Weak Solution and Invariant Probability Measure for McKean-Vlasov SDEs with Integrable Drifts*

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Abstract

In this paper, by utilizing Wang's Harnack inequality with power and the Banach fixed point theorem, the weak well-posedness for McKean-Vlasov SDEs with integrable drift is investigated. In addition, by Banach's fixed theorem, the existence and uniqueness of invariant probability measure for symmetric McKean-Vlasov SDEs and stochastic Hamiltonian system with integrable drifts are obtained.

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1 Introduction

Invariant probability measure is the equilibrium state in physics. There are plentiful results on the invariant probability measure for linear semigroup P_t associated to classical

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diffusion process in $\mathbb{R}^d$:

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t,$$

the infinitesimal generator of which is defined as

$$L = \frac{1}{2}\mathrm{Tr}(\sigma\sigma^*\nabla^2) + b\cdot\nabla.$$

The existence of invariant probability measure can be studied by investigating the tightness of the sequence of probability measures

$$\frac{1}{n}\int_0^n P_t^*\delta_x\mathrm{d}t, \quad n\geq 1,$$

see [12]. Meanwhile, a useful sufficient condition to obtain the existence of invariant probability measure is Lyapunov's condition, i.e.

$$LW_1 \leq C - W_2$$

for some positive function $W_1\in C^2(\mathbb{R}^d)$, positive compact function W_2 and some constant $C>0$ can derive that P_t has an invariant probability measure μ satisfying $\mu(W_2)\leq C$, see [4] and references therein.

For the uniqueness, the classical principle is strong Feller property together with irreducibility, see [12, Theorem 4.2.1]. Moreover, by Wang's Harnack inequality [25, Theorem 1.4.1], the uniqueness can also be ensured. Furthermore, using couplings or generalized couplings, [21] proved the uniqueness of the invariant measures. One can also refer to [6, 5, 7] for conditions on the existence and uniqueness of invariant probability measure by Lyapunov function $V\in C^2(\mathbb{R}^d)$ with $\lim_{|x|\to\infty}V(x)=\infty$ and constants $C,R>0$ such that

$$LV\leq -C, \quad |x|\geq R.$$

Recently, in [23], the existence and uniqueness as well as the regularity such as relative entropy and Sobolev's estimate are derived by hyperboundedness or log-Sobolev's inequality.

However, all the above methods are invalid to obtain the existence and uniqueness for distribution dependent SDEs (McKean-Vlasov SDEs or mean field SDEs), where the associated semigroup P_t^* is nonlinear [17]. In [22], Wang obtained the existence and uniqueness of invariant probability measure and the exponential ergodicity in Wasserstein distance by the method of basic coupling ([10, Definition 2.4]), see [16] for the McKean-Vlasov SDEs with Lévy noise. Quite recently, [27] investigated the existence of invariant probability measure by Schauder's fixed point theorem, see also [1] for the existence of invariant probability measure of functional McKean-Vlasov SDEs by Kakutani's fixed point theorem. In addition, [24] proved the existence and uniqueness of invariant probability measure for (reflecting) McKean-Vlasov SDEs by exponential ergodicity of the decoupled

SDEs and Banach's fixed point theorem. What's more, by using log-Sobolev's inequality or Poincáre's inequality for the invariant probability measure of decoupled SDE and the Banach fixed point theorem, [3, 4] investigated the existence and uniqueness of the solution to stationary nonlinear and non-degenerate Fokker-Planck-Kolmogorov equations.

In this paper, we will prove the weak well-posedness for McKean-Vlasov SDEs with drift being integrable in the spacial component with respect to some reference probability measure by Banach's fixed point theorem. The result extends the one in [23]. Since the invariant probability measure of decoupled SDE is in general unknown when the drift is only assumed to be integrable, the conditions in [3, 4] such as log-Sobolev's inequality and Poincaré's inequality are not explicit. As a result, we only investigate the existence and uniqueness of the invariant probability measure for symmetric and non-degenerate McKean-Vlasov SDEs as well as distribution dependent stochastic Hamiltonian system, where the drift is assumed to be of gradient form and integrable in the spacial component with respect to some reference probability measure.

Let $\mathscr{P}(\mathbb{R}^d)$ be the space of all probability measures on $\mathbb{R}^d$ equipped with the weak topology. Consider the following distribution dependent SDE on $\mathbb{R}^d$:

$$\text{(1.1)} \qquad \mathrm{d}X_t = \{Z_0(X_t) + \sigma(X_t)Z(X_t, \mathscr{L}_{X_t})\}\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t,$$

where $(W_t)_{t\geq 0}$ is an n-dimensional Brownian motion on a complete filtration probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$, $\mathscr{L}_{X_t}$ is the law of X_t,

$$Z : \mathbb{R}^d \times \mathscr{P}(\mathbb{R}^d) \to \mathbb{R}^n, \quad Z_0 : \mathbb{R}^d \to \mathbb{R}^d, \quad \sigma : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^n$$

are measurable. Compared with [23], Z can depend on the distribution of the solution, see (A_Z) below for the condition of Z on the measure component. When a different probability measure $\tilde{\mathbb{P}}$ is concerned, we use $\mathscr{L}_\xi|\tilde{\mathbb{P}}$ to denote the law of a random variable ξ under the probability $\tilde{\mathbb{P}}$, and use $\mathbb{E}_{\tilde{\mathbb{P}}}$ to stand for the expectation under $\tilde{\mathbb{P}}$.

Definition 1.1. (1) An adapted continuous process $(X_t)_{t\geq 0}$ on $\mathbb{R}^d$ is called a solution of (1.1), if X_0 is $\mathscr{F}_0$-measurable,

$$\text{(1.2)} \qquad \mathbb{E}\int_0^T \big\{|Z_0(X_t)| + |\sigma(X_t)Z(X_t, \mathscr{L}_{X_t})| + \|\sigma(X_t)\|^2\big\}\mathrm{d}t < \infty, \quad T > 0,$$

and $\mathbb{P}$-a.s.

$$\text{(1.3)} \quad X_t = X_0 + \int_0^t Z_0(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)Z(X_s, \mathscr{L}_{X_s})\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}W_s, \quad t \geq 0.$$

(2) For any $\mu_0 \in \mathscr{P}(\mathbb{R}^d)$, $((\tilde{X}_t)_{t\geq 0}, (\tilde{W}_t)_{t\geq 0})$ is called a weak solution to (1.1) starting at μ_0, if $(\tilde{W}_t)_{t\geq 0}$ is an n-dimensional Brownian motion under a complete filtration probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \{\tilde{\mathscr{F}}_t\}_{t\geq 0}, \tilde{\mathbb{P}})$, $(\tilde{X}_t)_{t\geq 0}$ is a continuous $\tilde{\mathscr{F}}_t$-adapted process on $\mathbb{R}^d$ with $\mathscr{L}_{\tilde{X}_0}|\tilde{\mathbb{P}} = \mu_0$ and $\tilde{X}_0 \in \tilde{\mathscr{F}}_0$, and (1.2)-(1.3) hold for $(\tilde{X}, \tilde{W}, \tilde{\mathbb{P}}, \mathbb{E}_{\tilde{\mathbb{P}}})$ replacing $(X, W, \mathbb{P}, \mathbb{E})$.

(3) We call (1.1) weakly well-posed for an initial distribution μ_0, if it has a weak solution starting at μ_0 and any weak solution with the same initial distribution is equal in law.

For the well-posedness of distribution dependent SDEs with singular drifts, one can refer to [2, 9, 8, 13, 14, 15, 18, 20, 28] and references within.

The remaining part of the paper is organized as follows: In Section 2, we investigate the weak well-posedness of (1.1) under integrable condition. The existence and uniqueness of invariant probability measure are provided in Section 3.

2 Weak Well-posedness

For any $\mu, \nu \in \mathscr{P}(\mathbb{R}^d)$, the total variation distance between μ and ν is defined as

$$\|\mu - \nu\|_{TV} = 2 \sup_{A \in \mathscr{B}(\mathbb{R}^d)} |\mu(A) - \nu(A)| = \sup_{\|f\|_\infty \le 1} |\mu(f) - \nu(f)|.$$

To obtain the weak well-posedness of (1.1), we make the following assumptions, see [23, Example 4.1, Example 4.3, Example 5.1, Example 5.2] and Section 3 below for the models where (**A**) holds.

(**A**) The reference SDE

$$\mathrm{d}X_t = Z_0(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t \tag{2.1}$$

is strongly well-posed and has a unique invariant probability measure μ^0.

(A_Z) There exist constants $\varepsilon > 0, K_Z > 0$ such that

$$\mu^0(\mathrm{e}^{\varepsilon |Z(\cdot,\mu^0)|^2}) < \infty, \tag{2.2}$$

and

$$|Z(x,\gamma) - Z(x,\tilde{\gamma})| \le K_Z \|\gamma - \tilde{\gamma}\|_{TV}, \quad x \in \mathbb{R}^d, \gamma, \tilde{\gamma} \in \mathscr{P}(\mathbb{R}^d). \tag{2.3}$$

Let π_t be the projection map from $C([0,\infty);\mathbb{R}^d)$ to $\mathbb{R}^d$, i.e.

$$\pi_t(w) = w_t, \quad w \in C([0,\infty);\mathbb{R}^d).$$

For any $\gamma \in \mathscr{P}(\mathbb{R}^d)$, we will prove that (1.1) has a unique weak solution with initial distribution γ and the distribution of the solution $\mathbb{P}^\gamma$ satisfying

$$\mathbb{P}^\gamma \left(w \in C([0,\infty);\mathbb{R}^d), \int_0^t |Z(w_s, \mathbb{P}^\gamma \circ \pi_s^{-1})|^2 \mathrm{d}s < \infty, \quad t \ge 0 \right) = 1. \tag{2.4}$$

Firstly, modifying the proof of [23, Theorem 2.1], we can extend it to the time inhomogeneous case. More precisely, consider

$$\mathrm{d}X_t = \{Z_0(X_t) + \sigma(X_t)\tilde{Z}_t(X_t)\}\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \tag{2.5}$$

here $\tilde{Z} : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^n$ is measurable. Then we have the following result.

Theorem 2.1. *Assume* (**A**) *and that there exists a constant* $\varepsilon > 0$ *such that*

$$\|\mathrm{e}^{\varepsilon|\tilde{Z}|^2}\|_{L^\infty([0,t];L^1(\mu^0))} < \infty, \quad t > 0. \tag{2.6}$$

(i) For any $\gamma \in \mathscr{P}(\mathbb{R}^d)$ *with* $\gamma \ll \mu^0$, (2.5) *has a unique weak solution starting at* γ *and the distribution* $\mathbb{P}^\gamma$ *satisfying*

$$\mathbb{P}^\gamma\left(w \in C([0,\infty);\mathbb{R}^d), \int_0^t |\tilde{Z}_s(w_s)|^2 \mathrm{d}s < \infty, \quad t \geq 0\right) = 1.$$

(ii) If in addition, the semigroup P_t^0 *associated to* (2.1) *satisfies the Harnack inequality, i.e. there exists* $p > 1$ *such that*

$$(P_t^0|f|)^p(z) \leq (P_t^0|f|^p)(\bar{z})\mathrm{e}^{\Phi_p(t,z,\bar{z})}, \quad f \in \mathscr{B}_b(\mathbb{R}^d), z, \bar{z} \in \mathbb{R}^d, t > 0 \tag{2.7}$$

with

$$\int_0^t \left\{\mu^0(\mathrm{e}^{-\Phi_p(s,z,\cdot)})\right\}^{-\frac{1}{p}} \mathrm{d}s < \infty, \quad t > 0, z \in \mathbb{R}^d, \tag{2.8}$$

then the assertion in (i) holds for any $\gamma \in \mathscr{P}(\mathbb{R}^d)$.

Proof. See the Appendix below. □

The main result in this section is the following theorem.

Theorem 2.2. *Assume* (**A**) *and* (A_Z).

(i) For any $\gamma \in \mathscr{P}(\mathbb{R}^d)$ *with* $\gamma \ll \mu^0$, (1.1) *has a unique weak solution starting at* γ *and satisfying* (2.4).

(ii) If in addition, P_t^0 *satisfies* (2.7) *and* (2.8), *then for any* $\gamma \in \mathscr{P}(\mathbb{R}^d)$, *then (i) holds for any* $\gamma \in \mathscr{P}(\mathbb{R}^d)$.

Remark 2.3. *Compared with the localized integrable condition of the drift on the spacial component for the weak well-posedness in [28, Theorem 3.9], the drift* Z *in Theorem 2.2 is allowed to be of some growth. For instance, taking* $n = d$, $Z_0 = -x$, $\sigma = \sqrt{2}I_{d\times d}$, *if* Z *satisfies*

$$|Z(x,\mu^0)| \leq c(1+|x|),$$

then (2.2) *holds with* $c^2\varepsilon < \frac{1}{2}$. *Moreover,* (1.1) *can be degenerate, see [23, Example 4.3] and Example 3.2 below.*

To prove Theorem 2.2, it is sufficient to prove that for any $T > 0$, (1.1) is weakly well-posed on $[0,T]$. So, we fix $T > 0$ in the following. For any $\gamma \in \mathscr{P}(\mathbb{R}^d)$, $\mu \in \mathscr{B}([0,T];\mathscr{P}(\mathbb{R}^d))$, consider

$$\mathrm{d}X_t = \{Z_0(X_t) + \sigma(X_t)Z(X_t,\mu_t)\}\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t \tag{2.9}$$

with initial distribution γ.

Proof of Theorem 2.2. Since the proofs of (i) and (ii) are completely the same, we only prove (ii).

Note that (A_Z) implies that for any $t \in [0,T]$,

$$\|e^{\frac{\varepsilon}{2}|Z(\cdot,\mu_\cdot)|^2}\|_{L^\infty([0,t];L^1(\mu^0))} \le \mu^0(e^{\varepsilon|Z(\cdot,\mu^0)|^2+4\varepsilon K_Z^2}) = \mu^0(e^{\varepsilon|Z(\cdot,\mu^0)|^2})e^{4\varepsilon K_Z^2} < \infty.$$

This together with (**A**), (2.7), (2.8) and Theorem 2.1(ii) for $\tilde{Z}_t = Z(\cdot,\mu_t)$ yields that for any $\gamma \in \mathscr{P}(\mathbb{R}^d)$, (2.9) has a unique weak solution starting at γ and the distribution $\mathbb{P}^{\gamma,\mu}$ satisfying

$$\mathbb{P}^{\gamma,\mu}\left(w \in C([0,T];\mathbb{R}^d), \int_0^T |Z(w_s,\mu_s)|^2 ds < \infty\right) = 1. \tag{2.10}$$

Let $(\{\tilde{X}_t^{\gamma,\mu}\}_{t\in[0,T]}, \{\tilde{W}_t\}_{t\in[0,T]})$ in $(\tilde{\Omega}, \tilde{\mathscr{F}}, \{\tilde{\mathscr{F}}_t\}_{t\in[0,T]}, \tilde{\mathbb{P}})$ be a weak solution to (2.9) with $\mathscr{L}_{\tilde{X}_0^{\gamma,\mu}}|\tilde{\mathbb{P}} = \gamma$. Then we have

$$d\tilde{X}_t^{\gamma,\mu} = \{Z_0(\tilde{X}_t^{\gamma,\mu}) + \sigma(\tilde{X}_t^{\gamma,\mu})Z(\tilde{X}_t^{\gamma,\mu},\mu_t)\}dt + \sigma(\tilde{X}_t^{\gamma,\mu})d\tilde{W}_t. \tag{2.11}$$

Define $\Phi_t^\gamma(\mu) = \mathscr{L}_{\tilde{X}_t^{\gamma,\mu}}|\tilde{\mathbb{P}}, \quad t \in [0,T]$. For $\nu \in \mathscr{B}([0,T];\mathscr{P}(\mathbb{R}^d))$, let

$$\begin{aligned}
\xi_s &= Z(\tilde{X}_s^{\gamma,\mu},\nu_s) - Z(\tilde{X}_s^{\gamma,\mu},\mu_s), \quad s \in [0,T],\\
R(t) &= \exp\left\{\int_0^t \langle \xi_s, d\tilde{W}_s\rangle - \frac{1}{2}\int_0^t |\xi_s|^2 ds\right\}, \quad t \in [0,T],\\
W_t^{\mu,\nu} &= \tilde{W}_t - \int_0^t \xi_s ds, \quad t \in [0,T].
\end{aligned}$$

Then (2.11) can be rewritten as

$$d\tilde{X}_t^{\gamma,\mu} = \{Z_0(\tilde{X}_t^{\gamma,\mu}) + \sigma(\tilde{X}_t^{\gamma,\mu})Z(\tilde{X}_t^{\gamma,\mu},\nu_t)\}dt + \sigma(\tilde{X}_t^{\gamma,\mu})dW_t^{\mu,\nu}.$$

Since $|\xi_s| \le 2K_Z$ due to (2.3), Girsanov's theorem yields that under the probability $d\mathbb{Q} = R(t)d\mathbb{P}$, the process $W_t^{\mu,\nu}$ is an n-dimensional Brownian motion. Then, we have

$$\Phi_t^\gamma(\nu)(f) = \mathbb{E}(R(t)f(\tilde{X}_t^{\gamma,\mu})), \quad f \in \mathscr{B}_b(\mathbb{R}^d), t \in [0,T].$$

Applying Pinsker's inequality [11, 19], we obtain

$$\|\Phi_t^\gamma(\nu) - \Phi_t^\gamma(\mu)\|_{TV}^2 \le \mathbb{E}(|R(t)-1|)^2 \le 2\mathbb{E}(R(t)\log R(t)).$$

This together with (2.3) implies

$$\|\Phi_t^\gamma(\nu) - \Phi_t^\gamma(\mu)\|_{TV}^2 \le \int_0^t K_Z^2\|\mu_s - \nu_s\|_{TV}^2 ds. \tag{2.12}$$

Take $\lambda = K_Z^2$ and consider the space $E_T := \{\mu \in \mathscr{B}([0,T];\mathscr{P}(\mathbb{R}^d)) : \mu_0 = \gamma\}$ equipped with the complete metric

$$\rho(\nu,\mu) := \sup_{t\in[0,T]} \mathrm{e}^{-\lambda t}\|\nu_t - \mu_t\|_{TV}.$$

It follows from (2.12) that

$$\begin{aligned}\sup_{t\in[0,T]} \mathrm{e}^{-2\lambda t}\|\Phi_t^\gamma(\nu) - \Phi_t^\gamma(\mu)\|_{TV}^2 &\le \sup_{t\in[0,T]} \int_0^t K_Z^2 \mathrm{e}^{-2\lambda(t-s)}\mathrm{e}^{-2\lambda s}\|\mu_s - \nu_s\|_{TV}^2 \mathrm{d}s\\ &\le \sup_{s\in[0,T]} \mathrm{e}^{-2\lambda s}\|\mu_s - \nu_s\|_{TV}^2 \sup_{t\in[0,T]} \int_0^t K_Z^2 \mathrm{e}^{-2\lambda(t-s)}\mathrm{d}s\\ &\le \frac{1}{2}\sup_{s\in[0,T]} \mathrm{e}^{-2\lambda s}\|\mu_s - \nu_s\|_{TV}^2.\end{aligned}$$

Then Φ^γ is a strictly contractive map on E_T, so it follows from the Banach fixed theorem that the equation

$$\Phi_t^\gamma(\mu) = \mu_t, \quad t \in [0,T]$$

has a unique solution $\mu \in E_T$. This combined with (2.10) completes the proof. □

3 Existence and Uniqueness of Invariant Probability Measure

In this section, we will consider two cases: one is the symmetric case and the other one is stochastic Hamiltonian system. We will consider the class of invariant probability measures of (1.1) absolutely continuous with respect to μ^0:

$$\tilde{\mathscr{P}}_Z = \{\nu = \rho_\nu\mu^0 : \nu \text{ is an invariant probability measure of } (1.1)\},$$

where ρ_ν is the Radon-Nikodym derivative. For any $\mu \in \mathscr{P}(\mathbb{R}^d)$, consider

$$\mathrm{d}X_t = \{Z_0(X_t) + \sigma(X_t)Z(X_t,\mu)\}\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \tag{3.1}$$

and denote

$$\mathscr{P}_Z^\mu = \{\nu = \rho_\nu\mu^0 : \nu \text{ is an invariant probability measure of } (3.1)\}.$$

3.1 Symmetric Case

In this section, let

$$Z_0 = \frac{1}{2}\sum_{i,j=1}^d \{\partial_j(\sigma\sigma^*)_{ij} - (\sigma\sigma^*)_{ij}\partial_j V\}e_i$$

for some $V \in C^2(\mathbb{R}^d)$. Define

$$\mathscr{E}_0(f,g) = \mu^0(\langle \sigma^* \nabla f, \sigma^* \nabla g \rangle), \quad f,g \in C_0^\infty(\mathbb{R}^d).$$

Let $H_\sigma^{1,2}(\mu^0)$ be the completion of $C_0^\infty(\mathbb{R}^d)$ under the norm

$$\sqrt{\mathscr{E}_1(f,f)} := \{\mu^0(|f|^2 + |\sigma^* \nabla f|^2)\}^{\frac{1}{2}}.$$

Then $(\mathscr{E}_0, H_\sigma^{1,2}(\mu^0))$ is a symmetric Dirichlet form on $L^2(\mu^0)$.

Moreover, we shall introduce the condition **(H)** in [23] and one can refer to [23, Example 5.1, Example 5.2] for the models satisfying **(H)**.

(H) Assume that $\mu^0(\mathrm{d}x) = \mathrm{e}^{-V}\mathrm{d}x$ is a probability measure. There exists $k \geq 2$ such that $\sigma \in C^k(\mathbb{R}^d, \mathbb{R}^d \otimes \mathbb{R}^n)$ and vector fields

$$U_i = \sum_{j=1}^d \sigma_{ji} \partial_j, \quad i = 1, \cdots, n$$

satisfy the Hörmander condition up to the k-th order of Lie brackets. Moreover, $1 \in H_\sigma^{1,2}(\mu_0)$ with $\mathscr{E}_0(1,1) = 0$, and defective log-Sobolev inequality

$$\mu^0(f^2 \log f^2) \leq \kappa \mu^0(|\sigma^* \nabla f|^2) + \beta, \quad f \in C_0^\infty(\mathbb{R}^d), \mu^0(f^2) = 1$$

holds for some $\kappa > 0$ and $\beta \geq 0$.

Theorem 3.1. *Let $F : \mathbb{R}^d \times \mathscr{P}(\mathbb{R}^d) \to \mathbb{R}$ and $\bar{F} : \mathbb{R}^d \to \mathbb{R}$ be measurable and differential in spatial variable. Let $Z(x,\mu) = \frac{\sqrt{2}}{2}(\nabla F(x,\mu) + \nabla \bar{F}(x)), x \in \mathbb{R}^d, \mu \in \mathscr{P}(\mathbb{R}^d)$ and $\sigma = \sqrt{2} I_{d\times d}$ in* (3.1)*. Assume* (**H**) *and that there exist constants $\varepsilon > \kappa, C > 0$ and $\delta \in (0, \frac{\log 2}{2})$ such that*

$$\mu^0(\mathrm{e}^{F(\cdot,\mu^0)+\bar{F}}) + \mu^0(\mathrm{e}^{\frac{\varepsilon}{2}|\nabla F(\cdot,\mu^0)+\nabla \bar{F}|^2}) < \infty, \tag{3.2}$$

$$|\nabla F(x,\mu) - \nabla F(x,\nu)| \leq C\|\mu - \nu\|_{TV}, \quad \mu,\nu \in \mathscr{P}(\mathbb{R}^d), x \in \mathbb{R}^d, \tag{3.3}$$

and

$$|F(x,\mu) - F(x,\nu)| \leq \delta \|\mu - \nu\|_{TV}, \quad \mu,\nu \in \mathscr{P}(\mathbb{R}^d), x \in \mathbb{R}^d.$$

Then (1.1) *has a unique invariant probability measure in $\tilde{\mathscr{P}}_Z$.*

Note that (3.2) and (3.3) imply (A_Z). Moreover, we give a simple example where the conditions in Theorem 3.1 holds. Taking $V = \frac{|x|^2}{2} + d\log(\sqrt{2\pi})$, then $\mu^0 = \frac{1}{(2\pi)^d}\mathrm{e}^{-\frac{|x|^2}{2}}\mathrm{d}x$ and **(H)** holds for $\kappa = 2$, $\beta = 0$. Let $\varepsilon > 2$ and $0 < c < (8\varepsilon)^{-\frac{1}{2}} < \frac{1}{4}$. Set

$$F(x,\mu) = \int_{\mathbb{R}^d} \tilde{F}(y)\mu(\mathrm{d}y), \quad \bar{F}(x) = c|x|^2, \quad x \in \mathbb{R}^d, \mu \in \mathscr{P}(\mathbb{R}^d),$$

where $\tilde{F} : \mathbb{R}^d \to \mathbb{R}$ is a bounded measurable function with $\sup_{x\in\mathbb{R}^d} |\tilde{F}|(x) < \frac{\log 2}{2}$.

Proof of Theorem 3.1. For any $\tilde{\varepsilon} \in (0, \varepsilon)$, it follows from (2.3) that

$$\begin{aligned}\mu^0(\mathrm{e}^{\tilde{\varepsilon}|Z(\cdot,\mu)|^2}) &\leq \mu^0(\mathrm{e}^{\tilde{\varepsilon}(|Z(\cdot,\mu^0)|+K_Z\|\mu-\mu^0\|_{TV})^2}) \\ &= \mu^0(\mathrm{e}^{\tilde{\varepsilon}|Z(\cdot,\mu^0)|^2+2\tilde{\varepsilon}\frac{K_Z\|\mu-\mu^0\|_{TV}}{\sqrt{\varepsilon-\tilde{\varepsilon}}}\sqrt{\varepsilon-\tilde{\varepsilon}}|Z(\cdot,\mu^0)|+\tilde{\varepsilon}K_Z^2\|\mu-\mu^0\|_{TV}^2}) \\ &\leq \mu^0(\mathrm{e}^{\varepsilon|Z(\cdot,\mu^0)|^2})\mathrm{e}^{K_Z^2\frac{\varepsilon\tilde{\varepsilon}}{\varepsilon-\tilde{\varepsilon}}\|\mu-\mu^0\|_{TV}^2} < \infty.\end{aligned} \tag{3.4}$$

[23, Theorem 5.2] implies that for any $\mu \in \mathscr{P}(\mathbb{R}^d)$, (3.1) has a unique invariant probability measure in $\mathscr{P}_Z^\mu$, which is denoted by $\Gamma(\mu)$. Therefore, Γ construct a map from $\mathscr{P}(\mathbb{R}^d)$ to $\mathscr{P}(\mathbb{R}^d)$. Moreover, it is clear that

$$\frac{\mathrm{d}\Gamma(\mu)}{\mathrm{d}\mu^0} = \frac{\mathrm{e}^{F(\cdot,\mu)+\bar{F}}}{\mu^0(\mathrm{e}^{F(\cdot,\mu)+\bar{F}})}.$$

In fact, in this case, under assumption **(H)**, (3.1) degenerate to

$$\mathrm{d}X_t = \nabla\{F(X_t,\mu) + \bar{F}(X_t) - V(X_t)\}\mathrm{d}t + \sqrt{2}\mathrm{d}W_t, \tag{3.5}$$

which has invariant probability measure with the following form

$$\begin{aligned}\Gamma(\mu)(\mathrm{d}x) &= \frac{\exp\{F(x,\mu)+\bar{F}(x)-V(x)\}\mathrm{d}x}{\int_{-\infty}^{\infty}\exp\{F(x,\mu)+\bar{F}(x)-V(x)\mathrm{d}x} \\ &= \frac{\exp\{F(x,\mu)+\bar{F}(x)-V(x)\}\mathrm{d}x}{\int_{-\infty}^{\infty}\mathrm{e}^{F(x,\mu)+\bar{F}(x)}\mu^0(dx)} \\ &= \frac{\mathrm{e}^{F(x,\mu)+\bar{F}(x)-V(x)}\mathrm{d}x}{\int_{-\infty}^{\infty}\mathrm{e}^{F(x,\mu)+\bar{F}(x)}\mu^0(\mathrm{d}x)} \\ &= \frac{\mathrm{e}^{F(x,\mu)+\bar{F}(x)}\mathrm{e}^{-V(x)}\mathrm{d}x}{\int_{-\infty}^{\infty}\mathrm{e}^{F(x,\mu)+\bar{F}(x)}\mu^0(\mathrm{d}x)}.\end{aligned}$$

Then

$$\frac{\Gamma(\mu)(\mathrm{d}x)}{\mu^0(\mathrm{d}x)} = \frac{\Gamma(\mu)(\mathrm{d}x)}{\mathrm{e}^{-V(x)}\mathrm{d}x} = \frac{\mathrm{e}^{F(x,\mu)+\bar{F}(x)}}{\int_{-\infty}^{\infty}\mathrm{e}^{F(x,\mu)+\bar{F}(x)}\mu^0(\mathrm{d}x)}.$$

By Taylor's expansion, we arrive at

$$\begin{aligned}\left|\mathrm{e}^{F(x,\mu)} - \mathrm{e}^{F(x,\nu)}\right| &\leq \mathrm{e}^{F(x,\nu)}\sum_{k=1}^{\infty}\frac{|F(x,\mu)-F(x,\nu)|^k}{k!} \\ &\leq \mathrm{e}^{F(x,\nu)}\sum_{k=1}^{\infty}\frac{\delta^k\|\mu-\nu\|_{TV}^k}{k!} \\ &\leq \mathrm{e}^{F(x,\nu)}\|\mu-\nu\|_{TV}\sum_{k=1}^{\infty}\frac{\delta^k 2^{k-1}}{k!}\end{aligned}$$

$$= \mathrm{e}^{F(x,\nu)}\|\mu-\nu\|_{TV}\frac{\mathrm{e}^{2\delta}-1}{2}.$$

As a result, it holds

$$\begin{aligned}
&\|\Gamma(\mu)-\Gamma(\nu)\|_{TV}\\
&=\int_{\mathbb{R}^d}\left|\frac{\mathrm{e}^{F(\cdot,\mu)+\bar F}}{\mu^0(\mathrm{e}^{F(\cdot,\mu)+\bar F})}-\frac{\mathrm{e}^{F(\cdot,\nu)+\bar F}}{\mu^0(\mathrm{e}^{F(\cdot,\nu)+\bar F})}\right|\mu^0(\mathrm{d}x)\\
&=\int_{\mathbb{R}^d}\left|\frac{\mathrm{e}^{F(\cdot,\mu)+\bar F}\mu^0(\mathrm{e}^{F(\cdot,\nu)+\bar F})-\mathrm{e}^{F(\cdot,\nu)+\bar F}\mu^0(\mathrm{e}^{F(\cdot,\mu)+\bar F})}{\mu^0(\mathrm{e}^{F(\cdot,\mu)+\bar F})\mu^0(\mathrm{e}^{F(\cdot,\nu)+\bar F})}\right|\mu^0(\mathrm{d}x)\\
&\le\int_{\mathbb{R}^d}\left|\frac{\mathrm{e}^{F(\cdot,\mu)+\bar F}\left(\mu^0(\mathrm{e}^{F(\cdot,\nu)+\bar F})-\mu^0(\mathrm{e}^{F(\cdot,\mu)+\bar F})\right)}{\mu^0(\mathrm{e}^{F(\cdot,\mu)+\bar F})\mu^0(\mathrm{e}^{F(\cdot,\nu)+\bar F})}\right|\mu^0(\mathrm{d}x)\\
&\quad+\int_{\mathbb{R}^d}\left|\frac{\left(\mathrm{e}^{F(\cdot,\mu)+\bar F}-\mathrm{e}^{F(\cdot,\nu)+\bar F}\right)\mu^0(\mathrm{e}^{F(\cdot,\mu)+\bar F})}{\mu^0(\mathrm{e}^{F(\cdot,\mu)+\bar F})\mu^0(\mathrm{e}^{F(\cdot,\nu)+\bar F})}\right|\mu^0(\mathrm{d}x)\\
&=\int_{\mathbb{R}^d}\frac{\left|\mu^0(\mathrm{e}^{F(\cdot,\nu)+\bar F})-\mu^0(\mathrm{e}^{F(\cdot,\mu)+\bar F})\right|}{\mu^0(\mathrm{e}^{F(\cdot,\nu)+\bar F})}\mu^0(\mathrm{d}x)+\int_{\mathbb{R}^d}\frac{\left|\mathrm{e}^{F(\cdot,\mu)+\bar F}-\mathrm{e}^{F(\cdot,\nu)+\bar F}\right|}{\mu^0(\mathrm{e}^{F(\cdot,\nu)+\bar F})}\mu^0(\mathrm{d}x)\\
&=\int_{\mathbb{R}^d}\frac{\left|\mu^0(\mathrm{e}^{F(\cdot,\nu)+\bar F}-\mathrm{e}^{F(\cdot,\mu)+\bar F})\right|}{\mu^0(\mathrm{e}^{F(\cdot,\nu)+\bar F})}\mu^0(\mathrm{d}x)+\int_{\mathbb{R}^d}\frac{\left|\mathrm{e}^{F(\cdot,\mu)+\bar F}-\mathrm{e}^{F(\cdot,\nu)+\bar F}\right|}{\mu^0(\mathrm{e}^{F(\cdot,\nu)+\bar F})}\mu^0(\mathrm{d}x)\\
&=2\frac{\left|\mu^0(\mathrm{e}^{F(\cdot,\nu)+\bar F}-\mathrm{e}^{F(\cdot,\mu)+\bar F})\right|}{\mu^0(\mathrm{e}^{F(\cdot,\nu)+\bar F})}\\
&\le 2\frac{\mu^0\left(\mathrm{e}^{F(\cdot,\nu)+\bar F}\|\mu-\nu\|_{TV}\frac{\mathrm{e}^{2\delta}-1}{2}\right)}{\mu^0(\mathrm{e}^{F(\cdot,\nu)+\bar F})}\\
&=\|\mu-\nu\|_{TV}(\mathrm{e}^{2\delta}-1).
\end{aligned}$$

So, when $\delta\in(0,\frac{\log 2}{2})$, Γ is a strictly contractive map on $(\mathscr{P}(\mathbb{R}^d),\|\cdot\|_{TV})$. By Banach's fixed point theorem, we prove that (1.1) has a unique invariant probability measure in $\tilde{\mathscr{P}}_Z$. □

3.2 Stochastic Hamiltonian system

Let $Z_0(x,y)=(y,-x-y)$, $\sigma=\begin{pmatrix}0_{d\times d} & 0_{d\times d}\\ 0_{d\times d} & \sqrt{2}I_{d\times d}\end{pmatrix}$ and $Z(x,\mu)=\frac{\sqrt{2}}{2}(0,\nabla H(\cdot,\mu)(x)+\nabla\bar H(x)),x,y\in\mathbb{R}^d$ with $H:\mathbb{R}^d\times\mathscr{P}(\mathbb{R}^{2d})\to\mathbb{R}$, $\bar H:\mathbb{R}^d\to\mathbb{R}$. Consider

$$(3.6)\qquad \begin{cases}\mathrm{d}X_t=Y_t\mathrm{d}t\\ \mathrm{d}Y_t=(-X_t-Y_t)\mathrm{d}t+\nabla H(\cdot,\mathscr{L}_{(X_t,Y_t)})(X_t)\mathrm{d}t+\nabla\bar H(X_t)\mathrm{d}t+\sqrt{2}\mathrm{d}W_t,\end{cases}$$

By [26, Example 5.1], ($\mathbf{A}$) holds and $\mu^0(\mathrm{d}x,\mathrm{d}y)=\frac{1}{(2\pi)^d}\mathrm{e}^{-\frac{|x|^2+|y|^2}{2}}\mathrm{d}x\mathrm{d}y$ and

$$\|P^0_{t_0}\|_{L^2(\mu^0)\to L^4(\mu^0)}=1$$

for some $t_0 > 0$.

Theorem 3.2. *Assume that there exist constants $\varepsilon > \frac{5}{2}t_0, C > 0$ and $\delta \in (0, \frac{\log 2}{2})$ such that*

$$\mu^0(\mathrm{e}^{H(\cdot,\mu^0)+\bar{H}}) + \mu^0(\mathrm{e}^{\frac{\varepsilon}{2}|\nabla H(\cdot,\mu^0)+\nabla\bar{H}|^2}) < \infty, \tag{3.7}$$

$$|\nabla H(x,\mu) - \nabla H(x,\nu)| \le C\|\mu-\nu\|_{TV}, \quad \mu,\nu \in \mathscr{P}(\mathbb{R}^{2d}), x \in \mathbb{R}^d, \tag{3.8}$$

and

$$|H(x,\mu) - H(x,\nu)| \le \delta\|\mu-\nu\|_{TV}, \quad \mu,\nu \in \mathscr{P}(\mathbb{R}^{2d}), x \in \mathbb{R}^d.$$

Then (3.6) *has a unique invariant probability measure in $\tilde{\mathscr{P}}_Z$.*

Observe that (3.7) and (3.8) imply (A_Z). Similar to Section 3.1, we give an example in which the conditions in Theorem 3.2 hold. Let $\varepsilon > \frac{5}{2}t_0$ and $0 < c < \min(\frac{1}{2}, (8\varepsilon)^{-\frac{1}{2}})$. Consider

$$H(x,\mu) = \int_{\mathbb{R}^{2d}} \tilde{H}(z)\mu(\mathrm{d}z), \quad \bar{H} = c|x|^2, \quad x \in \mathbb{R}^d, \mu \in \mathscr{P}(\mathbb{R}^{2d})$$

for some bounded measurable function $\tilde{H} : \mathbb{R}^{2d} \to \mathbb{R}$ with $\sup_{z\in\mathbb{R}^{2d}} |\tilde{H}|(z) < \frac{\log 2}{2}$.

Proof of Theorem 3.2. For any $\mu \in \mathscr{P}(\mathbb{R}^d)$, substituting $\mathscr{L}_{(X_t,Y_t)}$ with μ in (3.6), we rewrite (3.6) as following

$$\begin{cases} \mathrm{d}X_t = Y_t\mathrm{d}t \\ \mathrm{d}Y_t = (-X_t - Y_t)\mathrm{d}t + \nabla H(\cdot,\mu)(X_t)\mathrm{d}t + \nabla\bar{H}(X_t)\mathrm{d}t + \sqrt{2}\mathrm{d}W_t. \end{cases} \tag{3.9}$$

Due to (3.9) is a special case of (3.1), using (3.4) for $Z(x,\mu) = \nabla H(x,\mu) + \nabla\bar{H}(x)$ and $\tilde{\varepsilon} \in (\kappa_0, \varepsilon)$ with $\kappa_0 = \frac{5}{2}t_0$ and applying [23, Theorem 3.1, Theorem 4.1], we conclude that (3.9) has a unique invariant probability measure in $\mathscr{P}_Z^\mu$ and we denote it as $\Gamma(\mu)$, which satisfies

$$\frac{\mathrm{d}\Gamma(\mu)}{\mathrm{d}\mu^0} = \frac{\mathrm{e}^{H(\cdot,\mu)+\bar{H}}}{\mu^0(\mathrm{e}^{H(\cdot,\mu)+\bar{H}})}. \tag{3.10}$$

In fact, the infinitesimal generator of (3.9) is

$$Lf(x,y) = y\nabla_x f + (-x - y + \nabla H(x,\mu) + \nabla\bar{H}(x))\nabla_y f + \nabla_y^2 f, \quad f \in C_0^\infty(\mathbb{R}^{2d}),$$

which yields

$$\begin{aligned} &\int_{\mathbb{R}^{2d}} Lf(x,y)\exp\left\{-\frac{|x|^2}{2} - \frac{|y|^2}{2} + H(x,\mu) + \bar{H}(x)\right\}\mathrm{d}x\mathrm{d}y \\ &= -\int_{\mathbb{R}^{2d}} \left\langle \nabla_x f, \nabla_y \exp\left\{-\frac{|x|^2}{2} - \frac{|y|^2}{2} + H(x,\mu) + \bar{H}(x)\right\}\right\rangle \mathrm{d}x\mathrm{d}y \end{aligned}$$

$$+\int_{\mathbb{R}^{2d}}\left\langle \nabla_y f, \nabla_x \exp\left\{-\frac{|x|^2}{2}-\frac{|y|^2}{2}+H(x,\mu)+\bar{H}(x)\right\}\right\rangle \mathrm{d}x\mathrm{d}y=0,\quad f\in C_0^\infty(\mathbb{R}^{2d}).$$

Therefore,

$$\begin{aligned}\Gamma(\mu)(\mathrm{d}x,\mathrm{d}y)&=\frac{\exp\left\{-\frac{|x|^2}{2}-\frac{|y|^2}{2}+H(x,\mu)+\bar{H}(x)\right\}\}\mathrm{d}x}{\int_{\mathbb{R}^{2d}}\exp\{-\frac{|x|^2}{2}-\frac{|y|^2}{2}+H(x,\mu)+\bar{H}(x)\mathrm{d}x\mathrm{d}y}\\&=\frac{(2\pi)^d\mu^0(\mathrm{d}x,\mathrm{d}y)\mathrm{e}^{H(x,\mu)+\bar{H}(x)}}{(2\pi)^d\int_{\mathbb{R}^{2d}}\mathrm{e}^{H(x,\mu)+\bar{H}(x)}\mu^0(\mathrm{d}x,\mathrm{d}y)}\\&=\frac{\mu^0(\mathrm{d}x,\mathrm{d}y)\mathrm{e}^{H(x,\mu)+\bar{H}(x)}}{\int_{\mathbb{R}^{2d}}\mathrm{e}^{H(x,\mu)+\bar{H}(x)}\mu^0(\mathrm{d}x,\mathrm{d}y)}.\end{aligned}$$

Then the claim (3.10) holds. Repeating the proof of Theorem 3.1, we know that Γ is a strictly contractive map on $(\mathscr{P}(\mathbb{R}^d),\|\cdot\|_{TV})$, which proves that (3.6) has a unique invariant probability measure in $\tilde{\mathscr{P}}_Z$. □

4 Appendix

In this section, we give the proof of Theorem 2.1.

Proof of Theorem 2.1. Since the proof of the weak uniqueness can be completely repeated according to [23, Proof of Theorem 2.1(2)], we only prove the weak existence in the following.

(i) For any $t_0\geq 0$, let

$$R_{t_0}^{\mu^0}(t)=\exp\left\{\int_0^t\langle\tilde{Z}_{s+t_0}(X_s^{\mu^0}),\mathrm{d}W_s\rangle-\frac{1}{2}\int_0^t|\tilde{Z}_{s+t_0}(X_s^{\mu^0})|^2\mathrm{d}s\right\},\quad t\geq 0,$$

where μ^0 is the unique invariant probability measure of the reference SDE (2.1), $X_t^{\mu^0}$ is the solution of (2.5) started from the initial distribution μ^0. For any $\xi\in\mathbb{R}^d$, $R_{t_0}^{\xi}(t)$ is defined in the same way by replacing μ^0 with ξ. For simplicity, we denote $R^{\mu^0}(t)=R_0^{\mu^0}(t), R^\xi(t)=R_0^\xi(t), t\geq 0$.

We first prove that for μ^0-a.e. ξ, $(R^\xi(t))_{t\geq 0}$ is a martingale and divide the proof into the following three steps.

Step 1: For μ^0-a.e. ξ, $(R^\xi(t))_{t\geq 0}$ is a supmartingale.

Note that (2.6) implies that

$$\mathbb{E}\int_0^T|\tilde{Z}_s(X_s^{\mu^0})|^2\mathrm{d}s=\int_0^T\mathbb{E}|\tilde{Z}_s(X_s^{\mu^0})|^2\mathrm{d}s=\int_0^T\mu^0(|\tilde{Z}_s|^2)\mathrm{d}s<\infty,\ T>0.$$

Let

$$\tau_n:=\inf\{t\geq 0:\int_0^t|\tilde{Z}_s(X_s^{\mu^0})|^2\mathrm{d}s\geq n\},\quad n\geq 1.$$

Then $\mathbb{P}$-a.s. $\lim_{n\to\infty}\tau_n=\infty$. By Girsanov's theorem, for each $n\geq 1$,

$$R^{\mu^0}(t\wedge\tau_n):=\exp\left\{\int_0^{t\wedge\tau_n}\langle\tilde{Z}_s(X_s^{\mu^0}),\mathrm{d}W_s\rangle-\frac{1}{2}\int_0^{t\wedge\tau_n}|\tilde{Z}_s(X_s^{\mu^0})|^2\mathrm{d}s\right\},\quad t\geq 0,$$

is a martingale. Applying Fatou's lemma, we arrive at

$$\begin{aligned}\mathbb{E}(R^{\mu^0}(t)|\mathscr{F}_s)&=\mathbb{E}\left(\liminf_{n\to\infty}R^{\mu^0}(t\wedge\tau_n)|\mathscr{F}_s\right)\leq\liminf_{n\to\infty}\mathbb{E}\left(R^{\mu^0}(t\wedge\tau_n)|\mathscr{F}_s\right)\\&=\liminf_{n\to\infty}R^{\mu^0}(s\wedge\tau_n)=R^{\mu^0}(s),\ t\geq s\geq 0.\end{aligned}$$

Then, $(R^{\mu^0}(t))_{t\geq 0}$ is a supmartingale. Since

$$\mathbb{E}\int_0^T|\tilde{Z}_s(X_s^{\mu^0})|^2\mathrm{d}s=\int_{\mathbb{R}^d}\mathbb{E}\int_0^T|\tilde{Z}_s(X_s^{\xi})|^2\mathrm{d}s\mu^0(\mathrm{d}\xi)<\infty,\quad T>0,$$

then $\mathbb{E}\int_0^T|\tilde{Z}_s(X_s^{\xi})|^2\mathrm{d}s<\infty,T>0$ for μ^0-a.e. ξ. By the same argument as in step 1, we conclude that the claim holds.

Step 2: $\mathbb{E}R^{\mu^0}(t)=1$**,** $t\geq 0$.

For any $t_0\geq 0$, by Jensen's inequality and (2.6), we obtain

$$\begin{aligned}\mathbb{E}\mathrm{e}^{\frac{1}{2}\int_0^{2\varepsilon}|\tilde{Z}_{s+t_0}(X_s^{\mu^0})|^2\mathrm{d}s}&\leq\frac{1}{2\varepsilon}\int_0^{2\varepsilon}\mathbb{E}\mathrm{e}^{\varepsilon|\tilde{Z}_{s+t_0}(X_s^{\mu^0})|^2}\mathrm{d}s=\frac{1}{2\varepsilon}\int_0^{2\varepsilon}\mu^0(\mathrm{e}^{\varepsilon|\tilde{Z}_{s+t_0}|^2})\mathrm{d}s\\&=\frac{1}{2\varepsilon}\int_{t_0}^{2\varepsilon+t_0}\mu^0(\mathrm{e}^{\varepsilon|\tilde{Z}_s|^2})\mathrm{d}s<\infty.\end{aligned}$$

It follows from Novikov's condition that for any $t_0\geq 0$, $(R_{t_0}^{\mu^0}(t))_{t\in[0,2\varepsilon]}$ is a martingale. Note that

$$\mathbb{E}\mathrm{e}^{\frac{1}{2}\int_0^{2\varepsilon}|\tilde{Z}_{s+t_0}(X_s^{\mu^0})|^2\mathrm{d}s}=\int_{\mathbb{R}^d}\mathbb{E}\mathrm{e}^{\frac{1}{2}\int_0^{2\varepsilon}|\tilde{Z}_{s+t_0}(X_s^{\xi})|^2\mathrm{d}s}\mu^0(\mathrm{d}\xi)<\infty,$$

a same argument yields that for any $t_0\geq 0$, μ^0-a.e. ξ, $(R_{t_0}^{\xi}(t))_{t\in[0,2\varepsilon]}$ is a martingale. As a result, for any $t_0\geq 0$, we get μ^0-a.e. ξ,

$$\mathbb{E}R_{t_0}^{\mu^0}(t)=\mathbb{E}R_{t_0}^{\xi}(t)=1,\ t\in[0,2\varepsilon].\tag{4.1}$$

Now we assume that $\mathbb{E}R^{\mu^0}(t)=1,t\in[0,2k\varepsilon]$ for some $k\geq 1$, it remains to prove $\mathbb{E}R^{\mu^0}(t)=1,t\in[2k\varepsilon,2(k+1)\varepsilon]$. Indeed, let $t_1=2k\varepsilon$ and

$$\tilde{W}_t=W_{t+t_1}-W_{t_1},\ t\geq 0.$$

Then $\tilde{W}(t)$ is a Brownian motion on the same probability space $(\Omega,\mathscr{F},\mathbb{P})$ with respect to filtration $(\mathscr{F}_{t+t_1})_{t\geq 0}$. Consider

$$\mathrm{d}\tilde{X}_t=Z_0(\tilde{X}_t)\mathrm{d}t+\sigma(\tilde{X}_t)\mathrm{d}\tilde{W}_t.\tag{4.2}$$

By Assumption (**A**), (4.2) is strongly well-posed. Let $\tilde{X}_t^\xi$ be the unique solution with $\tilde{X}_0^\xi = \xi$. Observing that

$$\begin{aligned} X^\xi_{t_1+t-t_1} &= X^\xi_{t_1} + \int_{t_1}^t Z_0(X^\xi_s)\mathrm{d}s + \int_{t_1}^t \sigma(X^\xi_s)\mathrm{d}W_s \\ &= X^\xi_{t_1} + \int_0^{t-t_1} Z_0(X^\xi_{s+t_1})\mathrm{d}s + \int_0^{t-t_1} \sigma(X^\xi_{s+t_1})\mathrm{d}\tilde{W}_s, \quad t \geq t_1, \end{aligned}$$

we conclude that $X_t^\xi = \tilde{X}_{t-t_1}^{X_{t_1}^\xi}$ by the strong well-posedness of (4.2). Therefore, we derive

$$\begin{aligned} \Gamma(t) :&= \mathbb{E}\left(\mathrm{e}^{\int_{t_1}^t \langle \tilde{Z}_s(X_s^{\mu^0}), \mathrm{d}W_s\rangle - \frac{1}{2}\int_{t_1}^t |\tilde{Z}_s(X_s^{\mu^0})|^2 \mathrm{d}s} \middle| \mathscr{F}_{t_1} \right) \\ &= \mathbb{E}\left(\mathrm{e}^{\int_0^{t-t_1} \langle \tilde{Z}_{r+t_1}(X_{r+t_1}^{\mu^0}), \mathrm{d}\tilde{W}_r\rangle - \frac{1}{2}\int_0^{t-t_1} |\tilde{Z}_{r+t_1}(X_{r+t_1}^{\mu^0})|^2 \mathrm{d}r} \middle| \mathscr{F}_{t_1} \right) \\ &= \mathbb{E}\left(\mathrm{e}^{\int_0^{t-t_1} \langle \tilde{Z}_{r+t_1}(\tilde{X}_r^{X_{t_1}^{\mu^0}}), \mathrm{d}\tilde{W}_r\rangle - \frac{1}{2}\int_0^{t-t_1} |\tilde{Z}_{r+t_1}(\tilde{X}_r^{X_{t_1}^{\mu^0}})|^2 \mathrm{d}r} \middle| \mathscr{F}_{t_1} \right) \\ &= \left\{ \mathbb{E}\left(\mathrm{e}^{\int_0^{t-t_1} \langle \tilde{Z}_{r+t_1}(\tilde{X}_r^\xi), \mathrm{d}\tilde{W}_r\rangle - \frac{1}{2}\int_0^{t-t_1} |\tilde{Z}_{r+t_1}(\tilde{X}_r^\xi)|^2 \mathrm{d}r} \right) \right\} \Bigg|_{\xi = X_{t_1}^{\mu^0}} \\ &= \mathbb{E} R_{t_1}^\xi (t-t_1)\big|_{\xi = X_{t_1}^{\mu^0}}, \; t \in [t_1, t_1 + 2\varepsilon]. \end{aligned}$$

Since the law of $X_{t_1}^{\mu^0}$ is μ^0, (4.1) with $t_0 = t_1$ implies that $\mathbb{P}$-a.s. $\Gamma(t) = 1, t \in [2k\varepsilon, 2(k+1)\varepsilon]$, which yields $\mathbb{P}$-a.s.

$$\mathbb{E}[R^{\mu^0}(t)|\mathscr{F}_{t_1}] = R^{\mu^0}(t_1)\Gamma(t) = R^{\mu^0}(t_1), \quad t \in [2k\varepsilon, 2(k+1)\varepsilon]. \tag{4.3}$$

This together with $\mathbb{E}R^{\mu^0}(t) = 1, t \in [0, t_1]$ implies

$$\mathbb{E}R^{\mu^0}(t) = \mathbb{E}(\mathbb{E}[R^{\mu^0}(t)|\mathscr{F}_{t_1}]) = \mathbb{E}(R^{\mu^0}(t_1)) = 1, \; t \in [2k\varepsilon, 2(k+1)\varepsilon].$$

Step 3: For μ^0-a.e. ξ, $(R^\xi(t))_{t\geq 0}$ is a martingale.

Combining Step 1 and Step 2, we conclude that $(R^{\mu^0}(t))_{t\geq 0}$ is a martingale. Since for μ^0-a.e. ξ, $(R^\xi(t))_{t\geq 0}$ is a supmartingale due to Step 1, then μ^0-a.e. ξ, $\mathbb{E}(R^\xi(t))$ is decreasing in t and $\mathbb{E}(R^\xi(t)) \leq \mathbb{E}(R^\xi(0)) = 1$. Noting that

$$1 = \mathbb{E}R^{\mu^0}(t) = \int_{\mathbb{R}^d} \mathbb{E}(R^\xi(t))\mu^0(\mathrm{d}\xi), \quad t \geq 0,$$

we conclude that for any $t \geq 0$, μ^0-a.e. ξ, $\mathbb{E}R^\xi(t) = 1$, which together with the fact that $\mathbb{E}(R^\xi(t))$ is decreasing in t implies that for μ^0-a.e. ξ, $\mathbb{E}R^\xi(t) = 1, t \geq 0$. So, for μ^0-a.e. ξ, $(R^\xi(t))_{t\geq 0}$ is a martingale.

As a result, applying Girsanov's theorem, for μ^0-a.e. ξ,

$$\bar{W}_t^\xi = W_t - \int_0^t \tilde{Z}_s(X_s^\xi)\mathrm{d}s, \ t \in [0,T]$$

is a d-dimensional Brownian motion under the probability measure $\mathbb{Q}^\xi$ on $\mathscr{F}_\infty$ defined by

$$\mathbb{Q}^\xi(A) := \mathbb{E}(1_A R^\xi(T)), \quad A \in \mathscr{F}_T, T > 0,$$

which is well-defined in view of the martingale property of $R^\xi(t)$. Now, we rewrite (2.1) as

$$\mathrm{d}X_t^\xi = (Z_0(X_t^\xi) + \sigma(X_t^\xi)\tilde{Z}_t(X_t^\xi))\mathrm{d}t + \sigma(X_t^\xi)\mathrm{d}\bar{W}_t^\xi.$$

Therefore, for μ^0-a.e. ξ, $(\{X_t^\xi, \bar{W}_t^\xi\}_{t\geq 0}, \mathbb{Q}^\xi)$ is a weak solution of (2.5), which tells that for any $\gamma \in \mathscr{P}(\mathbb{R}^d)$ with $\gamma \ll \mu^0$, γ-a.e. ξ, $(\{X_t^\xi, \bar{W}_t^\xi\}_{t\geq 0}, \mathbb{Q}^\xi)$ is a weak solution of (2.5), then (2.5) has a weak solution starting at γ. Combining the weak uniqueness of (2.1) as mentioned at the beginning of the proof, we complete the proof.

(ii) Adopting the same argument of obtaining (4.3) in the proof of (i), it is sufficient to find some constant $\tau > 0$ such that for any $t_0 \geq 0$, $\xi \in \mathbb{R}^d$, $\{R_{t_0}^\xi(t)\}_{t\in[0,\tau]}$ is a martingale, which can be ensured if

$$\mathbb{E}\mathrm{e}^{\frac{1}{2}\int_0^\tau |\tilde{Z}_{s+t_0}(X_s^\xi)|^2\mathrm{d}s} < \infty. \tag{4.4}$$

By the Harnack inequality (2.7), we have

$$\mu^0(\mathrm{e}^{-\Phi_p(s,\xi,\cdot)})(P_s^0 \mathrm{e}^{\frac{\varepsilon}{p}|\tilde{Z}_{t_0+s}|^2})^p(\xi) \leq \mu^0(\mathrm{e}^{\varepsilon|\tilde{Z}_{t_0+s}|^2}).$$

Combining this with (2.6) and (2.8), for any $\lambda > 0$, it holds

$$\begin{aligned}
&\mathbb{E}\mathrm{e}^{\lambda\int_0^{\frac{\varepsilon}{p\lambda}} |\tilde{Z}_{t_0+s}(X_s^\xi)|^2\mathrm{d}s} \\
&\leq \frac{p\lambda}{\varepsilon}\int_0^{\frac{\varepsilon}{p\lambda}} \mathbb{E}\mathrm{e}^{\frac{\varepsilon}{p}|\tilde{Z}_{t_0+s}(X_s^\xi)|^2}\mathrm{d}s \\
&\leq \frac{p\lambda}{\varepsilon}\int_0^{\frac{\varepsilon}{p\lambda}} \{\mu^0(\mathrm{e}^{-\Phi_p(s,\xi,\cdot)})\}^{-\frac{1}{p}}\{\mu^0(\mathrm{e}^{\varepsilon|\tilde{Z}_{t_0+s}|^2})\}^{\frac{1}{p}}\mathrm{d}s \\
&\leq \left(\|\mathrm{e}^{\varepsilon|\tilde{Z}|^2}\|_{L^\infty([0,\frac{\varepsilon}{p\lambda}+t_0],L^1(\mu^0))}\right)^{\frac{1}{p}} \frac{p\lambda}{\varepsilon}\int_0^{\frac{\varepsilon}{p\lambda}} \{\mu^0(\mathrm{e}^{-\Phi_p(s,\xi,\cdot)})\}^{-\frac{1}{p}}\mathrm{d}s < \infty, \quad t_0 \geq 0, \xi \in \mathbb{R}^d.
\end{aligned}$$

This yields that (4.4) holds for some $\tau > 0$. □

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