Observability for heat equations with time-dependent analytic memory

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Abstract

This paper presents a complete analysis of the observability property of heat equations with time-dependent real analytic memory kernels. More precisely, we characterize the geometry of the space-time measurable observation sets ensuring sharp observability inequalities, which are relevant both for control and inverse problems purposes.

Despite the abundant literature on the observation of heat-like equations, existing methods do not apply to models involving memory terms.

We present a new methodology and observation strategy, relying on the decomposition of the flow, the time-analyticity of solutions and the propagation of singularities. This allows us to obtain a sufficient and necessary geometric condition on the measurable observation sets for sharp two-sided observability inequalities. In addition, some applications to control and relevant open problems are presented.

Keywords. Heat equations with memory, analytic kernels, observability inequalities, flow decomposition, propagation of singularities, geometric characterization of observation sets.

1 Introduction

1.1 Equation and aim

Let $\Omega \subset \mathbb{R}^n$ ($n \in \mathbb{N}^+ := \{1, 2, 3, \cdots\}$) be a bounded domain with a C^2-boundary $\partial\Omega$ and consider the heat equation with memory:

$$\begin{cases} \partial_t y(t,x) - \Delta y(t,x) + \displaystyle\int_0^t M(t-s) y(s,x) ds = 0, & (t,x) \in \mathbb{R}^+ \times \Omega, \\ y(t,x) = 0, & (t,x) \in \mathbb{R}^+ \times \partial\Omega, \\ y(0,\cdot) = y_0(\cdot) \in L^2(\Omega), \end{cases} \tag{1.1}$$

where $\mathbb{R}^+ := (0, +\infty)$ and M satisfies the following assumption:

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Mathematics Subject Classification (2020): 93B07 93B05 45K05

($\mathfrak{H}$) The time-dependent memory kernel M is a non-trivial real analytic function over $[0, +\infty)$.

Equation (1.1) has a unique solution, denoted by $y(\cdot, \cdot; y_0)$, in $C([0, +\infty); L^2(\Omega))$.

Equations with memory have been the object of intensive study since the early works of Maxwell [30], Boltzmann [5, 6], and Volterra [36, 37] and the literature on the topic is broad (e.g., [1, 7, 10, 11, 12, 13, 19, 31, 32, 40, 41] and the references therein).

The observability problem we analyze in this paper is relevant in the context of inverse problems and control. Roughly it aims to recover the full information on solutions out of partial measurements.

The observability problem has been intensively investigated for the pure heat equation, corresponding to the null memory kernel (i.e., (1.1) with $M = 0$), the main goal being to recover full information about the solution at the final time $T > 0$ out of its restriction to a space-time open subdomain or measurable set (see, for instance, [2, 15, 17, 21, 24, 25, 26, 33, 43]).

Compared with the pure heat equation, the study on the observability for heat equations with memory is lagging behind, despite some key works ([3, 9, 16, 18, 20, 32, 42] and the references therein). To our best knowledge, [23] is the first work showing that the controllability to rest does not hold for a class of one-dimensional heat equations with memory in the principal part, controlled from the boundary or the interior of the physical domain where the dynamics evolves. This is then extended in [20] to the lack of the classical null controllability property for boundary controlled heat equations with memory (see also [18]). A similar result is obtained in [42] for equation (1.1) when equipped with interior control, where the observation set is time-invariant (of the form $(0, T) \times \omega$ for some open subset $\omega \subset \Omega$) and proper. As a remedy, in [9], the authors proposed a moving or time-varying observation mechanism to ensure the observability property. However, the necessity of the aforementioned geometric condition was not shown.

Observability and controllability problems have also been considered for other PDEs involving memory terms. See, for example, [28, 29] for wave equations and [14] for Stokes equations with memory, respectively. Here we focus on the heat equation with memory (1.1).

This paper is devoted to proving a necessary and sufficient geometric condition on measurable subsets $Q \subset \mathbb{R}^+ \times \Omega$ guaranteeing the observability of equation (1.1), so that the restriction to Q suffices to fully recover the solution. The geometry of the observation sets and the specific form of the observability inequalities studied in the current paper are motivated by the following property $(\mathcal{P})$ of the dynamics :

$(\mathcal{P})$ The singularities of the solutions of equation (1.1) propagate along the t-axis as follows:

- For $t > 0$, singularities propagate without any change on the regularity order.
- The backward propagation of singularities to the initial data results in the loss of 4 space-derivatives.

This property exhibits and reflects both, the singularity propagation properties for $t > 0$, proper to hyperbolic systems, but also a boundary layer at $t = 0$, which does not occur in the hyperbolic setting, being rather of a parabolic nature. Note however that for classical heat-like equations, the backward propagation of singularities to the initial data results in the loss of infinitely many space-derivatives, while for our model the loss is exactly 4, and this is true for all non-trivial analytic memory kernels.

These two fundamental aspects of the dynamics under consideration, constitute a manifestation of the hybrid parabolic-hyperbolic nature of equation (1.1). The numerical simulations in Section 6 confirm and provide computational evidence of this hybrid character of the dynamics.

Property $(\mathcal{P})$ is induced by the decomposition of the flow generated by equation (1.1), recently developed in [40, Theorem 1.1] (and recalled in Theorem 8.1 in the Appendix at the end of this paper for the sake of completeness). It states that the flow

$$\Phi(t)y_0 := y(t,\cdot;y_0) \in L^2(\Omega),\ y_0 \in L^2(\Omega),\ t \geq 0 \tag{1.2}$$

can be decomposed into three components: a heat-like one $\mathcal{P}_N(t)$, a wave-like one $\mathcal{W}_N(t)$, and a remainder $\mathfrak{R}_N(t)$.

1.2 Main results

We start by introducing the following definitions:

(D1) (*Space* $\mathcal{H}^s$) The Laplacian is denoted by

$$Af := \Delta f, \quad \text{with its domain} \quad D(A) := H^2(\Omega) \cap H_0^1(\Omega). \tag{1.3}$$

Let η_j be the j^{th} eigenvalue of $-A$ and e_j be the corresponding normalized eigenfunction in $L^2(\Omega)$. For each $s \in \mathbb{R}$, we define the real Hilbert space

$$\mathcal{H}^s := \left\{ f = \sum_{j=1}^{\infty} a_j e_j \ :\ (a_j)_{j\geq 1} \subset \mathbb{R},\ \sum_{j=1}^{\infty} |a_j|^2 \eta_j^s < +\infty \right\}, \tag{1.4}$$

equipped with the inner product

$$\langle f, g\rangle_{\mathcal{H}^s} := \sum_{j=1}^{\infty} a_j b_j \eta_j^s, \qquad f = \sum_{j=1}^{\infty} a_j e_j \in \mathcal{H}^s \ \text{ and } \ g = \sum_{j=1}^{\infty} b_j e_j \in \mathcal{H}^s, \tag{1.5}$$

where $(a_j)_{j\geq 1}, (b_j)_{j\geq 1} \subset \mathbb{R}$.

(D2) (*Moving observation condition*, MOC for short) The triplet (Q, S, T), where $T > S \geq 0$ and Q is a nonempty measurable subset of $\mathbb{R}^+ \times \Omega$, is said to satisfy the MOC if

$$\mathcal{T}_\Omega(Q, S, T) := \operatorname*{ess-inf}_{x\in\Omega} \int_S^T \chi_Q(t,x)dt > 0 \tag{1.6}$$

(here and below, for each set E, χ_E denotes its characteristic function).

Remark 1.1. In the definition (D2), the following points should be noted:

- (i) The MOC condition is natural in view of the wave-like nature of the flow and, more precisely, is dictated by the propagation of singularities in the t-direction as described in the aforementioned property $(\mathcal{P})$. Indeed, for each $x \in \Omega$, the slice $Q_x := \{(t,x) \in Q : t \in [S,T]\}$ lies on the half-line $\{(t,x) : t \geq 0\}$, which is the characteristic line issued from x at time $t = 0$ (see [40, (iv) and (ii) of Theorem 1.2], where the propagation of singularities was introduced). We write

$$\mathcal{T}_{x,S,T} := |Q_x| = \int_S^T \chi_Q(t,x)dt,$$

  where $|Q_x|$ denotes the measure of Q_x in $\mathbb{R}$. Then, $\mathcal{T}_{x,S,T}$ can be viewed as the cumulative time that the characteristic line spends in the observation set Q during the time period $[S,T]$. Consequently, (1.6) means that the cumulative time $\mathcal{T}_{x,S,T}$ has a uniform positive lower bound with respect to a.e. $x \in \Omega$. This guarantees that all propagating singularities of the solutions (to equation (1.1)) spend an uniform percentage of time in the observation set Q.

- (ii) The MOC was introduced in [9, Assumption 4.1] (see also [8]). But in these works some extra technical assumptions on Q were imposed for the employment of Carleman inequalities. In the most general definition of the MOC considered here, Q can be a measurable set, while in the definition of [9, Assumption 4.1], Q was an open set. Our MOC is strictly weaker than [9, Assumption 4.1] (see Example 4.4 for a concrete example of the MOC), because of the absence of any other geometric technical conditions, that in [9] were motivated by the use of Carleman inequalities.

The main result of this paper is as follows.

Theorem 1.2. *Let M be a nonzero real analytic function over $[0,+\infty)$. Let $T > S > 0$ and Q be a nonempty measurable subset of $(0,+\infty) \times \Omega$. The following two statements are equivalent:*

(i) The triplet (Q,S,T) satisfies the MOC.

(ii) There is a constant $C > 0$ such that

$$\frac{1}{C}\|y_0\|_{\mathcal{H}^{-4}} \leq \int_S^T \left\|\chi_Q(t,\cdot)y(t,\cdot;y_0)\right\|_{L^2(\Omega)}dt \leq C\|y_0\|_{\mathcal{H}^{-4}} \tag{1.7}$$

for every solution of equation (1.1) with $y_0 \in L^2(\Omega)$.

Remark 1.3. The following comments are worthy of consideration:

- (a1) Theorem 1.2 guarantees that when $S > 0$, the MOC satisfied by (Q,S,T) is a sufficient and necessary condition to ensure the two-sided observability inequality (1.7).

  The first inequality in (1.7) guarantees that, by measuring a solution of equation (1.1) over Q in the space $L^1(S,T;L^2(\Omega))$, one can recover its initial datum in $\mathcal{H}^{-4}$, while the second inequality in (1.7) shows its optimality.

(a2) By density, the inequalities in (1.7) hold for all $y_0 \in \mathcal{H}^{-4}$. In fact, the regularity estimate in [40, Theorem 1.4] states that when $T > S > 0$, there is a $C > 0$ such that

$$\|y(\cdot, \cdot; y_0)\|_{C([S,T];L^2(\Omega))} \leq C\|y_0\|_{\mathcal{H}^{-4}}, \ \forall\, y_0 \in \mathcal{H}^{-4}.$$

Moreover, the fact that the norm $\mathcal{H}^{-4}$ of the initial datum is the one observed is natural in view of the discussion above and the emergence of singularities of order 4 when solving the dynamics backwards in time at $t = 0$. In fact, away from $t = 0$ the dynamics can be decomposed as

$$y(t, \cdot; y_0) = -M(t)A^{-2}y_0 + \text{"smooth rest terms"}, \quad 0 < S \leq t \leq T \tag{1.8}$$

(see (6.2) in the later section, for a rigorous analysis).

(a3) To ensure Theorem 1.2, it is necessary that $S > 0$ (see Theorem 4.1 for the case $S = 0$). The treatment of the more delicate case $S = 0$ requires involving the weight function t^α (with $\alpha > 1$) into the integrand in (1.7) (see also Theorem 4.1).

(a4) The proof of Theorem 1.2 employs the wave-like aspects of the dynamics (1.1). Inspired by the observability for wave equations (see for instance [4, 21, 34]), we develop the following modified three-step strategy to prove Theorem 1.2:

Step 1. We establish a relaxed observability inequality (see Lemma 2.8), based on the simplified equality (1.8), which is a consequence of the decomposition in Theorem 8.1, and especially of the hyperbolic-like component.

Step 2. We obtain a qualitative unique continuation property for solutions of equation (1.1) (see Lemma 2.9). Our proof relies on the time-analyticity property (stated in Proposition 2.4) of the solutions of equation (1.1).

Step 3. We conclude the exact observability inequality (2.19), by means of the classical compactness-uniqueness argument.

(a5) Theorem 1.2 remains true, as can be proven by similar arguments, when the norm in $L^1_t L^2_x$ in (1.7) is replaced by that in $L^p_t L^2_x$ (with $1 < p \leq +\infty$), i.e., when $T > S > 0$, the triplet (Q, S, T) satisfies the MOC if and only if there is a $C > 0$ such that

$$\frac{1}{C}\|y_0\|_{\mathcal{H}^{-4}} \leq \|\chi_Q y(\cdot, \cdot; y_0)\|_{L^p(S,T;L^2(\Omega))} \leq C\|y_0\|_{\mathcal{H}^{-4}} \quad \text{for all } y_0 \in L^2(\Omega).$$

(a6) In Section 4, we present some further developments on Theorem 1.2, and in Section 5, we discuss some applications of Theorem 1.2 to the control of system (1.1).

1.3 Discussion on the contributions

The main contributions of the results of this paper are as follows:

(b1) The two-sided observability inequality (1.7) and the necessary and sufficient MOC condition on (Q, S, T).

(b2) The optimality and minimality of our MOC condition, involving measurable (not necessarily open) observation sets.

(b3) The methods in this paper themselves are also new and can be of independent use to tackle other problems related with parabolic memory models and in particular inverse problems and long time asymptotics.

1.4 Organization of the paper

The rest of this paper is organized as follows. In Section 2 we present some preliminary results. In Section 3, Theorem 1.2 is proven. Section 4 presents some extensions of Theorem 1.2. Section 5 shows some applications in control. Section 6 provides numerical simulations for the hybrid parabolic-hyperbolic nature of equation (1.1). Section 7 is devoted to discussing several open problems. Section 8 is devoted to the technical Appendix.

2 Preliminaries

2.1 Properties of the flow

According to Theorem 8.1 in the Appendix of this paper (see also [40, Theorem 1.1]), we know that for each $t \geq 0$, the operator $\Phi(t)$ (given in (1.2)) constitutes an element of the space $\mathcal{L}(\mathcal{H}^s)$ for any $s \in \mathbb{R}$.

In this subsection, we present some properties of the flow $\Phi(t)$, which are consequences of Theorem 8.1.

Proposition 2.1. *There is an $\mathcal{R}_c \in C\big(\mathbb{R}^+; C(\mathbb{R}^+)\big)$ such that*

$$\Phi(t) = e^{tA}\Big(1 - tM(0)A^{-1} + M(0)A^{-2}\Big) - M(t)A^{-2} + \mathcal{R}_c(t, -A)A^{-3}, \quad t > 0, \tag{2.1}$$

and such that for each $s \in \mathbb{R}$, $\mathcal{R}_c(\cdot, -A)$ belongs to $C(\mathbb{R}^+; \mathcal{L}(\mathcal{H}^s))$ and satisfies

$$\|\mathcal{R}_c(t, -A)\|_{\mathcal{L}(\mathcal{H}^s)} \leq \exp\Big(2(1+t)\big(1 + \|M\|_{C^2([0,t])}\big)\Big), \quad t > 0. \tag{2.2}$$

Proof. Apply Theorem 8.1 (with $N = 2$) to get

$$\Phi(t) = e^{tA} + e^{tA}\Big(- p_0(t)A^{-1} + p_1(t)A^{-2}\Big) + h_1(t)A^{-2} - R_2(t, -A)A^{-3}, \quad t > 0. \tag{2.3}$$

At the same time, it follows from (8.1) that for each $t > 0$,

$$\begin{cases} p_0(t) = M(0)t, \\ p_1(t) = M(0) - M'(0)t + \frac{1}{2}M(0)^2 t^2, \\ h_1(t) = -M(t). \end{cases} \tag{2.4}$$

From equations (2.3) and (2.4), we find

$$\begin{aligned}\Phi(t) =& e^{tA}\Big(1 - M(0)tA^{-1} + M(0)A^{-2}\Big) - M(t)A^{-2}\\ &+ e^{tA}\Big(- M'(0)t + \frac{1}{2}M(0)^2t^2\Big)A^{-2} - R_2(t,-A)A^{-3},\ t>0.\end{aligned}\tag{2.5}$$

Next, we define

$$\mathcal{R}_c(t,\tau) := e^{-t\tau}\Big(- M'(0)t + \frac{1}{2}M(0)^2t^2\Big)(-\tau) - R_2(t,\tau),\ \ t>0,\ \tau>0.$$

Then, by spectral functional calculus, we have

$$\mathcal{R}_c(t,-A) = e^{tA}\Big(- M'(0)t + \frac{1}{2}M(0)^2t^2\Big)A - R_2(t,-A),\ \ t>0.\tag{2.6}$$

At the same time, we can directly check that for each $s \in \mathbb{R}$,

$$\|Ae^{tA}\|_{\mathcal{L}(\mathcal{H}^s)} \le t^{-1},\ \ t>0.\tag{2.7}$$

Now, (2.1) follows from (2.5) and (2.6), while (2.2) follows from (2.6), (2.7), and (8.6) (where $N=2$). This completes the proof. □

The proofs of the following two corollaries are presented in Subsection 8.3 in the Appendix.

Corollary 2.2. *Let $\beta \in [2,3]$. Then, there is an $\widehat{\mathcal{R}}_c \in C\big(\mathbb{R}^+; C(\mathbb{R}^+)\big)$ such that*

$$\Phi(t) = -M(t)A^{-2} + \widehat{\mathcal{R}}_c(t,-A)(-A)^{-\beta},\ \ t>0.\tag{2.8}$$

Moreover, for each $s \in \mathbb{R}$, $\widehat{\mathcal{R}}_c(\cdot,-A)$ belongs to $C(\mathbb{R}^+;\mathcal{L}(\mathcal{H}^s))$ and satisfies for some $C>0$,

$$\|\widehat{\mathcal{R}}_c(t,-A)\|_{\mathcal{L}(\mathcal{H}^s)} \le \frac{C}{t^\beta}\exp\Big(2(1+t)\big(1+\|M\|_{C^2([0,t])}\big)\Big),\ \ t>0.\tag{2.9}$$

Corollary 2.3. *There is an $\widetilde{\mathcal{R}}_c \in C\big(\mathbb{R}^+; C(\mathbb{R}^+)\big)$ such that*

$$\Phi(t) = e^{tA} + \widetilde{\mathcal{R}}_c(t,-A)A^{-2},\ \ t>0.\tag{2.10}$$

Moreover, for each $s \in \mathbb{R}$, $\widetilde{\mathcal{R}}_c(\cdot,-A)$ belongs to $C(\mathbb{R}^+;\mathcal{L}(\mathcal{H}^s))$ and satisfies for some $C_1>0$,

$$\|\widetilde{\mathcal{R}}_c(t,-A)\|_{\mathcal{L}(\mathcal{H}^s)} \le C_1\exp\Big(2(1+t)\big(1+\|M\|_{C^2([0,t])}\big)\Big),\ \ t>0.$$

2.2 Time-analyticity of solutions

This subsection is concerned with the analyticity of the solutions to equation (1.1) with respect to the time variable.

Proposition 2.4. *Let $y_0 \in \cup_{s\in\mathbb{R}}\mathcal{H}^s$ be such that $(A^{-2}y_0)|_\omega \in L^2_{loc}(\omega)$ for some nonempty open subset $\omega \subset \Omega$. Then, the following statements are true:*

(i) The restriction of the solution $y(\cdot,\cdot;y_0)$ over $(0,+\infty)\times\omega$ belongs to $L^2_{loc}((0,+\infty)\times\omega)$.

(ii) For a.e. $x \in \omega$*, the function* $t \mapsto y(t, x; y_0)$ *is real analytic over* $(0, +\infty)$.

Remark 2.5. We remark that in general, the solutions of (1.1) are not analytic in the space variable because of the finite-order regularizing effect of $\Phi(t)$ (see [40, Theorem 1.4]). This shows a difference between the heat semigroup $\{e^{tA}\}_{t\geq 0}$ and $\{\Phi(t)\}_{t\geq 0}$ from the perspective of the analyticity of solutions.

Proof of Proposition 2.4. Since

$$(A^{-2}y_0)|_\omega \in L^2_{loc}(\omega) \text{ and } y_0 \in \mathcal{H}^{-2m} \text{ for some } m \in \mathbb{N}^+, \tag{2.11}$$

it follows from (8.3) and (8.4) with $N = n + m$ (n is the space dimension) in the Appendix that for each $t > 0$,

$$\begin{aligned} y(t, \cdot; y_0) = \Phi(t)y_0 =& e^{tA}y_0 + e^{tA}\sum_{l=0}^{n+m-1} p_l(t)(-A)^{-l-1}y_0 \\ &+ \sum_{l=1}^{n+m-1} h_l(t)(-A)^{-l-1}y_0 + \mathfrak{R}_{n+m}(t)y_0, \end{aligned} \tag{2.12}$$

where p_l, h_l, and $\mathfrak{R}_{n+m}$ are given by (8.1) and (8.4)-(8.5), respectively. By (2.12), we see that to prove conclusions (i) and (ii), it suffices to show the following three assertions:

(A1) For each $l \in \mathbb{N}$, both p_l and h_l are real analytic over $(0, +\infty)$;

(A2) For each $l \in \mathbb{N}^+$, $(-A)^{-l-1}y_0 \in \mathcal{H}^{-2m}$ and $[(-A)^{-l-1}y_0]|_\omega \in L^2_{loc}(\omega)$;

(A3) For each $z \in \mathcal{H}^{-2m}$, the functions $t \mapsto e^{tA}z$ $(t > 0)$ and $t \mapsto \mathfrak{R}_{n+m}(t)z$ $(t > 0)$ belong to $C(\mathbb{R}^+; L^2(\Omega))$. Moreover, for each $z \in \mathcal{H}^{-2m}$ and for a.e. $x \in \Omega$, the functions $t \mapsto (e^{tA}z)(x)$ $(t > 0)$ and $t \mapsto (\mathfrak{R}_{n+m}(t)z)(x)$ $(t > 0)$ are real analytic.

We now prove (A1)–(A3). First, (A1) follows from (8.1) and the analyticity of M over $[0, +\infty)$. Second, (A2) can be checked directly from (2.11) and the iterative use of the following property:

$$h \in \mathcal{H}^{-2m} \cap L^2_{loc}(\omega) \Rightarrow A^{-1}h \in \mathcal{H}^{-2m} \cap L^2_{loc}(\omega)$$

(the above property is due to the ellipticity of the operator $A = \Delta$ and is a direct consequence of [22, Theorem 18.1.29]).

The remaining task is to show (A3). To this end, we arbitrarily fix $z \in \mathcal{H}^{-2m}$. Then, we have

$$z = \sum_{j\geq 1} a_j \eta_j^m e_j \;\text{ for some } (a_j)_{j\geq 1} \in \ell^2 \tag{2.13}$$

(recall that η_j is the j^{th} eigenvalue of $-A$ and e_j is the corresponding normalized eigenfunction in $L^2(\Omega)$). From (2.13), (8.4), and (8.5) in the Appendix, we find

$$e^{tA}z = \sum_{j\geq 1} f_j(t)\eta_j^{-n}a_j e_j \text{ and } \mathfrak{R}_{n+m}(t)z = \sum_{j\geq 1} g_j(t)\eta_j^{-n}a_j e_j,\; t > 0, \tag{2.14}$$

where

$$f_j(t) := \eta_j^{n+m} e^{-\eta_j t} \text{ and } g_j(t) := \int_0^t e^{-\eta_j s} \partial_s^{n+m} K_M(t,s) ds,\ t > 0. \tag{2.15}$$

Next, we will examine the analyticity of f_j and g_j ($j \in \mathbb{N}^+$). We claim the following:

There is an open subset $\mathcal{O}$ in $\mathbb{C}$ with $\mathcal{O} \supset (0, +\infty)$ such that each g_j (resp., f_j) can be extended to be an analytic function over $\mathcal{O}$, denoted by $\tilde{g}_j$ (resp., $\tilde{f}_j$), with the following estimate:

$$\sup_{j\geq 1} \|\tilde{g}_j\|_{C(G)} < +\infty \ \Big(\textit{resp., } \sup_{j\geq 1} \|\tilde{f}_j\|_{C(G)} < +\infty\Big) \textit{ for each } G \Subset \mathcal{O}. \tag{2.16}$$

For this purpose, we use the real analyticity of K_M over the set $S_+ := \{(t,s) \in \mathbb{R}^2 \ : \ t \geq s\}$ (see Proposition 8.3 in the Appendix) to obtain an open subset $\mathcal{D} \supset S_+$ in $\mathbb{C}^2$ such that K_M can be extended to an analytic function over $\mathcal{D}$. We still use K_M to denote this extension. We define

$$\mathcal{O} := \Big\{ t \in \mathbb{C} \ : \ \mathrm{Re}\, t > 0 \ \text{ and } \ (t, t\tau) \in \mathcal{D} \text{ for each } \tau \in [0,1] \Big\} \supset (0, +\infty),$$

and for each $j \in \mathbb{N}^+$, we define the following function over $\mathcal{O}$:

$$\tilde{g}_j(t) := t \int_0^1 e^{-\eta_j \tau t} \partial_s^{n+m} K_M(t, \tau t) d\tau,\ t \in \mathcal{O}. \tag{2.17}$$

Several properties on $\tilde{g}_j$ are given. First, it follows from (2.17) and the analyticity of K_M that $\tilde{g}_j$ is analytic over $\mathcal{O}$. Second, it follows from (2.17) and (2.15) that $\tilde{g}_j|_{(0,+\infty)} = g_j$. Third, it follows from (2.17) that

$$|\tilde{g}_j(t)| \leq |t| \Big(\sup_{0<\tau<1} |\partial_s^{n+m} K_M(t, \tau t)| \Big),\ t \in \mathcal{O}.$$

From the above properties, we see that the above claim is true for g_j. Similarly, we can show that it also holds for f_j. Thus, we have proven the above claim.

Finally, by (2.13), we have

$$\int_\Omega \sum_{j\geq 1} |a_j e_j(x)|^2 dx = \sum_{j\geq 1} |a_j|^2 < +\infty,$$

which shows that

$$\sum_{j\geq 1} |a_j e_j(x)|^2 < +\infty \text{ for a.e. } x \in \Omega. \tag{2.18}$$

Meanwhile, by Weyl's asymptotic formula for the eigenvalues of the Laplace operator (see for instance [35, Theorem XIII.78, pp. 271]), we find that $\lim_{j\to+\infty} \eta_j j^{-2/n} > 0$, showing that $(\eta_j^{-n})_{j\geq 1} \in \ell^2$. This, together with (2.18) and the Cauchy–Schwarz inequality, yields

$$\sum_{j\geq 1} \eta_j^{-n} |a_j e_j(x)| \leq \Big(\sum_{j\geq 1} \eta_j^{-2n} \Big)^{\frac{1}{2}} \Big(\sum_{j\geq 1} |a_j e_j(x)|^2 \Big)^{\frac{1}{2}} < +\infty \text{ for a.e. } x \in \Omega.$$

Then, by (2.16), we see that for a.e. $x \in \Omega$, both series

$$\sum_{j\geq 1} \tilde{f}_j(t) \eta_j^{-n} a_j e_j(x) \text{ and } \sum_{j\geq 1} \tilde{g}_j(t) \eta_j^{-n} a_j e_j(x),\ t \in \mathcal{O}$$

absolutely converge over each nonempty compact subset $G \subset \mathcal{O}$, and thus, their sums are analytic over $\mathcal{O}$. Thus, by (2.14), the assertion (A3) follows at once. This completes the proof. □

2.3 Exact observability

Throughout this subsection, *we suppose that* $Q \subset \mathbb{R}^+ \times \Omega$ *is a nonempty measurable subset.* We first present the following observability estimate.

Theorem 2.6. *Let* $T > S \geq 0$*. We suppose that* (Q, S, T) *satisfies the MOC. Then, for some* $C > 0$*,*

$$\|y_0\|_{\mathcal{H}^{-4}} \leq C \int_S^T \|\chi_Q(t, \cdot) y(t, \cdot; y_0)\|_{L^2(\Omega)} dt \ \text{ for each } y_0 \in L^2(\Omega). \tag{2.19}$$

We will prove Theorem 2.6 through a modified three-step strategy that was originally designed for the observability of the wave equation (see, for instance, [4, 34]). The proof requires several technical lemmas. The first one presents estimates about the analytic functions and the MOC.

Lemma 2.7. *Let* $T > 0$ *and let* f *be a non-trivial real analytic function over* $[0, T]$*. Then, there are two constants* $C, \beta > 0$ *(dependent on* f*) such that for each* $S \in [0, T)$*,*

$$\int_S^T \chi_Q(t, x)|f(t)| dt \geq C \left(\int_S^T \chi_Q(t, x) dt \right)^{\beta+1} \ \text{ for a.e. } x \in \Omega. \tag{2.20}$$

If we further assume that (Q, S, T) *satisfies the MOC, then*

$$\underset{x \in \Omega}{\text{ess-inf}} \int_{S+\varepsilon}^T \chi_Q(t, x)|f(t)| dt > 0 \ \text{ when } 0 \leq \varepsilon < \mathcal{T}_\Omega(Q, S, T), \tag{2.21}$$

where $\mathcal{T}_\Omega(Q, S, T)$ *is given by* (1.6)*.*

Proof. Because f is real analytic, it has at most a finite number of distinct zeros over $[0, T]$, denoted by $\{t_j\}_{j=1}^m$. Let $d_j \in \mathbb{N}^+$ be the order of t_j ($j = 1, \ldots, m$). Then, $f(t)/\min_{1 \leq j \leq m} |t - t_j|^{d_j}$ can be extended to be a continuous function over $[0, T]$ without zeros. Thus, there is a $C_1 > 0$ such that

$$|f(t)| \geq C_1 \min_{1 \leq j \leq m} |t - t_j|^{d_j}, \ \text{ when } 0 \leq t \leq T.$$

We set $\beta := \max_{1 \leq j \leq m} d_j$. Then, there is a $C_2 > 0$ such that

$$|f(t)| \geq C_1 \min_{1 \leq j \leq m} T^{d_j} \Big(|t - t_j|/T \Big)^{d_j} \geq C_2 \min_{1 \leq j \leq m} |t - t_j|^{\beta}, \ \text{ when } \ 0 \leq t \leq T. \tag{2.22}$$

We now show that (2.20) is satisfied. To this end, we arbitrarily fix $S \in [0, T)$. For each $x \in \Omega$, we define the following set:

$$I(x) := \Big\{ t \in [S, T] \ : \ (t, x) \in Q \Big\}. \tag{2.23}$$

Because $Q \subset \mathbb{R}^+ \times \Omega$ is measurable, we see that for a.e. $x \in \Omega$, $I(x)$ is measurable. We arbitrarily fix $x \in \Omega$ with $I(x)$ measurable. We define

$$E_j(x) := \Big\{ t \in I(x) \ : \ |t - t_j|^{\beta} = \min_{1 \leq k \leq m} |t - t_k|^{\beta} \Big\}, \ \ 1 \leq j \leq m. \tag{2.24}$$

The following facts hold:

Fact 1: It follows from (2.23) and (2.22) that

$$\int_S^T \chi_Q(t,x)|f(t)|dt = \int_{I(x)} |f(t)|dt \geq C_2 \int_{I(x)} \min_{1\leq j\leq m} |t-t_j|^\beta dt. \tag{2.25}$$

Fact 2: It follows from (2.23) and (2.24) that $I(x) = \bigcup_{j=1}^m E_j(x)$. Thus, there is a $j_0 \in \{1,\ldots,m\}$ such that

$$|E_{j_0}(x)| \geq \frac{|I(x)|}{m} = \frac{1}{m}\int_S^T \chi_Q(t,x)dt. \tag{2.26}$$

Fact 3: It follows from (2.24) (where $j=j_0$) that

$$\int_{I(x)} \min_{1\leq j\leq m} |t-t_j|^\beta dt \geq \int_{E_{j_0}(x)} |t-t_{j_0}|^\beta dt. \tag{2.27}$$

Fact 4: We have

$$\int_{E_{j_0}(x)} |t-t_{j_0}|^\beta dt \geq \int_{|t-t_{j_0}|\leq \frac{|E_{j_0}(x)|}{2}} |t-t_{j_0}|^\beta dt. \tag{2.28}$$

To show that (2.28) is satisfied, we set $E_- := E_{j_0}(x)\cap(-\infty,t_{j_0})$ and $E_+ := E_{j_0}(x)\setminus E_-$. We fix an open interval $I \Subset (-\infty,t_{j_0})$. Since the function $t\mapsto |t-t_{j_0}|^\beta$ ($t\in\mathbb{R}$) is decreasing over $(-\infty,t_{j_0})$, we have

$$\int_I |t-t_{j_0}|^\beta dt \geq \int_{t_{j_0}-|I|}^{t_{j_0}} |t-t_{j_0}|^\beta dt.$$

Because any subset of positive measure in $\mathbb{R}$ differs from a countable intersection of open sets by a null set, the above inequality, where I is replaced by a subset of positive measure in $(-\infty,t_{j_0})$, still holds. In particular, we have $\int_{E_-} |t-t_{j_0}|^\beta dt \geq \int_{t_{j_0}-|E_-|}^{t_{j_0}} |t-t_{j_0}|^\beta dt$. Similarly, we can show that $\int_{E_+} |t-t_{j_0}|^\beta dt \geq \int_{t_{j_0}}^{t_{j_0}+|E_+|} |t-t_{j_0}|^\beta dt$. Thus, we have

$$\int_{E_{j_0}} |t-t_{j_0}|^\beta dt = \int_{E_-} |t-t_{j_0}|^\beta dt + \int_{E_+} |t-t_{j_0}|^\beta dt \geq \int_{t_{j_0}-|E_-|}^{t_{j_0}+|E_+|} |t-t_{j_0}|^\beta dt. \tag{2.29}$$

Since $|E_{j_0}| = |E_+|+|E_-|$ and the function $t\mapsto |t-t_{j_0}|^\beta$ ($t\in\mathbb{R}$) is symmetric about $t=t_{j_0}$, we obtain (2.28) from (2.29) at once.

Fact 5: It follows from (2.26) that

$$\int_{|t-t_{j_0}|\leq \frac{|E_{j_0}(x)|}{2}} |t-t_{j_0}|^\beta dt \geq \frac{2}{\beta+1}\left(\frac{|E_{j_0}(x)|}{2}\right)^{\beta+1}. \tag{2.30}$$

Hence, it follows from (2.25), (2.27), (2.28), and (2.30) that

$$\int_S^T \chi_Q(t,x)|f(t)|dt \geq \frac{2C_2}{\beta+1}\left(\frac{1}{2m}\int_S^T \chi_Q(t,x)dt\right)^{\beta+1},$$

which leads to (2.20).

Finally, we prove (2.21). To this end, we arbitrarily fix (Q,S,T) with the MOC and $\varepsilon \in [0, \mathcal{T}_\Omega(Q,S,T))$. It is clear that

$$\int_{S+\varepsilon}^{T} \chi_Q(t,x)dt \geq \int_S^T \chi_Q(t,x)dt - \varepsilon \text{ for a.e. } x \in \Omega.$$

The above, along with (2.20) (where S is replaced by $S+\varepsilon$) and (1.6), leads to (2.21). This completes the proof. □

The following lemma corresponds to the first step of the above-mentioned three-step strategy, which is a relaxed observability inequality.

Lemma 2.8. *Let $T > S \geq 0$. We assume that (Q,S,T) satisfies the MOC. Then, there is a $C > 0$ such that*

$$C\|y_0\|_{\mathcal{H}^{-4}} \leq \int_S^T \|\chi_Q(t,\cdot)y(t,\cdot;y_0)\|_{L^2(\Omega)}dt + \|y_0\|_{\mathcal{H}^{-6}} \text{ for each } y_0 \in L^2(\Omega). \tag{2.31}$$

Proof. Since (Q,S,T) satisfies the MOC, it follows from (1.6) that

$$\varepsilon_0 := \frac{1}{2}\mathcal{T}_\Omega(Q,S,T) > 0. \tag{2.32}$$

We arbitrarily fix a $y_0 \in L^2(\Omega)$. We recall that $y(t,\cdot;y_0) = \Phi(t)y_0$. By Corollary 2.2 (with $\beta = 3$), using the triangle inequality, we can find a $C_1 > 0$ (independent of z) such that

$$\|\chi_Q(t,\cdot)y(t,\cdot;y_0)\|_{L^2(\Omega)} \geq \|\chi_Q(t,\cdot)M(t)A^{-2}y_0\|_{L^2(\Omega)} - C_1 t^{-3}\|A^{-3}y_0\|_{L^2(\Omega)},\ t \in (S+\varepsilon_0, T).$$

This yields

$$\begin{aligned}&\int_{S+\varepsilon_0}^T \|\chi_Q(t,\cdot)M(t)A^{-2}y_0\|_{L^2(\Omega)}dt \\ &\leq \int_{S+\varepsilon_0}^T \|\chi_Q(t,\cdot)y(t,\cdot;y_0)\|_{L^2(\Omega)}dt + C_1(S+\varepsilon_0)^{-3}\|y_0\|_{\mathcal{H}^{-6}}.\end{aligned} \tag{2.33}$$

Meanwhile, we have

$$\left\|\Big(\int_{S+\varepsilon_0}^T \chi_Q(t,\cdot)|M(t)|dt\Big)A^{-2}y_0\right\|_{L^2(\Omega)} \leq \int_{S+\varepsilon_0}^T \|\chi_Q(t,\cdot)M(t)A^{-2}y_0\|_{L^2(\Omega)}dt.$$

From this and (2.33), we obtain

$$\begin{aligned}&\left(\int_\Omega \Big(\int_{S+\varepsilon_0}^T \chi_Q(t,x)|M(t)|dt\Big)^2 |(A^{-2}y_0)(x)|^2 dx\right)^{\frac{1}{2}} \\ &\leq \int_{S+\varepsilon_0}^T \|\chi_Q(t,\cdot)y(t,\cdot;y_0)\|_{L^2(\Omega)}dt + C_1(S+\varepsilon_0)^{-3}\|y_0\|_{\mathcal{H}^{-6}}.\end{aligned} \tag{2.34}$$

At the same time, since (Q, S, T) satisfies the MOC, it follows from (2.21) (with $(f, \varepsilon) = (M, \varepsilon_0)$) and (2.32) that for a.e. $x \in \Omega$,

$$\int_{S+\varepsilon_0}^{T} \chi_Q(t,x)|M(t)|dt \geq \operatorname*{ess-inf}_{x\in\Omega} \left(\int_{S+\varepsilon_0}^{T} \chi_Q(t,x)|M(t)|dt \right) > 0.$$

The above, along with (2.34), yields (2.31). This completes the proof. $\square$

The next lemma gives a unique continuation property for the solutions to equation (1.1). It corresponds to the second step in the three-step strategy mentioned before.

Lemma 2.9. *Let $T > S \geq 0$ and let $y_0 \in \mathcal{H}^{-4}$. We assume that (Q, S, T) satisfies the MOC. If*

$$y(t, x; y_0) = 0 \text{ for a.e. } (t, x) \in Q \cap \big((S, T) \times \Omega\big), \tag{2.35}$$

then $y_0 = 0$ in $\mathcal{H}^{-4}$.

Proof. First, note that $A^{-2}y_0 \in L^2(\Omega)$ since $y_0 \in \mathcal{H}^{-4}$. Second, notice that for a.e. $x \in \Omega$, $y(\cdot, x; y_0)$ vanishes over a subset of positive measure in $\mathbb{R}$. Indeed, since (Q, S, T) satisfies the MOC, we can use (1.6) to see that for a.e. $x \in \Omega$, the set

$$I(x) := \Big\{ t > 0 \ : \ (t, x) \in Q \cap \big((S, T) \times \Omega\big) \Big\}$$

is a subset of positive measure in $\mathbb{R}$. This, along with (2.35), yields the second fact.

Finally, from the above two facts, we can use Proposition 2.4 to determine that for a.e. $x \in \Omega$, $y(\cdot, x; y_0) \equiv 0$ over $(0, +\infty)$. Since $y_0 \in \mathcal{H}^{-4}$, $y(\cdot, \cdot; y_0) \in C((0, +\infty); L^2(\Omega))$ is obtained from Corollary 2.2 (with $\beta = 2$). Therefore, $y(\cdot, \cdot; y_0) = 0$ in $C((0, +\infty); L^2(\Omega))$. On the other hand, given that $\{\Phi(t)\}_{t\geq 0}$ is also a C_0 semigroup over $\mathcal{H}^{-4}$, we have

$$y_0(\cdot) = \lim_{t\to 0^+} y(t, \cdot; y_0) \text{ in } \mathcal{H}^{-4}.$$

Then $y_0 = 0$ in $\mathcal{H}^{-4}$ and this completes the proof. $\square$

Remark 2.10. In [42] it was shown that, when M is an exponential function, if $y = 0$ over $(0, T) \times \omega$ (where ω is an open nonempty subset of Ω), then y is identically zero. We do not know if such a unique continuation property holds for general analytic memory kernels. But the unique continuation property shown above suffices for our purposes since we will be dealing with observation sets fulfilling the MOC condition.

We are now in a position to prove Theorem 2.6 combining the compactness–uniqueness argument and Lemmas 2.8–2.9.

Proof of Theorem 2.6. By contradiction, we suppose that (2.19) is not true. Then, there is a $\{\hat{z}_k\}_{k=1}^{\infty} \subset L^2(\Omega)$ such that

$$\|\hat{z}_k\|_{\mathcal{H}^{-4}} = 1 \ \text{ for each } \ k \in \mathbb{N}^+; \qquad \lim_{k\to\infty} \int_S^T \|\chi_Q(t,\cdot)y(t,\cdot;\hat{z}_k)\|_{L^2(\Omega)}dt = 0. \tag{2.36}$$

From the first equality in (2.36), we can find a subsequence of $\{\hat{z}_k\}_{k=1}^{\infty}$, denoted in the same manner, and $\hat{z} \in \mathcal{H}^{-4}$ such that

$$\hat{z}_k \rightharpoonup \hat{z} \text{ weakly in } \mathcal{H}^{-4}, \text{ as } k \to \infty. \tag{2.37}$$

According to Corollary 2.2 (with $\beta = 3$ and $s = 0$) as well as (2.37), there is a subsequence of $\{\hat{z}_k\}_{k=1}^{\infty}$, denoted in the same manner, such that

$$y(\cdot,\cdot;\hat{z}_k) \rightharpoonup y(\cdot,\cdot;\hat{z}) \text{ weakly in } L^2_{loc}((0,+\infty);L^2(\Omega)),$$

i.e., for each $\varepsilon \in (0,1)$ and $f \in L^2((\varepsilon,1/\varepsilon);L^2(\Omega))$,

$$\int_{\varepsilon}^{1/\varepsilon} \langle y(t,\cdot;\hat{z}), f(t)\rangle_{L^2(\Omega)} dt = \lim_{k\to+\infty} \int_{\varepsilon}^{1/\varepsilon} \langle y(t,\cdot;\hat{z}_k), f(t)\rangle_{L^2(\Omega)} dt.$$

This, along with the second equality in (2.36), yields

$$y(\cdot,\cdot;\hat{z}) = 0 \text{ over } Q \cap \big((S,T)\times\Omega\big). \tag{2.38}$$

Since $\hat{z} \in \mathcal{H}^{-4}$ and (Q,S,T) satisfies the MOC, we can apply Lemma 2.9 and (2.38) to determine that $\hat{z} = 0$ in $\mathcal{H}^{-4}$. This, together with (2.37), implies that $\lim_{k\to\infty} \|\hat{z}_k\|_{\mathcal{H}^{-6}} = 0$. Thus, we can use Lemma 2.8, as well as the second equality in (2.36), to find that $\lim_{k\to\infty} \|\hat{z}_k\|_{\mathcal{H}^{-4}} = 0$, which contradicts the first equality in (2.36). Hence, (2.19) is true. This ends the proof. □

3 Proof of the main theorem

This section aims to prove Theorem 1.2.

Proof of Theorem 1.2. We first aim to show that (i) implies (ii). For this purpose, we assume that the statement (i) is true, i.e., the triplet (Q,S,T) satisfies the MOC. Then, by Theorem 2.6, we obtain the first inequality in (1.7). To show the second inequality in (1.7), we apply Corollary 2.2 (with $\beta = 2$ and $s = 0$) to find a $C_1 > 0$ such that for each $y_0 \in L^2(\Omega)$,

$$\begin{aligned}\int_S^T \|\Phi(t)y_0\|_{L^2(\Omega)} dt &\le \int_S^T \|M(t)A^{-2}y_0\|_{L^2(\Omega)} dt + \int_S^T C_1 t^{-2} \|A^{-2}y_0\|_{L^2(\Omega)} dt \\ &\le \left(\int_S^T |M(t)| dt + C_1 \int_S^T t^{-2} dt\right) \|A^{-2}y_0\|_{L^2(\Omega)}.\end{aligned}$$

Since $S > 0$, the second inequality in (1.7) follows from the inequality above. Thus, (1.7) is proven, i.e., the statement (ii) is true.

Next, we aim to verify that (ii) implies (i). We suppose that the statement (ii) is true, i.e., (1.7) holds for some $C > 0$. It follows from Corollary 2.2 (with $\beta = 3$ and $s = 0$) that for some $C_2 > 0$,

$$\|\Phi(t)y_0 + M(t)A^{-2}y_0\|_{L^2(\Omega)} \le C_2 t^{-3} \|(-A)^{-3}y_0\|_{L^2(\Omega)},\ y_0 \in L^2(\Omega),\ t \in [S,T].$$

Then, we obtain from (1.7) that for each $y_0 \in L^2(\Omega)$,

$$\|A^{-2}y_0\|_{L^2(\Omega)} = \|y_0\|_{\mathcal{H}^{-4}} \leq C\int_S^T \|\chi_Q(t,\cdot)\Phi(t)y_0\|_{L^2(\Omega)}dt$$
$$\leq C\|M\|_{C([0,T])}\int_S^T \|\chi_Q(t,\cdot)A^{-2}y_0\|_{L^2(\Omega)}dt + CC_2S^{-2}\|(-A)^{-3}y_0\|_{L^2(\Omega)}. \tag{3.1}$$

By a standard density argument, we replace $A^{-2}y_0$ by z in (3.1) to determine that for some $C_3 > 0$,

$$C_3\|z\|_{L^2(\Omega)} \leq \int_S^T \|\chi_Q(t,\cdot)z\|_{L^2(\Omega)}dt + \|(-A)^{-1}z\|_{L^2(\Omega)} \ \text{ for all } \ z \in L^2(\Omega). \tag{3.2}$$

Now, we will use (3.2) to derive that

$$(Q,S,T) \text{ satisfies the MOC.} \tag{3.3}$$

In what follows, we use $B(x,r)$ to denote the closed ball in $\mathbb{R}^n$, centered at x with radius r. We arbitrarily fix an $x_0 \in \Omega$ and set

$$z_k := |B(x_0,1/k)|^{-\frac{1}{2}}\chi_{B(x_0,1/k)}, \ k \in \mathbb{N}^+. \tag{3.4}$$

It is clear that as $k \in \mathbb{N}^+$ is large,

$$\text{supp}\, z_k \subset \Omega, \ \|z_k\|_{L^2(\Omega)} = 1 \text{ and } z_k \rightharpoonup 0 \text{ weakly in } L^2(\Omega). \tag{3.5}$$

The last equality in (3.5) implies that $\lim_{k\to+\infty}(-A)^{-1}z_k = 0$ in $L^2(\Omega)$. This, along with (3.2) (where $z = z_k$) and the second equality in (3.5), yields

$$C_3 \leq \limsup_{k\to+\infty}\int_0^T \|\chi_Q(t,\cdot)z_k\|_{L^2(\Omega)}dt. \tag{3.6}$$

Applying Hölder's inequality to (3.6) leads to

$$C_3 \leq \limsup_{k\to+\infty}\left[\left(\int_S^T 1dt\right)^{\frac{1}{2}}\cdot\left(\int_S^T \|\chi_Q(t,\cdot)z_k\|^2_{L^2(\Omega)}dt\right)^{\frac{1}{2}}\right].$$

This, along with (3.4), implies that

$$C_3^2T^{-1} \leq \limsup_{k\to+\infty}\left(\frac{1}{|B(x_0,1/k)|}\int_{B(x_0,1/k)}\left(\int_S^T \chi_Q(t,x)dt\right)dx\right).$$

Because $x_0 \in \Omega$ was arbitrarily taken, the above inequality gives

$$\int_S^T \chi_Q(t,x)dt \geq C_3^2T^{-1} \ \text{ for a.e. } x \in \Omega.$$

Thus, (1.6) holds for the current (Q,S,T), which leads to (3.3), i.e., the statement (i) is true.

The proof of Theorem 1.2 is now completed. □

4 Extension of the main theorem

This section aims to extend Theorem 1.2 to the case with $S = 0$ in the following two directions: First, we show that the MOC satisfied by $(Q, 0, T)$ is still a sufficient and necessary condition to ensure the two-sided observability inequality with a weight t^α where $\alpha > 1$ (see Theorem 4.1 below). Second, we show that the MOC satisfied by $(Q, 0, T)$ is a sharp sufficient condition to ensure the two-sided observability inequality without weight (see Theorem 4.2 below).

Theorem 4.1. *Let $T > 0$. Then, for each $\alpha > 1$ and each nonempty measurable subset $Q \subset (0, +\infty)\times\Omega$, the following two statements are equivalent:*

(i) The triplet $(Q, 0, T)$ satisfies the MOC.

(ii) There is a constant $C > 0$ such that

$$\frac{1}{C}\|y_0\|_{\mathcal{H}^{-4}} \leq \int_0^T \big\|\chi_Q(t,\cdot)y(t,\cdot;y_0)\big\|_{L^2(\Omega)} t^\alpha dt \leq C\|y_0\|_{\mathcal{H}^{-4}} \text{ for all } y_0 \in L^2(\Omega). \tag{4.1}$$

Furthermore, when $\alpha \leq 1$, (i) and (ii) are not equivalent if Q contains a set of the form $(0, \varepsilon) \times B_\varepsilon$ (where $\varepsilon > 0$ is small enough such that Ω contains an open ball B_ε of radius ε).

Theorem 4.2. *Let $T > 0$ and let $Q \subset (0, +\infty) \times \Omega$ be a nonempty measurable subset. Then, for the following statements, the former leads to the latter:*

(i) The triplet $(Q, 0, T)$ satisfies the MOC.

(ii) There is a $C > 0$ such that

$$\|y_0\|_{\mathcal{H}^{-4}} \leq C\|\chi_Q y(\cdot,\cdot;y_0)\|_{L^1(0,T;L^2(\Omega))} \text{ for all } y_0 \in L^2(\Omega). \tag{4.2}$$

(iii) There is a $C > 0$ such that

$$\|y(T,\cdot;y_0)\|_{L^2(\Omega)} \leq C\|\chi_Q y(\cdot,\cdot;y_0)\|_{L^1(0,T;L^2(\Omega))} \text{ for all } y_0 \in L^2(\Omega). \tag{4.3}$$

(iv) The triplet $(Q, 0, T)$ satisfies

$$\inf_{x_0\in\Omega} ⨍_{B(x_0,r)} \left(\int_0^T \chi_Q(t,x)dt\right)dx > 0 \text{ for each } r > 0, \tag{4.4}$$

where $⨍_{B(x_0,r)}$ denotes the average value of the integral over the closed ball $B(x_0, r) \subset \mathbb{R}^n$, centered at x_0 with radius r.

Remark 4.3. (i) Theorem 4.1 shows that the equivalence in Theorem 1.2 remains true for the case with $S = 0$ by inserting the weight function t^α (with $\alpha > 1$) into the integrand in (1.7).

Theorem 4.1 is mainly motivated by the following facts. First, the wave-like effect in equation (1.1) determines the geometry of the observable set Q and the space $\mathcal{H}^{-4}$ for the initial data. Second, the

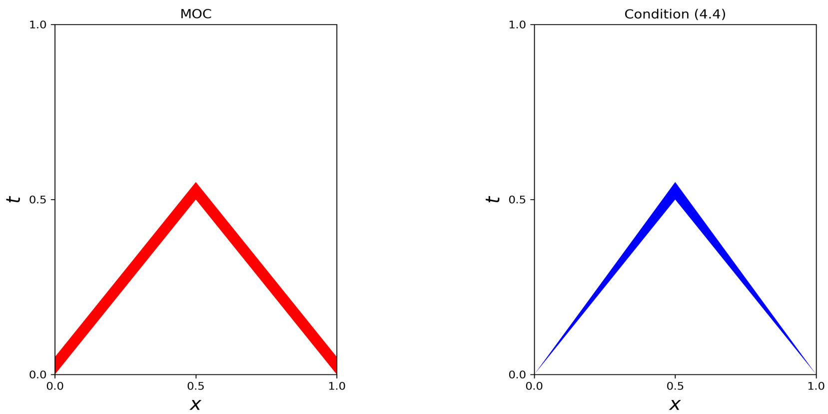

Figure 1: The red and blue sets correspond to the MOC and condition (4.4), respectively.

condition that $\alpha > 1$ is determined by the heat-like effect in equation (1.1) (see Remark 4.6 for more explanations).

(ii) Theorem 4.2 provides a sharp sufficient geometric condition (i.e., $(Q, 0, T)$ satisfies the MOC in (1.6)) ensuring the observability inequalities (4.2) and (4.3).

The gap between (i) and (iv) in this theorem is rather thin for the following reasons (see also Example 4.4 for further details on these two conditions): first, (i) implies (iv) directly; second, (i) ensures that the time for almost all characteristic lines to pass through the observation set has a positive lower bound (see also (i) of Remark 1.1); third, (iv) states that the aforementioned time, averaged over any small ball of a fixed radius, has a positive lower bound.

Example 4.4. Here we give concrete examples about the MOC (given by (1.6)) and its average version (4.4). Let $\Omega := (0, 1)$ and $\varepsilon \in (0, 1)$. We define the following two subsets in $\mathbb{R}^+ \times (0, 1)$:

$$Q_1 := \Big\{(t, x) \in \mathbb{R}^+ \times (0, 1) \ : \ f(x; 1) < t < f(x; 1) + \frac{\varepsilon}{2}\Big\},$$
$$Q_2 := \Big\{(t, x) \in \mathbb{R}^+ \times (0, 1) \ : \ f(x; 1) < t < f(x; 1 + \varepsilon)\Big\},$$

where for each $a \in \mathbb{R}$,

$$f(x; a) := \begin{cases} ax, \ x \in (0, 1/2), \\ a(1 - x), \ x \in [1/2, 1). \end{cases}$$

See Figure 1 for their images (where $\varepsilon = 0.1$). In this figure, the red set is Q_1 and the blue set is Q_2. One can directly check that $(Q_1, 0, 1)$ satisfies both the MOC and condition (4.4), and $(Q_2, 0, 1)$ satisfies condition (4.4) but not the MOC.

The proof of Theorem 4.1 needs the following proposition whose proof is put in Section 8.3 in the Appendix.

Proposition 4.5. *Let $Q \supset (0, \varepsilon_0) \times \omega$ be measurable, where $\varepsilon_0 > 0$ and $\omega \subset \Omega$ is a nonempty open subset. Let $\alpha \in \mathbb{R}$. Assume that the second inequality in (4.1) holds for the above Q, i.e., there is a constant $C > 0$ such that*

$$\int_0^T \big\|\chi_Q(t,\cdot)y(t,\cdot;y_0)\big\|_{L^2(\Omega)} t^\alpha dt \le C\|y_0\|_{\mathcal{H}^{-4}} \text{ for all } y_0 \in L^2(\Omega). \tag{4.5}$$

Then, it holds that $\alpha > 1$.

Remark 4.6. The condition $\alpha > 1$ in Proposition 4.5 is essentially determined by the heat-like effect in equation (1.1). In fact, generally speaking, we cannot obtain that the pure heat solution $e^{tA}y_0$ $(t > 0)$ is in the space $L^1(0, T; L^2(\Omega))$ from the fact that $y_0 \in \mathcal{H}^{-4}$, unless this solution is multiplied by a weight t^α with $\alpha > 1$. The same can be said about the solution $\Phi(t)y_0$ $(t > 0)$, due to its heat-like nature. This is the essential idea in the proof of Proposition 4.5.

At last, $\alpha = 1$ is critical because $\int_0^T e^{-\lambda t} t^\alpha dt \sim \Gamma(\alpha+1)\lambda^{-\alpha-1}$ as $\lambda \to +\infty$ when $\alpha \ge 0$ (here $\Gamma(\cdot)$ is the Gamma function defined by Euler's integral of second kind), and we have

$$\int_0^T e^{tA} y_0\, t^\alpha dt \ \sim\ \Gamma(\alpha+1)\,(-A)^{-\alpha-1}y_0$$

for any y_0 involving only high frequency spectral components. Then $\alpha = 1$ is crucial when the above function $(-A)^{-\alpha-1}y_0$ is compared to the recovered term $(-A)^{-2}y_0$ in (4.1). A similar idea can be applied to the integral $\int_0^T \|\chi_Q e^{tA} y_0\|_{L^2(\Omega)} t^\alpha dt$, as it is actually done in the proof of Proposition 4.5.

We are now in the position to prove Theorem 4.1.

Proof of Theorem 4.1. The proof is divided into the following two parts.

Part 1. *We prove $(i) \Leftrightarrow (ii)$ when $\alpha > 1$.*

Let $\alpha > 1$. We first prove that $(i) \Rightarrow (ii)$. For this purpose, we suppose that (i) is true, i.e., $(Q, 0, T)$ satisfies the MOC. Then, from (1.6) it follows that

$$\varepsilon_0 := \frac{1}{2}\operatorname*{ess-inf}_{x\in\Omega} \int_0^T \chi_Q(t,x)dt \in (0, T).$$

This implies that for a.e. $x \in \Omega$,

$$\int_{\varepsilon_0}^T \chi_Q(t,x)dt \ge \int_0^T \chi_Q(t,x)dt - \varepsilon_0 \ge \frac{1}{2}\operatorname*{ess-inf}_{x\in\Omega}\int_0^T \chi_Q(t,x)dt > 0,$$

which shows that (Q, ε_0, T) also satisfies the MOC. Then, by Theorem 2.6, we obtain (2.19) with $S = \varepsilon_0$. Meanwhile, it is clear that for each $y_0 \in L^2(\Omega)$,

$$\int_{\varepsilon_0}^T \|\chi_Q(t,\cdot)\Phi(t)y_0\|_{L^2(\Omega)} dt \le \varepsilon_0^{-\alpha} \int_{\varepsilon_0}^T \|\chi_Q(t,\cdot)\Phi(t)y_0\|_{L^2(\Omega)} t^\alpha dt.$$

This, along with (2.19) (where $S = \varepsilon_0$), leads to the first inequality in (4.1). To show the second inequality in (4.1), we apply Corollary 2.2 (with $\beta = 2$ and $s = 0$) to find a $C_1 > 0$ such that for each $y_0 \in L^2(\Omega)$,

$$\begin{aligned}\int_0^T \|\Phi(t)y_0\|_{L^2(\Omega)} t^\alpha dt &\leq \int_0^T \|M(t)A^{-2}y_0\|_{L^2(\Omega)} t^\alpha dt + \int_0^T C_1 t^{-2} \|A^{-2}y_0\|_{L^2(\Omega)} t^\alpha dt \\ &\leq \left(\int_0^T |M(t)| t^\alpha dt + C_1 \int_0^T t^{\alpha-2} dt \right) \|A^{-2}y_0\|_{L^2(\Omega)}.\end{aligned}$$

Since $\alpha > 1$, the second inequality in (4.1) follows from the inequality above. In conclusion, (4.1) is proven, i.e., the statement (ii) is true.

Next, we aim to show that $(ii) \Rightarrow (i)$. To this end, we assume that (ii) is true, i.e., (4.1) holds for some $C > 0$. Set

$$\beta_0 := \min\left\{2 + (\alpha - 1)/2, 3\right\} \in (2, 3] \cap (2, \alpha + 1) \tag{4.6}$$

(here, the fact that $\alpha > 1$ is used). It follows from Corollary 2.2 (with $\beta = \beta_0$ and $s = 0$) that for some $C_2 > 0$,

$$\|\Phi(t)y_0 + M(t)A^{-2}y_0\|_{L^2(\Omega)} \leq C_2 t^{-\beta_0} \|(-A)^{-\beta_0} y_0\|_{L^2(\Omega)},\ y_0 \in L^2(\Omega),\ t \in (0, T]. \tag{4.7}$$

Because $2 < \beta_0 < \alpha + 1$ (see (4.6)), we obtain from (4.1) and (4.7) that for each $y_0 \in L^2(\Omega)$,

$$\begin{aligned}&\|A^{-2}y_0\|_{L^2(\Omega)} = \|y_0\|_{\mathcal{H}^{-4}} \leq C \int_0^T \|\chi_Q(t,\cdot)\Phi(t)y_0\|_{L^2(\Omega)} t^\alpha dt \\ \leq& CT^\alpha \|M\|_{C([0,T])} \int_0^T \|\chi_Q(t,\cdot)A^{-2}y_0\|_{L^2(\Omega)} dt + CC_2 \frac{T^{\alpha+1-\beta_0}}{\alpha + 1 - \beta_0} \|(-A)^{-\beta_0} y_0\|_{L^2(\Omega)}.\end{aligned}$$

Then, by a standard density argument, we obtain that for some $C_3, \beta > 0$,

$$C_3 \|z\|_{L^2(\Omega)} \leq \int_0^T \|\chi_Q(t,\cdot)z\|_{L^2(\Omega)} dt + \|(-A)^{-\beta} z\|_{L^2(\Omega)} \ \text{ for all } \ z \in L^2(\Omega). \tag{4.8}$$

Now, we observe that (4.8) is similar to (3.2). In a similar way as we used (3.2) to prove (3.3), we can deduce from (4.8) that (3.3) holds with $S = 0$ (i.e., $(Q, 0, T)$ satisfies the MOC). Thus, the statement (i) is proven.

In conclusion, when $\alpha > 1$, the equivalence between (i) and (ii) is proven.

Part 2. *We prove that when $\alpha \leq 1$, (i) and (ii) are not equivalent if Q contains a set of the form $(0, \varepsilon) \times B_\varepsilon$, where $B_\varepsilon \subset \Omega$ is a small ball.*

Let $\alpha \leq 1$. First of all, the statement (ii) can not hold. Otherwise, since $Q \supset (0, \varepsilon) \times B_\varepsilon$, by the second inequality in (ii), we can apply Proposition 4.5 to determine that $\alpha > 1$, which leads to a contradiction (since it was assumed that $\alpha \leq 1$).

Next, when $(Q, 0, T)$ satisfies the MOC and Q contains the form $(0, \varepsilon) \times B_\varepsilon$ (the existence of such Q is easily guaranteed), the statement (i) holds but the statement (ii) does not. Thus, (i) and (ii) are not equivalent if Q contains the form $(0, \varepsilon) \times B_\varepsilon$.

In conclusion, we have completed the proof of Theorem 4.1. □

Next, we will prove Theorem 4.2.

Proof of Theorem 4.2. We arbitrarily fix a nonempty measurable subset $Q \subset \mathbb{R}^+ \times \Omega$. We organize the proof in several steps.

Step 1. We show $(i) \Rightarrow (ii)$.

We assume that (i) holds. Then, by Theorem 4.1, we have (4.1) with $\alpha = 2$. This, along with the fact that $\sup\limits_{0<t<T} t^\alpha = T^\alpha$, yields (ii) of Theorem 4.2.

Step 2. We show $(ii) \Rightarrow (iii)$.

According to Corollary 2.2 (with $\beta = 2$, $s = 0$, and $t = T$), there is a $C_1 > 0$ such that

$$\|\Phi(T)y_0\|_{L^2(\Omega)} \le C_1\|A^{-2}y_0\|_{L^2(\Omega)} = C_1\|y_0\|_{\mathcal{H}^{-4}}.$$

This, along with (ii) of this theorem, implies (iii) of this theorem.

Step 3. We show $(iii) \Rightarrow (iv)$.

Let (iii) of this theorem hold. By contradiction, we suppose that (iv) is not true. Then, there is an $r_1 > 0$ such that

$$\inf_{x_1\in\Omega} \fint_{B(x_1,r_1)} \left(\int_0^T \chi_Q(t,x)dt \right) dx = 0. \tag{4.9}$$

We define the following function over $\overline{\Omega}$ via

$$F(x_1) := \fint_{B(x_1,r_1)} \left(\int_0^T \chi_Q(t,x)dt \right) dx, \quad x_1 \in \overline{\Omega}.$$

We can directly check that F is a non-negative and continuous function over $\overline{\Omega}$. Then, by (4.9), there is a minimizer $\hat{x}_1 \in \overline{\Omega}$ such that $F(\hat{x}_1) = 0$. From this, we can easily show that there is an $x_0 \in \Omega$ and an $r > 0$ such that

$$\chi_Q(t,x) = 0 \ \text{ for a.e. } (t,x) \in (0,T) \times B(x_0,r). \tag{4.10}$$

At the same time, we can apply (iii) (in this theorem) to find some $C > 0$ such that

$$\|\Phi(T)y_0\|_{L^2(\Omega)} \le C\|\chi_Q\Phi(\cdot)y_0\|_{L^\infty(0,T;L^2(\Omega))} \ \text{ for each } \ y_0 \in L^2(\Omega).$$

This, together with (4.10), yields

$$\|\Phi(T)z\|_{L^2(\Omega)} \le C\|\Phi(\cdot)z\|_{L^\infty(0,T;L^2(\Omega\setminus B(x_0,r)))} \ \text{ for each } \ z \in L^2(\Omega). \tag{4.11}$$

Now, we will present a contradiction to (4.11) by choosing a suitable sequence $\{z_k\}_{k\ge1}$ in $L^2(\Omega)$. According to Theorem 8.1 (in the Appendix), $\{h_l(T)\}_{l\ge1}$ (given by (8.1)) is not the zero sequence. Thus, we have

$$J := \min\left\{ l \in \mathbb{N}^+ \ : \ h_l(T) \neq 0 \right\} < +\infty. \tag{4.12}$$

We select $\{\varepsilon_k\}_{k\geq 1} \subset (0, r/2)$ and $\rho \in C_0^\infty(\mathbb{R}^n)$ such that

$$\lim_{k\to\infty} \varepsilon_k = 0; \quad \|\rho\|_{L^2(\mathbb{R}^n)} = 1; \quad \rho(x) = 0, \ |x|_{\mathbb{R}^n} \geq 1. \tag{4.13}$$

We define $\{w_k\}_{k\geq 1} \subset C_0^\infty(\Omega)$ by

$$w_k(x) := \varepsilon_k^{-n/2} \rho\Big(\frac{x - x_0}{\varepsilon_k}\Big), \ x \in \Omega.$$

From this and (4.13), we can directly check that

$$\operatorname{supp} w_k \subset B(x_0, r/2), \ \forall\, k \in \mathbb{N}^+; \ \|w_k\|_{L^2(\Omega)} = 1, \ \forall\, k \in \mathbb{N}^+; \ \text{w-}\lim_{k\to\infty} w_k = 0 \text{ in } L^2(\Omega). \tag{4.14}$$

Now we define the sequence $\{z_k\}_{k\geq 1} \subset L^2(\Omega)$ by

$$z_k := A^{J+1} w_k, \ \ k \geq 1. \tag{4.15}$$

From (4.15) and (4.14), as well as (1.3), we see that for each $l \in \{0, 1, \ldots, J+1\}$,

$$\operatorname{supp}(A^{-l} z_k) \subset B(x_0, r/2), \ \ \forall\, k \in \mathbb{N}^+; \quad \lim_{k\to\infty} z_k = 0 \ \text{ in } \ \mathcal{H}^{-2J-4}. \tag{4.16}$$

Meanwhile, by Theorem 8.1 (in the Appendix) with $N = J + 1$, we have

$$\begin{aligned} \varPhi(t) =& e^{tA}\Big(Id + \sum_{l=0}^{J} p_l(t)(-A)^{-l-1}\Big) + \sum_{l=1}^{J} h_l(t)(-A)^{-l-1} + R_{J+1}(t, -A)(-A)^{-J-2} \\ :=& \mathcal{P}(t) + \mathcal{W}(t) + \widetilde{\mathcal{R}}(t), \ \ t > 0, \end{aligned} \tag{4.17}$$

where $R_{J+1}(\cdot, -A) \in C(\mathbb{R}^+; \mathcal{L}(\mathcal{H}^0)) \bigcap L^\infty_{loc}(\overline{\mathbb{R}^+}; \mathcal{L}(\mathcal{H}^0))$ is given in Theorem 8.1 with $N = J + 1$.

With regard to three terms on the right hand side of (4.17), we have the following: First, from the second conclusion in (4.16), as well as the regularity of $R_{J+1}(\cdot, -A)$, we find that

$$\lim_{k\to\infty} \sup_{0<t\leq T} \|\widetilde{\mathcal{R}}(t) z_k\|_{L^2(\Omega)} = 0. \tag{4.18}$$

Second, by the smoothing effect of $\{e^{tA}\}_{t\geq 0}$, we determine that

$$\lim_{k\to\infty} \mathcal{P}(T) z_k = 0 \ \text{ in } \ L^2(\Omega). \tag{4.19}$$

Third, by (4.12), we have

$$\mathcal{W}(T) = h_J(T)(-A)^{-J-1},$$

which, along with (4.15) and the second equality in (4.14), yields

$$\|\mathcal{W}(T) z_k\|_{L^2(\Omega)} = |h_J(T)| \neq 0, \ \ \forall\, k \in \mathbb{N}^+. \tag{4.20}$$

Now from (4.17), (4.18), (4.19), and (4.20), it follows that

$$\lim_{k\to\infty} \|\varPhi(T) z_k\|_{L^2(\Omega)} = |h_J(T)| \neq 0. \tag{4.21}$$

Finally, by (4.16) and the iterative use of Lemma 8.4 (with $z = A^{-l}z_k$, $l \in \{0, \cdots, J+1\}$), we find that

$$\lim_{k\to\infty} \sup_{0<t\leq T} \|\mathcal{P}(t)z_k\|_{L^2(\Omega\setminus B(x_0,r))} = 0. \tag{4.22}$$

Meanwhile, from the first conclusion in (4.16) and the definition of $\mathcal{W}(t)$ (see (4.17)), we see that for each $k \in \mathbb{N}^+$,

$$\mathcal{W}(t)z_k = 0 \ \text{ over } \ \Omega \setminus B(x_0, r), \ \ 0 \leq t \leq T. \tag{4.23}$$

From (4.17), (4.22), (4.23), and (4.18), we obtain

$$\lim_{k\to\infty} \sup_{0<t\leq T} \|\Phi(t)z_k\|_{L^2(\Omega\setminus B(x_0,r))} = 0. \tag{4.24}$$

Now, the combination of (4.24) and (4.21) contradicts (4.11). Thus, (iv) is true.

Finally, the proof of Theorem 4.2 is completed. □

5 Applications to control problems

In this section, we denote by $Q \subset \mathbb{R}^+ \times \Omega$ a nonempty measurable subset of positive measure that will play the role of support of the control, and let $p \in [1, +\infty]$. We consider the following controlled heat equation with memory:

$$\begin{cases} \partial_t y(t,x) - \Delta y(t,x) + \displaystyle\int_0^t M(t-s)y(s,x)ds = \chi_Q(t,x)u(t,x), & (t,x) \in \mathbb{R}^+ \times \Omega, \\ y(t,x) = 0, & (t,x) \in \mathbb{R}^+ \times \partial\Omega, \\ y(0,x) = y_0(x), & x \in \Omega, \end{cases} \tag{5.1}$$

where $y_0 \in L^2(\Omega)$ and $u \in L^p(\mathbb{R}^+; L^2(\Omega))$. We treat the solution of the control system (5.1) as a function from $[0, +\infty)$ to $L^2(\Omega)$ and denote it by $y(\cdot; y_0, u)$.

Inspired by the classical null controllability property of semigroups, it is natural to address the following controllability problem: Given a $T > 0$, for each $y_0 \in L^2(\Omega)$, show the existence of a control $u \in L^2(0, +\infty; L^2(\Omega))$, with $u = 0$ over $(T, +\infty)$, such that

$$y(t; y_0, u) = 0 \ \text{ for each } t \geq T. \tag{5.2}$$

We refer to (5.2) as the memory-type controllability.

We were not able to solve this problem so far, and we turn our attention to the weaker goal of controlling the state at time $t = T$, i.e.,

$$y(T; y_0, u) = 0, \tag{5.3}$$

instead of the whole trajectory for $t \geq T$ (i.e., $y(\cdot; y_0, u)|_{[T,+\infty)}$).

This, i.e., (5.3), constitutes a partial controllability problem. Indeed, even if $y(T; y_0, u) \equiv 0$, due to the memory effects of the system, (5.2) will not be guaranteed. This is the main difference and added difficulty of the control of heat-like equations involving memory terms.

In the sequel we limit the discussion to the partial controllability (5.3) of $y(T;y_0,u)$. The analysis of the full control of $y(\cdot;y_0,u)|_{[T,+\infty)}$ constitutes an interesting open problem. Note that the methods in [9], imposing stronger geometric conditions on the control sets and restricting the analysis to specific kernels, like polynomial ones, in particular, allow to ensure the full control of the system. Whether these results can be extended to general analytic kernels under the sharp MOC of this paper is an interesting open problem. We will further discuss this issue in the next section.

In what follows, we present several applications of Theorem 1.2 (as well as Theorems 4.1 and 4.2) to the control system (5.1).

5.1 Main results

To state the first theorem, we introduce the following definitions:

(D3) Given $T>0$ and $y_0\in L^2(\Omega)$, let

$$\mathcal{R}_M^p(T,y_0):=\left\{y(T;y_0,u)\ :\ u\in L^p(\mathbb{R}^+;L^2(\Omega))\right\} \tag{5.4}$$

be the reachable set for the control system (5.1) at time T.

(D4) Given $T>0$ and $y_0\in L^2(\Omega)$, let

$$\mathcal{R}_0^p(T,y_0):=\left\{z(T;y_0,u)\ :\ u\in L^p(\mathbb{R}^+;L^2(\Omega))\right\}$$

be the reachable set for the pure heat equation at time T, where $z(\cdot;y_0,u)$ is the solution of the system (5.1) with $M=0$.

The first theorem refers to the reachable set of the control system (5.1). It shows that, under the MOC, the reachable set of the control system (5.1) is the sum of the space $\mathcal{H}^4$ and the reachable set for the pure heat equation.

Theorem 5.1. *Let $T>0$, $p\in[1,+\infty]$, and $y_0\in L^2(\Omega)$. We assume that $(Q,0,T)$ satisfies the MOC. Then,*

$$\mathcal{H}^4\subset\mathcal{R}_M^p(T,y_0)=\mathcal{R}_0^p(T,y_0)+\mathcal{H}^4. \tag{5.5}$$

Moreover, for each $\alpha>1$,

$$\mathcal{H}^4=\mathcal{R}_M^\infty(T,y_0,\alpha):=\Big\{y(T;y_0,u)\in L^2(\Omega)\ :\ \underset{0<t<T}{\text{ess-sup}}\,\|(T-t)^{-\alpha}u(t)\|_{L^2(\Omega)}<+\infty\Big\}. \tag{5.6}$$

In particular, for every $y_0\in L^2(\Omega)$ and target $y_1\in\mathcal{H}^4$, there is a control $u\in L^\infty(\mathbb{R}^+;L^2(\Omega))$ (with $\operatorname{supp}u\subset[0,T]\times\Omega$) such that $y(T;y_0,u)=y_1$.

Furthermore, the MOC satisfied by $(Q,0,T)$ is sharp to ensure the structure (5.5). The following theorem, provides a necessary geometric condition on the control region for the partial controllability of the control system (5.1) to hold. The gap between the MOC satisfied by $(Q,0,T)$ and this necessary condition is thin, as discussed in Remark 4.3.

Theorem 5.2. *Let $T > 0$ and $p \in [1, +\infty]$. We assume that the system (5.1) is partially controllable over $[0, T]$ with L^p controls (i.e., for each $y_0 \in L^2(\Omega)$, there is a $u \in L^p(\mathbb{R}^+; L^2(\Omega))$ such that $y(T; y_0, u) = 0$). Then, condition (4.4) holds.*

Remark 5.3. (i) Theorem 5.1 presents the exact difference between the reachable set of the control system (5.1) with and without memory, and clarifies the role of the space $\mathcal{H}^4$.

(ii) As mentioned above, Theorem 5.1 ensures only the partial controllability of the system (5.1). In that sense the result is weaker than the ones in [9] about the memory-type null controllability of the system where the memory term is also controlled.

But in Theorem 5.1 the control region Q is measurable, while in most related works, it is required to be open. We refer to [2] for the null controllability of the heat equation with measurable control regions.

(iii) Theorem 5.2 shows that the MOC is nearly sharp for the partial controllability to hold.

5.2 Proofs of main results

We start with the following technical lemma.

Lemma 5.4. *Assume that $(Q, 0, T)$ satisfies the MOC and let $y_0 \in L^2(\Omega)$. Then*

$$\mathcal{R}_M^\infty(T, y_0, \alpha) = \mathcal{H}^4 \subset \mathcal{R}_M^\infty(T, y_0) \quad \text{for all } \alpha > 1, \tag{5.7}$$

where $\mathcal{R}_M^\infty(T, y_0, \alpha)$ is as in (5.6).

Proof. We fix $\alpha > 1$. Since $\Phi(t)y_0 = y(t, \cdot; y_0)$ ($t \geq 0$), it follows from the definition of $\mathcal{R}_M^\infty(T, y_0, \alpha)$ (given in (5.6)) and (5.4) that

$$\mathcal{R}_M^\infty(T, 0, \alpha) + \Phi(T)y_0 = \mathcal{R}_M^\infty(T, y_0, \alpha) \subset \mathcal{R}_M^\infty(T, y_0) = \mathcal{R}_M^\infty(T, 0) + \Phi(T)y_0.$$

Accordingly, since $\Phi(T)y_0 \in \mathcal{H}^4$ (see Corollary 2.2), it suffices to show the equality in (5.7) with $y_0 = 0$.

For this purpose, we write

$$\begin{aligned} L_\alpha^1(0, T; L^2(\Omega)) &:= \Big\{ f : (0, T) \to L^2(\Omega) \ \Big| \ \int_0^T \|f(t)\|_{L^2(\Omega)} (T - t)^\alpha dt < +\infty \Big\}, \\ L_{-\alpha}^\infty(0, T; L^2(\Omega)) &:= \Big\{ g : (0, T) \to L^2(\Omega) \ \Big| \ \operatorname*{ess-sup}_{0<t<T} \|(T - t)^{-\alpha} g(t)\|_{L^2(\Omega)} < +\infty \Big\}. \end{aligned} \tag{5.8}$$

We define three Banach spaces as follows: $X := \mathcal{H}^{-4}$, $Y := L_\alpha^1(0, T; L^2(\Omega))$, $Z := L^2(\Omega)$. We can directly check that

$$X^* = \mathcal{H}^4, \ Y^* = L_{-\alpha}^\infty(0, T; L^2(\Omega)), \ Z^* = L^2(\Omega). \tag{5.9}$$

Then, we define two operators $\mathcal{R} : Z \to X$ and $\mathcal{O} : Z \to Y$ in the following manner:

$$\mathcal{R}z := z, \ z \in Z \ \text{ and } \ \mathcal{O}z := \chi_Q \Phi(T - \cdot)z, \ z \in Z. \tag{5.10}$$

By Proposition 8.5 (in the Appendix) and the self-adjointness of $\Phi(\cdot)$ (see (8.8) in the Appendix), we can directly check that

$$\mathcal{R}^* z = z,\ z \in X^* \ \text{ and } \ \mathcal{O}^* u := y(T; 0, u),\ u \in Y^*. \tag{5.11}$$

Meanwhile, since $(Q, 0, T)$ satisfies the MOC, it follows by (1.6) that $(\widehat{Q}, 0, T)$ also satisfies the MOC, where

$$\widehat{Q} := \big\{(t, x) \in (0, T) \times \Omega \ :\ (T - t, x) \in Q\big\}.$$

Thus, we can use Theorem 4.1 to see that (4.1) holds with Q replaced by $\widehat{Q}$. This, along with (5.10), yields that for some $C > 0$,

$$\frac{1}{C}\|\mathcal{R}z\|_X \le \|\mathcal{O}z\|_Y \le C\|\mathcal{R}z\|_X,\ \ z \in Z. \tag{5.12}$$

We now claim that

$$\text{Range}\,\mathcal{R}^* = \text{Range}\,\mathcal{O}^*. \tag{5.13}$$

Indeed, from the first inequality in (5.12), we apply Corollary 8.7 to see that for each $z \in X^*$, there is a $u \in Y^*$ such that $\mathcal{R}^* z = \mathcal{O}^* u$, which yields that Range $\mathcal{R}^* \subset$ Range $\mathcal{O}^*$. Similarly, from the second inequality in (5.12) and Corollary 8.7, we can see that Range $\mathcal{R}^* \supset$ Range $\mathcal{O}^*$. Hence, (5.13) is true.

Finally, from (5.11), (5.13), (5.8), and (5.9), we see that

$$\mathcal{H}^4 = \Big\{y(T; 0, u) \in L^2(\Omega) \ :\ u \in Y^* = L^\infty_{-\alpha}(0, T; L^2(\Omega))\Big\}.$$

This, along with the definition of $\mathcal{R}^\infty_M(T, y_0, \alpha)$ (given in (5.6)), shows that the equality in (5.7) (with $y_0 = 0$) is true. This finishes the proof of Lemma 5.4. □

We are now in a position to prove Theorem 5.1.

Proof of Theorem 5.1. First, we recall that $y(\cdot; y_0, u)$ denotes the solution to (5.1), while $z(\cdot; y_0, u)$ denotes the solution to (5.1) where $M = 0$. We arbitrarily fix the initial datum $y_0 \in L^2(\Omega)$. We notice that (5.6) directly follows from Lemma 5.4. It remains to show (5.5), i.e.,

$$\mathcal{H}^4 \subset \mathcal{R}^p_M(T, y_0) = \mathcal{R}^p_0(T, y_0) + \mathcal{H}^4. \tag{5.14}$$

The proof of (5.14) is organized in several steps.

Step 1. We show that for each $u \in L^1_{loc}(\overline{\mathbb{R}^+}; L^2(\Omega))$,

$$f_u(\cdot) \in C(\mathbb{R}^+; \mathcal{H}^4), \tag{5.15}$$

where

$$f_u(t) := y(t; y_0, u) - z(t; y_0, u),\ \ t > 0. \tag{5.16}$$

To this end, we arbitrarily fix a $u \in L^1_{loc}(\overline{\mathbb{R}^+}; L^2(\Omega))$. Then, by (5.16), Proposition 8.5 (in the Appendix), and Corollary 2.3, we find that

$$f_u(t) = \widetilde{\mathcal{R}}_c(t, -A)A^{-2}y_0 + \int_0^t \widetilde{\mathcal{R}}_c(t-s, -A)A^{-2}(\chi_Q u)(s)ds, \quad t > 0. \tag{5.17}$$

Meanwhile, it follows by Corollary 2.3 that $\widetilde{\mathcal{R}_c} \in C(\mathbb{R}^+; \mathcal{L}(\mathcal{H}^0)) \cap L^\infty_{loc}(\overline{\mathbb{R}^+}; \mathcal{L}(\mathcal{H}^0))$. This, along with (5.17), yields that when $t_2 \geq t_1 > 0$,

$$\begin{aligned} \|f_u(t_1) - f_u(t_2)\|_{\mathcal{H}^4} \leq & \|\widetilde{\mathcal{R}}_c(t_1, -A) - \widetilde{\mathcal{R}}_c(t_2, -A)\|_{\mathcal{L}(\mathcal{H}^0)}\|y_0\|_{\mathcal{H}^0} + \|\widetilde{\mathcal{R}}_c(\cdot, -A)\|_{L^\infty(0,t_2;\mathcal{L}(\mathcal{H}^0))} \\ & \times \Big(\int_0^{t_1} \|u(t_1 - s) - u(t_2 - s)\|_{\mathcal{H}^0} ds + \int_{t_1}^{t_2} \|u(t_2 - s)\|_{\mathcal{H}^0} ds \Big), \end{aligned}$$

which leads to (5.15).

Step 2. We show that

$$\mathcal{R}^p_M(T, y_0) \subset \mathcal{R}^p_0(T, y_0) + \mathcal{H}^4. \tag{5.18}$$

We arbitrarily fix a $y_1 \in \mathcal{R}^p_M(T, y_0)$. By (5.4), there is a $u_1 \in L^p(\mathbb{R}^+; L^2(\Omega))$ such that

$$y_1 = y(T; y_0, u_1) = z(T; y_0, u_1) + \Big(y(T; y_0, u_1) - z(T; y_0, u_1) \Big).$$

Since $z(T; y_0, u_1) \in \mathcal{R}^p_0(T, y_0)$, the above, along with (5.15), leads to (5.18).

Step 3. We show that

$$\mathcal{R}^p_M(T, y_0) \supset \mathcal{R}^p_0(T, y_0) + \mathcal{H}^4. \tag{5.19}$$

We arbitrarily fix two functions $\hat{y}_1 \in \mathcal{R}^p_0(T, y_0)$ and $\hat{y}_2 \in \mathcal{H}^4$. According to the definition of $\mathcal{R}^p_0(T, y_0)$ (see (5.4) with $M = 0$), there is a $\hat{u}_1 \in L^p(\mathbb{R}^+; L^2(\Omega))$ such that

$$\hat{y}_1 = z(T; y_0, \hat{u}_1). \tag{5.20}$$

Since $\hat{y}_2 \in \mathcal{H}^4$, we see from (5.15) that

$$\hat{y}_3 := \hat{y}_2 - \Big(y(T; y_0, \hat{u}_1) - z(T; y_0, \hat{u}_1) \Big) \in \mathcal{H}^4. \tag{5.21}$$

Since $\hat{y}_3 \in \mathcal{H}^4$, we can apply Lemma 5.4 (where $y_0 = 0$) to find a control $\hat{u}_2 \in L^\infty(\mathbb{R}^+; L^2(\Omega))$, with $\hat{u}_2|_{(T,+\infty)} = 0$, such that $\hat{y}_3 = y(T; 0, \hat{u}_2)$. This, together with (5.21) and (5.20), yields

$$y(T; y_0, \hat{u}_1 + \hat{u}_2) = y(T; y_0, \hat{u}_1) + \hat{y}_3 = \hat{y}_2 + z(T; y_0, \hat{u}_1) = \hat{y}_2 + \hat{y}_1.$$

Since $\hat{y}_1$ and $\hat{y}_2$ were arbitrarily taken from $\mathcal{R}^p_0(T, y_0)$ and $\mathcal{H}^4$, respectively, the above leads to (5.19).

Step 4. We check (5.14).

Because $(Q, 0, T)$ satisfies the MOC, by Lemma 5.4, it follows that $\mathcal{H}^4 \subset \mathcal{R}^\infty_M(T, y_0)$. At the same time, $\mathcal{R}^\infty_M(T, y_0) \subset \mathcal{R}^p_M(T, y_0)$ by their definitions. Therefore, $\mathcal{H}^4 \subset \mathcal{R}^p_M(T, y_0)$. This, along with (5.18) and (5.19), yields (5.14).

Hence, we have completed the proof of Theorem 5.1. □

We end this section by proving Theorem 5.2.

Proof of Theorem 5.2. Assume that the control system (5.1) is L^p-partially controllable over $[0,T]$. By contradiction, we suppose that (4.4) is not true. Then, there is an $r_1>0$ such that

$$\inf_{x_1\in\Omega} \fint_{B(x_1,r_1)} \Big(\int_0^T \chi_Q(t,x)dt\Big)dx = 0.$$

Using the same arguments used for (4.9) and (4.10), we can find $x_0\in\Omega$ and $r>0$ such that

$$\chi_Q(t,x)=0 \text{ for a.e. } (t,x)\in(0,T)\times B(x_0,r). \tag{5.22}$$

We now claim that there is a constant $C>0$ such that for each $y_0\in L^2(\Omega)$, there exists $u_{y_0}\in L^1(0,T;L^2(\Omega))$ satisfying

$$y(T;y_0,\tilde{u}_{y_0})=0 \text{ and } \|u_{y_0}\|_{L^1(0,T;L^2(\Omega))}\le C\|y_0\|_{L^2(\Omega))} \tag{5.23}$$

(here and in what follows, given a control v over $[0,T]$, we use $\tilde{v}$ to denote its zero extension over $\mathbb{R}^+$).

To show (5.23), we define the operator

$$L_T(u):=y(T;0,\tilde{u}),\ \ u\in L^p(0,T;L^2(\Omega)).$$

Then, by the assumption of the L^p-partial controllability, we have Range $\Phi(T)\subset$ Range L_T. Without loss of generality, we can assume that L_T is injective; otherwise, we can replace L_T by the operator $\widetilde{L}_T$ (from the quotient space $L^p(0,T;L^2(\Omega))/\ker L_T$ to $L^2(\Omega)$), which is uniquely induced by L_T. Then, for each $y_0\in L^2(\Omega)$, there is a unique u_{y_0} such that

$$\Phi(T)y_0=L_Tu_{y_0},\ \text{i.e.,}\ y(T;y_0,-\tilde{u}_{y_0})=0.$$

According to the closed graph theorem, we can directly check that the map $y_0\mapsto u_{y_0}$ is continuous from $L^2(\Omega)$ to the space $L^p(0,T;L^2(\Omega))$. This yields (5.23), since $p\ge 1$.

We next claim that there is a $C>0$ such that

$$\|\Phi(T)z\|_{L^2(\Omega)}\le C\|\Phi(\cdot)z\|_{L^\infty(0,T;L^2(\Omega\setminus B(x_0,r)))} \text{ for each } z\in L^2(\Omega). \tag{5.24}$$

Indeed, by (5.23), using the classical duality argument (see for instance [39, Theorem 1.18]), we can obtain the following observability inequality: there is a constant $C>0$ such that

$$\|\Phi(T)z\|_{L^2(\Omega)}\le C\|\chi_Q\Phi(T-\cdot)z\|_{L^\infty(0,T;L^2(\Omega))} \text{ for each } z\in L^2(\Omega). \tag{5.25}$$

Now (5.24) follows from (5.25) and (5.22).

Finally, we notice that (5.24) is the same as (4.11). Thus, we can use the same arguments as those after (4.11) (in the proof of Theorem 4.2) to get to a contradiction. Hence, the conclusion in Theorem 5.2 is true. This completes the proof of Theorem 5.2. □

6 Numerical experiments

The hybrid parabolic-hyperbolic effect of equation (1.1) was shown in [40], and plays a key role in the study of this paper. Here we present some numerical experiments in one space dimension confirming this hybrid behaviour.

Let $\Omega = (0,1)$ and consider the following equation with the constant memory kernel:

$$y'(t) - Ay(t) + \int_0^t y(s)ds = 0,\ t > 0;\ y(0) = y_0. \tag{6.1}$$

Recall that $\{\eta_j\}_{j\geq 1}$ and $\{e_j\}_{j\geq 1}$ are the eigenvalues and the corresponding eigenvectors (normalized in $L^2(\Omega)$) of the operator $-A$, respectively. By the spectral method, we have that for each $t \geq 0$ and $x \in (0,1)$,

$$y(t,x;\delta_{0.3}) = \sum_{j\geq 1} \frac{1}{2}\Big[\Big(1 + \frac{\eta_j}{\sqrt{\eta_j^2-4}}\Big)e^{\left(-\eta_j - \sqrt{\eta_j^2-4}\right)\frac{t}{2}} + \Big(1 - \frac{\eta_j}{\sqrt{\eta_j^2-4}}\Big)e^{\left(-\eta_j + \sqrt{\eta_j^2-4}\right)\frac{t}{2}}\Big] e_j(0.3)e_j(x).$$

We discretize the interval $\Omega = (0,1)$ with the mesh size $h = 10^{-3}$, keep the first 10^3 frequency components in the last expression, and then draw the solution $y(\cdot,\cdot;\delta_{0.3})$ and its 4-th space derivative in black in Figure 2.

In this figure, the red curves (in both rows) represent the solution of the pure heat equation (with the same initial datum) and its 4-th space derivative, respectively. The blue curves (marked as "leading wave") in the first row represent the first nontrivial term $-M(t)A^{-2}y_0$ in the wave-like component $\mathcal{W}_N$ of the decomposition (8.3). The black curves represent the complete solution of equation (6.1) and its fourth-order derivative.

In these figures we confirm that: (i) at both two time instants, the 4-th space derivative of $y(\cdot,\cdot;\delta_{0.3})$ reproduces the singularity of the initial datum $\delta_{0.3}$. This confirms the propagation of singularities along time of the solutions of (1.1); (ii) the solution $y(\cdot,\cdot;\delta_{0.3})$ evolves gradually from the pure heat solution to the vicinity of the first nonzero term $-M(t)A^{-2}y_0$ in the wave-like component $\mathcal{W}_N$ in the decomposition (8.3).

Next, we focus on the simplified equality (1.8): when $T > S > 0$,

$$y(t,\cdot;y_0) = -M(t)A^{-2}y_0 + \text{"smooth rest terms"},\quad t \in [S,T], \tag{6.2}$$

which is used to highlight the wave-like effect in the decomposition (8.3) for equation (1.1).

Let us first motivate the decomposition (6.2). Equation (6.1) for the constant memory kernel, by the spectral method, leads to the following one, depending on the parameter $\eta > 0$:

$$x'(t) + \eta x(t) + \int_0^t x(s)ds = 0,\ t > 0;\ x(0) = 1. \tag{6.3}$$

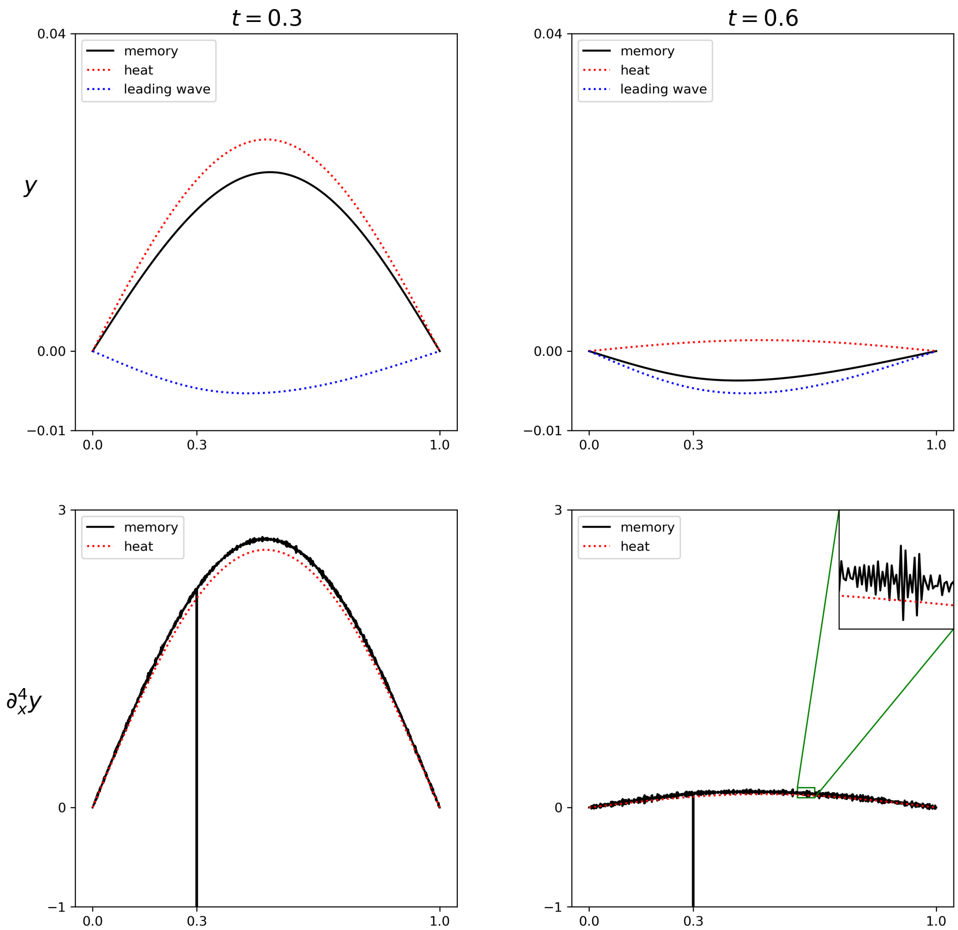

Figure 2: The curves in the first and second columns correspond to the functions (drawn in different colors) at the time instants $t = 0.3$ and $t = 0.6$, respectively. The curves in the first and second rows correspond to the functions (drawn in different colors) and their 4-th space derivatives, respectively.

The solution takes the form

$$x(t) = \big(1 + o(1)\big) \operatorname{exp}\big[-\big(1 + o(1)\big)\eta t\big] - \big(1 + o(1)\big)\eta^{-2} \operatorname{exp}\big[-\big(1 + o(1)\big)\eta^{-1}t\big],$$

where each $o(1)$ stands for infinitesimal perturbations as $\eta \to +\infty$. In particular, when $T > S > 0$,

$$x(t) = -\eta^{-2} + o(\eta^{-2}),\ S \le t \le T,$$

which is then transformed to (6.2) for $M \equiv 1$ by the spectral calculus (that allows to replace the parameter η by the operator $-A$).

The same analysis can be extended to more general memory kernels,

$$x'(t) + \eta x(t) + \int_0^t M(t-s)x(s)ds = 0,\ t > 0;\ x(0) = 1. \tag{6.4}$$

Let $T > S > 0$. We adopt the following ansatz with coefficients $\{a_k(\cdot)\}_{k=0}^2$ and $\{b_k(\cdot)\}_{k=0}^2$ to be determined:

$$x(t) = e^{-\eta t}\Big(\sum_{k=0}^{2} a_k(t)\eta^{-k} + o(\eta^{-2})\Big) + \sum_{k=0}^{2} b_k(t)\eta^{-k} + o(\eta^{-2}),\ t \in [0, T],$$

and take it into (6.4) to get that $b_0 = b_1 = 0$ and $b_2 = -M$. Then, we obtain

$$x(t) = -M(t)\eta^{-2} + o(\eta^{-2}),\ t \in [S, T] \subset (0, T],$$

to justify (6.2). Theorem 8.1 is a consequence of this kind of analysis.

Another way to justify (6.2) can be briefly mentioned without extensive details as follows: by taking the time derivative of the original equation (1.1), we obtain the related equation

$$\partial_{tt} y(t) - A\partial_t y(t) = -M(t)y(0) - \int_0^t M(s)\partial_t y(t-s)ds,\ t > 0.$$

Next, we establish estimates for $\partial_t y$, and subsequently for y, by referring back to the original equation (1.1).

7 Open problems

A number of interesting issues could be considered in connection with the results and methods developed in this paper. Here, we briefly present some of them.

- *Memory-type controllability.* We have analysed the partial controllability (5.3). The controllability problem (5.2) under the MOC for general analytic kernels, as considered here, is open.

  Note that the methods in [9], imposing stronger geometric conditions on the control sets and restricting the analysis to specific kernels, like polynomial ones, in particular, allow to ensure the full control of the system (5.1). Let us explain why those methods are insufficient to handle the more general setting in this paper.

  When the memory kernel is polynomial, $M(t) = \sum_{j=0}^{m} a_k t^k$, a key idea in [9] is to rewrite (5.1) as a hybrid control system of PDEs and ODEs which takes the form

$$\begin{aligned} &\partial_t y - \Delta y + z_0 = \chi_Q u, \\ &\partial_t z_k = z_{k+1} + M^{(k)}(0)y,\ k = 0, \ldots, m-1, \\ &\partial_t z_m = M^{(m)}(0)y, \end{aligned}$$

by introducing the following extra state variables $\{z_k\}_{k=0}^m$:

$$z_k(t,x) := \int_0^t M^{(k)}(t-s)y(s,x)ds,\ t \geq 0,\ x \in \Omega,\ k = 0, \dots, m.$$

In [9] the control u is designed so that the ensemble of the state variables $y, z_0, \dots, z_m$ are controlled at the final time T, which guarantees the memory-type controllability of the system (5.1). The null controllability of the above augmented system is proven by duality, as a consequence of an observability inequality from a moving observation set, employing Carleman inequalities. This requires however some stronger MOC conditions.

The methods in [9] seem insufficient to deal both with the sharp MOC condition in this paper and general analytic kernels. Note in particular that, in the case of general analytic memory kernels, the above strategy of adding auxiliary state variables leads to the coupling of a heat equation with an infinite number of ODEs, which makes it hard to implement methods inspired by Carleman inequalities. More precisely, the controllability problem (5.2) is open in the following three cases:

(i) under the stronger moving geometric conditions as in [9] for general analytic memory kernels;

(ii) under the sharp MOC condition of the present paper for polynomial memory kernels;

(iii) under the sharp MOC condition of the present paper for general analytic memory kernels.

- *Smooth memory kernels.* It would be interesting to investigate whether Theorem 1.2 holds when $M \in C^\infty([0, +\infty))$. Extending the method we developed for the real analytic memory kernels to this C^∞-case, we need the following two results: first, an analog of the decomposition in Theorem 8.1; second, a unique continuation theorem similar to Lemma 2.9 (there we used the analyticity in the time variable of the solutions to equation (1.1), presented in Proposition 2.4). The first result holds for smooth memory kernels (see [40, Theorem 4.10] for the details). However, without the analyticity property of the memory kernel, solutions are unlikely to have the time-analyticity, and then the unique continuation property is a challenging open problem.

- *Space-dependent memory kernels.* The extension of the results of this paper to the space-dependent memory kernels $M = M(t,x)$ is also open.

  In particular, we do not know how to reveal the hybrid parabolic-hyperbolic property, with a decomposition similar to Theorem 8.1. The unique continuation property requires further analysis as well.

- *Memory kernels in the principal part of the model.* It would be interesting to extend Theorem 1.2 to the following two types of heat equations with memory kernels:

  (i) $\partial_t y - \Delta y - \int_0^t M(t-s)\Delta y(s)ds = 0$;

  (ii) $\partial_t y - \int_0^t M(t-s)\Delta y(s)ds = 0$.

These models are more relevant than (1.1) from an applied perspective. The memory kernel enters within the principal part of equations, and this may bring new phenomena. It would be interesting to establish a similar decomposition as in Theorem 8.1 to reveal the possible hybrid parabolic-hyperbolic character of these models and its control theoretical consequences.

8 Appendix

This section reviews the decomposition theorem for the flow $\Phi(t)$ and some other properties established in [40].

Next, an estimate for the heat equation, as well as the proofs of several results developed before, is provided. Then, the variation of the constant formula for the control system (5.1) is presented. Finally, an abstract framework for observability/controllability (in [38]) is introduced.

8.1 Review of decomposition of flow

We start by recalling the definition of the following functions from [40]. First, for each $l \in \mathbb{N}$, we let

$$\begin{cases} h_l(t) := & (-1)^l \sum_{j=0}^{l} C_l^{l-j} \frac{d^{(l-j)}}{dt^{(l-j)}} \underbrace{M * \cdots * M}_{j}(t), \ t \geq 0, \\ p_l(t) := & -h_l(0) + (-1)^{l+1} \sum_{\substack{m,\, j \in \mathbb{N}^+, \\ 2j-l-1 \leq m \leq j}} \left(C_l^{l-j+m} \frac{d^{(l-j+m)}}{dt^{(l-j+m)}} \underbrace{M * \cdots * M}_{j}(0) \right) \frac{(-t)^m}{m!}, \ t \geq 0, \end{cases} \tag{8.1}$$

where $C_\beta^m := \frac{\beta!}{m!(\beta-m)!}$ and $\underbrace{M * \cdots * M}_{j} := 0$ if $j = 0$. Second, we let

$$K_M(t,s) := \sum_{j=1}^{+\infty} \frac{(-s)^j}{j!} \underbrace{M * \cdots * M}_{j}(t-s), \ \ t \geq s. \tag{8.2}$$

The next decomposition theorem and Proposition 8.3 below are consequences of [40, Theorems 1.1 and 1.2] and [40, Propositions 2.3 and 4.8], respectively.

Theorem 8.1. *For each integer $N \geq 2$, it holds that*

$$\Phi(t) = \mathcal{P}_N(t) + \mathcal{W}_N(t) + \mathfrak{R}_N(t), \ \ t \geq 0, \tag{8.3}$$

with

$$\left\{ \begin{array}{lcl} \mathcal{P}_N(t) & := & e^{tA} + e^{tA} \sum_{l=0}^{N-1} p_l(t)(-A)^{-l-1}, \\ \mathcal{W}_N(t) & := & \sum_{l=1}^{N-1} h_l(t)(-A)^{-l-1}, \\ \mathfrak{R}_N(t) & := & R_N(t,-A)(-A)^{-N-1}, \end{array} \right. \qquad t \geq 0, \tag{8.4}$$

where p_l and h_l are given by (8.1) and

$$R_N(t,\tau) := \int_0^t \tau e^{-\tau s} \partial_s^N K_M(t,s) ds, \ \ t \geq 0, \ \ \tau \geq 0, \tag{8.5}$$

with K_M given by (8.2). Moreover, for each $t \geq 0$, $\{h_l(t)\}_{l\geq 1}$ is not the null sequence, while for each integer $N \geq 2$ and each $s \in \mathbb{R}$, $R_N(\cdot, -A)|_{\mathbb{R}^+}$ belongs to $C(\mathbb{R}^+; \mathcal{L}(\mathcal{H}^s))$ and satisfies

$$\|R_N(t,-A)\|_{\mathcal{L}(\mathcal{H}^s)} \leq e^t \left\{ \exp\left[N(1+t) \left(\sum_{j=0}^{N} \max_{0\leq s\leq t} \left| \frac{d^j}{ds^j} M(s) \right| \right) \right] - 1 \right\}, \ \ t \geq 0. \tag{8.6}$$

Remark 8.2. When $N \geq 3$, the hyperbolic-like component $\mathcal{W}_N$ in (8.4) is of the form :

$$\mathcal{W}_N(t) = -M(t)A^{-2} + \sum_{l=2}^{N-1} h_l(t)(-A)^{-l-1}, \ \ t \geq 0. \tag{8.7}$$

This can be directly checked from (8.4) and (8.1) (where $h_1 = -M$).

Proposition 8.3. *Let K_M be given by (8.2). Then, K_M is real analytic over $S_+ := \{(t,s) \in \mathbb{R}^2 \ : \ t \geq s\}$. Moreover,*

$$\Phi(t)^* = \Phi(t) = e^{tA} + \int_0^t K_M(t,\tau)e^{\tau A} d\tau, \ \ t \geq 0. \tag{8.8}$$

8.2 Estimate for the heat equation

The following technical result provides an estimate for the heat equation that we present for the sake of completeness.

Lemma 8.4. *Let $B(x_0, r) \subset \Omega$ be a closed ball centered at $x_0 \in \Omega$ with radius $r > 0$. Then for each $s \in \mathbb{R}$, there exists a constant $C = C(s) > 0$ such that*

$$\sup_{t\geq 0} \|e^{tA}z\|_{L^2(\Omega\setminus B(x_0,r))} \leq C\|z\|_{\mathcal{H}^s} \ \ \text{for each} \ \ z \in \mathcal{H}^s\big(B(x_0, r/2)\big), \tag{8.9}$$

where $\mathcal{H}^s\big(B(x_0,r/2)\big) := \big\{ z \in \mathcal{H}^s \ : \ \text{supp}\, z \subset B(x_0, r/2)\big\}$.

Proof. It suffices to prove (8.9) for each $s = -m$ with $m \in \mathbb{N}^+$. For this purpose, we arbitrarily fix a $z \in \mathcal{H}^{-m}\big(B(x_0, r/2)\big)$. We select $\{r_l\}_{l=1}^{2m} \subset \mathbb{R}^+$ such that

$$r/2 < r_1 < \cdots < r_{2m} < r. \tag{8.10}$$

Then, we take a sequence of functions $\{\rho_l\}_{l=1}^{2m} \subset C_0^\infty(\mathbb{R}^n)$ such that for each $l \in \{1, \cdots, 2m\}$,

$$\rho_l = 0 \ \text{ over } \ B(x_0, r_{l-1}) \ \text{ and } \ \rho_l = 1 \ \text{ over } \ \mathbb{R}^n \setminus B(x_0, r_l). \tag{8.11}$$

We define a sequence of functions $\{f_l\}_{l=0}^{2m}$ in the following manner:

$$f_0(t) := e^{tA}z, \ t > 0; \ \ f_l(t) := \rho_l e^{tA}z, \ t > 0, \ l \in \{1, \cdots, 2m\}. \tag{8.12}$$

The rest of the proof is organized in several steps.

Step 1. We prove that for each $\chi_1 \in C_0^\infty(\Omega)$, $\chi_2 \in C_0^\infty(\Omega;\mathbb{R}^n)$, and $\alpha \geq 0$, there is a $C = C(\chi_1,\chi_2,\alpha) > 0$ such that

$$\|\chi_1 g\|_{\mathcal{H}^{-\alpha-1}} + \|\chi_2 \cdot \nabla g\|_{\mathcal{H}^{-\alpha-1}} \leq C\|g\|_{\mathcal{H}^{-\alpha}}, \text{ when } g \in \mathcal{H}^{-\alpha}. \tag{8.13}$$

We arbitrarily fix $\chi_1 \in C_0^\infty(\Omega)$ and $\chi_2 \in C_0^\infty(\Omega;\mathbb{R}^n)$. We claim that for each $\alpha \geq 0$, there is a $C_1 = C_1(\chi_1,\chi_2,\alpha) > 0$ such that

$$\|\chi_1 f\|_{\mathcal{H}^{\alpha}} + \|\text{div}\,(\chi_2 f)\|_{\mathcal{H}^{\alpha}} \leq C_1\|f\|_{\mathcal{H}^{\alpha+1}}, \text{ when } f \in \mathcal{H}^{\alpha+1}. \tag{8.14}$$

When (8.14) is proven, (8.13) follows by the standard duality argument.

By the interpolation theorem in [27, Theorem 5.1], we see that in order to show (8.14), it suffices to prove it for $\alpha = 2k$ with $k \in \mathbb{N}$. To this end, we arbitrarily fix $\alpha = 2k$ (with $k \in \mathbb{N}$) and $f \in \mathcal{H}^{2k+1}$. Since χ_1 and χ_2 are compactly supported in Ω, we have

$$\Omega_1 := \text{supp}\,\chi_1 \cup \text{supp}\,\chi_2 \subset\subset \Omega. \tag{8.15}$$

We claim that there is a $C_2 > 0$ (independent of f) such that

$$\|f\|_{H^{2k+1}(\Omega_1)} \leq C_2\|f\|_{\mathcal{H}^{2k+1}}. \tag{8.16}$$

In fact, given $h \in \mathcal{H}^{2k+1}$, we have that $A^k h \in \mathcal{H}^1$. From this, (1.3), and (1.4), we see that $\Delta^k h \in \mathcal{H}^1 = H_0^1(\Omega)$. Since Δ^k is an elliptic operator of order $2k$, the above shows that $h \in H^{2k+1}_{loc}(\Omega)$ (see for instance [22, Theorem 18.1.29]). Consequently, we have $h|_{\Omega_1} \in H^{2k+1}(\Omega_1)$. Thus, we can define a linear map $\mathcal{T}$ from $\mathcal{H}^1$ to $H^{2k+1}(\Omega_1)$ in the following manner:

$$\mathcal{T}(A^k\bar{h}) := \bar{h}|_{\Omega_1}, \ \ \bar{h} \in \mathcal{H}^{2k+1}. \tag{8.17}$$

By using the closed graph theorem to $\mathcal{T}$, we deduce that it is bounded. Then, by (8.17), there is a $C_3 > 0$ such that

$$\|\bar{h}\|_{H^{2k+1}(\Omega_1)} \leq C_3\|A^k\bar{h}\|_{\mathcal{H}^1} = C_3\|\bar{h}\|_{\mathcal{H}^{2k+1}} \ \text{ for each } \ \bar{h} \in \mathcal{H}^{2k+1},$$

which leads to (8.16).

Now, by (1.4), (1.3), and (8.15), there is a $C_4 > 0$ (independent of f) such that

$$\|\chi_1 f\|_{\mathcal{H}^{2k}} + \|\text{div}\,(\chi_2 f)\|_{\mathcal{H}^{2k}} = \|\Delta^k(\chi_1 f)\|_{L^2(\Omega)} + \|\Delta^k\text{div}\,(\chi_2 f)\|_{L^2(\Omega)} \leq C_4\|f\|_{H^{2k+1}(\Omega_1)}.$$

The above, along with (8.16), yields (8.14) with $\alpha = 2k$. This ends the proof of Step 1.

Step 2. We prove that for each $l \in \{1,\cdots,2m\}$, there exists a $C_l > 0$ (independent of z) such that

$$\|f_l\|_{L^\infty(\mathbb{R}^+;\mathcal{H}^{l/2-m})} \leq C_l\|f_{l-1}\|_{L^\infty(\mathbb{R}^+;\mathcal{H}^{(l-1)/2-m})}. \tag{8.18}$$

We will show that (8.18) is satisfied by induction. To this end, we first show that (8.18) is satisfied with $l = 1$. Indeed, from (8.12), we know that $f_1 = \rho_1 f_0$. This, along with (1.3), yields

$$\frac{d}{dt}f_1(t) - Af_1(t) = F_1(t), \ t > 0; \ f_1(0) = 0, \tag{8.19}$$

where

$$F_1(t) := (-\Delta\rho_1)f_0 - 2\nabla\rho_1\cdot\nabla f_0,\ t>0. \tag{8.20}$$

Meanwhile, it follows from (8.11) that $\Delta\rho_1 \in C_0^\infty(\Omega)$ and $\nabla\rho_1 \in C_0^\infty(\Omega;\mathbb{R}^n)$. From these and (8.20), we can apply (8.13) (with $(\chi_1,\chi_2,\alpha) = (\Delta\rho_1,\nabla\rho_1,m)$) to find a constant $\widehat{C}_1 > 0$ (independent of z) such that

$$\|F_1\|_{L^\infty(\mathbb{R}^+;\mathcal{H}^{-1-m})} \le \widehat{C}_1\|f_0\|_{L^\infty(\mathbb{R}^+;\mathcal{H}^{-m})}. \tag{8.21}$$

We now claim that there is a $\widehat{C}_2 > 0$ (independent of z) such that

$$\|f_1\|_{L^\infty(\mathbb{R}^+;\mathcal{H}^{1/2-m})} \le \widehat{C}_2\|f_0\|_{L^\infty(\mathbb{R}^+;\mathcal{H}^{-m})}. \tag{8.22}$$

Indeed, from (8.19), we can find a $C>0$ (independent of z) such that for each $t>0$,

$$\begin{aligned}\|f_1(t)\|_{\mathcal{H}^{1/2-m}} &\le \int_0^t \|e^{(t-s)A}\|_{\mathcal{L}(\mathcal{H}^{-1-m};\mathcal{H}^{1/2-m})}\|F_1(s)\|_{\mathcal{H}^{-1-m}}ds\\ &\le \left(\int_0^t \left\|\left[(-A)^{3/4}e^{(t-s)A/2}\right]e^{(t-s)A/2}\right\|_{\mathcal{L}(\mathcal{H}^{-1-m})}ds\right)\|F_1\|_{L^\infty(\mathbb{R}^+;\mathcal{H}^{-1-m})}\\ &\le \left(\int_0^t C(t-s)^{-3/4}e^{-(t-s)\eta_1/2}ds\right)\|F_1\|_{L^\infty(\mathbb{R}^+;\mathcal{H}^{-1-m})}.\end{aligned}$$

This, along with (8.21), leads to (8.22). Thus, (8.18) holds for $l=1$.

Next, we assume that for some $l_0 \in \{1,\cdots,2m-1\}$, (8.18) holds for all $l \le l_0$. We aim to prove (8.18) with $l = l_0+1$. In fact, from (8.12) and (8.11), we have $f_{l_0+1} = \rho_{l_0+1}f_{l_0}$. By this and using a similar method to that used in the proof of (8.18) with $l=1$, we can determine (8.18) with $l=l_0+1$. This ends the proof of Step 2.

Step 3. We verify (8.9).

Since $z\in\mathcal{H}^{-m}$, it follows from (8.12) and (1.5) that

$$\|f_0(t)\|_{\mathcal{H}^{-m}} = \|e^{tA}z\|_{\mathcal{H}^{-m}} \le \|z\|_{\mathcal{H}^{-m}},\ t\ge 0.$$

This, together with (8.18) and (8.10)–(8.12), yields (8.9) with $s=-m$.

Hence, we have completed the proof of Lemma 8.4. □

8.3 Technical proofs

In this subsection, we present the proofs of several results stated before. We start with the proof of Corollary 2.2.

Proof of Corollary 2.2. We arbitrarily fix a $\beta\in[2,3]$. Let $\mathcal{R}_c$ be given by Proposition 2.1. We define

$$\widehat{\mathcal{R}}_c(t,\tau) := e^{-t\tau}\Big(1+M(0)t\tau^{-1}+M(0)\tau^{-2}\Big)\tau^\beta - \mathcal{R}_c(t,\tau)\tau^{\beta-3},\ \ t>0,\ \tau>0.$$

Then, by spectral functional calculus, we have

$$\widehat{\mathcal{R}}_c(t,-A) = e^{tA}\Big(1 - M(0)tA^{-1} + M(0)A^{-2}\Big)(-A)^{\beta} - \mathcal{R}_c(t,-A)(-A)^{\beta-3},\ \ t>0. \tag{8.23}$$

Now, (2.8) follows from (2.1) and (8.23).

Next, because $2 \le \beta \le 3$, for each $j \in \{0,1,2\}$ and each $s \in \mathbb{R}$,

$$\|(-A)^{\beta-j}e^{tA}\|_{\mathcal{L}(\mathcal{H}^s)} \le \sup_{\lambda>0} \lambda^{\beta-j}e^{-t\lambda} \le 2t^{j-\beta},\ \ t>0. \tag{8.24}$$

Hence, (2.9) follows from (8.23), (8.24), and (2.2) directly. This completes the proof. □

The proof of Corollary 2.3 is as follows.

Proof of Corollary 2.3. Let $\mathcal{R}_c$ be given by Proposition 2.1. We define

$$\widetilde{\mathcal{R}}_c(t,\tau) := e^{-t\tau}\big(tM(0)\tau + M(0)\big) - M(t) - \mathcal{R}_c(t,\tau)\tau^{-1},\ t>0,\ \tau>0.$$

Then, by spectral functional calculus, we find

$$\widetilde{\mathcal{R}}_c(t,-A) = e^{tA}\big(-tM(0)A + M(0)\big) - M(t) + \mathcal{R}_c(t,-A)A^{-1},\ t>0.$$

This, along with (2.1), yields (2.10).

Next, we can directly check that for each $s \in \mathbb{R}$, $\|tAe^{tA}\|_{\mathcal{L}(\mathcal{H}^s)} \le 1$ when $t \ge 0$. This, along with (2.2), yields the desired estimate of the above $\widetilde{\mathcal{R}}_c$ and completes the proof. □

The proof of Proposition 4.5 is presented as follows.

Proof of Proposition 4.5. Without loss of generality, we can assume that $\alpha \ge 0$. Otherwise, we can replace $\alpha < 0$ by $\alpha = 0$. It follows from Corollary 2.3 that the map $t \mapsto \Phi(t) - e^{tA}$ $(t>0)$ belongs to $L^\infty(\mathbb{R}^+;\mathcal{L}(\mathcal{H}^{-4},L^2(\Omega)))$. Meanwhile, note that $\Phi(t)y_0 = y(t,\cdot;y_0)$, $t \ge 0$. Then, by the triangle inequality and (4.5), we determine that

$$\begin{aligned}
&\sup_{y_0\in L^2(\Omega),\|y_0\|_{\mathcal{H}^{-4}}\le 1} \int_0^T \|\chi_Q e^{tA}y_0\|_{L^2(\Omega)}t^\alpha dt \\
\le &\sup_{y_0\in L^2(\Omega),\|y_0\|_{\mathcal{H}^{-4}}\le 1} \int_0^T \|\chi_Q y(t,\cdot;y_0)\|_{L^2(\Omega)}t^\alpha dt \\
&+ \sup_{y_0\in L^2(\Omega),\|y_0\|_{\mathcal{H}^{-4}}\le 1} \int_0^T \|\chi_Q\big(\Phi(t)-e^{tA}\big)y_0\|_{L^2(\Omega)}t^\alpha dt < +\infty.
\end{aligned}$$

Since $Q \supset (0,\varepsilon_0)\times\omega$ and ω contains a closed ball $B(x_0,r)$ (centered at x_0 with radius r), from the above inequality and Lemma 8.4 in the Appendix, we see that

$$\sup_{y_0\in C_0^\infty(B(x_0,r/2)),\ \|y_0\|_{\mathcal{H}^{-4}}\le 1} \int_0^{\varepsilon_0} \|e^{tA}y_0\|_{L^2(\Omega)}t^\alpha dt < +\infty,$$

which shows that for some $C_1 > 0$,

$$\int_0^{\varepsilon_0} \|e^{tA}A^2 z\|_{L^2(\Omega)} t^\alpha dt \le C_1 \|z\|_{L^2(\Omega)} \text{ for all } z \in C_0^\infty(B(x_0, r/2)). \tag{8.25}$$

Next, we take a sequence $\{y_k\}_{k=1}^{+\infty} \subset C_0^\infty(B(x_0, r/2))$ such that

$$\|y_k\|_{L^2(\Omega)} = 1 \text{ for all } k \in \mathbb{N}^+; \ \ y_k \rightharpoonup 0 \text{ weakly in } L^2(\Omega). \tag{8.26}$$

We are going to show that there is a subsequence $\{k_l\}_{l=1}^{+\infty} \subset \mathbb{N}^+$ (with $k_1 < k_2 < \cdots$) such that

$$\limsup_{m \to +\infty} \sum_{l=m}^{m+m_0-1} \int_0^{+\infty} \|e^{tA}A^2 y_{k_l}\|_{L^2(\Omega)} t^\alpha dt \le C_1 \sqrt{m_0} \text{ for each } m_0 \in \mathbb{N}^+ \setminus \{1\}. \tag{8.27}$$

The proof of (8.27) is organized in the following two steps.

Step 1. We claim that there is a subsequence $\{k_l\}_{l=1}^{+\infty} \subset \mathbb{N}^+$ and a decreasing sequence $\{t_l\}_{l=1}^{+\infty} \subset (0, \varepsilon_0]$ such that for each $l \in \mathbb{N}^+$,

$$\sum_{m<l} |\langle y_{k_l}, y_{k_m} \rangle_{L^2(\Omega)}| + \int_{\mathbb{R}^+ \setminus (t_{l+1}, t_l)} \|e^{tA}A^2 y_{k_l}\|_{L^2(\Omega)} t^\alpha dt \le 2^{-l}. \tag{8.28}$$

Here, $\sum_{m<l} |\langle y_{k_l}, y_{k_m} \rangle_{L^2(\Omega)}| := 0$ when $l = 1$.

For this purpose, we first take $t_1 = \varepsilon_0$. We next take $k_1 \in \mathbb{N}^+$ sufficiently large that

$$\int_{t_1}^{+\infty} \|e^{tA}A^2 y_{k_1}\|_{L^2(\Omega)} t^\alpha dt \le \frac{1}{4} \tag{8.29}$$

(the existence of such k_1 is ensured by the exponential decay property of e^{tA} and the weak convergence in (8.26)). We then take $t_2 \in (0, t_1)$ sufficiently small that

$$\int_0^{t_2} \|e^{tA}A^2 y_{k_1}\|_{L^2(\Omega)} t^\alpha dt \le \frac{1}{4} \tag{8.30}$$

(the existence of such a t_2 can be obtained directly from (8.25)). Thus, it follows from (8.29) and (8.30) that the above k_1 and $\{t_l\}_{l=1}^2$ satisfy (8.28) with $l = 1$.

Next, we suppose inductively that for some $N \in \mathbb{N}^+$, there are $k_1 < \cdots < k_N$ and $t_1 > t_2 > \cdots > t_N > t_{N+1}$ satisfying (8.28) with $l = N$. We aim to find $k_{N+1} > k_N$ and $t_{N+2} < t_{N+1}$ such that $\{k_l\}_{l=1}^{N+1}$ and $\{t_l\}_{l=1}^{N+2}$ satisfy (8.28) with $l = N+1$. To this end, we use the weak convergence in (8.26) and the exponential decay property of e^{tA} to find $\hat{k} > k_N$ such that

$$\sum_{m<N+1} |\langle y_{\hat{k}}, y_{k_m} \rangle_{L^2(\Omega)}| + \int_{t_{N+1}}^{+\infty} \|e^{tA}A^2 y_{\hat{k}}\|_{L^2(\Omega)} t^\alpha dt \le 2^{-1}2^{-N-1}. \tag{8.31}$$

Meanwhile, since $y_{\hat{k}} \in C_0^\infty(\Omega)$, we have $\int_0^{\varepsilon_0} \|e^{tA}A^2 y_{\hat{k}}\|_{L^2(\Omega)} t^\alpha dt < +\infty$. Thus, there is a $\hat{t} \in (0, t_{N+1})$ sufficiently small that

$$\int_0^{\hat{t}} \|e^{tA}A^2 y_{\hat{k}}\|_{L^2(\Omega)} t^\alpha dt \le 2^{-1}2^{-N-1}. \tag{8.32}$$

Let $k_{N+1} := \hat{k}$ and $t_{N+2} := \hat{t}$. Then, it follows from (8.32) and (8.31) that $\{k_l\}_{l=1}^{N+1}$ and $\{t_l\}_{l=1}^{N+2}$ satisfy (8.28) with $l = N+1$. Thus, by the induction, the claim in *Step 1* is true.

Step 2. We show that (8.27) *is satisfied.*

We arbitrarily fix $m_0 \in \mathbb{N}^+ \setminus \{1\}$ and $m \in \mathbb{N}^+$. We define

$$z_m := \sum_{l \in \mathcal{J}_m^{m_0}} y_{k_l} \in C_0^\infty(\omega) \text{ where } \mathcal{J}_m^{m_0} := \{m, \cdots, m+m_0-1\}. \tag{8.33}$$

It follows from (8.33), (8.26), and (8.28) that

$$\begin{aligned}\|z_m\|_{L^2(\Omega)}^2 =& \sum_{l=m}^{m+m_0-1} \Big(\langle y_{k_l}, y_{k_l} \rangle_{L^2(\Omega)} + 2 \sum_{m \le l' < l} \langle y_{k_l}, y_{k_{l'}} \rangle_{L^2(\Omega)} \Big) \\ &\in \Big(m_0 - 2^{2-m}, m_0 + 2^{2-m} \Big).\end{aligned} \tag{8.34}$$

Meanwhile, we can directly check that for each $t > 0$,

$$\begin{aligned}\chi_{(t_{m+m_0}, t_m)}(t) z_m =& \Big(\sum_{l' \in \mathcal{J}_m^{m_0}} \chi_{(t_{l'+1}, t_{l'})}(t) \Big) \sum_{l \in \mathcal{J}_m^{m_0}} y_{k_l} \\ =& \sum_{l \in \mathcal{J}_m^{m_0}} \chi_{(t_{l+1}, t_l)}(t) y_{k_l} + \sum_{l, l' \in \mathcal{J}_m^{m_0}, l \neq l'} \chi_{(t_{l'+1}, t_{l'})}(t) y_{k_l}.\end{aligned}$$

From the above, we can directly verify that

$$\begin{aligned}\int_0^{\varepsilon_0} \|e^{tA} A^2 z_m\|_{L^2(\Omega)} t^\alpha dt \ge& \int_0^{\varepsilon_0} \|e^{tA} A^2 \chi_{(t_{m+m_0}, t_m)}(t) z_m\|_{L^2(\Omega)} t^\alpha dt \\ \ge& \int_0^{\varepsilon_0} \Big\| e^{tA} A^2 \sum_{l \in \mathcal{J}_m^{m_0}} \chi_{(t_{l+1}, t_l)}(t) y_{k_l} \Big\|_{L^2(\Omega)} t^\alpha dt \\ & - \sum_{l, l' \in \mathcal{J}_m^{m_0}, l \neq l'} \int_0^{\varepsilon_0} \|e^{tA} A^2 \chi_{(t_{l'+1}, t_{l'})}(t) y_{k_l}\|_{L^2(\Omega)} t^\alpha dt,\end{aligned}$$

which, together with the definition of $\mathcal{J}_m^{m_0}$ (in (8.33)), implies that

$$\begin{aligned}\int_0^{\varepsilon_0} \|e^{tA} A^2 z_m\|_{L^2(\Omega)} t^\alpha dt \ge& \sum_{l \in \mathcal{J}_m^{m_0}} \int_{t_{l+1}}^{t_l} \|e^{tA} A^2 y_{k_l}\|_{L^2(\Omega)} t^\alpha dt \\ & - \sum_{l \in \mathcal{J}_m^{m_0}} m_0 \int_{\mathbb{R}^+ \setminus (t_{l+1}, t_l)} \|e^{tA} A^2 y_{k_l}\|_{L^2(\Omega)} t^\alpha dt.\end{aligned}$$

The above, along with (8.28) and (8.25), implies that

$$\begin{aligned}& \sum_{l \in \mathcal{J}_m^{m_0}} \int_0^{+\infty} \|e^{tA} A^2 y_{k_l}\|_{L^2(\Omega)} t^\alpha dt \le \sum_{l \in \mathcal{J}_m^{m_0}} \int_{t_l}^{t_{l+1}} \|e^{tA} A^2 y_{k_l}\|_{L^2(\Omega)} t^\alpha dt + \sum_{l \in \mathcal{J}_m^{m_0}} 2^{-l} \\ \le& \Big(\int_0^{\varepsilon_0} \|e^{tA} A^2 z_m\|_{L^2(\Omega)} t^\alpha dt + m_0 \sum_{l \in \mathcal{J}_m^{m_0}} 2^{-l} \Big) + \sum_{l \in \mathcal{J}_m^{m_0}} 2^{-l} \\ \le& C_1 \|z_m\|_{L^2(\Omega)} + (m_0+1) \sum_{l \in \mathcal{J}_m^{m_0}} 2^{-l}.\end{aligned}$$

This, together with (8.34) and the definition of $\mathcal{J}_m^{m_0}$ (see (8.33)), leads to (8.27).

Finally, we will use (8.26) and (8.27) to show $\alpha > 1$. We observe that for each $z = \sum_{j\geq 1} a_j e_j \in C_0^\infty(\Omega)$,

$$\begin{aligned}&\int_0^{+\infty} \|e^{tA}A^2 z\|_{L^2(\Omega)} t^\alpha dt = \int_0^{+\infty} \left\| \left(e^{-t\eta_j}\eta_j^2 a_j\right)_{j\geq 1}\right\|_{\ell^2} t^\alpha dt \\ \geq& \sup_{\|(b_j)_{j\geq 1}\|_{\ell^2}\leq 1} \int_0^{+\infty} \left[\sum_{j\geq 1}\left(e^{-t\eta_j}\eta_j^2 a_j\right) b_j\right] t^\alpha dt \\ =& \left\|\left(\eta_j^{1-\alpha} a_j\right)_{j\geq 1}\right\|_{\ell^2} \int_0^{+\infty} e^{-t} t^\alpha dt = \|(-A)^{1-\alpha} z\|_{L^2(\Omega)} \int_0^{+\infty} e^{-t}t^\alpha dt.\end{aligned}$$

From this and (8.27), it follows that

$$\limsup_{m\to+\infty} \sum_{l=m}^{m+m_0-1} \|(-A)^{1-\alpha} y_{k_l}\|_{L^2(\Omega)} \leq C_1\sqrt{m_0} \Big/ \int_0^{+\infty} e^{-t}t^\alpha dt \ \text{ for all } \ m_0 \in \mathbb{N}^+ \setminus \{1\}.$$

Then, by (8.26), after some direct computations, we determine that $\alpha > 1$. This ends the proof of Proposition 4.5. □

8.4 Variation of constant formula

This subsection provides a variation of the constant formula for equation (5.1). We did not find this in the literature and present it here for the sake of completeness.

Proposition 8.5. *When $y_0 \in L^2(\Omega)$ and $u \in L^1_{loc}([0, +\infty); L^2(\Omega))$,*

$$y(t; y_0, u) = \Phi(t)y_0 + \int_0^t \Phi(t-s)(\chi_Q u)(s)ds, \ \ t \geq 0. \tag{8.35}$$

Proof. We arbitrarily fix $y_0 \in L^2(\Omega)$ and $u \in L^1_{loc}([0, +\infty); L^2(\Omega))$. We simply write $y(\cdot)$ for the solution $y(\cdot; y_0, u)$. First, (8.35) is clearly true for $t = 0$. We now fix $t > 0$ and $z \in \mathcal{H}^2$. We write

$$\varphi(s; z) := \Phi(t-s)z, \ \ s \in [0, t]. \tag{8.36}$$

Then, by (8.36), (1.2), and (1.1), we see that $\varphi(\cdot; z)$ satisfies

$$-\varphi'(s; z) - A\varphi(s; z) + \int_s^t M(\tau - s)\varphi(\tau; z)d\tau = 0, \ \ s \in (0, t); \quad \varphi(t; z) = z. \tag{8.37}$$

By (5.1) and (8.37), we find that

$$\begin{aligned}&\big\langle y(t), \varphi(t; z)\big\rangle_{L^2(\Omega)} - \big\langle y_0, \varphi(0; z)\big\rangle_{L^2(\Omega)} \\ =& \int_0^t \frac{d}{ds}\big\langle y(s; y_0, u), \varphi(s; z)\big\rangle_{L^2(\Omega)} ds \\ =& \int_0^t \Big\langle Ay(s) - \int_0^s M(s-\tau)y(\tau)d\tau + \chi_Q u(s), \varphi(s; z)\Big\rangle_{\mathcal{H}^{-2},\mathcal{H}^2} ds + \int_0^t \big\langle y(s), \varphi'(s; z)\big\rangle_{L^2(\Omega)} ds.\end{aligned} \tag{8.38}$$

Meanwhile, by the Fubini theorem, it follows that

$$\int_0^t \left\langle \int_0^s M(s-\tau)y(\tau)d\tau, \varphi(s;z) \right\rangle_{\mathcal{H}^{-2},\mathcal{H}^2} ds = \int_0^t \int_\tau^t M(s-\tau)\langle y(\tau), \varphi(s;z)\rangle_{\mathcal{H}^{-2},\mathcal{H}^2} ds d\tau$$
$$= \int_0^t \left\langle y(s), \int_s^t M(\tau - s)\varphi(\tau;z)d\tau \right\rangle_{L^2(\Omega)} ds.$$

This, along with (8.38), yields

$$\begin{aligned}&\langle y(t), \varphi(t;z)\rangle_{L^2(\Omega)} - \langle y_0, \varphi(0;z)\rangle_{L^2(\Omega)} \\ &= \int_0^t \left\langle y(s), \varphi'(s;z) + A\varphi(s;z) - \int_s^t M(\tau - s)\varphi(\tau;z)d\tau \right\rangle_{L^2(\Omega)} ds + \int_0^t \langle \chi_Q u(s), \varphi(s;z)\rangle_{L^2(\Omega)} ds.\end{aligned}$$

The above, together with (8.37) and (8.36), indicates that

$$\langle y(t), z\rangle_{L^2(\Omega)} - \langle y_0, \Phi(t)z\rangle_{L^2(\Omega)} = \int_0^t \langle \chi_Q u(s), \Phi(t-s)z\rangle_{L^2(\Omega)} ds.$$

Since z was arbitrarily selected from $\mathcal{H}^2$, we can use a standard density argument in the above equality, as well as the first equality in (8.8), to obtain (8.35). This completes the proof. □

8.5 Functional framework

The following lemma is cited from [38, Lemma 5.1].

Lemma 8.6. *Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$, and let X, Y, and Z be three Banach spaces over $\mathbb{K}$, with their dual spaces X^*, Y^* and Z^*, respectively. Let $R \in \mathcal{L}(Z, X)$ and $O \in \mathcal{L}(Z, Y)$. Then, the following two propositions are equivalent:*
(i) There exists a $\widehat{C}_0 > 0$ and an $\hat{\varepsilon}_0 > 0$ such that

$$\|Rz\|_X^2 \le \widehat{C}_0 \|Oz\|_Y^2 + \hat{\varepsilon}_0 \|z\|_Z^2 \ \text{ for all } \ z \in Z. \tag{8.39}$$

(ii) There exists a $C_0 > 0$ and an $\varepsilon_0 > 0$ such that for each $x^* \in X^*$, there is a $y^* \in Y^*$ that satisfies

$$\frac{1}{C_0}\|y^*\|_{Y^*}^2 + \frac{1}{\varepsilon_0}\|R^* x^* - O^* y^*\|_{Z^*}^2 \le \|x^*\|_{X^*}^2. \tag{8.40}$$

Furthermore, when one of the above two propositions holds, the constant pairs (C_0, ε_0) and $(\widehat{C}_0, \hat{\varepsilon}_0)$ can be chosen to be the same.

The following result is a consequence of Lemma 8.6.

Corollary 8.7. *With the notation in Lemma 8.6, the following two propositions are equivalent:*
(i) There exists a $C_1 > 0$ such that

$$\|Rz\|_X \le C_1 \|Oz\|_Y \ \text{ for all } \ z \in Z.$$

(ii) There exists a $C_2 > 0$ such that for each $x^* \in X^*$, there is a $y^* \in Y^*$ that satisfies

$$R^* x^* - O^* y^* = 0 \text{ in } Z^* \text{ and } \|y^*\|_{Y^*} \le C_2 \|x^*\|_{X^*}.$$

Furthermore, when one of the above two propositions holds, the constants C_1 and C_2 can be chosen to be the same.

Proof. We first show $(i) \Rightarrow (ii)$. Suppose that (i) holds (with $C_1 > 0$). Then, for each $\hat{\varepsilon}_0 > 0$, we have (8.39) (with $\widehat{C}_0 = C_1^2$). This, along with Lemma 8.6, leads to (8.40), where y^* and (C_0, ε_0) are replaced by $y^*_{\hat{\varepsilon}_0}$ and $(C_1^2, \hat{\varepsilon}_0)$, respectively. Hence, the family $\{y^*_{\hat{\varepsilon}_0}\}_{\hat{\varepsilon}_0>0}$ is bounded in Y^*. Thus, there is a subsequence $\{y^*_{\varepsilon_k}\}_{k\geq 1}$ that converges to $\hat{y}^*$ weakly star in Y^*. Thus, (ii) holds for $y^* = \hat{y}^*$ and $C_2 = C_1$.

We next show $(ii) \Rightarrow (i)$. We suppose that (ii) holds (with $C_2 > 0$). Then, for each $\varepsilon_0 > 0$, we have (8.40), with $C_0 = C_2^2$. This, together with Lemma 8.6, yields (8.39) where $(\widehat{C}_0, \hat{\varepsilon}_0)$ is replaced by (C_2^2, ε_0). Hence, (i) holds for $C_1 = C_2$. This completes the proof. □

Acknowledgments.

The authors thank the anonymous referees for their valuable feedback, which allowed authors to significantly improve the presentation. The first two authors also extend their gratitude to Dr. Huaiqiang Yu for his valuable comments and suggestions.

The first two authors were partially supported by the National Natural Science Foundation of China under grants 12371450 and 12171359, respectively. The second author was also funded by the Humboldt Research Fellowship for Experienced Researchers program from the Alexander von Humboldt Foundation.

The third author has been funded by the Alexander von Humboldt-Professorship program, the Transregio 154 Project "Mathematical Modelling, Simulation and Optimization Using the Example of Gas Networks" of the DFG, the ModConFlex Marie Curie Action, HORIZON-MSCA-2021-DN-01, the COST Action MAT-DYN-NET, AFOSR Proposal 24IOE027, grants PID2020-112617GB-C22 and TED2021-131390B-I00 of MINECO (Spain), and by the Madrid Goverment – UAM Agreement for the Excellence of the University Research Staff in the context of the V PRICIT (Regional Programme of Research and Technological Innovation).

Data Availability Statement Data sharing is not applicable to this paper because no dataset was analysed or generated during the study.

Statements and Declarations

Competing Interests The authors declare that they have no financial interests nor any conflict of interest.

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