A Low-regularity Fourier Integrator for the Davey-Stewartson II System with Almost Mass Conservation * [1]

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Abstract. In this work, we propose a low-regularity Fourier integrator with almost mass conservation to solve the Davey-Stewartson II system (hyperbolic-elliptic case). Arbitrary order mass convergence can be achieved by suitably adding correction terms while keeping the first order accuracy in $H^\gamma \times H^{\gamma+1}$ for initial data in $H^{\gamma+1} \times H^{\gamma+1}$ with $\gamma > 1$. The main theorem is that, up to some fixed time T, there exist constants τ_0 and C depending only on T and $\|u\|_{L^\infty((0,T);H^{\gamma+1})}$, such that for any $0 < \tau \leq \tau_0$, we have

$$\|u(t_n,\cdot) - u^n\|_{H^\gamma} \leq C\tau, \quad \|v(t_n,\cdot) - v^n\|_{H^{\gamma+1}} \leq C\tau,$$

where u^n and v^n denote the numerical solutions at $t_n = n\tau$. Moreover, the mass of the numerical solution $M(u^n)$ satisfies

$$|M(u^n) - M(u_0)| \leq C\tau^5.$$

Key words: Davey-Stewartson II system, Low-regularity, Exponential integrator, First order accuracy, Mass conservation

2010 Mathematics Subject Classification: 65M12, 65M15, 35Q41

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[1]* C. Hao is partially supported by NSFC 12171356. C. Ning is partially supported by NSFC 11901120.

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1. Introduction

Recently, Kou, Ning and Wang [4] constructed a first-order low-regularity integrator for Davey-Stewartson II (DS-II) system, which showed the first order accuracy in H^γ for initial data in $H^{\gamma+1}$. However, it is difficult to maintain the geometric structure of underlying PDEs for low-regularity integrators. The geometric structure is not only important property, but also becomes a standard to judge the effectiveness of the numerical methods. In generally, the conservative schemes perform much better than the nonconservative schemes.

Ismail and Taha [2] proposed a linearly implicit scheme with mass conservation for solving the coupled nonlinear Schrödinger equation, and the proposed scheme conserves the mass exactly ruling out any possibility of blowing up of the numerical solution. Wu and Yao [7] proposed a first-order Fourier integrator with almost mass conservation for solving the cubic nonlinear Schrödinger equation in one dimension. To the best of our knowledge, this is the first attempt to consider the conservation laws of the numerical solution for the exponential-type integrators. For the Korteweg-de Vries equation, Maierhofer and Schratz [5] proposed a implicit scheme with mass conservation. For the Davey-Stewartson systems, Frauendiener and Klein [1] presented a detailed numerical study of the Davey-Stewartson I system and obtained the relative conservation of the mass.

In this work, inspired by the works of [7] and [4], we construct a new scheme such that it could almost conserve the mass and require as low regularity as possible while keeping the first-order convergence for the DS-II system under the rough initial data on a torus. Due to the complexity of the phase function, we shall fully exploit the structure of the DS-II system and employ delicate Fourier analysis.

The DS-II system with the rough initial data on a torus studied in this work is

$$\begin{cases} i\partial_t u(t,\boldsymbol{x}) + \partial_{x_1}^2 u(t,\boldsymbol{x}) - \partial_{x_2}^2 u(t,\boldsymbol{x}) = \mu_1 |u(t,\boldsymbol{x})|^2 u(t,\boldsymbol{x}) + \mu_2 u(t,\boldsymbol{x})\partial_{x_1} v(t,\boldsymbol{x}), \\ \partial_{x_1}^2 v(t,\boldsymbol{x}) + \partial_{x_2}^2 v(t,\boldsymbol{x}) = \partial_{x_1}\left(|u(t,\boldsymbol{x})|^2\right), \quad t>0,\ \boldsymbol{x}=(x_1,x_2)\in\mathbb{T}^2, \end{cases} \tag{1.1}$$

where $\mu_1,\ \mu_2 \in \mathbb{R}$, $\mathbb{T} = (0, 2\pi)$, $u = u(t,\boldsymbol{x}) : \mathbb{R}^+ \times \mathbb{T}^2 \to \mathbb{C}$, $u_0 = u(0,\boldsymbol{x}) \in H^\gamma(\mathbb{T}^2)\,(\,\gamma \geq 0\,)$ is unknown.

A variable substitution is introduced:

$$\begin{cases} \xi_1 = \dfrac{1}{2}(x_1 + x_2), \\ \xi_2 = \dfrac{1}{2}(x_1 - x_2), \end{cases}$$

that is

$$\begin{cases} \phi(\xi_1,\xi_2) = u(x_1,x_2), \\ \psi(\xi_1,\xi_2) = v(x_1,x_2). \end{cases} \tag{1.2}$$

then, the equations (1.1) can be rewritten as

$$\begin{cases} i\phi_t + \phi_{\xi_1\xi_2} = \mu_1|\phi|^2\phi + \dfrac{1}{2}\mu_2\phi(\psi_{\xi_1} + \psi_{\xi_2}), \\ \psi_{\xi_1\xi_1} + \psi_{\xi_2\xi_2} = (\partial_{\xi_1} + \partial_{\xi_2})(|\phi|^2). \end{cases} \tag{1.3}$$

To avoid confusion of subsequent symbols, the equations (1.3) are rewritten as equations with respect to x_1, x_2 to obtain

$$\begin{cases} i\phi_t + \partial_{x_1x_2}\phi - \phi E(|\phi|^2) = 0, \\ \psi = -(-\Delta)^{-1}(\partial_{x_1} + \partial_{x_2})(|\phi|^2), \end{cases} \tag{1.4}$$

where $Ef = (\tilde{\mu}_1 + \mu_2\frac{\partial_{x_1x_2}}{\Delta})f, \tilde{\mu}_1 = \mu_1 + \frac{1}{2}\mu_2, \phi_0 = u\big(0, x_1+x_2, x_1-x_2\big)$. The derivation process can be found in [4].

Based on the above variable substitution, it can be seen that we only need to analyze the system (1.4). The DS-II system is completely integrable and thus has an infinite number of formally conserved quantities. For the solution ϕ of the system (1.4) in L^2, we have the following law of mass conservation

$$M(\phi(t)) = \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} |\phi(t, \boldsymbol{x})|^2 \, d\boldsymbol{x} = M(\phi_0) = M_0. \tag{1.5}$$

To this purpose, we define a modified numerical scheme of (1.4) as follows.

First, we define the function

$$\omega(z) = \begin{cases} \dfrac{e^z - 1}{z}, & \text{if } z \neq 0, \\ 1, & \text{if } z = 0. \end{cases} \tag{1.6}$$

Here we denote $\Pi_0(f)$ to be the zero mode of the function f, that is

$$\Pi_0(f) = \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} f(\boldsymbol{x})d\boldsymbol{x},$$

so

$$M_0 = \Pi_0(|\phi_0|^2).$$

And we define a function

$$\Psi(f) = e^{i\partial_{x_1x_2}\tau}f - i\tau e^{i\partial_{x_1x_2}\tau}\Big[f \cdot E\big(\omega(-2i\partial_{x_1x_2}\tau)\bar{f} \cdot f\big)\Big]. \tag{1.7}$$

Then we denote the functionals I, J_1, J_2 to be

$$I(U) = \Psi(U) - e^{i\partial_{x_1x_2}\tau}U; \tag{1.8}$$

$$J_1(U) = H(U)e^{i\partial_{x_1x_2}\tau}U; \tag{1.9}$$

$$J_2(U) = -\frac{1}{2}\big(H(U)\big)^2 e^{i\partial_{x_1x_2}\tau}U - \Pi_0(|U|^2)^{-1}H(U)\mathrm{Re}\big(\Pi_0(I(U)e^{-i\partial_{x_1x_2}\tau}\bar{U})\big)e^{i\partial_{x_1x_2}\tau}U; \tag{1.10}$$

and

$$H(U) = -\Pi_0(|U|^2)^{-1}\Big[\mathrm{Re}\big(\Pi_0(I(U)e^{-i\partial_{x_1x_2}\tau}\bar{U})\big) + \frac{1}{2}\Pi_0(|I(U)|^2)\Big]. \tag{1.11}$$

Now the modified numerical scheme of ϕ is defined by

$$\phi^n = \Psi(\phi^{n-1}) + J_1(\phi^{n-1}) + J_2(\phi^{n-1}), \tag{1.12}$$

based on the DS systems (1.4), we can write the numerical solution of ψ

$$\psi^n = -(-\Delta)^{-1}(\partial_{x_1} + \partial_{x_2})(|\phi^n|^2), \tag{1.13}$$

where $n = 1, 2, \dots, \frac{T}{\tau}$; $\quad \phi^0 = \phi_0$.

Then we obtain that

Theorem 1.1. *Let ϕ^n and ψ^n be the numerical solution of the DS-II system (1.4) obtained from the LRI schemes (1.12) and (1.13) up to some fixed time $T > 0$. Under the assume $\phi_0 \in H^{\gamma+1}(\mathbb{T}^2)$, for some $\gamma > 1$, there exist constants τ_0 and $C > 0$, such that for any $0 < \tau \le \tau_0$, we have*

$$\|\phi(t_n, \cdot) - \phi^n\|_{H^\gamma} \le C\tau, \quad \|\psi(t_n, \cdot) - \psi^n\|_{H^{\gamma+1}} \le C\tau, \quad n = 0, 1, \dots, \frac{T}{\tau}; \tag{1.14}$$

moreover,

$$|M(\phi^n) - M(\phi_0)| \le C\tau^5, \tag{1.15}$$

where the constants τ_0 and C depend only on T and $\|\phi\|_{L^\infty((0,T);H^{\gamma+1})}$.

By variable substitution (1.2), we can obtain the scheme of u and v for $n = 1, 2, \dots, \frac{T}{\tau}$, $u^0 = u_0$:

$$u^n = \tilde{\Psi}(u^{n-1}) + \tilde{J}_1(u^{n-1}) + \tilde{J}_2(u^{n-1}), \tag{1.16}$$

and

$$v^n = -(-\Delta)^{-1}\partial_{x_1}(|u^n|^2), \tag{1.17}$$

where

$$\tilde{\Psi}(f) = e^{i(\partial_{x_1}^2 - \partial_{x_2}^2)\tau} f - i\tau e^{i(\partial_{x_1}^2 - \partial_{x_2}^2)\tau}\Big[f \cdot E\big(\omega(-2i(\partial_{x_1}^2 - \partial_{x_2}^2)\tau)\bar{f} \cdot f\big)\Big],$$

$$\tilde{I}(U) = \tilde{\Psi}(U) - e^{i(\partial_{x_1}^2 - \partial_{x_2}^2)\tau}U; \qquad \tilde{J}_1(U) = \tilde{H}(U)e^{i(\partial_{x_1}^2 - \partial_{x_2}^2)\tau}U;$$

$$\tilde{J}_2(U) = -\frac{1}{2}\big(\tilde{H}(U)\big)^2 e^{i(\partial_{x_1}^2 - \partial_{x_2}^2)\tau}U - \Pi_0(|U|^2)^{-1}\tilde{H}(U)\mathrm{Re}\big(\Pi_0(\tilde{I}(U)e^{-i(\partial_{x_1}^2 - \partial_{x_2}^2)\tau}\bar{U})\big)e^{i(\partial_{x_1}^2 - \partial_{x_2}^2)\tau}U;$$

and

$$\tilde{H}(U) = -\Pi_0(|U|^2)^{-1}\Big[\mathrm{Re}\big(\Pi_0(I(U)e^{-i(\partial_{x_1}^2 - \partial_{x_2}^2)\tau}\bar{U})\big) + \frac{1}{2}\Pi_0(|\tilde{I}(U)|^2)\Big].$$

Then we get that

Corollary 1.2. *Let u^n and v^n be the numerical solution of the DS-II system (1.1) obtained from the LRI schemes (1.16) and (1.17) up to some fixed time $T > 0$. Under the assume $u_0 \in H^{\gamma+1}(\mathbb{T}^2)$, for some $\gamma > 1$, there exist constants τ_0 and $C > 0$, such that for any $0 < \tau \le \tau_0$, we have*

$$\|u(t_n, \cdot) - u^n\|_{H^\gamma} \le C\tau, \quad \|v(t_n, \cdot) - v^n\|_{H^{\gamma+1}} \le C\tau, \quad n = 0, 1, \dots, \frac{T}{\tau}; \tag{1.18}$$

moreover,

$$|M(u^n) - M(u_0)| \le C\tau^5; \tag{1.19}$$

where the constants τ_0 *and* C *depend only on* T *and* $\|u\|_{L^\infty((0,T);H^{\gamma+1})}$.

Remark 1.3. The almost mass convergence scheme proposed in this work, together with the first-order scheme proposed in [4], has the lowest regularity requirement among all schemes for the DS-II system so far. For example, the Strang splitting method requires the loss of two derivatives.

Remark 1.4. With respect to the Fourier integrator, our scheme also achieves first-order convergence in $H^\gamma \times H^{\gamma+1}$ compared to the first-order scheme of [4], and maintains fifth-order accuracy for mass. Thus the physical properties of the solution to the equation can be well preserved. Using the same method, we can obtain arbitrarily order mass convergence by suitably adding correction terms.

2. Preliminary

2.1. **Some notations.** Firstly, we present some notations and tools for future derivation and analysis. We use $A \lesssim B$ or $B \lesssim A$ to denote the statement that $A \leq CB$ for some absolute constant $C > 0$ which may vary from line to line but is independent of τ or n, and we denote $A \sim B$ for $A \lesssim B \lesssim A$. We use $O(Y)$ to denote any quantity X such that $X \lesssim Y$.

For $\boldsymbol{k} := (k_1, k_2) \in \mathbb{T}^2$, $\boldsymbol{x} := (x_1, x_2) \in \mathbb{T}^2$, we denote

$$\boldsymbol{k} \cdot \boldsymbol{x} = k_1 x_1 + k_2 x_2, \quad |\boldsymbol{k}|^2 = |k_1|^2 + |k_2|^2 .$$

We denote $\langle \cdot, \cdot \rangle$ to be the L^2-inner product, that is

$$\langle f, g \rangle = \mathrm{Re} \int_{\mathbb{T}^2} f(\boldsymbol{x}) \overline{g(\boldsymbol{x})} d\boldsymbol{x}.$$

The Fourier transform of a function f on $\mathbb{T}^2$ is defined by

$$\hat{f}_{\boldsymbol{k}} = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x})\, d\boldsymbol{x}.$$

and thus the Fourier inversion formula

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^2} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \hat{f}_{\boldsymbol{k}}.$$

Then the following usual properties of the Fourier transform hold

$$\|f\|_{L^2}^2 = (2\pi)^2 \sum_{\boldsymbol{k}\in\mathbb{Z}^2} |\hat{f}_{\boldsymbol{k}}|^2 = (2\pi)^2 \|\hat{f}_{\boldsymbol{k}}\|_{L^2(\mathbb{Z}^2)}^2, \tag{2.1}$$

$$\widehat{(fg)}(\boldsymbol{k}) = \sum_{\boldsymbol{k}_1\in\mathbb{Z}^d} \hat{f}_{\boldsymbol{k}-\boldsymbol{k}_1} \hat{g}_{\boldsymbol{k}_1}. \tag{2.2}$$

The Sobolev space $H^\gamma(\mathbb{T}^2)$ for $\gamma \geq 0$ has the equivalent norm

$$\|f\|_{H^\gamma(\mathbb{T}^d)}^2 = \|J^\gamma f\|_{L^2(\mathbb{T}^2)}^2 = (2\pi)^2 \sum_{\boldsymbol{k}\in\mathbb{Z}^2} (1+|\boldsymbol{k}|^2)^\gamma |\hat{f}_{\boldsymbol{k}}|^2, \tag{2.3}$$

where we denote the operator

$$J^s = (1-\Delta)^{\frac{s}{2}}, \quad \forall s \in \mathbb{R}. \tag{2.4}$$

we denote $(-\Delta)^{-1}$ to be the operator defined by

$$\widehat{(-\Delta)^{-1}}\, f(k) = \begin{cases} |\boldsymbol{k}|^{-2}\hat{f}_{\boldsymbol{k}}, & \text{if } \boldsymbol{k} \neq 0, \\ 0, & \text{if } \boldsymbol{k} = 0. \end{cases} \tag{2.5}$$

Moreover, we denote $|\nabla|^{-1}$ to be the operator defined by

$$\widehat{|\nabla|^{-1}f}(\boldsymbol{k}) = \begin{cases} |\boldsymbol{k}|^{-1}\hat{f}_{\boldsymbol{k}}, & \text{if } \boldsymbol{k} \neq 0, \\ 0, & \text{if } \boldsymbol{k} = 0. \end{cases} \tag{2.6}$$

We denote $T_m(M;\varphi)$ to be a class of qualities which is defined in the Fourier space by

$$\mathcal{F}T_m\left(M;\varphi\right)(\boldsymbol{k}) = O\Big(\sum_{\boldsymbol{k}=\boldsymbol{k}_1+\cdots+\boldsymbol{k}_m} |M(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_m)|\,|\hat{\varphi}_{\boldsymbol{k}_1}(t)|\cdots|\hat{\varphi}_{\boldsymbol{k}_m}(t)|\Big), \tag{2.7}$$

where $\boldsymbol{k}_j = (k_{j1}, k_{j2}) \in \mathbb{Z}^2$, $j = \{1,\cdots,m\}$; M is a function about $\boldsymbol{k}_1,\cdots,\boldsymbol{k}_m$.

Furthermore, we will make frequent use of the isometric property of the operator $e^{i\partial_{x_1x_2}t}$

$$\|e^{i\partial_{x_1x_2}t}f\|_{H^\gamma} = \|f\|_{H^\gamma}. \tag{2.8}$$

for all $f \in H^\gamma$, $\gamma > 1$ and $t \in \mathbb{R}$.

2.2. **Some preliminary estimates.** First, we will frequently apply the following Kato-Ponce inequality.

Lemma 2.1. [8]*(Kato-Ponce inequality) For any $\gamma > 1$, f, $g \in H^\gamma(\mathbb{T}^d)$, then the following inequality holds:*

$$\|fg\|_{H^\gamma} \lesssim \|f\|_{H^\gamma}\|g\|_{H^\gamma}. \tag{2.9}$$

To prove our main result below, we need the following specific estimate.

Lemma 2.2. *Let $\gamma > 1$ and $\varphi \in H^\gamma$, then the following inequality holds:*

$$\|T_3\left(k_{11}(k_{22}+k_{32}) + k_{21}(k_{12}+k_{32}) + k_{31}(k_{12}+k_{22});\varphi\right)\|_{H^{-\gamma}} \lesssim \|\varphi\|^3_{L^\infty((0,T);H^\gamma)}.$$

Proof. We assume that $\hat{\varphi}_{\boldsymbol{k}_j} > 0$, $j = \{1,2,3\}$ for any $\boldsymbol{k}_j$, otherwise one may replace them by $|\hat{\varphi}_{\boldsymbol{k}_j}|$.

Using the definition in (2.7) and Sobolev's embedding theorem, we have

$$\begin{aligned} &\|T_3(k_{11}(k_{22}+k_{32}) + k_{21}(k_{12}+k_{32}) + k_{31}(k_{12}+k_{22});\varphi)\|_{H^{-\gamma}} \\ &\lesssim \Big\|\sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3} \left(|\boldsymbol{k}_1||\boldsymbol{k}_2| + |\boldsymbol{k}_1||\boldsymbol{k}_3| + |\boldsymbol{k}_2||\boldsymbol{k}_3|\right)\hat{\varphi}_{\boldsymbol{k}_1}\hat{\varphi}_{\boldsymbol{k}_2}\hat{\varphi}_{\boldsymbol{k}_3}\Big\|_{H^{-\gamma}} \\ &\lesssim \|\left(|\nabla|\varphi\right)^2\varphi\|_{H^{-\gamma}} \lesssim \|\left(|\nabla|\varphi\right)^2\varphi\|_{L^1}. \end{aligned}$$

By Lemma 2.1, we get

$$\begin{aligned}&\|T_3(k_{11}(k_{22}+k_{32})+k_{21}(k_{12}+k_{32})+k_{31}(k_{12}+k_{22});\varphi)\|_{H^{-\gamma}}\\&\lesssim \|\nabla\varphi\|_{L^2}^2\|\varphi\|_{L^\infty}\lesssim \|\varphi\|^3_{L^\infty((0,T);H^\gamma)}.\end{aligned}$$

Therefore, the Lemma 2.2 is proved. □

2.3. **Review of first-order numerical scheme construction.** The article [4] solved the DS-II system (1.4) using Duhamel's formula

$$\phi(t)=e^{i\partial_{x_1x_2}t}\phi(t_0)-i\int_{t_0}^{t}e^{i\partial_{x_1x_2}(t-s)}\big[\phi\cdot E(|\phi|^2)\big]ds. \tag{2.10}$$

By the twisted variable $\varphi(t)=e^{-i\partial_{x_1x_2}t}\phi(t)$, we obtain

$$\varphi(t)=\varphi(t_0)-i\int_{t_0}^{t}e^{i\partial_{x_1x_2}s}\Big[e^{i\partial_{x_1x_2}t}\varphi\cdot E\big(|e^{i\partial_{x_1x_2}t}\varphi|^2\big)\Big]ds.$$

By Fourier transformation, we get

$$\begin{aligned}\hat{\varphi}_{\boldsymbol{k}}(t_{n+1})=&\hat{\varphi}_{\boldsymbol{k}}(t_n)-i\int_0^{\tau}\sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3}e^{i\alpha(t_n+s)}\big[\tilde{\mu}_1+\mu_2\frac{(k_{11}+k_{21})(k_{12}+k_{22})}{|\boldsymbol{k}_1+\boldsymbol{k}_2|^2}\big]\\&\cdot\hat{\bar{\varphi}}_{\boldsymbol{k}_1}(t_n+s)\hat{\varphi}_{\boldsymbol{k}_2}(t_n+s)\hat{\varphi}_{\boldsymbol{k}_3}(t_n+s)ds,\end{aligned}\tag{2.11}$$

where $\alpha=k_1k_2+k_{11}k_{12}-k_{21}k_{22}-k_{31}k_{32}$. And let $\beta=k_{11}(k_{22}+k_{32})+k_{21}(k_{12}+k_{32})+k_{31}(k_{12}+k_{22})$. Then, we have $\alpha=2k_{11}k_{12}+\beta$.

For the integration, we only choose the dominant quadratic term $2isk_{11}k_{12}$, so that the integration can be carried out fully in Fourier space as

$$\int_0^{\tau}e^{2isk_{11}k_{12}}ds=\tau\omega(2i\tau k_{11}k_{12}).$$

Hence, we have

$$\begin{aligned}\varphi(t_{n+1})&=\varphi(t_n)-i\tau e^{-i\partial_{x_1x_2}t_n}\Big[e^{i\partial_{x_1x_2}t_n}\varphi(t_n)E\big(\omega(-2i\partial_{x_1x_2}\tau)\overline{e^{i\partial_{x_1x_2}t_n}\varphi(t_n)}\cdot e^{i\partial_{x_1x_2}t_n}\varphi(t_n)\big)\Big]+\mathcal{R}_1^n+\mathcal{R}_2^n\\&:=\Phi^n(\varphi(t_n))+\mathcal{R}_1^n+\mathcal{R}_2^n,\end{aligned}\tag{2.12}$$

where

$$\begin{aligned}\mathcal{R}_1^n=-i\int_0^{\tau}e^{-i\partial_{x_1x_2}(t_n+s)}&\Big[e^{i\partial_{x_1x_2}(t_n+s)}\varphi(t_n+s)\cdot E\big(|e^{i\partial_{x_1x_2}(t_n+s)}\varphi(t_n+s)|^2\big)\\&-e^{i\partial_{x_1x_2}(t_n+s)}\varphi(t_n)\cdot E\big(|e^{i\partial_{x_1x_2}(t_n+s)}\varphi(t_n)|^2\big)\Big]ds,\end{aligned}$$

and

$$\mathcal{R}_2^n=-i\sum_{\boldsymbol{k}\in\mathbf{z}^2}\sum_{\substack{\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3\in\mathbf{z}^2\\ \boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3}}e^{it_n\alpha}\big[\tilde{\mu}_1+\mu_2\frac{(k_{11}+k_{21})(k_{12}+k_{22})}{|\boldsymbol{k}_1+\boldsymbol{k}_2|^2}\big]\hat{\bar{\varphi}}_{\boldsymbol{k}_1}\hat{\varphi}_{\boldsymbol{k}_2}\hat{\varphi}_{\boldsymbol{k}_3}e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\int_0^{\tau}e^{2isk_{11}k_{12}}(e^{is\beta}-1)ds.$$

In summary, we know that the scheme of the first order low-regularity integrator (LRI) for solving the DS-II system (1.4): $\phi^n = \phi^n(\boldsymbol{x})$ as the numerical solution, for $n = 1, 2, 3, \ldots$,

$$\phi^n = e^{i\partial_{x_1x_2}\tau}\phi^{n-1} - i\tau e^{i\partial_{x_1x_2}\tau}\Big[\phi^{n-1}\cdot E\big(\omega(-2i\partial_{x_1x_2}\tau)\overline{\phi^{n-1}}\cdot\phi^{n-1}\big)\Big]. \tag{2.13}$$

Based on the DS systems (1.4), we can write the numerical solution of ψ: denote $\psi^n = \psi^n(\boldsymbol{x})$as the numerical solution, for $n = 1, 2, 3, \ldots$,

$$\psi^n = -(-\Delta)^{-1}(\partial_{x_1} + \partial_{x_2})|\phi^n|^2. \tag{2.14}$$

Meanwhile, [4] also proved that the scheme can reach first order accuracy.

Theorem 2.3. [4] *Let ϕ^n and ψ^n be the numerical solution of the DS-II system (1.4) obtained from the schemes (2.13) and (2.14) up to some fixed time $T > 0$. Under the assume $\phi_0 \in H^{\gamma+1}(\mathbb{T}^2)$ for some $\gamma > 1$ there exist constants $\tau_0 > 0$ and $C > 0$, such that for any $0 < \tau \le \tau_0$, we have*

$$\|\phi(t_n) - \phi^n\|_{H^\gamma} \le C\tau, \quad \|\psi(t_n) - \psi^n\|_{H^{\gamma+1}} \le C\tau, \quad n = 0, 1, \ldots, \frac{T}{t};$$

where the constants τ_0 and C depend only on T and $\|\phi\|_{L^\infty((0,T);H^{\gamma+1})}$.

In the following we list some estimates without proofs, and the relevant proofs can be found in [4]. Firstly, the estimates for $\mathcal{R}_1^n$ and $\mathcal{R}_2^n$ are

Lemma 2.4. [4] *Let $\gamma > 1$. Assume that $\phi_0 \in H^{\gamma+1}(\mathbb{T}^2)$, then there exist constants $\tau_0 > 0$ and $C > 0$, such that for any $0 < \tau \le \tau_0$, the following estimate hold:*

$$\|\mathcal{R}_1^n\|_{H^\gamma} \le C\tau^2,$$

where τ_0 and C depend only on T and $\|\phi\|_{L^\infty((0,T);H^\gamma)}$.

Lemma 2.5. [4] *Let $\gamma > 1$. Assume that $\phi_0 \in H^{\gamma+1}(\mathbb{T}^2)$, then there exist constants $\tau_0 > 0$ and $C > 0$, such that for any $0 < \tau \le \tau_0$, the following estimate holds:*

$$\|\mathcal{R}_2^n\|_{H^\gamma} \le C\tau^2,$$

where τ_0 and C depend only on T and $\|\phi\|_{L^\infty((0,T);H^{\gamma+1})}$.

Combining Lemma 2.4 and Lemma 2.5, the local error estimate of the numerical propagator is obtained.

Lemma 2.6. [4]*(Local error) Let $\gamma > 1$. Assume that $\phi_0 \in H^{\gamma+1}(\mathbb{T}^2)$, then there exist constants $\tau_0 > 0$ and $C > 0$, such that for any $0 < \tau \le \tau_0$, the following estimate holds:*

$$\|\varphi(t_{n+1}) - \Phi^n(\varphi(t_n))\|_{H^\gamma} \le C\tau^2,$$

where τ_0 and C depend only on T and $\|\phi\|_{L^\infty((0,T);H^{\gamma+1})}$.

Finally, we introduce the stability result.

Lemma 2.7. [4]*(Stability) Let $f,g\in H^\gamma$, then for $\gamma>1$, the following estimate holds:*
$$\|\Phi^n(f)-\Phi^n(g)\|_{H^\gamma}\le(1+C\tau)\|f-g\|_{H^\gamma}+C\tau\|f-g\|_{H^\gamma}^3,$$
where C depends only on $\|f\|_{H^\gamma}$.

3. The almost mass-conserved scheme

3.1. **Construction of the numerical integrator.** Let $\varphi^n=e^{-i\partial_{x_1x_2}t_n}\phi^n$. Accordingly, from (1.8)-(1.11), we have that
$$\varphi^{n+1}=\varphi^n+I^n(\varphi^n)+J_1^n(\varphi^n)+J_2^n(\varphi^n), \tag{3.1}$$
where Φ^n is defined in (2.12),
$$I^n(\varphi^n)=\Phi^n(\varphi^n)-\varphi^n, \tag{3.2}$$
and the functionals J_1^n, J_2^n are given by
$$J_1^n(\varphi^n)=H^n(\varphi^n)\varphi^n, \tag{3.3}$$
$$J_2^n(\varphi^n)=-\frac{1}{2}\big(H^n(\varphi^n)\big)^2\varphi^n-\big(\|\varphi^n\|_{L^2}^2\big)^{-1}H^n(\varphi^n)\big\langle I^n(\varphi^n),\varphi^n\big\rangle\varphi^n, \tag{3.4}$$
and
$$H^n(\varphi^n)=-\big(\|\varphi^n\|_{L^2}^2\big)^{-1}\Big(\langle I^n(\varphi^n),\varphi^n\rangle+\frac{1}{2}\|I^n(\varphi^n)\|_{L_2}^2\Big). \tag{3.5}$$
The proof of Theorem 1.1 depends on the following key lemmas. Firstly, the convergence order of $I^n(\varphi)$ is given.

Lemma 3.1. *Let $\gamma>1$. Assume that $\varphi\in H^\gamma$, then there exists a constant $C>0$, such that*
$$\|I^n(\varphi)\|_{L^2}\lesssim C\tau, \tag{3.6}$$
where C depends on $\|\varphi\|_{L^\infty((0,T);H^\gamma)}$.

Proof. By (2.12) and (3.2), we have
$$I^n(\varphi)=-i\tau e^{-i\partial_{x_1x_2}t_n}\Big[e^{i\partial_{x_1x_2}t_n}\varphi E\big(\omega(-2i\partial_{x_1x_2}\tau)\overline{e^{i\partial_{x_1x_2}t_n}\varphi}\cdot e^{i\partial_{x_1x_2}t_n}\varphi\big)\Big].$$
Hence, we have
$$\begin{aligned}\|I^n(\varphi)\|_{L^2}&=\big\|-i\tau e^{-i\partial_{x_1x_2}t_n}\big[e^{i\partial_{x_1x_2}t_n}\varphi E\big(\omega(-2i\partial_{x_1x_2}\tau)\overline{e^{i\partial_{x_1x_2}t_n}\varphi}\cdot e^{i\partial_{x_1x_2}t_n}\varphi\big)\big]\big\|_{L^2}\\&\le\tau\big\|e^{i\partial_{x_1x_2}t_n}\varphi E\big(\omega(-2i\partial_{x_1x_2}\tau)\overline{e^{i\partial_{x_1x_2}t_n}\varphi}\cdot e^{i\partial_{x_1x_2}t_n}\varphi\big)\big\|_{L^2}\\&\le\tau\big\|e^{i\partial_{x_1x_2}t_n}\varphi\big\|_{L^\infty}\big\|E\big(\omega(-2i\partial_{x_1x_2}\tau)\overline{e^{i\partial_{x_1x_2}t_n}\varphi}\cdot e^{i\partial_{x_1x_2}t_n}\varphi\big)\big\|_{L^2}.\end{aligned}$$
By $\|Ef\|_{L^2}\le C\|f\|_{L^2}$, we have
$$\|I^n(\varphi)\|_{L^2}\lesssim\tau\big\|\varphi\big\|_{L^\infty}\big\|\omega(-2i\partial_{x_1x_2}\tau)\overline{e^{i\partial_{x_1x_2}t_n}\varphi}\cdot e^{i\partial_{x_1x_2}t_n}\varphi\big\|_{L^2},$$
Together with Lemma 2.1 and $\|\omega(-2i\partial_{x_1x_2}\tau)f\|_{H^\gamma}\le C\|f\|_{H^\gamma}$, we obtain
$$\begin{aligned}\|I^n(\varphi)\|_{L^2}&\lesssim\tau\big\|\varphi\big\|_{H^\gamma}\big\|\overline{e^{i\partial_{x_1x_2}t_n}\varphi}\big\|_{H^\gamma}\big\|\varphi\big\|_{H^\gamma}\\&\lesssim\tau\|\varphi\|_{L^\infty((0,T);H^\gamma)}^3.\end{aligned}$$

This proves this lemma. □

Next, we give the convergence order of $\langle I^n(\varphi), \varphi\rangle$.

Lemma 3.2. *Let $\gamma > 1$. Assume that $\varphi \in H^\gamma$, then there exists a constant $C > 0$, such that*

$$|\langle I^n(\varphi), \varphi\rangle| \lesssim C\tau^2, \tag{3.7}$$

where C depends on $\|\varphi\|_{L^\infty((0,T);H^\gamma)}$.

Proof. We perform a Fourier transformation on $I^n(\varphi)$ to obtain

$$\widehat{I^n}(\varphi) = -i \sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3} \int_0^\tau e^{i\alpha t_n} e^{2ik_{11}k_{12}s} ds \big[\tilde{\mu}_1 + \mu_2 \frac{(k_{11}+k_{21})(k_{12}+k_{22})}{|\boldsymbol{k}_1+\boldsymbol{k}_2|^2}\big] \hat{\bar{\varphi}}_{\boldsymbol{k}_1} \hat{\varphi}_{\boldsymbol{k}_2} \hat{\varphi}_{\boldsymbol{k}_3}.$$

Together with $\alpha = 2k_{11}k_{12} + \beta$, we get

$$\begin{aligned} \widehat{I^n}(\varphi) = -i &\sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3} \int_0^\tau e^{i\alpha t_n} \left[e^{i\alpha s} - e^{2ik_{11}k_{12}s}(e^{2i\beta s} - 1)\right] ds \\ &\cdot \big[\tilde{\mu}_1 + \mu_2 \frac{(k_{11}+k_{21})(k_{12}+k_{22})}{|\boldsymbol{k}_1+\boldsymbol{k}_2|^2}\big] \hat{\bar{\varphi}}_{\boldsymbol{k}_1} \hat{\varphi}_{\boldsymbol{k}_2} \hat{\varphi}_{\boldsymbol{k}_3}. \end{aligned} \tag{3.8}$$

Therefore, we can write $\widehat{I^n}(\varphi)$ as

$$\widehat{I^n}(\varphi) = -i \sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3} \int_0^\tau e^{i\alpha(t_n+s)} ds W \hat{\bar{\varphi}}_{\boldsymbol{k}_1} \hat{\varphi}_{\boldsymbol{k}_2} \hat{\varphi}_{\boldsymbol{k}_3} + (\widehat{\mathcal{R}_2^n})_{\boldsymbol{k}}, \tag{3.9}$$

where $\widehat{(\mathcal{R}_2^n)}_{\boldsymbol{k}}$ is defined as

$$\widehat{(\mathcal{R}_2^n)}_{\boldsymbol{k}} = i \sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3} \int_0^\tau e^{i\alpha t_n} e^{2ik_{11}k_{12}s}(e^{2i\beta s} - 1) ds \big[\tilde{\mu}_1 + \mu_2 \frac{(k_{11}+k_{21})(k_{12}+k_{22})}{|\boldsymbol{k}_1+\boldsymbol{k}_2|^2}\big] \hat{\bar{\varphi}}_{\boldsymbol{k}_1} \hat{\varphi}_{\boldsymbol{k}_2} \hat{\varphi}_{\boldsymbol{k}_3}.$$

By making a Fourier inversion of the (3.8) equation, we obtain

$$I^n(\varphi) = -i \int_0^\tau e^{-i\partial_{x_1x_2}(t_n+s)} \Big(e^{i\partial_{x_1x_2}(t_n+s)}\varphi \cdot E\big(|e^{i\partial_{x_1x_2}(t_n+s)}\varphi|^2\big)\Big) ds + \mathcal{R}_2^n. \tag{3.10}$$

By taking the inner product of I^n and substituting (3.10) into the equation, we get

$$\begin{aligned} \langle I^n(\varphi), \varphi\rangle &= \Big\langle -i \int_0^\tau e^{-i\partial_{x_1x_2}(t_n+s)} \big(e^{i\partial_{x_1x_2}(t_n+s)}\varphi \cdot E\big(|e^{i\partial_{x_1x_2}(t_n+s)}\varphi|^2\big)\big) ds, \varphi \Big\rangle + \langle \mathcal{R}_2^n, \varphi\rangle \\ &= \int_0^\tau \left\langle -ie^{i\partial_{x_1x_2}(t_n+s)}\varphi \cdot E\big(|e^{i\partial_{x_1x_2}(t_n+s)}\varphi|^2\big), e^{i\partial_{x_1x_2}(t_n+s)}\varphi \right\rangle ds + \langle \mathcal{R}_2^n, \varphi\rangle. \end{aligned}$$

Since

$$\Big\langle -if \cdot E\big(|f|^2\big), f \Big\rangle = Re \int_{\mathbb{T}^d} -if \cdot E\big(|f|^2\big) \cdot \bar{f} dx = 0,$$

we get

$$\int_0^\tau \left\langle -ie^{i\partial_{x_1x_2}(t_n+s)}\varphi \cdot E\big(|e^{i\partial_{x_1x_2}(t_n+s)}\varphi|^2\big), e^{i\partial_{x_1x_2}(t_n+s)}\varphi \right\rangle ds = 0.$$

Hence, we have

$$\langle I^n(\varphi), \varphi\rangle = \langle \mathcal{R}_2^n, \varphi\rangle .$$

According to the Lemma 2.2, we get

$$|\langle I^n(\varphi), \varphi\rangle| = |\langle \mathcal{R}_2^n, \varphi\rangle| \le \|\mathcal{R}_2^n\|_{H^{-\gamma}}\|\varphi\|_{H^\gamma} \lesssim C\tau^2,$$

where C depends on $\|\varphi\|_{L^\infty((0,T);H^\gamma)}$.

This proves Lemma 3.2. □

3.2. **The proof of the Theorem 1.1.** Since $\varphi^n = e^{-i\partial_{x_1x_2}t_n}\phi^n$, $\varphi(t_n) = e^{-i\partial_{x_1x_2}t_n}\phi(t_n)$, we only need to prove the conclusion of Theorem 1.1 holds for φ^n and $\varphi(t_n)$.

From (3.1), we have

$$\varphi^{n+1} = \varphi^n + I^n(\varphi^n) + J_1^n(\varphi^n) + J_2^n(\varphi^n).$$

Then, we get

$$\varphi^{n+1} - \varphi(t_{n+1}) = \Phi^n(\varphi^n) - \Phi^n(\varphi(t_n)) + \Phi^n(\varphi(t_n)) - \varphi(t_{n+1}) + J_1^n(\varphi^n) + J_2^n(\varphi^n).$$

By Lemma 2.6 and Lemma 2.7, we find

$$\|\varphi(t_{n+1}) - \Phi^n(\varphi(t_n))\|_{H^\gamma} \le C\tau^2,$$

and

$$\|\Phi^n(\varphi^n) - \Phi^n(\varphi(t_n))\|_{H^\gamma} \le (1 + C\tau)\|\varphi^n - \varphi(t_n)\|_{H^\gamma} + C\tau\|\varphi^n - \varphi(t_n)\|_{H^\gamma}^3 .$$

From (3.5), Lemma 3.1 and Lemma 3.2, we have

$$\begin{aligned}|H^n(\varphi^n)| &\le \big(\|\varphi^n\|_{L^2}^2\big)^{-1}\Big(|\langle I^n(\varphi^n), \varphi^n\rangle| + \frac{1}{2}\|I^n(\varphi^n)\|_{L^2}^2\Big)\\ &\le C\tau^2\big(\|\varphi^n\|_{H^\gamma}^2 + \|\varphi^n\|_{H^\gamma}^4\big) \le C\tau^2(1 + \|\varphi^n - \varphi(t_n)\|_{H^\gamma}^4).\end{aligned} \tag{3.11}$$

This yields that

$$\|J_1^n(\varphi^n)\|_{H^\gamma} = |H^n(\varphi^n)|\,\|\varphi^n\|_{H^\gamma} \le C\tau^2(1 + \|\varphi^n - \varphi(t_n)\|_{H^\gamma}^5). \tag{3.12}$$

Similarly, from (3.4), (3.11), Lemma 3.1 and Lemma 3.2, we have

$$\begin{aligned}\|J_2^n(\varphi^n)\|_{H^\gamma} &\le \frac{1}{2}|H^n(\varphi^n)|^2\|\varphi^n\|_\gamma + \big(\|\varphi^n\|_{L^2}^2\big)^{-1}|H^n(\varphi^n)||\big\langle I^n(\varphi^n), \varphi^n\big\rangle|\|\varphi^n\|_\gamma\\ &\le C\tau^4(1 + \|\varphi^n - \varphi(t_n)\|_{H^\gamma}^9).\end{aligned}$$

Putting together with the above estimates, we conclude that for any $\tau \le 1$,

$$\|\varphi^{n+1} - \varphi(t_{n+1})\|_{H^\gamma} \le C\tau^2 + (1 + C\tau)\|\varphi^n - \varphi(t_n)\|_{H^\gamma} + C\tau\|\varphi^n - \varphi(t_n)\|_{H^\gamma}^9, \tag{3.13}$$

where the constant C depends only on $\|\varphi\|_{L^\infty((0,T);H^\gamma)}$.

By the iteration and Gronwall inequalities, we get

$$\big\|\varphi(t_n) - \varphi^n\big\|_{H^\gamma} \le C\tau^2\sum_{j=0}^{n}(1 + C\tau)^j \le C\tau, \quad n = 0, 1\ldots, \frac{T}{\tau}. \tag{3.14}$$

This implies the first-order convergence and the following a prior estimate:

$$\|\varphi^n\|_{H^\gamma} \leq C, \quad n = 0, 1 \ldots, \frac{T}{\tau}. \tag{3.15}$$

Here the positive constant C depends only on T and $\|\varphi\|_{L^\infty((0,T);H^\gamma)}$.

From the DS-II system (1.4), we know that $\psi^n = -(-\Delta)^{-1}(\partial_x + \partial_y)|\phi^n|^2$. Meanwhile, using the first estimate in (1.14), we have

$$\begin{aligned}\|\psi(t_n) - \psi^n\|_{H^{\gamma+1}} &\leq \big\|-(-\Delta)^{-1}(\partial_{x_1} + \partial_{x_2})\big(|\phi(t_n)|^2 - |\phi^n|^2\big)\big\|_{H^{\gamma+1}} \\ &\leq C\big\||\phi(t_n)|^2 - |\phi^n|^2\big\|_{H^\gamma} \\ &\leq C\,\|\phi(t_n) - \phi^n\|_{H^\gamma}\left(\|\phi(t_n) - \phi^n\|_{H^\gamma} + \|\phi(t_n\|_{H^\gamma}\right) \\ &\leq C\tau,\end{aligned}$$

where the constant C depends only on $\|\phi\|_{L^\infty((0,T);H^{\gamma+1})}$. This proves (1.14).

Next we prove the almost mass conservation law. From (3.1), we have

$$\begin{aligned}\|\varphi^{n+1}\|_{L^2}^2 = \langle\varphi^{n+1}, \varphi^{n+1}\rangle &= \|\varphi^n\|_{L^2}^2 \\ &+ 2\langle I^n(\varphi^n), \varphi^n\rangle + 2\langle J_1^n(\varphi^n), \varphi^n\rangle + \|I^n(\varphi^n)\|_{L^2}^2 \\ &+ 2\langle J_2^n(\varphi^n), \varphi^n\rangle + 2\langle I^n(\varphi^n), J_1^n(\varphi^n)\rangle + \|J_1^n(\varphi^n)\|_{L^2}^2 \\ &+ 2\langle I^n(\varphi^n), J_2^n(\varphi^n)\rangle + 2\langle J_1^n(\varphi^n), J_2^n(\varphi^n)\rangle + \|J_2^n(\varphi^n)\|_{L^2}^2 \\ &:= \|\varphi^n\|_{L^2}^2 + I + II + III.\end{aligned} \tag{3.16}$$

Combine (3.3), (3.4) and(3.5) , we get that

$$I = 0, \quad II = 0.$$

By Lemma 3.2, (3.3), (3.4), (3.11) and (3.14), we obtain

$$\begin{aligned}2\big|\langle I^n(\varphi^n), J_2^n(\varphi^n)\rangle\big| &\leq \big|2\big(\|\varphi^n\|_{L^2}^2\big)^{-1} H^n(\varphi^n)\langle I^n(\varphi^n), \varphi^n\rangle^2\big| \\ &\quad + \big|\big(H^n(\varphi^n)\big)^2\langle I^n(\varphi^n), \varphi^n\rangle\big| \leq C\tau^6,\end{aligned}$$

$$2\big|\langle J_1^n(\varphi^n), J_2^n(\varphi^n)\rangle\big| \leq \big|\big[\big(H^n(\varphi^n)\big)^3\|\varphi^n\|_{L^2}^2 + 2\big(H^n(\varphi^n)\big)^2\langle I^n(\varphi^n), \varphi^n\rangle\big]\big| \leq C\tau^6,$$

and

$$\begin{aligned}\|J_2^n(\varphi^n)\|_{L^2}^2 &\leq \big(\|\varphi^n\|_{L^2}^2\big)^{-1}\big(H^n(\varphi^n)\big)^2\langle I^n(\varphi^n), \varphi^n\rangle \\ &\quad + \big(H^n(\varphi^n)\big)^3\langle I^n(\varphi^n), \varphi^n\rangle + \frac{1}{4}\big(H^n(\varphi^n)\big)^4\|\varphi^n\|_{L^2}^2 \leq C\tau^8.\end{aligned}$$

Then, we have

$$III \leq C\tau^6.$$

Therefore, we conclude that

$$\big|\|\varphi^{n+1}\|_{L^2}^2 - \|\varphi^n\|_{L^2}^2\big| \leq C\tau^6, \tag{3.17}$$

that is

$$|M(\varphi^{n+1}) - M(\varphi^n)| \leq C\tau^6. \tag{3.18}$$

Then by the iteration, we get

$$|M(\varphi^n) - M(\phi_0)| \leq C\tau^5.$$

This finishes the proof of Theorem 1.1.

4. Numerical Experiments

In this section, we present the numerical experiments of the scheme to justify the main theorem. Since ψ^n is calculated via equation (1.4), we only need to test ϕ^n in this section. To get an initial data with the desired regularity, we construct $\phi_0(x)$ through the following strategy [23]. Choose $N = 2^6$ as an even integer and discrete the spatial domain $\mathbb{T}^2$ with grid points $x_{j,k} = (\frac{2j\pi}{N}, \frac{2k\pi}{N}), \quad j, k = 0, \ldots, N$. Take a uniformly distributed random array rand $(N, N) \in [0,1]^{N\times N}$ and an $N \times N$ vector $\boldsymbol{\Phi}$ whose elements are defined as

$$\boldsymbol{\Phi}_{j,k,l} = \mathrm{rand}(N,N) + i\,\mathrm{rand}(N,N), \quad (j, k = 0, \ldots, N-1).$$

In our numerical experiments, we set

$$\phi_0(\boldsymbol{x}) := \frac{|\partial_{\boldsymbol{x},N}|^{-\gamma}\boldsymbol{\Phi}}{\||\partial_{\boldsymbol{x},N}|^{-\gamma}\boldsymbol{\Phi}\|_{L^\infty}}, \quad \boldsymbol{x} \in \mathbb{T}^2, \tag{4.1}$$

where the pseudo-differential operator $|\partial_{\boldsymbol{x},N}|^{-\gamma}$ for $\gamma \geq 0$ reads as follows: for Fourier modes $\boldsymbol{k} = (k_1, k_2)$ and $k_j = -N/2, \ldots, N/2-1$, for $j = 1, 2$, and

$$\left(|\partial_{\boldsymbol{x},N}|^{-\gamma}\right)_{\boldsymbol{k}} = \begin{cases} |\boldsymbol{k}|^{-\gamma}, & \text{if } \boldsymbol{k} \neq 0, \\ 0, & \text{if } \boldsymbol{k} = 0. \end{cases}$$

Since the almost mass conservation scheme given in this work has high accuracy, direct numerical calculation may not capture the convergence order of the mass error, in this experiment we enlarge the initial value by 10^5 times and calculate the relative errors. The MATLAB software is used to implement the numerical experiments, where the final time $T = 2.0$, the results are shown in Fig. 1 and Fig. 2. These illustrate that the scheme (3.1) achieves first-order convergence in H^γ and fifth-order convergence for the initial data in $H^{\gamma+1}, \gamma = 2, 3$, and compares with the scheme given in [4] has higher order convergence of mass.

5. Conclusion

In this work, we constructed a first-order Fourier integrator with almost mass conservation for solving the DS-II system on a torus under rough initial data. Based on the numerical scheme in [4], we designed a modified numerical scheme to obtain the first-order convergence in $H^\gamma \times H^{\gamma+1}$ with rough initial data in $H^{\gamma+1} \times H^{\gamma+1}$ and the fifth-order mass convergence. In addition, the scheme can be readily extent to constructed to obtain the arbitrary high-order mass convergence.

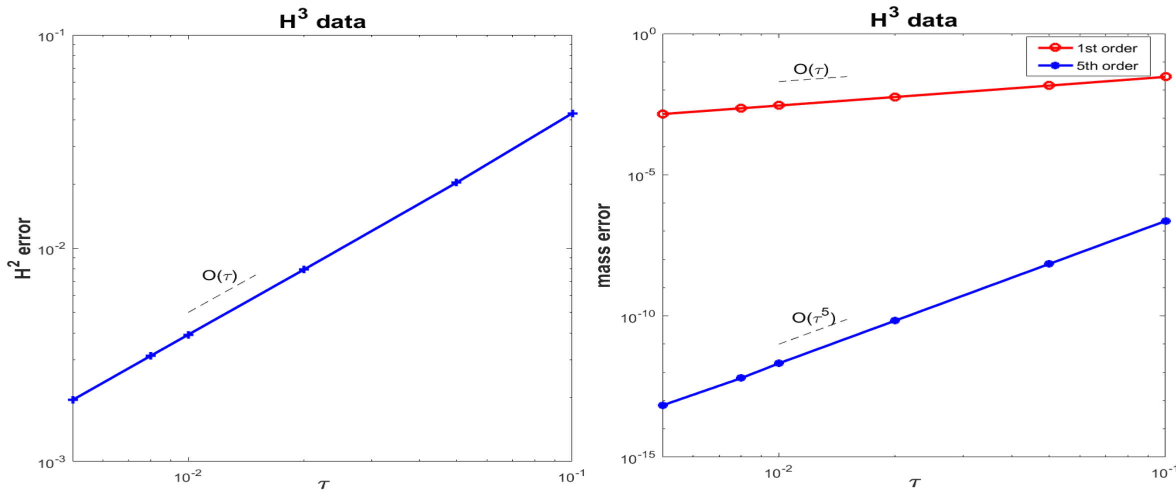

FIGURE 1. Convergence of (3.1): error $\frac{\|\phi_{ref}-\phi^n\|_{H^\gamma}}{\|\phi_{ref}\|_{H^\gamma}}$ (left) and error $\frac{|M(\phi^n)-M_0|}{|M_0|}$ (right) when $\gamma = 2$

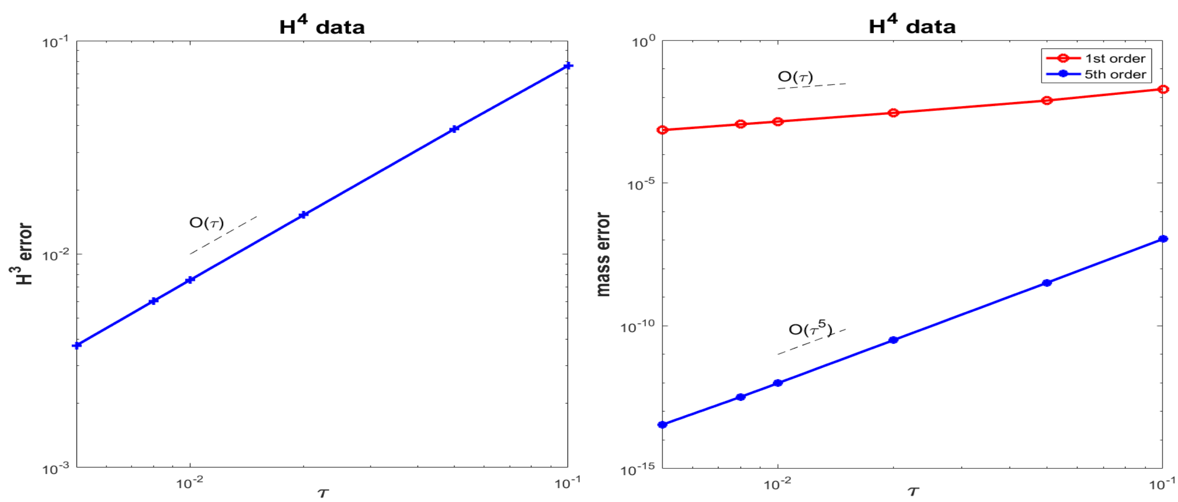

FIGURE 2. Convergence of (3.1): error $\frac{\|\phi_{ref}-\phi^n\|_{H^\gamma}}{\|\phi_{ref}\|_{H^\gamma}}$ (left) and error $\frac{|M(\phi^n)-M_0|}{|M_0|}$ (right) when $\gamma = 3$

COI STATEMENT

The authors declared that they have no conflict of interest

REFERENCES

[1] J. Frauendiener, C. Klein and U. Muhammad and N. Stoilov, Numerical study of Davey-Stewartson I systems, arXiv:2109.09169, 2021.

[2] M. S. Ismail and T. R. Taha, A linearly implicit conservative scheme for the coupled nonlinear Schrödinger equation, Mathematics and Computers in Simulation, 2007, 74(4-5): 302-311.

[3] T. Kato and G. Ponce, Commutator estimates and the Euler and Navier-Stokes equations, Communications on Pure and Applied Mathematics, 1988, 41(7): 891-907.

[4] X. Kou, C. Ning and Y. Wang, Low-regularity integrator for the Davey-Stewartson II system, preprint, 2022.

[5] G. Maierhofer and K. Schratz, Bridging the gap: symplecticity and low regularity on the example of the KdV equation, arXiv:2205.05024, 2022.

[6] Y. Wu and F. Yao, A first-order Fourier integrator for the nonlinear Schrödinger equation on $\mathbb{T}$ without loss of regularity, Math. Comp., 91(335), 1213-1235, 2022.

[7] Y. Wu and F. Yao, A first-order Fourier integrator for the nonlinear Schrödinger equation on $\mathbb{T}$ without loss of regularity, Math. Comp., 91(335), 1213-1235, 2022.

[8] T. Kato and G. Ponce, Commutator estimates and the Euler and Navier-Stokes equations, Communications on Pure and Applied Mathematics, 1988, 41(7): 891-907.