepsilon)],\quad t\in[0,T].$$

By **(A1)**-**(A2)**, (2.4) for $k=1$, (3.12) and (3.26), there exist constants $c_1, c_2 > 0$ such that

$$\begin{aligned}|\eta_t^{\varepsilon,r}| &\le c_1\big\{\mathbb{W}_{\beta,\varphi}(P_t^*\gamma^\varepsilon, P_t^*\gamma^{\varepsilon+r}) + \mathbb{W}_2(P_t^*\gamma^\varepsilon, P_t^*\gamma^{\varepsilon+r})\big\}\\ &\le c_2(1+\|\gamma\|_2+\|\tilde\gamma\|_2),\quad r,\varepsilon\in[0,1], t\in[0,T].\end{aligned}\tag{3.29}$$

It follows from Girsanov's theorem that

$$W_t^{\varepsilon,r} = W_t - \int_0^t \eta_s^{\varepsilon,r}\mathrm{d}s,\quad t\in[0,T]$$

is a d-dimensional Brownian motion under the probability $\mathbb{Q}^{\varepsilon,r} := R_T^{\varepsilon,r}\mathbb{P}$ with

$$R_t^{\varepsilon,r} := \exp\left\{\int_0^t \langle \eta_s^{\varepsilon,r}, \mathrm{d}W_s\rangle - \frac{1}{2}\int_0^t |\eta_s^{\varepsilon,r}|^2\mathrm{d}s\right\},\quad t\in[0,T].$$

Therefore, (3.28) can be reformulated as

$$\begin{cases}\mathrm{d}(X_t^{x,\gamma^\varepsilon})^{(1)} = (X_t^{x,\gamma^\varepsilon})^{(2)}\mathrm{d}t,\\ \mathrm{d}(X_t^{x,\gamma^\varepsilon})^{(2)} = B_t(X_t^{x,\gamma^\varepsilon}, P_t^*\gamma^{\varepsilon+r})\mathrm{d}t + \sigma_t\mathrm{d}W_t^{\varepsilon,r},\quad X_0^{x,\gamma^\varepsilon} = x\in\mathbb{R}^{2d}, t\in[0,T].\end{cases}$$

Recall that $\theta_t(x,\gamma^\varepsilon)$ solves (3.15) with γ^ε in place of γ. So, for any $f\in\mathscr{B}_b(\mathbb{R}^{2d})$, it holds

$$\begin{aligned}&P_t^{\gamma^{\varepsilon+r}}f(x) - P_t^{\gamma^\varepsilon}f(x)\\ &= \mathbb{E}\left[f(X_t^{x,\gamma^\varepsilon})(R_t^{\varepsilon,r}-1)\right]\\ &= \mathbb{E}\left[[f(X_t^{x,\gamma^\varepsilon}) - f(\theta_t(x,\gamma^\varepsilon))](R_t^{\varepsilon,r}-1)\right],\quad \varepsilon,r\in(0,1], t\in[0,T], x\in\mathbb{R}^{2d}.\end{aligned}$$

Moreover, by (3.29), (3.18) and (3.27), we obtain

$$\begin{aligned}\mathbb{E}|R_t^{\varepsilon,r}-1|^2 &= \mathbb{E}\big[(R_t^{\varepsilon,r})^2-1\big] \le \mathrm{esssup}_\Omega(\mathrm{e}^{\int_0^t|\eta_s^{\varepsilon,r}|^2\mathrm{d}s}-1)\\ &\le \mathrm{esssup}_\Omega\left(\mathrm{e}^{\int_0^t|\eta_s^{\varepsilon,r}|^2\mathrm{d}s}\int_0^t|\eta_s^{\varepsilon,r}|^2\mathrm{d}s\right)\\ &\le \psi(\varepsilon,r)\int_0^t\big\{\mathbb{W}_{\beta,\varphi}(P_s^*\gamma^\varepsilon, P_s^*\gamma^{\varepsilon+r})^2 + \mathbb{W}_2(P_s^*\gamma^\varepsilon, P_s^*\gamma^{\varepsilon+r})^2\big\}\mathrm{d}s\\ &\le c_3\psi(\varepsilon,r)\left(r^2\mathbb{W}_2(\gamma,\tilde\gamma)^2 + \int_0^t\mathbb{W}_{\beta,\varphi}(P_s^*\gamma^\varepsilon, P_s^*\gamma^{\varepsilon+r})^2\mathrm{d}s\right),\quad t\in[0,T]\end{aligned}\tag{3.30}$$

for some constant $c_3>0$. By (3.29), we have

$$\bar\psi := \sup_{\varepsilon,r\in[0,1]}\psi(\varepsilon,\gamma) < \infty.\tag{3.31}$$

Combining (2.3) for $k=1$, the Cauchy-Schwarz inequality and (3.30), we arrive at

$$\sup_{f\in\mathscr{B}_b(\mathbb{R}^{2d}),[f]_{\beta,\varphi}\le 1}|\gamma^{\varepsilon+r}(P_t^{\gamma^{\varepsilon+r}}f - P_t^{\gamma^\varepsilon}f)|^2$$

$$\leq \left(\int_{\mathbb{R}^{2d}} \sup_{f\in\mathscr{B}_b(\mathbb{R}^{2d}),[f]_{\beta,\varphi}\leq 1}\left|\mathbb{E}\left[(f(X_t^{x,\gamma^\varepsilon})-f(\theta_t(x,\gamma^\varepsilon)))(R_t^{\varepsilon,r}-1)\right]\right|\gamma^{\varepsilon+r}(\mathrm{d}x)\right)^2$$
$$\leq \left(\int_{\mathbb{R}^{2d}}\{2(2(\varphi(1)+1))^2\mathbb{E}(1+|X_t^{x,\gamma^\varepsilon}-\theta_t(x,\gamma^\varepsilon)|^2)\}^{\frac{1}{2}}\gamma^{\varepsilon+r}(\mathrm{d}x)\right)^2$$
$$\times\sup_x \mathbb{E}[|R_t^{\varepsilon,r}-1|^2]$$
$$\leq \left(\int_{\mathbb{R}^{2d}}\{2(2(\varphi(1)+1))^2\mathbb{E}(1+|X_t^{x,\gamma^\varepsilon}-\theta_t(x,\gamma^\varepsilon)|^2)\}^{\frac{1}{2}}\gamma^{\varepsilon+r}(\mathrm{d}x)\right)^2$$
$$\times c_3\psi(\varepsilon,r)\left(r^2\mathbb{W}_2(\gamma,\tilde\gamma)^2+\int_0^t \mathbb{W}_{\beta,\varphi}(P_s^*\gamma^\varepsilon,P_s^*\gamma^{\varepsilon+r})^2\mathrm{d}s\right),\quad t\in[0,T].$$

This together with (3.16) and (3.17) implies that (3.24) holds for some constant $c>0$. □

Lemma 3.7. *Assume* **(A1)-(A2)**. *Then there exists a constant $c>0$ such that*

$$I_2(t)\leq cr^2\mathbb{W}_2(\gamma,\tilde\gamma)^2\left(t^{3(\beta-1)}+\frac{\varphi(t^{\frac{1}{2}})^2}{t}\right),\quad t\in(0,T]. \tag{3.32}$$

Proof. Note that by Theorem 3.1, for any $p>1$, there exists some constant $c_1>0$ such that for $h\in\mathbb{R}^{2d}$,

$$\left(\mathbb{E}|N_t(h^{(1)},0)|^p\right)^{\frac{1}{p}}\leq c_1|h^{(1)}|t^{-\frac{3}{2}},\quad \left(\mathbb{E}|N_t(0,h^{(2)}|^p\right)^{\frac{1}{p}}\leq c_1|h^{(2)}|t^{-\frac{1}{2}},\quad t\in(0,T]. \tag{3.33}$$

Recall that $\theta_t(x,\gamma^\varepsilon)$ solves (3.15) with γ^ε in place of γ. For any $x_0\in\mathbb{R}^{2d},\varepsilon\in[0,1],t\in[0,T]$ and any $f\in\mathscr{B}_b(\mathbb{R}^{2d})$, let

$$\begin{aligned}
f^{(1)}_{x_0,t,\gamma^\varepsilon}(x)&=f(x)-f(\theta_t^{(1)}(x_0,\gamma^\varepsilon),x^{(2)}),\\
\tilde f^{(1)}_{x_0,t,\gamma^\varepsilon}(x)&=f(\theta_t^{(1)}(x_0,\gamma^\varepsilon),x^{(2)})-f(\theta_t(x_0,\gamma^\varepsilon)),\\
f^{(2)}_{x_0,t,\gamma^\varepsilon}(x)&=f(x)-f(\theta_t(x_0,\gamma^\varepsilon)),\quad x\in\mathbb{R}^{2d}.
\end{aligned}$$

By Theorem 3.1(1), (3.16), the first inequality in (3.33) and Hölder's inequality, for any $p>1$, we have

$$\begin{aligned}
&\sup_{f\in\mathscr{B}_b(\mathbb{R}^{2d}),[f]_{\beta,\varphi}\leq 1}|\nabla^{(1)}P_t^{\gamma^\varepsilon}f^{(1)}_{x_0,t,\gamma^\varepsilon}|(x_0)\\
&\leq c_1\left\{\mathbb{E}\left|(X_t^{x_0,\gamma^\varepsilon})^{(1)}-\theta_t^{(1)}(x_0,\gamma^\varepsilon)\right|^{\frac{\beta p}{p-1}}\right\}^{\frac{p-1}{p}}t^{-\frac{3}{2}}\leq c_2t^{\frac{3(\beta-1)}{2}},\quad t\in(0,T]
\end{aligned} \tag{3.34}$$

for some constant $c_2>0$. By (3.4), (3.17) and (2.2), we can find constants $c_3,c_4>0$ such that for any $t\in[0,T]$,

$$\sup_{f\in\mathscr{B}_b(\mathbb{R}^{2d}),[f]_{\beta,\varphi}\leq 1}|\nabla^{(1)}P_t^{\gamma^\varepsilon}\tilde f^{(1)}_{x_0,t,\gamma^\varepsilon}|(x_0)\leq c_3\{1+\mathbb{E}|(X_t^{x_0,\gamma^\varepsilon})^{(2)}-\theta_t^{(2)}(x_0,\gamma^\varepsilon)|^2\}^{\frac{1}{2}}t^{\frac{1}{2}}\leq c_4. \tag{3.35}$$

Moreover, by Theorem 3.1(1), the second inequality in (3.33), (3.16), (3.17), (3.20) and (2.1), we conclude

$$\begin{aligned}
&\sup_{f\in\mathscr{B}_b(\mathbb{R}^{2d}),[f]_{\beta,\varphi}\le 1}|\nabla^{(2)}P_t^{\gamma^\varepsilon}f^{(2)}_{x_0,t,\gamma^\varepsilon}|(x_0)\\
&\le \sup_{v\in\mathbb{R}^d,|v|\le 1}\mathbb{E}\left\{\left|(X_t^{x_0,\gamma^\varepsilon})^{(1)}-\theta_t^{(1)}(x_0,\gamma^\varepsilon)\right|^\beta |N_t(0,v|\right\}\\
&+\sup_{v\in\mathbb{R}^d,|v|\le 1}\mathbb{E}\left\{\varphi(|(X_t^{x_0,\gamma^\varepsilon})^{(2)}-\theta_t^{(2)}(x_0,\gamma^\varepsilon)|)|N_t(0,v|\right\}\\
&\le c_5\left(t^{\frac{3\beta-1}{2}}+\frac{\varphi(t^{\frac12})}{t^{\frac12}}\right),\quad t\in(0,T]
\end{aligned}\tag{3.36}$$

for some constant $c_5>0$. Since

$$\nabla^{(1)}P_t^{\gamma^\varepsilon}f=\nabla^{(1)}P_t^{\gamma^\varepsilon}f^{(1)}_{x_0,t,\gamma^\varepsilon}+\nabla^{(1)}P_t^{\gamma^\varepsilon}\tilde f^{(1)}_{x_0,t,\gamma^\varepsilon},\quad \nabla^{(2)}P_t^{\gamma^\varepsilon}f=\nabla^{(2)}P_t^{\gamma^\varepsilon}f^{(2)}_{x_0,t,\gamma^\varepsilon},\quad f\in\mathscr{B}_b(\mathbb{R}^{2d}),$$

we derive from (3.34) and (3.35) and $\beta\in(\frac23,1]$ that

$$\sup_{f\in\mathscr{B}_b(\mathbb{R}^{2d}),[f]_{\beta,\varphi}\le 1}|\nabla^{(1)}P_t^{\gamma^\varepsilon}f|(x_0)\le c_6 t^{\frac{3(\beta-1)}{2}},\quad t\in(0,T]\tag{3.37}$$

and from (3.36) that

$$\sup_{f\in\mathscr{B}_b(\mathbb{R}^{2d}),[f]_{\beta,\varphi}\le 1}|\nabla^{(2)}P_t^{\gamma^\varepsilon}f|(x_0)\le c_5\left(t^{\frac{3\beta-1}{2}}+\frac{\varphi(t^{\frac12})}{t^{\frac12}}\right)\tag{3.38}$$

for some constant $c_6>0$. Observe that

$$\begin{aligned}
I_2(t)&=2\sup_{f\in\mathscr{B}_b(\mathbb{R}^{2d}),[f]_{\beta,\varphi}\le 1}\left|\mathbb{E}\int_0^r\frac{\mathrm{d}}{\mathrm{d}\theta}P_t^{\gamma^\varepsilon}f(X_0^{\gamma^{\varepsilon+\theta}})\mathrm{d}\theta\right|^2\\
&=2\sup_{f\in\mathscr{B}_b(\mathbb{R}^{2d}),[f]_{\beta,\varphi}\le 1}\left|\mathbb{E}\int_0^r\left\{\nabla_{X_0^{\tilde\gamma}-X_0^{\gamma}}P_t^{\gamma^\varepsilon}f(X_0^{\gamma^{\varepsilon+\theta}})\right\}\mathrm{d}\theta\right|^2.
\end{aligned}$$

Combining this with (3.37), (3.38), (3.19) and the fact $t^{\frac{3\beta-1}{2}}\le Tt^{\frac{3(\beta-1)}{2}}, t\in(0,T]$, we find a constant $c_7>0$ such that

$$\begin{aligned}
I_2(t)&\le 2\sup_{f\in\mathscr{B}_b(\mathbb{R}^{2d}),[f]_{\beta,\varphi}\le 1}\left(\mathbb{E}\left[|X_0^{\gamma}-X_0^{\tilde\gamma}|\int_0^r|\nabla P_t^{\gamma^\varepsilon}f(X_0^{\gamma^{\varepsilon+\theta}})|\mathrm{d}\theta\right]\right)^2\\
&\le c_7\left\{\mathbb{E}|X_0^\gamma-X_0^{\tilde\gamma}|\int_0^r\left(t^{\frac{3(\beta-1)}{2}}+\frac{\varphi(t^{\frac12})}{t^{\frac12}}\right)\mathrm{d}\theta\right\}^2\\
&\le c_7 r^2\mathbb{W}_2(\gamma,\tilde\gamma)^2\left(t^{3(\beta-1)}+\frac{\varphi(t^{\frac12})^2}{t}\right).
\end{aligned}$$

Then (3.32) holds. □

Now, we are in the position to prove Lemma 3.5.

Proof of Lemma 3.5. Firstly, by $\varphi \in \mathscr{A}$ and $\beta \in (\frac{2}{3}, 1]$, we conclude that

$$\int_0^T \frac{\varphi(t^{\frac{1}{2}})^2}{t}\mathrm{d}t = 2\int_0^{T^{\frac{1}{2}}} \frac{\varphi(s)^2}{s}\mathrm{d}s < \infty, \quad \int_0^T t^{3(\beta-1)}\mathrm{d}t < \infty. \tag{3.39}$$

(3.32) together with (3.24) and (3.23) yields

$$\begin{aligned}&\mathbb{W}_{\beta,\varphi}(P_t^*\gamma^\varepsilon, P_t^*\gamma^{\varepsilon+r})^2\\ &\le c\psi(\varepsilon,r)\int_0^t \mathbb{W}_{\beta,\varphi}(P_s^*\gamma^\varepsilon, P_s^*\gamma^{\varepsilon+r})^2\mathrm{d}s\\ &+ cr^2\mathbb{W}_2(\gamma,\tilde{\gamma})^2\bigg(\psi(\varepsilon,r) + t^{3(\beta-1)} + \frac{\varphi\big(t^{\frac{1}{2}}\big)^2}{t}\bigg), \quad t \in (0,T].\end{aligned} \tag{3.40}$$

Let

$$\Gamma_t(\varepsilon,r) := \int_0^t \mathbb{W}_{\beta,\varphi}(P_s^*\gamma^\varepsilon, P_s^*\gamma^{\varepsilon+r})^2\mathrm{d}s.$$

So, it follows from (3.40) that

$$\begin{aligned}&\Gamma_t(\varepsilon,r) \le cr^2\mathbb{W}_2(\gamma,\tilde{\gamma})^2H(\varepsilon,r) + c\psi(\varepsilon,r)\int_0^t \Gamma_s(\varepsilon,r)\mathrm{d}s, \quad t\in[0,T],\\ &H(\varepsilon,r) := \int_0^T \left[\psi(\varepsilon,r) + t^{3(\beta-1)} + \frac{\varphi\big(t^{\frac{1}{2}}\big)^2}{t}\right]\mathrm{d}t, \quad \varepsilon,r\in[0,1].\end{aligned} \tag{3.41}$$

By Gronwall's inequality and (3.41), for any $\varepsilon, r \in [0,1]$ we have

$$\Gamma_t(\varepsilon,r) \le cr^2\mathbb{W}_2(\gamma,\tilde{\gamma})^2H(\varepsilon,r)\mathrm{e}^{c\psi(\varepsilon,r)t}, \quad t\in[0,T]. \tag{3.42}$$

Substituting this into (3.40), we get

$$\begin{aligned}&\mathbb{W}_{\beta,\varphi}(P_t^*\gamma^\varepsilon, P_t^*\gamma^{\varepsilon+r})^2\\ &\le cr^2\mathbb{W}_2(\gamma,\tilde{\gamma})^2\left[c\psi(\varepsilon,r)H(\varepsilon,r)\mathrm{e}^{c\psi(\varepsilon,r)t} + \psi(\varepsilon,r) + t^{3(\beta-1)} + \frac{\varphi\big(t^{\frac{1}{2}}\big)^2}{t}\right].\end{aligned} \tag{3.43}$$

Note that (3.31), (3.25), (3.41) and (3.42) imply that $\psi(\varepsilon,r)$ is bounded in $(\varepsilon,r)\in[0,1]^2$ with $\psi(\varepsilon,r)\to 1$ as $r\to 0$, so that by (3.43) and the dominated convergence theorem we find a constant $C>1$ such that

$$\limsup_{r\downarrow 0}\frac{\mathbb{W}_{\beta,\varphi}(P_t^*\gamma^\varepsilon, P_t^*\gamma^{\varepsilon+r})}{r} \le C\mathbb{W}_2(\tilde{\gamma},\gamma)\bigg\{\frac{\varphi(t^{\frac{1}{2}})}{\sqrt{t}} + t^{\frac{3(\beta-1)}{2}}\bigg\}. \tag{3.44}$$

By the triangle inequality,

$$\big|\mathbb{W}_{\beta,\varphi}(P_t^*\gamma, P_t^*\gamma^\varepsilon) - \mathbb{W}_{\beta,\varphi}(P_t^*\gamma, P_t^*\gamma^{\varepsilon+r})\big| \le \mathbb{W}_{\beta,\varphi}(P_t^*\gamma^\varepsilon, P_t^*\gamma^{\varepsilon+r}), \quad \varepsilon,r\in[0,1],$$

(3.44) implies that for any $t \in (0,T]$, $\mathbb{W}_{\beta,\varphi}(P_t^*\gamma, P_t^*\gamma^\varepsilon)$ is Lipschitz continuous (hence a.e. differentiable) in $\varepsilon \in [0,1]$, and

$$\begin{aligned}\Big|\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathbb{W}_{\beta,\varphi}(P_t^*\gamma, P_t^*\gamma^\varepsilon)\Big| &\le \limsup_{r\downarrow 0}\frac{\mathbb{W}_{\beta,\varphi}(P_t^*\gamma^\varepsilon, P_t^*\gamma^{\varepsilon+r})}{r}\\ &\le C\mathbb{W}_2(\tilde\gamma,\gamma)\Big\{\frac{\varphi(t^{\frac12})}{\sqrt t}+t^{\frac{3(\beta-1)}{2}}\Big\},\quad \varepsilon\in[0,1].\end{aligned}$$

Noting that $\gamma^1=\tilde\gamma$, this implies the desired estimate (3.21), which combined with (3.18) yields (3.22). □

Finally, we intend to prove Theorem 2.1.

Proof of Theorem 2.1. Let $t \in (0,T]$ be fixed. Consider

$$\begin{cases}\mathrm{d}(X_s^{x,\gamma})^{(1)} = (X_s^{x,\gamma})^{(2)}\mathrm{d}s,\\ \mathrm{d}(X_s^{x,\gamma})^{(2)} = B_s(X_s^{x,\gamma}, P_s^*\gamma)\mathrm{d}s+\sigma_s\mathrm{d}W_s,\quad X_0^{x,\gamma}=x\in\mathbb{R}^{2d}, s\in[0,t].\end{cases}$$

Recall that $\gamma_{s,t}(h)$ is defined in (3.2). Let $\tilde X_s$ solve

$$\begin{cases}\mathrm{d}\tilde X_s^{(1)} = \tilde X_s^{(2)}\mathrm{d}s,\\ \mathrm{d}\tilde X_s^{(2)} = B_s(X_s^{x,\gamma}, P_s^*\gamma)\mathrm{d}s+\sigma_s\mathrm{d}W_s+\gamma_{s,t}'(y-x)\mathrm{d}s,\quad \tilde X_0=y\in\mathbb{R}^{2d}, s\in[0,t].\end{cases}\tag{3.45}$$

Then it holds

$$\tilde X_s = X_s^{x,\gamma}+\left(y^{(1)}-x^{(1)}+\int_0^s\gamma_{u,t}(y-x)\mathrm{d}u,\ \ \gamma_{s,t}(y-x)\right),\quad s\in[0,t].$$

In particular, $\tilde X_t = X_t^{x,\gamma}$ due to (3.2). Let

$$\begin{aligned}&\eta_s^{\gamma,\tilde\gamma} := \sigma_s^{-1}[B_s(\tilde X_s, P_s^*\tilde\gamma)-B_s(X_s^{x,\gamma},P_s^*\gamma)-\gamma_{s,t}'(y-x)],\quad s\in[0,t],\\ &R_t^{\gamma,\tilde\gamma} := \mathrm{e}^{\int_0^t\langle\eta_s^{\gamma,\tilde\gamma},\mathrm{d}W_s\rangle-\frac12\int_0^t|\eta_s^{\gamma,\tilde\gamma}|^2\mathrm{d}s}.\end{aligned}$$

(A1)-**(A2)** imply

$$\begin{aligned}|\eta_s^{\gamma,\tilde\gamma}|\le c_0\bigg(&|\gamma_{s,t}(y-x)|+|y^{(1)}-x^{(1)}|+\int_0^s|\gamma_{u,t}(y-x)|\mathrm{d}u\\ &+\mathbb{W}_2(P_s^*\gamma,P_s^*\tilde\gamma)+\mathbb{W}_{\beta,\varphi}(P_s^*\gamma,P_s^*\tilde\gamma)+|\gamma_{s,t}'(y-x)|\bigg),\quad s\in[0,t]\end{aligned}\tag{3.46}$$

for some constant $c_0>0$. By (3.39) and Lemma 3.5, there exists a constant $c_1>0$ such that

$$\int_0^T\big\{\mathbb{W}_{\beta,\varphi}(P_s^*\gamma,P_s^*\tilde\gamma)^2+\mathbb{W}_2(P_s^*\gamma,P_s^*\tilde\gamma)^2\big\}\mathrm{d}s\le c_1\mathbb{W}_2(\gamma,\tilde\gamma)^2.$$

This together with (3.46) and (3.8) gives

$$\int_0^t |\eta_s^{\gamma,\tilde{\gamma}}|^2 \mathrm{d}s \leq \frac{c_2|x-y|^2}{t^3} + c_2 \mathbb{W}_2(\gamma,\tilde{\gamma})^2 \tag{3.47}$$

for some constant $c_2 > 0$. As a result, Girsanov's theorem implies that

$$W_s^{\gamma,\tilde{\gamma}} = W_s - \int_0^s \eta_u^{\gamma,\tilde{\gamma}} \mathrm{d}u, \quad s \in [0,t]$$

is a d-dimensional Brownian motion under the probability measure $\mathbb{Q}_t^{\gamma,\tilde{\gamma}} = R_t^{\gamma,\tilde{\gamma}}\mathbb{P}$. So, (3.45) can be rewritten as

$$\begin{cases} \mathrm{d}\tilde{X}_s^{(1)} = \tilde{X}_s^{(2)} \mathrm{d}s, \\ \mathrm{d}\tilde{X}_s^{(2)} = B_s(\tilde{X}_s, P_s^*\tilde{\gamma})\mathrm{d}s + \sigma_s \mathrm{d}W_s^{\gamma,\tilde{\gamma}}, \quad \tilde{X}_0 = y, \end{cases}$$

which derives

$$P_t^{\tilde{\gamma}} f(y) = \mathbb{E}^{\mathbb{Q}_t^{\gamma,\tilde{\gamma}}} f(\tilde{X}_t) = \mathbb{E}^{\mathbb{Q}_t^{\gamma,\tilde{\gamma}}} f(X_t^{x,\gamma}) = \mathbb{E}[R_t^{\gamma,\tilde{\gamma}} f(X_t^{x,\gamma})], \quad f \in \mathscr{B}_b(\mathbb{R}^{2d}).$$

By Young's inequality, we have

$$\begin{aligned} P_t^{\tilde{\gamma}} \log f(y) &\leq \log P_t^{\gamma} f(x) + \mathbb{E}(R_t^{\gamma,\tilde{\gamma}} \log R_t^{\gamma,\tilde{\gamma}}) \\ &\leq \log P_t^{\gamma} f(x) + \mathbb{E}^{\mathbb{Q}_t^{\gamma,\tilde{\gamma}}} \int_0^t \langle \eta_s^{\gamma,\tilde{\gamma}}, \mathrm{d}W_s^{\gamma,\tilde{\gamma}} \rangle + \frac{1}{2}\mathbb{E}^{\mathbb{Q}_t^{\gamma,\tilde{\gamma}}} \int_0^t |\eta_s^{\gamma,\tilde{\gamma}}|^2 \mathrm{d}s \\ &\leq \log P_t^{\gamma} f(x) + \frac{1}{2}\mathbb{E}^{\mathbb{Q}_t^{\gamma,\tilde{\gamma}}} \int_0^t |\eta_s^{\gamma,\tilde{\gamma}}|^2 \mathrm{d}s, \quad f \in \mathscr{B}_b^+(\mathbb{R}^{2d}), f > 0, \end{aligned} \tag{3.48}$$

where in the last step we have used the fact that the stochastic integral is a martingale under the probability measure $\mathbb{Q}_t^{\gamma,\tilde{\gamma}}$ since $\{W_s^{\gamma,\tilde{\gamma}}\}_{s\in[0,t]}$ is a d-dimensional Brownian motion under $\mathbb{Q}_t^{\gamma,\tilde{\gamma}}$. Moreover, Hölder's inequality yields that for any $p > 1$,

$$\begin{aligned} (P_t^{\tilde{\gamma}} f(y))^p &\leq P_t^{\gamma} f^p(x) (\mathbb{E}(R_t^{\gamma,\tilde{\gamma}})^{\frac{p}{p-1}})^{p-1} \\ &\leq P_t^{\gamma} f^p(x) \mathrm{esssup}_{\Omega} \exp\left\{ \frac{p}{2(p-1)} \int_0^t |\eta_u^{\gamma,\tilde{\gamma}}|^2 \mathrm{d}u \right\}, \quad f \in \mathscr{B}_b^+(\mathbb{R}^{2d}). \end{aligned} \tag{3.49}$$

Applying (3.47), taking expectation in (3.48) and (3.49) with respect to any $\pi \in \mathscr{C}(\gamma,\tilde{\gamma})$ and then taking infimum in $\pi \in \mathscr{C}(\gamma,\tilde{\gamma})$, the proof is completed by Jensen's inequality and (1.2). □

4 Appendix: Well-posedness

In this section, we consider a general version of (1.1). For any $x \in \mathbb{R}^{m+d}$, let $x^{(1)}$ denote the first m components and $x^{(2)}$ denote the last d components, that is $x = (x^{(1)}, x^{(2)}) \in \mathbb{R}^{m+d}$

with $x^{(1)} \in \mathbb{R}^m$ and $x^{(2)} \in \mathbb{R}^d$. Fix $T > 0$ and let $k \geq 1$. Consider the distribution dependent SDEs on $\mathbb{R}^{m+d}$:

$$
\begin{cases} \mathrm{d}X_t^{(1)} = b_t(X_t)\mathrm{d}t, \\ \mathrm{d}X_t^{(2)} = B_t(X_t, \mathscr{L}_{X_t})\mathrm{d}t + \sigma_t(X_t)\mathrm{d}W_t, \end{cases} \tag{4.1}
$$

where $b : [0,T]\times\mathbb{R}^{m+d} \to \mathbb{R}^m$, $B : [0,T]\times\mathbb{R}^{m+d}\times\mathscr{P}(\mathbb{R}^{m+d}) \to \mathbb{R}^d$, $\sigma : [0,T]\times\mathbb{R}^{m+d} \to \mathbb{R}^d\otimes\mathbb{R}^d$ are measurable and W_t is a d-dimensional Brownian motion on some complete filtration probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$. Let $C([0,T]; \mathscr{P}_k)$ denote the continuous maps from $[0,T]$ to $(\mathscr{P}_k, \mathbb{W}_k)$. Recall that

$$
\|\mu - \nu\|_{k,var} = \sup_{|f|\leq 1+|\cdot|^k} |\mu(f) - \nu(f)|, \quad \mu, \nu \in \mathscr{P}_k(\mathbb{R}^{m+d}).
$$

Definition 4.1. The SDE (4.1) is called well-posed for distributions in $\mathscr{P}_k(\mathbb{R}^{m+d})$, if for any $\mathscr{F}_0$-measurable initial value X_0 with $\mathscr{L}_{X_0} \in \mathscr{P}_k(\mathbb{R}^{m+d})$ (respectively any initial distribution $\gamma \in \mathscr{P}_k(\mathbb{R}^{m+d})$), it has a unique strong solution (respectively weak solution) such that $\mathscr{L}_{X_\cdot} \in C([0,T]; \mathscr{P}_k(\mathbb{R}^{m+d}))$.

We make the following assumptions.

(C1) For any $t \in [0,T], x \in \mathbb{R}^{m+d}$, $\sigma_t(x)$ is invertible and $\|\sigma^{-1}\|_\infty$ is finite.

(C2) There exists $K > 0$ such that

$$
\begin{aligned}
&|b_t(x) - b_t(\bar{x})| + |\sigma_t(x) - \sigma_t(\bar{x})| \leq K|x - \bar{x}|, \\
&|B_t(x,\gamma) - B_t(\bar{x},\bar{\gamma})| \leq K(|x - \bar{x}| + \mathbb{W}_k(\gamma,\bar{\gamma}) + \|\gamma - \bar{\gamma}\|_{k,var}), \\
&|b_t(0)| + |\sigma_t(0)| \leq K, \quad |B_t(0,\gamma)| \leq K(1 + \|\gamma\|_k), \quad x, \bar{x} \in \mathbb{R}^{m+d}, \gamma, \bar{\gamma} \in \mathscr{P}_k(\mathbb{R}^{m+d}).
\end{aligned}
$$

For any $\mu \in C([0,T], \mathscr{P}_k(\mathbb{R}^{m+d}))$, consider

$$
\begin{cases} \mathrm{d}X_t^{(1)} = b_t(X_t)\mathrm{d}t, \\ \mathrm{d}X_t^{(2)} = B_t(X_t, \mu_t)\mathrm{d}t + \sigma_t(X_t)\mathrm{d}W_t. \end{cases} \tag{4.2}
$$

Under **(C2)**, for any $\mathscr{F}_0$-measurable random variable X_0 with $\mathscr{L}_{X_0} \in \mathscr{P}_k(\mathbb{R}^{m+d})$, let $X_t^{X_0,\mu}$ be the unique solution to (4.2) with initial value X_0. It is standard to derive from **(C2)** that

$$
\mathbb{E}(\sup_{t\in[0,T]} |X_t^{X_0,\mu}|^n|\mathscr{F}_0) \leq c(n)(1 + |X_0|^n), \quad n \geq 1. \tag{4.3}
$$

Define the mapping $\Phi^{X_0} : C([0,T], \mathscr{P}_k(\mathbb{R}^{m+d})) \to C([0,T], \mathscr{P}_k(\mathbb{R}^{m+d}))$ as

$$
\Phi_t^{X_0}(\mu) = \mathscr{L}_{X_t^{X_0,\mu}}, \quad t \in [0,T].
$$

The following theorem provides the well-posedness for (4.1) and the proof is similar to that in [19, Theorem 3.2].

Theorem 4.1. *Assume* **(C1)**-**(C2)**. *Then* (4.1) *is well-posed in* $\mathscr{P}_k(\mathbb{R}^{m+d})$. *Moreover, there exists a constant* $C > 0$ *such that*

$$\|P_t^*\gamma\|_k^k \le C(1 + \|\gamma\|_k^k), \quad t \in [0, T]. \tag{4.4}$$

Proof. Since (4.4) is standard by **(C2)** and the BDG inequality, it is sufficient to prove that (4.1) is well-posed in $\mathscr{P}_k(\mathbb{R}^{m+d})$. It follows from **(C2)** that

$$\begin{aligned} |X_t^{X_0,\nu} - X_t^{X_0,\mu}|^k &\le C_0 \left(\int_0^t [\mathbb{W}_k(\mu_s, \nu_s) + \|\mu_s - \nu_s\|_{k,var}] \mathrm{d}s \right)^k + C_0 \int_0^t |X_s^{X_0,\nu} - X_s^{X_0,\mu}|^k \mathrm{d}s \\ &\quad + C_0 \left| \int_0^t [\sigma_s(X_s^{X_0,\nu}) - \sigma_s(X_s^{X_0,\mu})] \mathrm{d}W_s \right|^k \end{aligned} \tag{4.5}$$

for some constant $C_0 > 0$. By **(C2)** and the BDG inequality, there exist constants $C_1, C_2 > 0$ such that

$$\begin{aligned} &C_0 \mathbb{E} \sup_{t\in[0,r]} \left| \int_0^t [\sigma_s(X_s^{X_0,\mu}) - \sigma_s(X_s^{X_0,\nu})] \mathrm{d}W_s \right|^k \\ &\le C_1 \mathbb{E} \left(\int_0^r |X_s^{X_0,\mu} - X_s^{X_0,\nu}|^2 \mathrm{d}s \right)^{\frac{k}{2}} \\ &\le \frac{1}{2} \mathbb{E} \sup_{t\in[0,r]} |X_t^{X_0,\mu} - X_t^{X_0,\nu}|^k + C_2 \mathbb{E} \int_0^r |X_s^{X_0,\mu} - X_s^{X_0,\nu}|^k \mathrm{d}s. \end{aligned} \tag{4.6}$$

(4.6) together with (4.5), (4.3) and Gronwall's inequality yields

$$\mathbb{W}_k(\Phi_t^{X_0}(\mu), \Phi_t^{X_0}(\nu)) \le (\mathbb{E} \sup_{s\in[0,t]} |X_s^{X_0,\mu} - X_s^{X_0,\nu}|^k)^{\frac{1}{k}} \le C_3 \int_0^t [\mathbb{W}_k(\mu_s, \nu_s) + \|\mu_s - \nu_s\|_{k,var}] \mathrm{d}s$$

for some constant $C_3 > 0$. Therefore, for any $\lambda > 0$, we have

$$\sup_{t\in[0,T]} \mathrm{e}^{-\lambda t} \mathbb{W}_k(\Phi_t^{X_0}(\mu), \Phi_t^{X_0}(\nu)) \le \frac{C_3}{\lambda} \sup_{t\in[0,T]} \mathrm{e}^{-\lambda t} [\mathbb{W}_k(\mu_t, \nu_t) + \|\mu_t - \nu_t\|_{k,var}]. \tag{4.7}$$

Next, let

$$\begin{aligned} \zeta_s &= \sigma_s^{-1}(X_s^{X_0,\mu})(B_s(X_s^{X_0,\mu}, \nu_s) - B_s(X_s^{X_0,\mu}, \mu_s)), \quad s \in [0, T], \\ R(t) &= \exp\left\{ \int_0^t \langle \zeta_s, \mathrm{d}W_s \rangle - \frac{1}{2} \int_0^t |\zeta_s|^2 \mathrm{d}s \right\}, \quad t \in [0, T], \\ W_t^{\mu,\nu} &= W_t - \int_0^t \zeta_s \mathrm{d}s, \quad t \in [0, T]. \end{aligned}$$

Then we have

$$\begin{cases} \mathrm{d}(X_t^{X_0,\mu})^{(1)} = b_t(X_t^{X_0,\mu}) \mathrm{d}t, \\ \mathrm{d}(X_t^{X_0,\mu})^{(2)} = B_t(X_t^{X_0,\mu}, \nu_t) \mathrm{d}t + \sigma_t(X_t^{X_0,\mu}) \mathrm{d}W_t^{\mu,\nu}. \end{cases}$$

Noting that $|\zeta_s| \le K\|\sigma^{-1}\|_\infty(\mathbb{W}_k(\mu_s,\nu_s)+\|\mu_s-\nu_s\|_{k,var})$ due to **(C1)-(C2)**, Girsanov's theorem yields

$$\Phi_t^{X_0}(\nu)(f)=\mathbb{E}(R(t)f(X_t^{X_0,\mu})),\quad f\in\mathscr{B}_b(\mathbb{R}^{m+d}),t\in[0,T].$$

Therefore, by the Cauchy-Schwarz inequality for conditional expectation, we obtain

$$\begin{aligned}
&\|\Phi_t^{X_0}(\nu)-\Phi_t^{X_0}(\mu)\|_{k,var}\\
&=\mathbb{E}[|R(t)-1|(1+|X_t^{X_0,\mu}|^k)]\\
&\le\mathbb{E}\left([\mathbb{E}(|R(t)-1|^2|\mathscr{F}_0)]^{\frac12}[\mathbb{E}((1+|X_t^{X_0,\mu}|^k)^2|\mathscr{F}_0)]^{\frac12}\right)
\end{aligned}\tag{4.8}$$

Observe that

$$\begin{aligned}
&[\mathbb{E}(|R(t)-1|^2|\mathscr{F}_0)]^{\frac12}\\
&\le\left[\exp\left\{c\int_0^t(\mathbb{W}_k(\mu_s,\nu_s)+\|\mu_s-\nu_s\|_{k,var})^2\mathrm{d}s\right\}-1\right]^{\frac12}\\
&\le\exp\left\{\frac{c}{2}\int_0^t(\mathbb{W}_k(\mu_s,\nu_s)+\|\mu_s-\nu_s\|_{k,var})^2\mathrm{d}s\right\}\\
&\quad\times\sqrt{c}\left(\int_0^t(\mathbb{W}_k(\mu_s,\nu_s)+\|\mu_s-\nu_s\|_{k,var})^2\mathrm{d}s\right)^{\frac12}
\end{aligned}\tag{4.9}$$

for some constant $c>0$. For any $N\ge1$, let

$$\mathscr{P}_{k,X_0}^{N,T}=\{\mu\in C([0,T],\mathscr{P}_k(\mathbb{R}^{m+d})),\mu_0:=\mathscr{L}_{X_0},\sup_{t\in[0,T]}\mathrm{e}^{-Nt}(1+\mu_t(|\cdot|^k))\le N\}.\tag{4.10}$$

Then it is clear that as $N\uparrow\infty$,

$$\mathscr{P}_{k,X_0}^{N,T}\uparrow\mathscr{P}_{k,X_0}^{T}=\{\mu\in C([0,T],\mathscr{P}_k(\mathbb{R}^{m+d})),\mu_0=\mathscr{L}_{X_0}\}.$$

So, it remains to prove that there exists a constant $N_0>0$ such that for any $N\ge N_0$, Φ^{X_0} is a contractive map on $\mathscr{P}_{k,X_0}^{N,T}$.

Firstly, it follows from **(C2)** and the BDG inequality that there exists a constant $c_1>0$ such that for any $\mu\in\mathscr{P}_{k,X_0}^{N,T}$,

$$\begin{aligned}
\mathrm{e}^{-Nt}\mathbb{E}(1+|X_t^{X_0,\mu}|^k)&\le\mathbb{E}(1+|Z_0|^k)+c_1\mathrm{e}^{-Nt}\int_0^t\mathbb{E}(1+|Z_s^{Z_0,\mu}|^k)\mathrm{d}s\\
&\quad+c_1\mathrm{e}^{-Nt}\int_0^t(1+\mu_s(|\cdot|^k))\mathrm{d}s\\
&\le\mathbb{E}(1+|Z_0|^k)+\frac{c_1}{N}\sup_{s\in[0,t]}\mathrm{e}^{-Ns}\mathbb{E}(1+|Z_s^{Z_0,\mu}|^k)+c_1.
\end{aligned}$$

Aa a result, there exists a constant $N_0 > 1$ such that for any $N \geq N_0$, Φ^{X_0} maps $\mathscr{P}^{N,T}_{k,X_0}$ to $\mathscr{P}^{N,T}_{k,X_0}$. Next, we derive from (4.8), (4.3) and (4.9) that

$$\|\Phi_t^{X_0}(\nu) - \Phi_t^{X_0}(\mu)\|_{k,var} \leq C_0(N) \left(\int_0^t (\mathbb{W}_k(\mu_s, \nu_s) + \|\mu_s - \nu_s\|_{k,var})^2 \mathrm{d}s \right)^{\frac{1}{2}}, \quad \mu, \nu \in \mathscr{P}^{N,T}_{k,X_0}$$

for some constant $C_0(N) > 0$, which implies that

$$\sup_{t \in [0,T]} \mathrm{e}^{-\lambda t} \|\Phi_t^{X_0}(\nu) - \Phi_t^{X_0}(\mu)\|_{k,var} \leq \frac{C(N)}{\sqrt{\lambda}} \tilde{\mathbb{W}}_{k,\lambda}(\mu, \nu), \tag{4.11}$$

here for any $\lambda > 0$,

$$\tilde{\mathbb{W}}_{k,\lambda}(\mu, \nu) := \sup_{t \in [0,T]} \mathrm{e}^{-\lambda t} (\|\nu_t - \mu_t\|_{k,var} + \mathbb{W}_k(\mu_t, \nu_t)), \quad \mu, \nu \in \mathscr{P}^T_{k,X_0}.$$

Combining (4.11) with (4.7), we conclude that for any $N \geq N_0$, there exists a constant $\lambda(N) > 0$ such that Φ^{X_0} is a strictly contractive map on $(\mathscr{P}^{N,T}_{k,X_0}, \tilde{\mathbb{W}}_{k,\lambda(N)})$. Therefore, the proof is completed by the Banach fixed point theorem and (4.10).

□

Declarations

Conflict of Interests The authors declare that they have no conflict of interest.
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