# FUNCTIONAL INEQUALITIES FOR BROWNIAN MOTION ON MANIFOLDS WITH STICKY-REFLECTING BOUNDARY DIFFUSION

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Abstract. We prove an upper bound for the Poincaré constant for Brownian motion on manifolds with sticky reflecting boundary diffusion under general curvature conditions. This corresponds to bounding from below the first nontrivial eigenvalue of the Laplace operator with Wentzell-type boundary condition. Additionally we give an upper bound on the logarithmic Sobolev constant for Brownian motion with sticky reflecting boundary diffusion.
We also apply our results to Brownian motion with sticky reflection but without boundary diffusion. Furthermore we obtain a lower bound on the first nontrivial Steklov eigenvalue and an upper bound for the norm of the Trace operator in terms of the geometry of the manifold.

## 1. Introduction

Let $\Omega$ be a smooth compact connected Riemannian manifold of dimension $d \geq 2$ with connected and piecewise smooth boundary $\partial\Omega$. We consider so-called Brownian motion with sticky reflecting boundary diffusion, i.e. a diffusion on $\Omega$ with Feller generator $(\mathcal{D}(A), A)$ on $C(\Omega)$ given by

$$\mathcal{D}(A) = \{f \in C(\Omega) \mid Af \in C(\Omega)\}$$
$$Af = \Delta f \mathbb{1}_\Omega + \left(\Delta^\tau f - \gamma \frac{\partial f}{\partial N}\right) \mathbb{1}_{\partial\Omega},$$

where $\frac{\partial f}{\partial N}$ is the outer normal derivative, $\Delta^\tau$ is the Laplace-Beltrami operator on $\partial\Omega$ and $\gamma > 0$, which corresponds to inward sticky reflection at $\partial\Omega$. A construction of the associated semigroup and diffusion process was given e.g. in [?]. We assume that $\Omega$ and $\partial\Omega$ have finite (Hausdorff) measure and denote it respectively by $|\Omega|$ and $|\partial\Omega|$. Furthermore by $\lambda_\Omega$ resp. $\lambda_{\partial\Omega}$ we denote the normalised volume measure on $\Omega$ resp. normalised Hausdorff measure on $\partial\Omega$ and choose $\alpha \in [0,1]$, such that

$$\frac{\alpha}{1-\alpha} \frac{|\partial\Omega|}{|\Omega|} = \gamma.$$

Moreover we set

$$\lambda_\alpha := \alpha\lambda_\Omega + (1-\alpha)\lambda_{\partial\Omega}.$$

and find that $-A$ is $\lambda_\alpha$-symmetric.
Our aim is to estimate the Poincaré and logarithmic Sobolev constants for such processes.
In section 2 we show an upper bound on the Poincaré constant for Brownian motion with sticky reflecting boundary diffusion by using an interpolation approach introduced in [?]. In [?] it was shown that as time goes to infinity the proportion of time spent on the boundary is positive and in particular approaches $\frac{(1-\alpha)|\partial\Omega|}{\alpha|\Omega|}$. This illustrates that the interplay of boundary and interior will be of central importance. The interpolation approach used in section 2 is in accordance with this. We generalise the results previously achieved in [?] by allowing less strict assumptions on the geometry of the manifold $\Omega$. Furthermore we consider examples in the Euclidean and hyperbolic plane and compare the bounds from the interpolation approach with exact values as well as bounds obtained from a more simple approach that does not use any interpolation. Lastly we also apply our results to Brownian motion with sticky reflection but without boundary diffusion.
In section 3 we point out a connection with the trace operator and state some more results relating boundary and interior properties that might be of independent interest. These are analogues of Poincaré and Sobolev-Poincaré inequalities between boundary and interior of $\Omega$.
In section 4 we generalise the interpolation approach and use it in order to give an upper bound on the

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logarithmic Sobolev constant for Brownian motion with sticky reflecting boundary diffusion. Again we also apply our results to Brownian motion with sticky reflection but without boundary diffusion.

## 2. Poincaré Inequality

We say that a Poincaré Inequality is fulfilled if there is a constant $C_\alpha$ such that for all $f \in C^1(\Omega)$

$$Var_{\lambda_\alpha}(f) \le C_\alpha \mathcal{E}_\alpha(f),$$

where

$$\begin{aligned} Var_{\lambda_\alpha}(f) &= \int_\Omega f^2 d\lambda_\alpha - \left(\int_\Omega f d\lambda_\alpha\right)^2 \\ \mathcal{E}_\alpha(f) &= \alpha \int_\Omega |\nabla f|^2 d\lambda_\Omega + (1-\alpha)\int_{\partial\Omega} |\nabla^\tau f|^2 d\lambda_{\partial\Omega},\ f \in C^1(\Omega) \end{aligned}$$

and $\nabla^\tau$ denotes the tangential derivative operator on $\partial\Omega$. In the following we denote by $C_\alpha$ the optimal such constant. By $C_\Omega$ and $C_{\partial\Omega}$ we denote the usual (Neumann) Poincaré constants of $\Omega$ and $\partial\Omega$ respectively. We assume that $C_\Omega$ and $C_{\partial\Omega}$ (or respective upper bounds for them) are known.
In [**?**][Proposition 2.1] the following statement was proved in the setting introduced above using an interpolation approach:

**Proposition 2.1.** *Assume there exist constants $K_{\partial\Omega,\Omega}, K_1, K_2$ such that for any $f \in C^1(\Omega)$*

$$Var_{\lambda_{\partial\Omega}} f \le K_{\partial\Omega,\Omega} \int_\Omega |\nabla f|^2 d\lambda_\Omega \tag{1}$$

*and*

$$\left(\int_\Omega f d\lambda_\Omega - \int_{\partial\Omega} f d\lambda_{\partial\Omega}\right)^2 \le K_1 \int_\Omega |\nabla f|^2 d\lambda_\Omega + K_2 \int_{\partial\Omega} |\nabla^\tau f|^2 d\lambda_{\partial\Omega}, \tag{2}$$

*then it holds for any $\alpha \in (0,1)$*

$$C_\alpha \le max\left(C_\Omega + (1-\alpha)K_1, \alpha K_2, \frac{(1-\alpha)K_{\partial\Omega,\Omega}C_{\partial\Omega} + \alpha C_\Omega C_{\partial\Omega} + \alpha(1-\alpha)(K_{\partial\Omega,\Omega}K_2 + C_{\partial\Omega}K_1)}{(1-\alpha)K_{\partial\Omega,\Omega} + \alpha C_{\partial\Omega}}\right).$$

In [**?**][section 3.2] constants $K_{\partial\Omega,\Omega}, K_1, K_2$ were found under the assumption of a positive lower bound on Ricci curvature and a positive lower bound on the second fundamental form on the boundary $\partial\Omega$ (i.e. a convex boundary). Our aim is to find $K_{\partial\Omega,\Omega}, K_1, K_2$ and thus an upper bound on $C_\alpha$ assuming any upper and lower bound on Sectional curvature and any upper and lower bound for the second fundamental form on the boundary and to thereby generalise section 3.2 in [**?**].
We first aim to find suitable constants $K_1, K_2$ fulfilling (2) and for that matter prove the following Proposition. We do not yet make any assumptions on the geometry of the manifold, but will later combine the following Proposition with assumptions on curvature and second fundamental form in order to obtain explicit values for $K_1, K_2$.

**Proposition 2.2.** *For any $\varphi \in C^1(\Omega)$ such that $\frac{\partial\varphi}{\partial N}|_{\partial\Omega} = 1$ and $\nabla\varphi$ is Lipschitz continuous on $\Omega$ equation (2) in Proposition 2.1 is fulfilled with $K_2 = 0$ and*

$$K_1 = \left(\frac{|\Omega|}{|\partial\Omega|}\right)^2 \inf_{\varepsilon\in(0,\infty)} \left[(1+\varepsilon)|\nabla\varphi|_2^2 + (1+\varepsilon^{-1})C_\Omega|\Delta\varphi|_2^2\right],$$

*where $|\cdot|_2$ denotes the $L^2$-norm on $\Omega$ with respect to $\lambda_\Omega$.*

*Proof.* Let $f \in C^1(\Omega)$. Without loss of generality we can assume that $\int_\Omega f d\lambda_\Omega = 0$. Now $\forall \varepsilon > 0$

$$\begin{aligned}
&\left(\int_\Omega f d\lambda_\Omega - \int_{\partial\Omega} f d\lambda_{\partial\Omega}\right)^2 \\
&= \left(\int_{\partial\Omega} f d\lambda_{\partial\Omega}\right)^2 = \left(\int_{\partial\Omega} f N\varphi d\lambda_{\partial\Omega}\right)^2 \\
&= \left(\frac{|\Omega|}{|\partial\Omega|}\int_\Omega \nabla f\cdot\nabla\varphi d\lambda_\Omega + \frac{|\Omega|}{|\partial\Omega|}\int_\Omega f\Delta\varphi d\lambda_\Omega\right)^2 \\
&\le \left(\frac{|\Omega|}{|\partial\Omega|}\right)^2\left[(1+\varepsilon)\left(\int_\Omega \nabla f\cdot\nabla\varphi d\lambda_\Omega\right)^2 + (1+\varepsilon^{-1})\left(\int_\Omega f\Delta\varphi d\lambda_\Omega\right)^2\right] \\
&\le \left(\frac{|\Omega|}{|\partial\Omega|}\right)^2\left[(1+\varepsilon)\int_\Omega |\nabla f|^2 d\lambda_\Omega \int_\Omega |\nabla\varphi|^2 d\lambda_\Omega + (1+\varepsilon^{-1})\int_\Omega f^2 d\lambda_\Omega \int_\Omega (\Delta\varphi)^2 d\lambda_\Omega\right] \\
&\le \left(\frac{|\Omega|}{|\partial\Omega|}\right)^2\left[(1+\varepsilon)\int_\Omega |\nabla f|^2 d\lambda_\Omega \int_\Omega |\nabla\varphi|^2 d\lambda_\Omega + (1+\varepsilon^{-1})C_\Omega\int_\Omega |\nabla f|^2 d\lambda_\Omega \int_\Omega (\Delta\varphi)^2 d\lambda_\Omega\right].
\end{aligned}$$

□

We next find $K_{\partial\Omega,\Omega}$ such that equation (1) is fulfilled by proceeding similarly as in the proof of Proposition 2.2:

**Proposition 2.3.** *For any $\rho \in C^1(\Omega)$ such that $\frac{\partial\rho}{\partial N}|_{\partial\Omega} = -1$ and $\nabla\rho$ is Lipschitz continuous on $\Omega$ equation (1) in Proposition 2.1 is fulfilled with*

$$K_{\partial\Omega,\Omega} = \frac{|\Omega|}{|\partial\Omega|}\left(2|\nabla\rho|_\infty C_\Omega^{1/2} + |(\Delta\rho)^-|_\infty C_\Omega\right),$$

*where $(\cdot)^-$ denotes the negative part of a function and $|\cdot|_\infty$ denotes the $L^\infty$-norm on $\Omega$ with respect to $\lambda_\Omega$.*

*Proof.* Let $f \in C^1(\Omega)$. Without loss of generality we can assume that $\int_\Omega f d\lambda_\Omega = 0$. Now we can calculate similarly as in the previous result

$$\begin{aligned}
Var_{\lambda_{\partial\Omega}}(f) &= \int_{\partial\Omega} f^2 d\lambda_{\partial\Omega} - \left(\int_{\partial\Omega} f d\lambda_{\partial\Omega}\right)^2 \le \int_{\partial\Omega} f^2 d\lambda_{\partial\Omega} = -\int_{\partial\Omega} f^2 \frac{\partial\rho}{\partial N} d\lambda_{\partial\Omega} \\
&= -\frac{|\Omega|}{|\partial\Omega|}\int_\Omega 2f\nabla f\cdot\nabla\rho d\lambda_\Omega - \frac{|\Omega|}{|\partial\Omega|}\int_\Omega f^2\Delta\rho d\lambda_\Omega \\
&\le 2\frac{|\Omega|}{|\partial\Omega|}\int_\Omega |f||\nabla f||\nabla\rho| d\lambda_\Omega + \frac{|\Omega|}{|\partial\Omega|}\int_\Omega f^2(\Delta\rho)^- d\lambda_\Omega \\
&\le 2\frac{|\Omega|}{|\partial\Omega|}|\nabla\rho|_\infty\int_\Omega |f||\nabla f| d\lambda_\Omega + \frac{|\Omega|}{|\partial\Omega|}|(\Delta\rho)^-|_\infty\int_\Omega f^2 d\lambda_\Omega \\
&\le 2\frac{|\Omega|}{|\partial\Omega|}|\nabla\rho|_\infty\left(\int_\Omega f^2 d\lambda_\Omega\int_\Omega |\nabla f|^2 d\lambda_\Omega\right)^{1/2} + \frac{|\Omega|}{|\partial\Omega|}|(\Delta\rho)^-|_\infty\int_\Omega f^2 d\lambda_\Omega \\
&\le 2\frac{|\Omega|}{|\partial\Omega|}|\nabla\rho|_\infty C_\Omega^{1/2}\int_\Omega |\nabla f|^2 d\lambda_\Omega + \frac{|\Omega|}{|\partial\Omega|}|(\Delta\rho)^-|_\infty C_\Omega\int_\Omega |\nabla f|^2 d\lambda_\Omega \\
&= \frac{|\Omega|}{|\partial\Omega|}\left(2|\nabla\rho|_\infty C_\Omega^{1/2} + |(\Delta\rho)^-|_\infty C_\Omega\right)\int_\Omega |\nabla f|^2 d\lambda_\Omega.
\end{aligned}$$

□

**Remark 2.1.** Denote by $\sigma$ the first nontrivial eigenvalue of the Steklov eigenvalue problem

$$\begin{cases} \Delta f = 0, & \text{in } \Omega \\ \frac{\partial f}{\partial N} = \sigma f, & \text{on } \partial\Omega, \end{cases}$$

which is characterised (using normalised measures) by

$$\sigma = \frac{|\Omega|}{|\partial\Omega|}\inf_{\substack{f\in C^1(\Omega)\\ \int_{\partial\Omega} f d\lambda_{\partial\Omega}=0}}\frac{\int_\Omega |\nabla f|^2 d\lambda_\Omega}{\int_{\partial\Omega} f^2 d\lambda_{\partial\Omega}}.$$

Thus we have for the optimal constant $K_{\partial\Omega,\Omega}$ in equation (1) in Proposition 2.1

$$K_{\partial\Omega,\Omega} = \sup_{f\in C^1(\Omega)} \frac{Var_{\lambda_{\partial\Omega}}(f)}{\int_\Omega |\nabla f|^2 d\lambda_\Omega} = \sup_{\substack{f\in C^1(\Omega)\\ \int_{\partial\Omega} f d\lambda_{\partial\Omega}=0}} \frac{\int_{\partial\Omega} f^2 d\lambda_{\partial\Omega}}{\int_\Omega |\nabla f|^2 d\lambda_\Omega} = \left( \inf_{\substack{f\in C^1(\Omega)\\ \int_{\partial\Omega} f d\lambda_{\partial\Omega}=0}} \frac{\int_\Omega |\nabla f|^2 d\lambda_\Omega}{\int_{\partial\Omega} f^2 d\lambda_{\partial\Omega}} \right)^{-1} = \frac{|\Omega|}{|\partial\Omega|}\sigma^{-1}.$$

Therefore by finding upper bounds for the optimal $K_{\partial\Omega,\Omega}$ we find lower bounds for the first nontrivial Steklov eigenvalue. We use this connection in the computations for Example 2.2 below.

To obtain an explicit constant it now remains to specify functions $\varphi$ and $\rho$ with the desired properties. Note that despite fulfilling the same assumptions, $\varphi$ and $\rho$ may be chosen independently in order to optimise the estimates. It seems natural to define both functions of the form $\psi \circ \rho_{\partial\Omega}$ for some appropriate function $\psi$, where $\rho_{\partial\Omega}$ denotes the distance to the boundary function.
We use for $k, \gamma \in \mathbb{R}$ the function

$$h : [0,\infty) \to \mathbb{R},\ h(t) := \begin{cases} cos(\sqrt{k}t) - \frac{\gamma}{\sqrt{k}} sin(\sqrt{k}t), & k \geq 0 \\ cosh(\sqrt{-k}t) - \frac{\gamma}{\sqrt{-k}} sinh(\sqrt{-k}t), & k < 0. \end{cases} \tag{3}$$

Let $h^{-1}(0) := \inf\{t \geq 0 : h(0) = 0\}$, where $h^{-1}(0) = \infty$ if $h(t) > 0$ for all $t \geq 0$. We denote by $Ric$ and $sect$ the Ricci and sectional curvatures of $\Omega$, and by $\Pi$ the second fundamental form on the boundary $\partial\Omega$, i.e.

$$\Pi(X,Y) := \langle \nabla_X N, Y\rangle,\ X, Y \in T_x\partial\Omega, x \in \partial\Omega,$$

where $N$ is the outward pointing unit normal vector field of $\partial\Omega$.

**Lemma 2.1.** *Let $k_1, k_2 \in \mathbb{R}$ such that $Ric \geq (d-1), sect \leq k_2$ and $\gamma_1, \gamma_2 \in \mathbb{R}$ such that $\gamma_1 id \leq \Pi \leq \gamma_2 id$. We construct a function $\varphi \in C^1(\Omega)$ such that $\frac{\partial\varphi}{\partial N}|_{\partial\Omega} = -1$ and $\nabla\varphi$ is Lipschitz continuous on $\Omega$ and for $t_0 \in (0, h_2^{-1}(0))$*

$$|\nabla\varphi|_2^2 \leq \frac{1}{|\Omega|}\int_0^{t_0} H_{d-1}(\{\rho_{\partial\Omega} = t\}) \left(1 - \frac{t}{t_0}\right)^2 dt,$$

$$|\Delta\varphi|_2^2 \leq \frac{1}{|\Omega|}\int_0^{t_0} H_{d-1}(\{\rho_{\partial\Omega} = t\}) \Bigg( \left( \left( (d-1)\frac{h_2'}{h_2}(t)\left(1-\frac{t}{t_0}\right) - \frac{1}{t_0}\right)^- \right)^2$$
$$+ \left( \left( (d-1)\frac{h_1'}{h_1}(t)\left(1-\frac{t}{t_0}\right) - \frac{1}{t_0}\right)^+ \right)^2 \Bigg) dt,$$

*where $h_i, i = 1, 2$ are as defined above in (3) with $k = k_i$ and $\gamma = \gamma_i$.*

*Proof.* It is easy to see that $h_2^{-1}(0) \leq h_1^{-1}(0)$. Let $\rho_{\partial\Omega}$ be the distance function to the boundary. By the Laplacian comparison theorem, we have

$$\Delta\rho_{\partial\Omega} \leq \frac{(d-1)h_1'}{h_1}(\rho_{\partial\Omega}) \text{ on } \{\rho_{\partial\Omega} < h_1^{-1}(0)\}, \tag{4}$$

$$\Delta\rho_{\partial\Omega} \geq \frac{(d-1)h_2'}{h_2}(\rho_{\partial\Omega}) \text{ on } \{\rho_{\partial\Omega} < h_2^{-1}(0)\}. \tag{5}$$

Indeed, (5) follows from [?][Theorem 3.1] for the Laplacian comparison theorem due to [?], and by [?][Corollary 3.2] which says that the injectivity radius of $\partial\Omega$ is larger than $h_2^{-1}(0)$. Next, for $x \in \Omega$ with $\rho_\partial(x) < h_2^{-1}(0)$, let $p \in \partial\Omega$ be the projection such that $\gamma(s) := \exp[-sN(p)], s \in [0, \rho_\partial(x)]$ be the minimal geodesic form $p$ to $x$. Let $\{X_i\}_{1\leq i\leq d-1}$ be orthonormal vector fields around $x$ orthogonal to $\nabla\rho_\partial(x)$. Let $J_i(s))_{s\in[0,\rho_\partial(x)]}$ be the Jacobi fields along the geodesic $\gamma$ such that $J_i(\rho_\partial(x)) = X_i(x)$ and

$$\langle \dot{J}_i(0), v\rangle = -\Pi(J_i(0), v), \quad v \in T_p\partial\Omega.$$

Le $\mathcal{R}$ be the Riemannian curvature tensor. By the second variational formula (see page 321 in [?]) we have

$$\text{Hess}_{\rho_\partial}(X_i, X_i)(x) = -\Pi(J_i(0), J_i(0)) + \int_0^{\rho_\partial(x)} \left( |\dot{J}_i(s)|^2 - \langle \mathcal{R}(\dot\gamma(s), J_i(s))\dot\gamma, J_i(s)\rangle \right) ds.$$

Let $(X_i(s))_{s\in[0,\rho_\partial(x)]}$ be the parallel displacement of $(X_i(0) := X_i(x))_{s\in[0,\rho_\partial(x)]}$, and denote

$$\tilde{J}_i(s) = \frac{h_2(s)}{h(\rho_\partial(x))} X_i(s), \quad 1 \leq i \leq d-1, s \in [0, \rho_\partial(x)].$$

Then the index lemma yields

$$\mathrm{Hess}_{\rho_\partial}(X_i,X_i)(x) \le -\Pi(\tilde J_i(0),\tilde J_i(0)) + \int_0^{\rho_\partial(x)} \left(|\dot J_i(s)|^2 - \left\langle \mathcal{R}(\dot\gamma(s),\tilde J_i(s))\dot\gamma, \tilde J_i(s)\right\rangle\right) ds.$$

Noting that $h_2"(s) = -k_2 h_2(s)$, this implies (4).

Now for $t_0 \in (0, h_2^{-1}(0))$ we define

$$\varphi = \int_0^{\rho_{\partial\Omega}} \left(1 - \frac{s}{t_0}\right)^+ ds.$$

We have

$$\nabla\varphi(x) = \begin{cases} \nabla\rho_{\partial\Omega}(x)\cdot\left(1 - \frac{\rho_{\partial\Omega}(x)}{t_0}\right), & \rho_{\partial\Omega}(x) \le t_0 \\ 0, & \text{else,} \end{cases}$$

and thus $\frac{\partial\varphi}{\partial N}|_{\partial\Omega} = -1$ and $\nabla\varphi$ is Lipschitz continuous. Furthermore

$$\Delta\varphi(x) = \begin{cases} \Delta\rho_{\partial\Omega}(x)\left(1 - \frac{\rho_{\partial\Omega}(x)}{t_0}\right) - \frac{1}{t_0}, & \rho_{\partial\Omega}(x) \le t_0 \\ 0, & \text{else.} \end{cases}$$

Now using the Coarea formula we get

$$\begin{aligned}\int_\Omega |\nabla\varphi|^2 d\lambda_\Omega = \int_{\{\rho_{\partial\Omega}\le t_0\}} \left(1 - \frac{\rho_{\partial\Omega}}{t_0}\right)^2 |\nabla\rho_{\partial\Omega}|^2 d\lambda_\Omega &= \int_{\{\rho_{\partial\Omega}\le t_0\}} \left(1 - \frac{\rho_{\partial\Omega}}{t_0}\right)^2 d\lambda_\Omega \\ &= \frac{1}{|\Omega|}\int_0^{t_0}\int_{\{\rho_{\partial\Omega}=t\}} \left(1 - \frac{t}{t_0}\right)^2 dH_{d-1}dt \\ &= \frac{1}{|\Omega|}\int_0^{t_0} H_{d-1}(\{\rho_{\partial\Omega}=t\})\left(1 - \frac{t}{t_0}\right)^2 dt,\end{aligned}$$

where $H_{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure. Furthermore

$$\int_\Omega (\Delta\varphi)^2 d\lambda_\Omega = \int_\Omega \left((\Delta\varphi)^+ + (\Delta\varphi)^-\right)^2 d\lambda_\Omega = \int_\Omega \left((\Delta\varphi)^+\right)^2 d\lambda_\Omega + \int_\Omega \left((\Delta\varphi)^-\right)^2 d\lambda_\Omega.$$

We see that by equations (4) and (5) on $\{\rho_{\partial\Omega} \le t_0\}$

$$\Delta\varphi \ge (d-1)\frac{h_2'}{h_2}(\rho_{\partial\Omega})\left(1 - \frac{\rho_{\partial\Omega}}{t_0}\right) - \frac{1}{t_0} \Rightarrow (\Delta\varphi)^- \le \left((d-1)\frac{h_2'}{h_2}(\rho_{\partial\Omega})\left(1 - \frac{\rho_{\partial\Omega}}{t_0}\right) - \frac{1}{t_0}\right)^-,$$

$$\Delta\varphi \le (d-1)\frac{h_1'}{h_1}(\rho_{\partial\Omega})\left(1 - \frac{\rho_{\partial\Omega}}{t_0}\right) - \frac{1}{t_0} \Rightarrow (\Delta\varphi)^+ \le \left((d-1)\frac{h_1'}{h_1}(\rho_{\partial\Omega})\left(1 - \frac{\rho_{\partial\Omega}}{t_0}\right) - \frac{1}{t_0}\right)^+.$$

Thus

$$\begin{aligned}\int_\Omega (\Delta\varphi)^2 d\lambda_\Omega \le& \int_{\{\rho_{\partial\Omega}\le t_0\}} \left(\left((d-1)\frac{h_2'}{h_2}(\rho_{\partial\Omega})\left(1 - \frac{\rho_{\partial\Omega}}{t_0}\right) - \frac{1}{t_0}\right)^-\right)^2 \\ &+ \left(\left((d-1)\frac{h_1'}{h_1}(\rho_{\partial\Omega})\left(1 - \frac{\rho_{\partial\Omega}}{t_0}\right) - \frac{1}{t_0}\right)^+\right)^2 d\lambda_\Omega \\ =& \frac{1}{|\Omega|}\int_0^{t_0} H_{d-1}(\{\rho_{\partial\Omega}=t\})\left(\left(\left((d-1)\frac{h_2'}{h_2}(t)\left(1 - \frac{t}{t_0}\right) - \frac{1}{t_0}\right)^-\right)^2\right. \\ &+ \left.\left(\left((d-1)\frac{h_1'}{h_1}(t)\left(1 - \frac{t}{t_0}\right) - \frac{1}{t_0}\right)^+\right)^2\right) dt.\end{aligned}$$

□

In the previous Lemma $t_0 \in (0, h_2^{-1}(0))$ may be chosen to either optimise $|\nabla\varphi|_2^2$ or $|\Delta\varphi|_2^2$.

**Lemma 2.2.** *Let $k_2 \in \mathbb{R}$ such that $\sec t \le k_2$ and $\gamma_2 \in \mathbb{R}$ such that $\Pi \le \gamma_2 id$. Then $k_2 > -\gamma_2^2$, and there exists a function $\rho \in C^1(\Omega)$ such that $\frac{\partial\rho}{\partial N}|_{\partial\Omega} = -1$ and $\nabla\rho$ is Lipschitz continuous on $\Omega$ and*

$$|\nabla\rho|_\infty \le 1,$$

$$|(\Delta\rho)^-|_\infty \le \inf_{t_1\in(0,h_2^{-1}(0))}\sup_{t\in(0,t_1)} \left((d-1)\frac{h_2'}{h_2}(t)\left(1 - \frac{t}{t_1}\right) - \frac{1}{t_1}\right)^-.$$

*Proof.* Let $h_2$ be the function defined as above in equation (3) with $k = k_2$ and $\gamma = \gamma_2$. If $k_2 \le -\gamma_2^2$, then $h_2^{-1}(0) = \infty$, so that [?][Corollary 3.2] implies that the cut locus of $\partial\Omega$ is empty, which is contractive to the fact that the maximum point of $\rho_\partial$ is in the cut locus. Hence, $k_2 > -\gamma_2^2$.

Let $t_1 \in (0, h_2^{-1}(0))$ to be chosen later. By (5)

$$\Delta\rho_{\partial\Omega} \ge (d-1)\frac{h_2'}{h_2}(\rho_{\partial\Omega}) \text{ on } \{\rho_{\partial\Omega} \le t_1\}. \tag{6}$$

Now, define

$$\rho = \int_0^{\rho_{\partial\Omega}} \left(1 - \frac{s}{t_1}\right)^+ ds.$$

We have

$$\nabla\rho(x) = \begin{cases} \nabla\rho_{\partial\Omega}(x)\cdot\left(1 - \frac{\rho_{\partial\Omega}(x)}{t_1}\right), & \rho_{\partial\Omega}(x) \le t_1 \\ 0, & \text{else,} \end{cases}$$

and thus $\frac{\partial\rho}{\partial N}|_{\partial\Omega} = -1$, $|\nabla\rho|_\infty \le 1$ and $\nabla\rho$ is Lipschitz continuous. Furthermore

$$\Delta\rho(x) = \begin{cases} \Delta\rho_{\partial\Omega}(x)\left(1 - \frac{\rho_{\partial\Omega}(x)}{t_1}\right) - \frac{1}{t_1}, & \rho_{\partial\Omega}(x) \le t_1 \\ 0, & \text{else,} \end{cases}$$

and thus by equation (6) on $\{\rho_{\partial\Omega} \le t_1\}$

$$\Delta\rho \ge (d-1)\frac{h_2'}{h_2}(\rho_{\partial\Omega})\left(1 - \frac{\rho_{\partial\Omega}}{t_1}\right) - \frac{1}{t_1} \Rightarrow (\Delta\rho)^- \le \left((d-1)\frac{h_2'}{h_2}(\rho_{\partial\Omega})\left(1 - \frac{\rho_{\partial\Omega}}{t_1}\right) - \frac{1}{t_1}\right)^-.$$

We can still choose $t_1 \in (0, h_2^{-1}(0))$ to obtain

$$|(\Delta\rho)^-|_\infty \le \inf_{t_1 \in (0, h_2^{-1}(0))} \sup_{t \in (0, t_1)} \left((d-1)\frac{h_2'}{h_2}(t)\left(1 - \frac{t}{t_1}\right) - \frac{1}{t_1}\right)^-.$$

□

Inserting $\varphi$ and $\rho$ as defined in Lemma 2.1 and Lemma 2.2 in Proposition 2.2 and Proposition 2.3 we now get explicit constants $K_1, K_2$ and $K_{\partial\Omega,\Omega}$ in terms of bounds on sectional curvature and second fundamental form on the boundary. We state these in the following Proposition.

**Proposition 2.4.** *Let $k_1, k_2 \in \mathbb{R}$ such that $Ric \ge (d-1)k_1$, $sect \le k_2$ and $\gamma_1, \gamma_2 \in \mathbb{R}$ such that $\gamma_1 id \le \Pi \le \gamma_2 id$. Then for $t_0 \in (0, h_2^{-1}(0))$ the assumptions in Proposition 2.1 are fulfilled with*

$$K_1 = \frac{|\Omega|}{|\partial\Omega|^2} \inf_{\varepsilon \in (0,\infty)} \Bigg[\int_0^{t_0} (1+\varepsilon) H_{d-1}(\{\rho_{\partial\Omega} = t\})\left(1 - \frac{t}{t_0}\right)^2$$

$$+ \left(1 + \frac{1}{\varepsilon}\right) C_\Omega H_{d-1}(\{\rho_{\partial\Omega} = t\}) \left(\left(\left((d-1)\frac{h_2'}{h_2}(t)\left(1 - \frac{t}{t_0}\right) - \frac{1}{t_0}\right)^-\right)^2 + \left(\left((d-1)\frac{h_1'}{h_1}(t)\left(1 - \frac{t}{t_0}\right) - \frac{1}{t_0}\right)^+\right)^2\right) dt\Bigg],$$

$$K_2 = 0,$$

$$K_{\partial\Omega,\Omega} = \frac{|\Omega|}{|\partial\Omega|}\left(2C_\Omega^{1/2} + C_\Omega \inf_{t_1 \in (0, h_2^{-1}(0))} \sup_{t \in (0, t_1)} \left((d-1)\frac{h_2'}{h_2}(t)\left(1 - \frac{t}{t_1}\right) - \frac{1}{t_1}\right)^-\right).$$

As explained in Remark 2.1, upper bounds for the optimal $K_{\partial\Omega,\Omega}$ correspond to lower bounds for the first nontrivial Steklov eigenvalue. Thus we now also get a lower bound on the first nontrivial Steklov eigenvalue $\sigma$ that is explicit in terms of upper bounds on sectional curvature and second fundamental form on the boundary:

**Corollary 2.1.** *Let $k_2 \in \mathbb{R}$ such that $sect \le k_2$ and $\gamma_2 \in \mathbb{R}$ such that $\Pi \le \gamma_2 id$. Then*

$$\sigma \ge \left(2C_\Omega^{1/2} + C_\Omega \inf_{t_1 \in (0, h_2^{-1}(0))} \sup_{t \in (0, t_1)} \left((d-1)\frac{h_2'}{h_2}(t)\left(1 - \frac{t}{t_1}\right) - \frac{1}{t_1}\right)^-\right)^{-1}.$$

By inserting the set of constants stated in Proposition 2.4 into Proposition 2.1 we get an explicit upper bound on the Poincaré constant again in terms of bounds on sectional curvature and second fundamental form.

For comparison we state the following obvious upper bound for the optimal Poincaré constant without the interpolation approach:

**Proposition 2.5.** *Assume there exist constants $K_1, K_2$ such that for any $f \in C^1(\Omega)$*

$$\left(\int_\Omega f d\lambda_\Omega - \int_{\partial\Omega} f d\lambda_{\partial\Omega}\right)^2 \leq K_1 \int_\Omega |\nabla f|^2 d\lambda_\Omega + K_2 \int_{\partial\Omega} |\nabla^\tau f|^2 d\lambda_{\partial\Omega},$$

*then it holds for any $\alpha \in (0,1)$*

$$C_\alpha \leq \max\left(C_\Omega + (1-\alpha)K_1, C_{\partial\Omega} + \alpha K_2\right).$$

*Proof.* Let $f \in C^1(\Omega)$, then

$$\begin{aligned} Var_{\lambda_\alpha}(f) =& \alpha Var_{\lambda_\Omega}(f) + (1-\alpha)Var_{\lambda_{\partial\Omega}}(f) + \alpha(1-\alpha)\left(\int_\Omega f d\lambda_\Omega - \int_{\partial\Omega} f d\lambda_{\partial\Omega}\right)^2 \\ \leq& \alpha C_\Omega \int_\Omega |\nabla f|^2 d\lambda_\Omega + (1-\alpha)C_{\partial\Omega} \int_{\partial\Omega} |\nabla^\tau f|^2 d\lambda_{\partial\Omega} \\ &+ \alpha(1-\alpha)\left(K_1 \int_\Omega |\nabla f|^2 d\lambda_\Omega + K_2 \int_{\partial\Omega} |\nabla^\tau f|^2 d\lambda_{\partial\Omega}\right) \\ =& \left(C_\Omega + (1-\alpha)K_1\right)\alpha \int_\Omega |\nabla f|^2 d\lambda_\Omega + \left(C_{\partial\Omega} + \alpha K_2\right)(1-\alpha) \int_{\partial\Omega} |\nabla^\tau f|^2 d\lambda_{\partial\Omega} \\ \leq& \max\left(C_\Omega + (1-\alpha)K_1, C_{\partial\Omega} + \alpha K_2\right)\mathcal{E}_\alpha(f). \end{aligned}$$

□

In the following we consider as examples balls in Euclidean plane and in hyperbolic plane and compare the results obtained above with or without the interpolation approach. While the former example has already been treated in [?], the latter was not included in the setting of [?] due to negative curvature.

**Example 2.1.** Let $\Omega := B_1$ be the unit ball in $\mathbb{R}^2$. In this case the sectional curvature equals $k = 0$, and for the second fundamental form on the boundary we have $\gamma = 1$. The constants $K_1, K_2$ and $K_{\partial\Omega,\Omega}$ are now to be computed for this specific example. The general results in [?][section 3.2] concerning the values for these constants are not applicable. However the same paper also contains a computation adapted to this specific example. In the following we will consider the values for the three constants obtained by computations adapted to this specific example as well as the values obtained by the above general results. For both sets of constants we will compare the upper bounds on Poincaré constants obtained by the interpolation approach with the bounds obtained without interpolation as in Proposition 2.5 and the exact values for the Poincaré constants.
We first recall the results of the computations adapted to the ball example made in [?][section 3.1]:

$$C_\Omega \approx \frac{1}{3.39},\ C_{\partial\Omega} = 1,\ K_1 = \frac{3}{16},\ K_2 = 0,\ K_{\partial\Omega,\Omega} = \frac{1}{2}.$$

The upper bound obtained from Proposition 2.1 is

$$C_\alpha \leq \frac{8(1-\alpha) + 16\alpha C_\Omega + 3\alpha(1-\alpha)}{8(1+\alpha)},$$

while the upper bound obtained for the same values of $K_1, K_2$ and $K_{\partial\Omega,\Omega}$ from Proposition 2.5 is $C_\alpha \leq 1$. Moreover we also refer to [?] for the procedure to calculate the exact values for $C_\alpha, \alpha \in (0,1)$.
Using the results from the previous pages instead, we find different constants: We have $h_1(t) = h_2(t) = 1-t$ from which follows by Lemma 2.1 and Lemma 2.2 that

$$\forall \varepsilon > 0\ \exists \varphi : |\nabla\varphi|_2^2 \leq \varepsilon,\ |\Delta\varphi|_2^2 \leq 1 + \varepsilon$$

and

$$\exists \rho :\ |\nabla\rho|_\infty \leq 1,\ |(\Delta\rho)^-|_\infty \leq 2.$$

Inserting this in Proposition 2.2 and Proposition 2.3 we get

$$K_1' = \frac{C_\Omega}{4},\ K_2' = 0,\ K_{\partial\Omega,\Omega}' \approx 0.8381.$$

Inserting this in Proposition 2.5 results in $C_\alpha \leq 1$ while inserting in Proposition 2.1 we get

$$C_\alpha \leq \max\left(C_\Omega + (1-\alpha)K_1', \frac{(1-\alpha)K_{\partial\Omega,\Omega}' + \alpha C_\Omega + \alpha(1-\alpha)K_1'}{(1-\alpha)K_{\partial\Omega,\Omega}' + \alpha}\right).$$

We depict these results in Figure 1. Note that the green and purple curves overlap. From this we see that the upper bounds obtained from the above general results are only slightly worse than the upper

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FIGURE 1. Exact Poincaré constants (blue), interpolation (yellow) and no interpolation (green) results using computations specifically adapted to the example, interpolation (red) and no interpolation (purple) results using computations not specifically adapted to the example.

bounds obtained by computing with the specific example in mind. Furthermore it is obvious from the proof of Proposition 2.1, that results obtained from the interpolation approach must be at least as good as results obtained without interpolation. However the figure shows for both sets of constants that the interpolation results clearly differ from the no interpolation results and give significantly better bounds than the approach without interpolation. In particular the interpolation approach allows to better meet the decreasing shape of the curve of exact values of $C_\alpha$.

**Example 2.2.** We consider the unit metric ball in the hyperbolic plane and compute the exact Poincaré constants $C_\alpha, \alpha \in (0,1)$ numerically. Again constants $K_1, K_2$ and $K_{\partial\Omega,\Omega}$ are to be computed for this specific example. The general results in [?][section 3.2] concerning the values for these constants are not applicable. Instead we conduct a computation adapted to this specific example to obtain a set of three constants. Furthermore we will consider the values obtained by the above general results. For both sets of constants we will compare the upper bounds on Poincaré constants obtained by the interpolation approach with the bounds obtained without interpolation as in Proposition 2.5 and the exact values for the Poincaré constant.
In more detail we consider the unit ball $B_1(0) \subset \mathbb{R}^2$ with the hyperbolic metric

$$g_h = \frac{4}{(1-|x|^2)^2} g, \tag{7}$$

where $g = (dx^1)^2 + (dx^2)^2$ is the standard metric in $\mathbb{R}^2$, resulting in the space form of constant sectional curvature $K = -1$. $\Omega$ will be a unit ball in this hyperbolic plane. We will start by computing the exact values for $C_\alpha, \alpha \in (0,1)$. Using that $|\Omega| = 2\pi(cosh(1) - 1)$ and $|\partial\Omega| = 2\pi sinh(1)$ the operator $A = A_\alpha$ associated with the Dirichlet form $\mathcal{E}_\alpha$ then becomes

$$A_\alpha f = \Delta f \mathbb{1}_\Omega + \left(\Delta^\tau f - \frac{\alpha}{1-\alpha}\frac{sinh(1)}{cosh(1)-1}\frac{\partial f}{\partial N}\right)\mathbb{1}_{\partial\Omega}.$$

An eigenvector of $-A_\alpha$ for eigenvalue $\lambda \geq 0$ is then a function $f \in D(A_\alpha)$ such that the following system of partial differential equations is fulfilled

$$\begin{cases} \Delta f = -\lambda f & \text{in } \Omega \\ \Delta^\tau f - \frac{\alpha}{1-\alpha}\frac{sinh(1)}{cosh(1)-1}\frac{\partial f}{\partial N} = -\lambda f & \text{on } \partial\Omega \end{cases}.$$

Since $f$ and $A_\alpha f$ are continuous on $\Omega$, this is equivalent to

$$\begin{cases} \Delta f = -\lambda f & \text{in } \Omega \\ \Delta^\tau f - \frac{\alpha}{1-\alpha}\frac{sinh(1)}{cosh(1)-1}\frac{\partial f}{\partial N} = \Delta f & \text{on } \partial\Omega \end{cases}. \tag{8}$$

Following the well-known procedure for the Laplacian with Neumann boundary conditions, see e.g. [?][Chapter 2.5], we introduce spherical coordinates about $x = 0$ by

$$x = r\xi, \ r = tanh(t/2),$$

where $r \in [0,1]$, $t \in [0,\infty)$, $\xi \in \mathbb{S}^1$. $\Omega$ is then characterised by restriction of $t$ to $[0,1]$. In these coordinates (7) reads

$$g_h = (dt)^2 + sinh^2(t)|d\xi|^2.$$

We then separate variables, i.e. $f(t,\xi) = T(t)G(\xi)$. Furthermore we denote by $\square$ the Laplacian on $\mathbb{S}^1$ and by $'$ differentiation with respect to $t$. Using the Laplacian in spherical coordinates, the first equation of (8) becomes

$$sinh(t)^{-1}(sinh(t)T'(t))'G(\xi) + sinh(t)^{-2}T(t)\square G(\xi) = -\lambda T(t)G(\xi).$$

and in terms of $G$ and $T$

$$\begin{cases} \Box G(\xi) + \gamma G(\xi) = 0 \\ (sinh(t)T'(t))' + (\lambda - \gamma sinh(t)^{-2}) sinh(t) T = 0, \end{cases}$$

where $\gamma = l^2,\ l \in \mathbb{N}$ are the eigenvalues of $\Box$ on $\mathbb{S}^1$.
According to [?][Chapter 12.5] the solution $T$ is given via

$$T(t) = P_\nu^\mu(cosh(t)),$$

where $P_\nu^\mu(\cdot)$ is the associated Legendre function of first kind with $\mu$ and $\nu$ given via

$$\mu = l,\ \nu = -\frac{1}{2} \pm \sqrt{-\lambda + \frac{1}{4}}.$$

We thus obtain a two parameter family of eigenfunctions $f_{n,l}(t,\xi) = P_n^l(cosh(t))G_l(\xi)$ where $G_l$ is the eigenfunction for eigenvalue $l$ and $n$ respectively $\lambda_n$ is constrained via the boundary condition as follows: Using that $\Delta^\tau f = \frac{1}{sinh^2(1)} T \Box G$, the second equation of (8) which holds on the boundary, i.e. for $t = 1$, becomes

$$\begin{aligned} \Delta(TG)(1,\xi) &= \frac{1}{sinh^2(1)} T(1) \Box G(\xi) - \frac{sinh(1)}{cosh(1) - 1} \frac{\alpha}{1-\alpha} T'(1) G(\xi) \\ \Leftrightarrow & T''(1) + T'(1) \left( \frac{cosh(1)}{sinh(1)} + \frac{sinh(1)}{cosh(1) - 1} \frac{\alpha}{1-\alpha} \right) = 0 \\ \Leftrightarrow & P_\nu^{l\,\prime\prime}(cosh(1)) sinh^2(1) + P_\nu^{l\,\prime}(cosh(1)) \left( 2cosh(1) + \frac{sinh^2(1)}{cosh(1) - 1} \frac{\alpha}{1-\alpha} \right) = 0. \end{aligned}$$

We consider the (countably many) zero points of this function as a function in $\nu$ and thus obtain a corresponding countable family of values $\lambda_{l,n}$. Thus for $\alpha \in (0,1)$ $\lambda_\alpha := \min_{l,n} \lambda_{l,n}$ is the desired spectral gap and $C_\alpha = 1/\lambda_\alpha$.

Furthermore we need to compute $C_\Omega$ as well as $C_{\partial\Omega}$ for Proposition 2.1. Following the same procedure as above including spherical coordinates and separation of variables we see that eigenfunctions $f$ of the Laplacian on $\Omega$ with Neumann boundary conditions on $\partial\Omega$ are again of the form

$$f(t,\xi) = T(t)G(\xi),\ \text{with}\ T(t) = P_\nu^\mu(cosh(t)),\ \mu = 1,\ \nu = -\frac{1}{2} \pm \sqrt{-\lambda + \frac{1}{4}},$$

where $G$ are eigenfunctions of the Laplacian on $\mathbb{S}^1$ for the eigenvalues $l^2,\ l \in \mathbb{N}$.
Now the boundary condition amounts to

$$\frac{\partial f}{\partial N} = 0 \text{ on } \partial\Omega \ \Leftrightarrow\ P_\nu^{\mu\,\prime}(cosh(1)) = 0.$$

Thus by considering the countably many zero points of this function as a function in $\nu$ we again obtain a corresponding countable family of values $\lambda_{l,n}$. Again $\lambda_1 := \min_{l,n} \lambda_{l,n} \approx 2.9614$ is the desired spectral gap and $C_\Omega = 1/\lambda_1 \approx 0.3377$.
We can furthermore derive $C_{\partial\Omega}$ from the well-known spectrum of the Laplacian on a unit sphere in Euclidean space: Define

$$f : (S_H^1, g_H) \to (S_{\mathbb{R}}^1, g),\ f(x) := \frac{1}{tanh(1/2)} \cdot x,$$

where $S_H^1$ denotes the unit sphere in the hyperbolic plane, and $S_{\mathbb{R}}^1$ denotes the unit sphere in the Euclidean plane. Then for a smooth function $h : S_{\mathbb{R}}^1 \to \mathbb{R}$

$$\Delta_{S_H^1}(h \circ f) = \frac{1}{sinh^2(1)} \Delta_{S_{\mathbb{R}}^1}(h) \circ f.$$

Thus $h$ being an eigenfunction of $\Delta_{S_{\mathbb{R}}^1}$ for the eigenvalue $\lambda$ corresponds to $h \circ f$ being an eigenfunction of $\Delta_{S_H^1}$ for the eigenvalue $\lambda/sinh^2(1)$:

$$\Delta_{S_H^1}(h \circ f) = \frac{1}{sinh^2(1)} \cdot (\Delta_{S_{\mathbb{R}}^1} h) \circ f = \frac{1}{sinh^2(1)} \cdot \lambda \cdot h \circ f.$$

Since the eigenvalues of $\Delta_{S^1_{\mathbb{R}}}$ are known to be $\lambda_k := k^2,\ k \in \mathbb{N}_0$ (see e.g. [?][Chapter 2.4]), we can derive that those of $\Delta_{S^1_H}$ are $\tilde{\lambda}_k = \frac{k^2}{sinh^2(1)},\ k \in \mathbb{N}_0$ and thus $C_{\partial\Omega} = sinh^2(1)$.

We now compute $K_1$ and $K_2$ in a fashion adapted to the specific example. The following computation is similar to the one referenced above in Example 2.1. Using that for $f \in L^1(\partial\Omega)$:

$$\int_{\partial\Omega} f(y)\lambda_{\partial\Omega}(dy) = \int_\Omega f(x/|x|)\lambda_\Omega(dx),$$

we get

$$\begin{aligned}
&\left(\int_\Omega f d\lambda_\Omega - \int_{\partial\Omega} f d\lambda_{\partial\Omega}\right)^2 \\
&\le \int_\Omega (f(x) - f(x/|x|)^2 \lambda_\Omega(dx) \\
&= \frac{1}{\tilde{\lambda}_\Omega(\Omega)} \int_{\partial\Omega}\int_0^1 (f(tanh(t/2)\xi) - f(tanh(1/2)\xi))^2 sinh(t)dtd\xi \\
&= \frac{1}{\tilde{\lambda}_\Omega(\Omega)} \int_{\partial\Omega}\int_0^1 \left(\int_t^1 \frac{d}{ds} f(tanh(s/2)\xi)ds\right)^2 sinh(t)dtd\xi \\
&\le \frac{1}{\tilde{\lambda}_\Omega(\Omega)} \int_{\partial\Omega}\int_0^1 (1-t)\int_t^1 \left(\frac{d}{ds} f(tanh(s/2)\xi)\right)^2 ds\ sinh(t)dtd\xi \\
&= \frac{1}{\tilde{\lambda}_\Omega(\Omega)} \int_{\partial\Omega}\int_0^1 \int_0^s (1-t)sinh(t)dt \left(\frac{d}{ds} f(tanh(s/2)\xi)\right)^2 dsd\xi \\
&= \frac{1}{\tilde{\lambda}_\Omega(\Omega)} \int_{\partial\Omega}\int_0^1 (sinh(s) - (s-1)cosh(s) - 1)\left(\langle \nabla f(tanh(s/2)\xi), \frac{d}{ds} tanh(s/2)\xi\rangle\right)^2 dsd\xi \\
&\le \frac{1}{\tilde{\lambda}_\Omega(\Omega)} \int_{\partial\Omega}\int_0^1 (sinh(s) - (s-1)cosh(s) - 1)\,|\nabla f(tanh(s/2)\xi)|^2 dsd\xi \\
&\le K_1 \frac{1}{\tilde{\lambda}_\Omega(\Omega)} \int_{\partial\Omega}\int_0^1 |\nabla f(tanh(s/2)\xi)|^2 sinh(s)dsd\xi \\
&= K_1 \int_\Omega |\nabla f(x)|^2 \lambda_\Omega(dx),
\end{aligned}$$

where

$$K_1 := \max_{s\in[0,1]} \frac{sinh(s) - (s-1)cosh(s) - 1}{sinh(s)} \le 0.1782.$$

Thus $K_2 = 0$.
As explained in Remark 2.1 we may obtain the optimal constant $K_{\partial\Omega,\Omega}$ in Proposition 2.1 as $\frac{|\Omega|}{|\partial\Omega|}\sigma^{-1}$, where $\sigma$ denotes the first nontrivial Steklov eigenvalue. In the present example the first Steklov eigenvalue is $coth(1) - tanh(1/2)$, cf. [?] and thus

$$K_{\partial\Omega,\Omega} = \frac{cosh(1) - 1}{sinh(1)}(coth(1) - tanh(1/2))^{-1} \approx 0.5431.$$

Inserting this in Proposition 2.5 results in $C_\alpha \le C_{\partial\Omega} \approx 1.3811$. Furthermore we insert the same set of constants in Proposition 2.1.
Using the results from the previous pages instead, we find different constants: We have $h_1(t) = h_2(t) = cosh(t) - coth(1)sinh(t)$ from which follows by Lemma 2.1 and Lemma 2.2 that

$$\forall \varepsilon > 0\ \exists \varphi :\ |\nabla\varphi|_2^2 \le \varepsilon,\ |\Delta\varphi|_2^2 \le \frac{1}{2(cosh(1)-1)} + \varepsilon,$$

and

$$\exists \rho :\ |\nabla\rho|_\infty \le 1,\ |(\Delta\rho)^-|_\infty \le 2.3131.$$

Inserting this in Proposition 2.2 and Proposition 2.3 we get

$$K_1' = \frac{cosh(1) - 1}{2(sinh(1))^2}\cdot C_\Omega \approx 0.0664,\ K_2' = 0,\ K'_{\partial\Omega,\Omega} \approx 0.8981.$$

Inserting this in Proposition 2.5 results in $C_\alpha \le C_{\partial\Omega}$. Furthermore we insert the same set of constants in Proposition 2.1.

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FIGURE 2. Exact Poincaré constants (blue), interpolation (yellow) and no interpolation (green) results using computations specifically adapted to the example, interpolation (red) and no interpolation (purple) results using computations not specifically adapted to the example.

The curves for actual Poincaré Constants and respective upper bounds via Proposition 2.1 as well as via Proposition 2.5 obtained by plugging in the quantities collected above are depicted in Figure 2. Note that the green and purple curves overlap. Again from the figure we may see that our general results are only slightly worse than the ones obtained from computations specifically adapted to the example, and that the interpolation approach results in a significant improvement compared to no interpolation. Our attempts to transfer the computations made above for the specific example to a more general setting have not been successful, as they lead to significantly worse results.

Finally instead of Brownian motion with sticky reflecting boundary diffusion we show that the above results may as well be used to give upper bounds for Brownian motion with sticky reflection from the boundary (but without boundary diffusion). I.e. under the same assumptions on $\Omega$ as above we consider a diffusion on $\Omega$ with Feller generator $(\mathcal{D}(\hat{A}), \hat{A})$

$$\mathcal{D}(\hat{A}) = \{f \in C(\Omega) \mid \hat{A}f \in C(\Omega)\}$$
$$\hat{A}f = \Delta f \mathbb{1}_\Omega - \gamma \frac{\partial f}{\partial N} \mathbb{1}_{\partial\Omega},$$

where $\frac{\partial f}{\partial N}$ is the outer normal derivative and $\gamma > 0$, which corresponds to inward sticky reflection at $\partial\Omega$. A construction was given again in [?] using Dirichlet forms. We choose $\alpha \in [0,1]$, such that

$$\frac{\alpha}{1-\alpha} \frac{|\partial\Omega|}{|\Omega|} = \gamma.$$

and set

$$\lambda_\alpha := \alpha\lambda_\Omega + (1-\alpha)\lambda_{\partial\Omega}.$$

We find that $-\hat{A}$ is $\lambda_\alpha$-symmetric with spectral gap $\hat{\sigma}_\alpha$ characterised by the Rayleigh quotient resp. Poincaré constant $\hat{C}_\alpha$

$$\hat{\sigma}_\alpha = \inf_{\substack{f \in C^1(\Omega) \\ Var_{\lambda_\alpha}(f) > 0}} \frac{\hat{\mathcal{E}}_\alpha(f)}{Var_{\lambda_\alpha}(f)}, \qquad \hat{C}_\alpha := \hat{\sigma}_\alpha^{-1} = \sup_{\substack{f \in C^1(\Omega) \\ \hat{\mathcal{E}}_\alpha(f) > 0}} \frac{Var_{\lambda_\alpha}(f)}{\hat{\mathcal{E}}_\alpha(f)}$$

where

$$Var_{\lambda_\alpha}(f) = \int_\Omega f^2 d\lambda_\alpha - \left(\int_\Omega f d\lambda_\alpha\right)^2, \quad \hat{\mathcal{E}}_\alpha(f) = \alpha \int_\Omega |\nabla f|^2 d\lambda_\Omega, \; f \in C^1(\Omega)$$

By $C_\Omega$ and $C_{\partial\Omega}$ we still denote the usual (Neumann) Poincaré constants of $\Omega$ and $\partial\Omega$ respectively, which we assume to be known.
The eigenvalue problem corresponding to the Poincaré constant may be stated as

$$\begin{cases} \Delta f = -\lambda f & \text{in } \Omega, \\ -\gamma \frac{\partial f}{\partial N} = -\lambda f & \text{on } \partial\Omega. \end{cases}$$

This type of eigenvalue problems with eigenvalue featured in the boundary condition has been of separate interest, see e.g. [?], [?], [?], [?], [?]. For Brownian motion with sticky reflection spectral asymptotics have been examined e.g. in [?], however we are not aware of results on the spectral gap.

**Proposition 2.6.** *Assume there exist constants $K_{\partial\Omega,\Omega}, K_1$ such that for any $f \in C^1(\Omega)$*

$$Var_{\lambda_{\partial\Omega}} f \le K_{\partial\Omega,\Omega} \int_\Omega |\nabla f|^2 d\lambda_\Omega$$

*and*

$$\left(\int_\Omega f d\lambda_\Omega - \int_{\partial\Omega} f d\lambda_{\partial\Omega}\right)^2 \le K_1 \int_\Omega |\nabla f|^2 d\lambda_\Omega,$$

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FIGURE 3. Exact Poincaré constants (blue), estimates using computations specifically adapted to the example (yellow), estimates using computations not specifically adapted to the example (green).

*then it holds for any* $\alpha \in (0,1)$

$$\hat{C}_\alpha \le C_\Omega + \frac{(1-\alpha)}{\alpha} K_{\partial\Omega,\Omega} + (1-\alpha)K_1.$$

*Proof.* Let $f \in C^1(\Omega)$

$$\begin{aligned} Var_{\lambda_\alpha}(f) &= \alpha Var_{\lambda_\Omega}(f) + (1-\alpha)Var_{\lambda_{\partial\Omega}}(f) + \alpha(1-\alpha)\left(\int_\Omega f d\lambda_\Omega - \int_{\partial\Omega} f d\lambda_{\partial\Omega}\right)^2 \\ &\le \alpha C_\Omega \int_\Omega |\nabla f|^2 d\lambda_\Omega + (1-\alpha)K_{\partial\Omega,\Omega}\int_\Omega |\nabla f|^2 d\lambda_\Omega + \alpha(1-\alpha)K_1 \int_\Omega |\nabla f|^2 d\lambda_\Omega \\ &= \left(C_\Omega + \frac{(1-\alpha)}{\alpha}K_{\partial\Omega,\Omega} + (1-\alpha)K_1\right)\alpha \int_\Omega |\nabla f|^2 d\lambda_\Omega. \end{aligned}$$

□

For $f \in C^1(\Omega)$ with $Var_{\lambda_{\partial\Omega}}(f) > 0$ the term $Var_{\lambda_\alpha}(f)$ stays positive as $\alpha$ tends to zero while $\hat{\mathcal{E}}_\alpha(f)$ vanishes. Thus $\hat{C}_\alpha$ blows up as $\alpha$ tends to zero and accordingly so does the bound on $\hat{C}_\alpha$ proven in Proposition 2.6.
Of course the interpolation approach from above is not of any use anymore in this setting.
As we have previously shown that equation (2) can be fulfilled with $K_2 = 0$, we may now insert $K_1$ and $K_{\partial\Omega,\Omega}$ as computed above.
Thus in sum the results obtained above for Brownian motion with sticky reflecting boundary diffusion may also be used for the case without boundary diffusion precisely for the reason that we were able to show that equation (2) is fulfilled with $K_2 = 0$
If we make the more strict assumption of respective positive lower bounds $k_R$ on Ricci curvature on $\Omega$ and $\gamma$ on second fundamental form on $\partial\Omega$ as in [?][section 3.2] we may as well insert the set of constants $K_1 = \frac{d-1}{dk_R}, K_2 = 0, K_{\partial\Omega,\Omega} = \frac{|\Omega|}{|\partial\Omega|}\frac{2}{\gamma}$ obtained there. Note that this is possible as we again have $K_2 = 0$.

We again consider as examples a unit ball in $\mathbb{R}^2$ and a unit metric ball in the hyperbolic plane.

**Example 2.3.** As in Example 2.1 we consider a unit ball in $\mathbb{R}^2$. In order to compute the exact values for $\hat{C}_\alpha, \alpha \in (0,1)$ we again proceed as described in [?][section 3.1] and only need to adapt the boundary condition. I.e. an eigenfunction $f$ fulfills

$$\begin{cases} \Delta f = -\lambda f & \text{in } \Omega \\ -\frac{2\alpha}{1-\alpha}\frac{\partial f}{\partial N} = \Delta f & \text{on } \partial\Omega \end{cases}$$

and by passing to polar coordinates and seperating variables we obtain a family $\hat{\lambda}_{m,l}, m, l \in \mathbb{N}_0$ characterised by

$$\sqrt{\lambda} J_m''(\sqrt{\lambda}) + J_m'(\sqrt{\lambda})\frac{1+\alpha}{1-\alpha} - J_m(\sqrt{\lambda})\frac{m}{\lambda} = 0,$$

where $J_m, m \in \mathbb{N}_0$ are the Bessel functions of the first kind. We then get $\hat{\lambda}_\alpha = \min_{m,l\in\mathbb{N}_0} \hat{\lambda}_{m,l}$ and $\hat{C}_\alpha = \frac{1}{\hat{\lambda}_\alpha}$.

In order to calculate the explicit values for the upper bound stated in Proposition 2.6 we need $C_\Omega, K_{\partial\Omega,\Omega}$ and $K_1$. All of these have been computed in Example 2.1, in particular $K_{\partial\Omega,\Omega}$ and $K_1$ have been computed once in a manner adapted to the specific example and once from the previously stated general results (the latter marked by $'$).
The curves for actual Poincaré Constants and respective upper bounds via Proposition 2.6 obtained by plugging in the quantities collected above are depicted in Figure 3. As mentioned before $\hat{C}_\alpha$ blows up as $\alpha$ tends to zero. We therefore only consider $\alpha \ge 0.2$ for the plot.

Figure 3 suggests that Proposition 2.6 offers a precise upper bound for $\hat{C}_\alpha$ that is (in particular for small $\alpha$) highly depended on how close the values of $K_1$ and $K_{\partial\Omega,\Omega}$ are to the optimal constants in hp wobd plot.png hp wobd plot.pdf hp wobd plot.jpg hp wobd plot.mps hp wobd plot.jpeg hp wobd plot.jbig2 hp wobd plot.jb2 hp wobd plot.PNG hp wobd plot.PDF hp wobd plot.JPG hp wobd plot.JPEG hp wobd plot.JBIG2 hp wobd plot.JB2

FIGURE 4. Exact Poincaré constants (blue), estimates using computations specifically adapted to the example (yellow) and estimates using computations not specifically adapted to the example (green).

equation (1) and equation (2). More precisely for small values of $\alpha$ it is mainly the precision of the value for $K_{\partial\Omega,\Omega}$ that is relevant. Note that the value $K'_{\partial\Omega,\Omega}$ obtained from our general results is worse than $K_{\partial\Omega,\Omega}$ obtained from computations adapted to the specific example, while the opposite is true for the values of $K'_1$ and $K_1$.

**Example 2.4.** As in Example 2.2 we consider a unit ball in the hyperbolic plane and use the notation introduced above. To calculate the exact values of the Poincaré constants we proceed as explained in Example 2.2 and only need to adapt the boundary condition (see the second equation in (8)) to

$$\frac{-\alpha}{1-\alpha}\frac{sinh(1)}{cosh(1)-1}\frac{\partial f}{\partial N} = \Delta f \text{ on } \partial\Omega.$$

Inserting $f(t,\xi) = T(t)G(\xi)$ and then using $T(t) = P^{\mu}_{\nu}(cosh(t))$ as before, this results in

$$\begin{aligned}
&\frac{-\alpha}{1-\alpha}\frac{sinh(1)}{cosh(1)-1}T'(1)G(\xi) = \left(\frac{cosh(1)}{sinh(1)}T'(1) + T''(1)\right)G(\xi) + \frac{T(1)}{sinh^2(1)}\square G(\xi)\\
\Leftrightarrow &T'(1)\left(\frac{\alpha}{1-\alpha}\frac{sinh(1)}{cosh(1)-1} + \frac{cosh(1)}{sinh(1)}\right) + T''(1) - \gamma\frac{T(1)}{sinh^2(1)} = 0\\
\Leftrightarrow &P^{\mu}_{\nu}{}''(cosh(1))sinh^2(1) + P^{\mu}_{\nu}{}'(cosh(1))\left(\frac{\alpha}{1-\alpha}\frac{sinh^2(1)}{cosh(1)-1} + 2cosh(1)\right) - \gamma\frac{P^{\mu}_{\nu}(cosh(1))}{sinh^2(1)} = 0.
\end{aligned}$$

By considering the (countably many) zero points of this function as a function in $\nu$ we obtain a corresponding countable family of values $\hat{\lambda}_{l,n}$. For $\alpha \in (0,1)$ $\hat{\lambda}_{\alpha} := \min_{l,n}\hat{\lambda}_{l,n}$ is the desired spectral gap and $\hat{C}_{\alpha} = 1/\hat{\lambda}_{\alpha}$.
In order to calculate the explicit values for the upper bound stated in Proposition 2.6 we may again use the values for $C_{\Omega}, K_{\partial\Omega,\Omega}$ and $K_1$ as computed in Example 2.2 in a manner adapted to the specific example or from the previously stated general results (the latter marked by $'$).
The curves for actual Poincaré Constants and respective upper bounds via Proposition 2.6 obtained by plugging in these quantities are depicted in Figure 4. Again we only consider $\alpha \geq 0.2$ for the plot, as $\hat{C}_{\alpha}$ blows up as $\alpha$ tends to zero.

From Figure 4 we may again see that the precision of the upper bound for $\hat{C}_{\alpha}$ offered in Proposition 2.6 depends particularly for small $\alpha$ highly on how close the values of $K_1$ and $K_{\partial\Omega,\Omega}$ are to the optimal constants in equation (1) and equation (2). Note that in this example again the value for $K'_{\partial\Omega,\Omega}$ obtained from our general results is worse than $K_{\partial\Omega,\Omega}$ obtained from computations adapted to the specific example, while the opposite is true for the values of $K'_1$ and $K_1$.

## 3. BOUNDARY-INTERIOR INEQUALITIES

In the following we present some Boundary-Interior Inequalities that can be proved in a similar fashion as Proposition 2.2. They may be seen as alternatives for Proposition 2.2 but might also be of independent interest.
In the proof of Proposition 2.3 we have seen that:

**Proposition 3.1.** *For any $\rho \in C^1(\Omega)$ such that $\frac{\partial\rho}{\partial N}|_{\partial\Omega} = -1$ and $\nabla\rho$ is Lipschitz continuous on $\Omega$ it holds*

$$\begin{aligned}
\int_{\partial\Omega} f^2 d\lambda_{\partial\Omega} &\leq \frac{|\Omega|}{|\partial\Omega|}\left\{2|\nabla\rho|_{\infty}\left(\int_{\Omega} f^2 d\lambda_{\Omega}\cdot\int_{\Omega}|\nabla f|^2 d\lambda_{\Omega}\right)^{1/2} + |(\Delta\rho)^{-}|_{\infty}\int_{\Omega} f^2 d\lambda_{\Omega}\right\}\\
&\leq \left\{2|\nabla\rho|_{\infty}C_{\Omega}^{1/2} + |(\Delta\rho)^{-}|_{\infty}C_{\Omega}\right\}\frac{|\Omega|}{|\partial\Omega|}\int_{\Omega}|\nabla f|^2 d\lambda_{\Omega}
\end{aligned}$$

*$\forall f \in C^1(\Omega)$ with $\int_{\Omega} f d\lambda_{\Omega} = 0$.*

The statement in Proposition 3.1 is stronger than needed for equation (2) in Proposition 2.1 because we bound from above the integral of the squared function as opposed to the square of the integral. Nevertheless the proof of the next corollary follows directly as we may assume for equation (2) without loss of generality that $\int_\Omega f d\lambda_\Omega = 0$ for any $f \in C^1(\Omega)$. We thus get an upper bound for $K_1$ in equation (2) that is alternative to Proposition 2.2.

**Corollary 3.1.** *For any $\rho \in C^1(\Omega)$ such that $\frac{\partial \rho}{\partial N}|_{\partial\Omega} = -1$ and $\nabla\rho$ is Lipschitz continuous on $\Omega$ equation* (2) *in Proposition 2.1 is fulfilled with $K_2 = 0$ and*

$$K_1 = \left\{2|\nabla\rho|_\infty C_\Omega^{1/2} + |(\Delta\rho)^-|_\infty C_\Omega\right\} \frac{|\Omega|}{|\partial\Omega|}.$$

Additionally we think that this computation is of independent interest for the following reason:

**Remark 3.1.** We may calculate as in the proof of Proposition 3.1 to obtain for $f \in C^1(\Omega)$ (but not necessarily centered on $\Omega$):

$$\begin{aligned}\int_{\partial\Omega} f^2 d\lambda_{\partial\Omega} &\le \frac{|\Omega|}{|\partial\Omega|}\left\{2|\nabla\rho|_\infty \left(\int_\Omega f^2 d\lambda_\Omega \cdot \int_\Omega |\nabla f|^2 d\lambda_\Omega\right)^{1/2} + |(\Delta\rho)^-|_\infty \int_\Omega f^2 d\lambda_\Omega\right\} \\ &\le \frac{|\Omega|}{|\partial\Omega|}\left\{|\nabla\rho|_\infty \int_\Omega |\nabla f|^2 d\lambda_\Omega + (|\nabla\rho|_\infty + |(\Delta\rho)^-|_\infty) \int_\Omega f^2 d\lambda_\Omega\right\}.\end{aligned}$$

From this it follows that for $K_3 := \frac{|\Omega|}{|\partial\Omega|}(|\nabla\rho|_\infty + |(\Delta\rho)^-|_\infty)$

$$|f|^2_{L^2(\partial\Omega,\lambda_{\partial\Omega})} \le K_3 |f|^2_{W^{1,2}(\Omega,\lambda_\Omega)} \Leftrightarrow |f|_{L^2(\partial\Omega,\lambda_{\partial\Omega})} \le \sqrt{K_3}|f|_{W^{1,2}(\Omega,\lambda_\Omega)}.$$

As $W^{1,2}(\Omega, \lambda_\Omega)$ is the completion of smooth functions whose derivatives up to degree 1 are in $L^2$, the inequality also holds for all functions in $W^{1,2}(\Omega)$. Thus via stating a specific constant $K_3$, as can be obtained from Lemma 2.2, we also give an upper bound for the norm of the Trace operator $|_{\partial\Omega} : W^{1,2}(\Omega) \to L^2(\partial\Omega)$ that is explicit in terms of upper bounds on sectional curvature and second fundamental form on the boundary. An optimal upper bound in terms of the geometry of $\Omega$ seems to be unknown in this form as of yet.

**Proposition 3.2.** *Let $k_2 \in \mathbb{R}$ such that sect $\le k_2$ and $\gamma_2 \in \mathbb{R}$ such that $\Pi \le \gamma_2 id$. Then the norm of the Trace operator $|_{\partial\Omega} : W^{1,2}(\Omega) \to L^2(\partial\Omega)$ is bounded from above by*

$$\left(\frac{|\Omega|}{|\partial\Omega|}\left(1 + \inf_{t_1 \in (0, h_2^{-1}(0))} \sup_{t \in (0,t_1)} \left((d-1)\frac{h_2'}{h_2}(t)\left(1 - \frac{t}{t_1}\right) - \frac{1}{t_1}\right)^-\right)\right)^{1/2}.$$

It is known that on a smooth, compact $d$-dimensional Riemannian manifold $(\Omega, g)$ for $q \in [1, d)$ and $\frac{1}{p} = \frac{1}{q} - \frac{1}{d}$ (and thus for all $p \in [1, \frac{qd}{d-q}]$) there is a constant $C_{p,q}$ such that for $f \in H^{1,q}(\Omega)$:

$$\left(\int_\Omega |f - \bar{f}|^p d\lambda_\Omega\right)^{1/p} \le C_{p,q}\left(\int_\Omega |\nabla f|^q d\lambda_\Omega\right)^{1/q},$$

where $\bar{f} = \int_\Omega f d\lambda_\Omega$.

In terms of these Sobolev-Poincaré constants we may also show a generalisation of Proposition 3.1:

**Proposition 3.3.** *Let $(\Omega, g)$ be a smooth, compact Riemannian manifold of dimension $d \ge 3$, with a connected boundary. For any $\rho \in C^1(\Omega)$ such that $\frac{\partial\rho}{\partial N}|_{\partial\Omega} = -1$ and $\nabla\rho$ is Lipschitz continuous on $\Omega$ it holds $\forall f \in C^1(\Omega)$ with $\int_\Omega f d\lambda_\Omega = 0$*

$$\left(\int_{\partial\Omega} |f|^p d\lambda_{\partial\Omega}\right)^{2/p} \le \left(\left(\frac{|\Omega|}{|\partial\Omega|}|\nabla\rho|_\infty p\right)^{2/p} C_{2(p-1),2}^{2(p-1)/p} + \left(\frac{|\Omega|}{|\partial\Omega|}|(\Delta\rho)^-|_\infty\right)^{2/p} C_{p,2}^2\right)\int_\Omega |\nabla f|^2 d\lambda_\Omega,\ p \in \left[\frac{3}{2}, \frac{2d-2}{d-2}\right].$$

*Proof.* We may calculate as in the previous proofs to obtain

$$
\begin{aligned}
\left(\int_{\partial\Omega}|f|^p d\lambda_{\partial\Omega}\right)^{2/p} &\leq \left(\frac{|\Omega|}{|\partial\Omega|}|\nabla\rho|_\infty p\int_\Omega |f|^{p-1}|\nabla f|d\lambda_\Omega + \frac{|\Omega|}{|\partial\Omega|}|(\Delta\rho)^-|_\infty\int_\Omega |f|^p d\lambda_\Omega\right)^{2/p}\\
&\leq \left(\frac{|\Omega|}{|\partial\Omega|}|\nabla\rho|_\infty p\left(\int_\Omega |f|^{2(p-1)}d\lambda_\Omega\right)^{1/2}\left(\int_\Omega|\nabla f|^2 d\lambda_\Omega\right)^{1/2} + \frac{|\Omega|}{|\partial\Omega|}|(\Delta\rho)^-|_\infty\int_\Omega |f|^p d\lambda_\Omega\right)^{2/p}\\
&\leq\left(\frac{|\Omega|}{|\partial\Omega|}|\nabla\rho|_\infty p\left(C_{2(p-1),2}\left(\int_\Omega|\nabla f|^2 d\lambda_\Omega\right)^{1/2}\right)^{(p-1)}\left(\int_\Omega|\nabla f|^2 d\lambda_\Omega\right)^{1/2}\right.\\
&\quad\left.+\frac{|\Omega|}{|\partial\Omega|}|(\Delta\rho)^-|_\infty\left(C_{p,2}\left(\int_\Omega|\nabla f|^2 d\lambda_\Omega\right)^{1/2}\right)^p\right)^{2/p}\\
&\leq\left(\frac{|\Omega|}{|\partial\Omega|}|\nabla\rho|_\infty p\left(C_{2(p-1),2}\left(\int_\Omega|\nabla f|^2 d\lambda_\Omega\right)^{1/2}\right)^{(p-1)}\left(\int_\Omega|\nabla f|^2 d\lambda_\Omega\right)^{1/2}\right)^{2/p}\\
&\quad+\left(\frac{|\Omega|}{|\partial\Omega|}|(\Delta\rho)^-|_\infty\right)^{2/p}C_{p,2}^2\int_\Omega|\nabla f|^2 d\lambda_\Omega\\
&=\left(\frac{|\Omega|}{|\partial\Omega|}|\nabla\rho|_\infty p\right)^{2/p}\left(C_{2(p-1),2}^2\int_\Omega|\nabla f|^2 d\lambda_\Omega\right)^{(p-1)/p}\left(\int_\Omega|\nabla f|^2 d\lambda_\Omega\right)^{1/p}\\
&\quad+\left(\frac{|\Omega|}{|\partial\Omega|}|(\Delta\rho)^-|_\infty\right)^{2/p}C_{p,2}^2\int_\Omega|\nabla f|^2 d\lambda_\Omega\\
&=\left(\frac{|\Omega|}{|\partial\Omega|}|\nabla\rho|_\infty p\right)^{2/p}C_{2(p-1),2}^{2(p-1)/p}\int_\Omega|\nabla f|^2 d\lambda_\Omega+\left(\frac{|\Omega|}{|\partial\Omega|}|(\Delta\rho)^-|_\infty\right)^{2/p}C_{p,2}^2\int_\Omega|\nabla f|^2 d\lambda_\Omega\\
&=\left(\left(\frac{|\Omega|}{|\partial\Omega|}|\nabla\rho|_\infty p\right)^{2/p}C_{2(p-1),2}^{2(p-1)/p}+\left(\frac{|\Omega|}{|\partial\Omega|}|(\Delta\rho)^-|_\infty\right)^{2/p}C_{p,2}^2\right)\int_\Omega|\nabla f|^2 d\lambda_\Omega.
\end{aligned}
$$

Here we have used the Sobolev-Poincaré inequalities associated with $C_{2(p-1),2}$ and $C_{p,2}$. Note therefor that for $p\in\left[\frac{3}{2},\frac{2d-2}{d-2}\right]$ it holds $p, 2(p-1)\in\left[1,\frac{2d}{d-2}\right]$. □

Again explicit constants may be obtained from Lemma 2.2 in terms of upper bounds on sectional curvature and second fundamental form on the boundary.

## 4. Logarithmic Sobolev Inequality

Using the notation from above we say that a (tight) logarithmic Sobolev inequality is fulfilled if

$$\exists L_\alpha\geq 0 \text{ s.t. } Ent_{\lambda_\alpha}(f^2)\leq L_\alpha\cdot\mathcal{E}_\alpha(f)\ \forall f\in C^1(\Omega).$$

where

$$
\begin{aligned}
Ent_{\lambda_\alpha}(f) &= \int_\Omega f^2 log(f^2)d\lambda_\alpha - \left(\int_\Omega f^2 d\lambda_\alpha\right)log\left(\int_\Omega f^2 d\lambda_\alpha\right)\\
\mathcal{E}_\alpha(f) &= \alpha\int_\Omega|\nabla f|^2 d\lambda_\Omega + (1-\alpha)\int_{\partial\Omega}|\nabla^\tau f|^2 d\lambda_{\partial\Omega},\ f\in C^1(\Omega)
\end{aligned}
$$

and $\nabla^\tau$ denotes the tangential derivative operator on $\partial\Omega$.
In the following by $L_\alpha$ we will denote the optimal such constant. By $L_\Omega$ respectively $L_{\partial\Omega}$ we will denote the optimal logarithmic Sobolev constant associated to the Laplace operator on $\Omega$ with Neumann boundary conditions and the logarithmic Sobolev constant associated to the Laplace-Beltrami operator on $\partial\Omega$. We assume that $L_\Omega$ and $L_{\partial\Omega}$ (or respective upper bounds for them) are known. We aim at bounding $L_\alpha$ for $\alpha\in(0,1)$ from above.
We consider here the entropy with respect to $\lambda_\alpha$ which is a mixture or more specifically a convex combination of the two measures $\lambda_\Omega$ and $\lambda_{\partial\Omega}$. The entropy with respect to mixtures of two measures such as $\lambda_\alpha$ has been considered previously e.g. in [?], [?], [?]. We first show an analogue of [?][Proposition 2.1] for the entropy with respect to $\lambda_\alpha$:

**Proposition 4.1.** *Assume there exist constants $K_{\partial\Omega,\Omega}, L_{\partial\Omega,\Omega}, K_1, K_2$ such that $\forall f \in C^1(\Omega)$:*

$$Var_{\lambda_{\partial\Omega}}(f) \leq K_{\partial\Omega,\Omega} \int_\Omega |\nabla f|^2 d\lambda_\Omega \tag{9}$$

$$Ent_{\lambda_{\partial\Omega}}(f^2) \leq L_{\partial\Omega,\Omega} \int_\Omega |\nabla f|^2 d\lambda_\Omega \tag{10}$$

$$\left(\int_\Omega f d\lambda_\Omega - \int_{\partial\Omega} f d\lambda_{\partial\Omega}\right)^2 \leq K_1 \int_\Omega |\nabla f|^2 d\lambda_\Omega + K_2 \int_{\partial\Omega} |\nabla^\tau f|^2 d\lambda_{\partial\Omega}. \tag{11}$$

*Then it holds for any $\alpha \in (0,1)$*

$$L_\alpha \leq \max\left\{a, d-e-\theta, \min\left[a+c+b\left(\frac{d-a-c-\theta}{e+b}\right), a+c\left(\frac{d-a}{c+\theta}\right), a+b\left(\frac{d-a}{e+b}\right), a+b+c\left(\frac{d-a-e-b}{c+\theta}\right)\right]\right\},$$

*where*

$$a = L_\Omega + \frac{(1-\alpha)(log(\alpha)-log(1-\alpha))}{2\alpha-1}(C_\Omega+K_1), \quad b = \frac{1-\alpha}{\alpha}L_{\partial\Omega,\Omega},$$
$$c = \frac{(1-\alpha)(log(\alpha)-log(1-\alpha))}{2\alpha-1}K_{\partial\Omega,\Omega}, \quad d = L_{\partial\Omega} + \frac{\alpha(log(\alpha)-log(1-\alpha))}{2\alpha-1}(C_{\partial\Omega}+K_2)$$
$$e = L_{\partial\Omega}, \quad \theta = \frac{\alpha(log(\alpha)-log(1-\alpha))}{2\alpha-1}C_{\partial\Omega}.$$

*Proof.* Applying (9) and (10) we can estimate for any $f \in C^1(\Omega)$:

$$Var_{\lambda_{\partial\Omega}}(f) \leq tK_{\partial\Omega,\Omega}\int_\Omega |\nabla f|^2 d\lambda_\Omega + (1-t)C_{\partial\Omega}\int_{\partial\Omega}|\nabla^\tau f|^2 d\lambda_{\partial\Omega}$$
$$Ent_{\lambda_{\partial\Omega}}(f^2) \leq sL_{\partial\Omega,\Omega}\int_\Omega |\nabla f|^2 d\lambda_\Omega + (1-s)L_{\partial\Omega}\int_{\partial\Omega}|\nabla^\tau f|^2 d\lambda_{\partial\Omega}$$

for any $s,t \in [0,1]$. We apply this in the following after first using a decomposition of the entropy with respect to the mixture of two measures as well as an optimal logarithmic Sobolev inequality for Bernoulli measures as described in [**?**][section 4]:

$$\begin{aligned}
Ent_{\lambda_\alpha}(f^2) \leq & \alpha Ent_{\lambda_\Omega}(f^2) + (1-\alpha)Ent_{\lambda_{\partial\Omega}}(f^2) \\
& + \frac{\alpha(1-\alpha)(log(\alpha)-log(1-\alpha))}{2\alpha-1}\left(Var_{\lambda_\Omega}(f) + Var_{\lambda_{\partial\Omega}}(f) + (\mathbb{E}_{\lambda_\Omega}(f) - \mathbb{E}_{\lambda_{\partial\Omega}}(f))^2\right) \\
\leq & \int_\Omega |\nabla f|^2 d\lambda_\Omega \left(\alpha L_\Omega + (1-\alpha)sL_{\partial\Omega,\Omega} + \frac{\alpha(1-\alpha)(log(\alpha)-log(1-\alpha))}{2\alpha-1}(C_\Omega + tK_{\partial\Omega,\Omega} + K_1)\right) \\
& + \int_{\partial\Omega}|\nabla^\tau f|^2 d\lambda_{\partial\Omega}\left((1-\alpha)(1-s)L_{\partial\Omega} + \frac{\alpha(1-\alpha)(log(\alpha)-log(1-\alpha))}{2\alpha-1}((1-t)C_{\partial\Omega}+K_2)\right).
\end{aligned}$$

And thus

$$\begin{aligned}
L_\alpha \leq & \inf_{s,t\in[0,1]} \max\{L_\Omega + \frac{(1-\alpha)}{\alpha}sL_{\partial\Omega,\Omega} + \frac{(1-\alpha)(log(\alpha)-log(1-\alpha))}{2\alpha-1}(C_\Omega + tK_{\partial\Omega,\Omega}+K_1), \\
& \qquad (1-s)L_{\partial\Omega} + \frac{\alpha(log(\alpha)-log(1-\alpha))}{2\alpha-1}((1-t)C_{\partial\Omega}+K_2)\} \\
= & \inf_{s,t\in[0,1]} \max\{L_\Omega + \frac{(1-\alpha)(log(\alpha)-log(1-\alpha))}{2\alpha-1}(C_\Omega+K_1) + s\cdot\frac{(1-\alpha)}{\alpha}L_{\partial\Omega,\Omega} + \\
& \qquad t\cdot\frac{(1-\alpha)(log(\alpha)-log(1-\alpha))}{2\alpha-1}\cdot K_{\partial\Omega,\Omega}, \\
& \qquad L_{\partial\Omega} + \frac{\alpha(log(\alpha)-log(1-\alpha))}{2\alpha-1}(C_{\partial\Omega}+K_2) - sL_{\partial\Omega} - t\cdot\frac{\alpha(log(\alpha)-log(1-\alpha))}{2\alpha-1}\cdot C_{\partial\Omega}\}.
\end{aligned}$$

For any $a,b,c,d,e,f \in \mathbb{R}_{\geq 0}$ it holds

$$\begin{aligned}
&\inf_{s,t\in[0,1]} \max(a+sb+tc, d-se-t\theta)\\
&=\begin{cases}
a, & \text{if } a>d\\
d-e-\theta, & \text{if } d-e-\theta > a+b+c\\
a+c+b\cdot\left(\frac{d-a-c-\theta}{e+b}\right), & \text{if } a\leq d, d-e-\theta\leq a+b+c, b-c\cdot\frac{e+b}{c+\theta}\geq 0, \frac{d-a-c-\theta}{e+b}\geq 0\\
a+c\cdot\left(\frac{d-a}{c+\theta}\right), & \text{if } a\leq d, d-e-\theta\leq a+b+c, b-c\cdot\frac{e+b}{c+\theta}\geq 0, \frac{d-a-c-\theta}{e+b}< 0\\
a+b\cdot\left(\frac{d-a}{e+b}\right), & \text{if } a\leq d, d-e-\theta\leq a+b+c, b-c\cdot\frac{e+b}{c+\theta}< 0, \frac{d-a}{e+b}\leq 1\\
a+b+c\cdot\left(\frac{d-a-e-b}{c+\theta}\right), & \text{if } a\leq d, d-e-\theta\leq a+b+c, b-c\cdot\frac{e+b}{c+\theta}< 0, \frac{d-a}{e+b}> 1
\end{cases}\\
&=\max\left\{a, d-e-\theta, \min\left[a+c+b\left(\frac{d-a-c-\theta}{e+b}\right), a+c\left(\frac{d-a}{c+\theta}\right), a+b\left(\frac{d-a}{e+b}\right), a+b+c\left(\frac{d-a-e-b}{c+\theta}\right)\right]\right\}.
\end{aligned}$$

□

In spite of its complicated structure this result allows to estimate variance as well as entropy on the boundary $\partial\Omega$ not only via the Poincaré resp. logarithmic Sobolev constants for $\partial\Omega$ but alternatively via $K_{\partial\Omega,\Omega}$ and $L_{\partial\Omega,\Omega}$ as needed. This interpolation approach is a direct generalisation of [?][Proposition 2.1]. For comparison we state the more simple result one can obtain based on [?][section 4] without using the interpolation approach:

**Proposition 4.2.** *Assume there exist constants $K_1, K_2$ such that $\forall f \in C^1(\Omega)$:*

$$\left(\int_\Omega f d\lambda_\Omega - \int_{\partial\Omega} f d\lambda_{\partial\Omega}\right)^2 \leq K_1 \int_\Omega |\nabla f|^2 d\lambda_\Omega + K_2 \int_{\partial\Omega} |\nabla^\tau f|^2 d\lambda_{\partial\Omega}.$$

*Then it holds for any $\alpha \in (0,1)$*

$$L_\alpha \leq \max\left\{L_\Omega + \frac{(1-\alpha)(log(\alpha)-log(1-\alpha))}{2\alpha-1}(C_\Omega+K_1), L_{\partial\Omega} + \frac{\alpha(log(\alpha)-log(1-\alpha))}{2\alpha-1}(C_{\partial\Omega}+K_2)\right\}.$$

*Proof.* As in the proof of Proposition 4.1 we use a decomposition of the entropy with respect to the mixture of two measures as well as an optimal logarithmic Sobolev inequality for Bernoulli measures as described in [?][section 4]:

$$\begin{aligned}
Ent_{\lambda_\alpha}(f^2) \leq& \alpha Ent_{\lambda_\Omega}(f^2) + (1-\alpha) Ent_{\lambda_{\partial\Omega}}(f^2)\\
&+ \frac{\alpha(1-\alpha)(log(\alpha)-log(1-\alpha))}{2\alpha-1}\left(Var_{\lambda_\Omega}(f) + Var_{\lambda_{\partial\Omega}}(f) + (\mathbb{E}_{\lambda_\Omega}(f) - \mathbb{E}_{\lambda_{\partial\Omega}}(f))^2\right)\\
\leq& \alpha L_\Omega \int_\Omega |\nabla f|^2 d\lambda_\Omega + (1-\alpha) L_{\partial\Omega} \int_{\partial\Omega} |\nabla^\tau f|^2 d\lambda_{\partial\Omega}\\
&+ \frac{\alpha(1-\alpha)(log(\alpha)-log(1-\alpha))}{2\alpha-1}\left((C_\Omega+K_1)\int_\Omega |\nabla f|^2 d\lambda_\Omega + (C_{\partial\Omega}+K_2)\int_{\partial\Omega} |\nabla^\tau f|^2 d\lambda_{\partial\Omega}\right)\\
=& \left(L_\Omega + \frac{(1-\alpha)(log(\alpha)-log(1-\alpha))}{2\alpha-1}(C_\Omega+K_1)\right)\alpha \int_\Omega |\nabla f|^2 d\lambda_\Omega\\
&+ \left(L_{\partial\Omega} + \frac{\alpha(log(\alpha)-log(1-\alpha))}{2\alpha-1}(C_{\partial\Omega}+K_2)\right)(1-\alpha)\int_{\partial\Omega} |\nabla^\tau f|^2 d\lambda_{\partial\Omega}\\
\leq& \max(L_\Omega + \frac{(1-\alpha)(log(\alpha)-log(1-\alpha))}{2\alpha-1}(C_\Omega+K_1),\\
&\quad L_{\partial\Omega} + \frac{\alpha(log(\alpha)-log(1-\alpha))}{2\alpha-1}(C_{\partial\Omega}+K_2)) \cdot \mathcal{E}_\alpha(f).
\end{aligned}$$

□

It has been discussed in [?] that the logarithmic Sobolev constant of a mixture of two measures may blow up as the mixture proportion goes to 0 or 1. Accordingly so may our bounds for the logarithmic Sobolev constant as $\alpha$ approaches 0 or 1. More specifically the upper bound in Proposition 4.2 always blows up as $\alpha$ approaches 0 or 1. The same is true for the bound in Proposition 4.1 as $\alpha$ tends to 0 but not necessarily as $\alpha$ tends to 1. On the contrary the Poincaré constant as well as the upper bound for it in Proposition 2.1 does not blow up as $\alpha$ tends to 0 or 1.
We consider again a ball in the Euclidean plane as an example:

r2 lsi plot.png r2 lsi plot.pdf r2 lsi plot.jpg r2 lsi plot.mps r2 lsi plot.jpeg r2 lsi plot.jbig2 r2 lsi plot.jb2 r2 lsi plot.PNG r2 lsi plot.PDF r2 lsi plot.JPG r2 lsi plot.JPEG r2 lsi plot.JBIG2 r2 lsi plot.JB2

FIGURE 5. Lower bound via exact Poincaré constants (blue), and upper bound via interpolation (yellow) and no interpolation (green) for the Logarithmic Sobolev Constant

**Example 4.1.** As in Example 2.1 we consider $\Omega = B_1$, the unit ball in $\mathbb{R}^2$. We evaluate the upper bounds for $L_\alpha, \alpha \in (0,1)$ obtained via Proposition 4.2 and Proposition 4.1. Instead of calculating the exact value for $L_\alpha$ for comparison, we recall that a logarithmic Sobolev inequality with constant $C$ implies a Poincaré inequality with constant $C/2$, cf. [?][Proposition 5.1.3]. Thus a lower bound for the optimal logarithmic Sobolev constants is given via

$$L_\alpha \geq 2 \cdot C_\alpha, \forall \alpha \in (0,1),$$

with $C_\alpha$ as computed in Example 2.1. Furthermore we use the following quantities collected in Example 2.1:

$$C_\Omega = \frac{1}{3.39}, C_{\partial\Omega} = 1, K_1 = \frac{C_\Omega}{4}, K_2 = 0, K_{\partial\Omega,\Omega} = \frac{1}{2}.$$

Added to that we need values for $L_\Omega, L_{\partial\Omega}$ and $L_{\partial\Omega,\Omega}$. It is known that $L_{\partial\Omega} = 1$, cf. [?]. Furthermore from [?] we obtain $L_{\partial\Omega,\Omega} = 1$ and from [?] we obtain $L_\Omega \leq 2.9305$.
We insert these quantities into Proposition 4.2 and Proposition 4.1 and depict the results in Figure 5. Note that the yellow and green curves partly overlap.

Again the figure shows that the interpolation results clearly differ from the no interpolation results and give significantly better bounds than the approach without interpolation. In particular the interpolation approach gives an upper bound that does not blow up as $\alpha$ tends to 1.

To obtain explicit bounds on $L_\alpha$ in the general setting we can use the results from section 2 for equation (9) and equation (11) and it remains to find $L_{\partial\Omega,\Omega}$ such that equation (10) is fulfilled.
As a preliminary result we cite the following lemma, cf. [?].

**Lemma 4.1.** *[Rothaus' Lemma] Let $f : \partial\Omega \to \mathbb{R}$ be measurable and such that $\int_{\partial\Omega} f^2 log(1+f^2) d\lambda_{\partial\Omega} < \infty$. For every $a \in \mathbb{R}$*

$$Ent_{\lambda_{\partial\Omega}}\left((f+a)^2\right) \leq Ent_{\lambda_{\partial\Omega}}\left(f^2\right) + 2\int_{\partial\Omega} f^2 d\lambda_{\partial\Omega}.$$

**Lemma 4.2.** *If $f \in C^1(\Omega)$ fulfills $\int_\Omega f d\lambda_\Omega = 0$ and if there are constants $\tilde{C}_{p,2}$ such that*

$$\left(\int_{\partial\Omega} |f|^p d\lambda_{\partial\Omega}\right)^{2/p} \leq \tilde{C}_{p,2}\int_\Omega |\nabla f|^2 d\lambda_\Omega, \forall p \in \left[2, \frac{2d-2}{d-2}\right], \tag{12}$$

*then it holds*

$$Ent_{\lambda_{\partial\Omega}}\left(f^2\right) \leq \inf_{p\in[2,\frac{2d-2}{d-2}]} \frac{p}{p-2}\frac{\tilde{C}_{p,2}}{e}\int_\Omega |\nabla f|^2 d\lambda_\Omega.$$

The proof of this Lemma is adapted from [?][Proposition 6.2.3], see also [?][Proposition 5.1.8] for details.

*Proof.* Without loss of generality we may assume $\int_{\partial\Omega} f^2 d\lambda_{\partial\Omega} = 1$ and define

$$\phi : (0,1] \to \mathbb{R},\ \phi(r) := log\left(\left(\int_{\partial\Omega} |f|^{1/r} d\lambda_{\partial\Omega}\right)^r\right).$$

$\phi$ is convex and $\phi'\left(\frac{1}{2}\right) = -Ent_{\lambda_{\partial\Omega}}(f^2)$. Now for $p \in \left[2, \frac{2d-2}{d-2}\right]$ via the convexity of $\phi$

$$\begin{aligned} d\left(\phi\left(\frac{1}{2}\right) - \phi\left(\frac{1}{p}\right)\right) = d\int_{1/p}^{1/2} \phi'(s)ds \leq d\phi'\left(\frac{1}{2}\right)\left(\frac{1}{2} - \frac{1}{p}\right) \\ \Leftrightarrow -Ent_{\lambda_{\partial\Omega}}(f^2) \geq \frac{2p}{p-2}\left(\phi\left(\frac{1}{2}\right) - \phi\left(\frac{1}{p}\right)\right) \\ \Leftrightarrow Ent_{\lambda_{\partial\Omega}}(f^2) \leq \frac{p}{p-2} log\left(\left(\int_{\partial\Omega} |f|^p d\lambda_{\partial\Omega}\right)^{2/p}\right). \end{aligned}$$

Inserting equation (12) we obtain

$$Ent_{\lambda_{\partial\Omega}}(f^2) \leq \frac{p}{p-2} log\left(\tilde{C}_{p,2}\int_\Omega |\nabla f|^2 d\lambda_\Omega\right).$$

We define $\tilde{\phi}:(0,\infty)\to\mathbb{R}, \tilde{\phi}(r):=\frac{p}{p-2}log(\tilde{C}_{p,2}r)$. $\tilde{\phi}$ is concave and we may thus compute

$$Ent_{\lambda_{\partial\Omega}}(f^2)\le\tilde{\phi}\left(\int_\Omega|\nabla f|^2d\lambda_\Omega\right)\le\tilde{\phi}(r)+\tilde{\phi}'(r)\left(\int_\Omega|\nabla f|^2d\lambda_\Omega-r\right)=\tilde{\phi}'(r)\int_\Omega|\nabla f|^2d\lambda_\Omega+\left(\tilde{\phi}(r)-r\tilde{\phi}'(r)\right).$$

Choosing $r=\frac{e}{\tilde{C}_{p,2}}$ the last term vanishes and we obtain

$$Ent_{\lambda_{\partial\Omega}}(f^2)\le\tilde{\phi}'\left(\frac{e}{\tilde{C}_{p,2}}\right)\int_\Omega|\nabla f|^2d\lambda_\Omega=\frac{p}{p-2}\frac{\tilde{C}_{p,2}}{e}\int_\Omega|\nabla f|^2d\lambda_\Omega.$$

□

**Proposition 4.3.** *Assume that $d\ge3$. For any $\rho\in C^1(\Omega)$ such that $\frac{\partial\rho}{\partial N}|_{\partial\Omega}=-1$ and $\nabla\rho$ is Lipschitz continuous on $\Omega$ equation* (10) *in Proposition 4.1 is fulfilled with*

$$L_{\partial\Omega,\Omega}=\inf_{p\in[2,\frac{2d-2}{d-2}]}\frac{p}{p-2}\frac{1}{e}\left(\left(\frac{|\Omega|}{|\partial\Omega|}|\nabla\rho|_\infty p\right)^{2/p}C_{2(p-1),2}^{2(p-1)/p}+\left(\frac{|\Omega|}{|\partial\Omega|}|(\Delta\rho)^-|_\infty\right)^{2/p}C_{p,2}^2\right)$$
$$+\left\{2|\nabla\rho|_\infty C_\Omega^{1/2}+|(\Delta\rho)^-|_\infty C_\Omega\right\}\frac{|\Omega|}{|\partial\Omega|},$$

*where $(\cdot)^-$ denotes the negative part of a function.*

*Proof.* Let $f\in C^1(\Omega)$ then for $a:=\int_\Omega fd\lambda_\Omega$ we define $\tilde{f}:=f-a$ and by Lemma 4.1 it holds

$$Ent_{\lambda_{\partial\Omega}}(f^2)=Ent_{\lambda_{\partial\Omega}}((\tilde{f}+a)^2)\le Ent_{\lambda_{\partial\Omega}}(\tilde{f}^2)+2\int_{\partial\Omega}\tilde{f}^2d\lambda_{\partial\Omega}.$$

As $\tilde{f}$ is centered on $\Omega$ the assumptions of Lemma 4.2 are fulfilled due to Proposition 3.3 and we obtain

$$Ent_{\lambda_{\partial\Omega}}(\tilde{f}^2)\le\inf_{p\in[2,\frac{2d-2}{d-2}]}\frac{p}{p-2}\frac{1}{e}\left(\left(\frac{|\Omega|}{|\partial\Omega|}|\nabla\rho|_\infty p\right)^{2/p}C_{2(p-1),2}^{2(p-1)/p}+\left(\frac{|\Omega|}{|\partial\Omega|}|(\Delta\rho)^-|_\infty\right)^{2/p}C_{p,2}^2\right)\int_\Omega|\nabla\tilde{f}|^2d\lambda_\Omega.$$

Furthermore by Proposition 3.1

$$\int_{\partial\Omega}\tilde{f}^2d\lambda_{\partial\Omega}\le\left\{2|\nabla\rho|_\infty C_\Omega^{1/2}+|(\Delta\rho)^-|_\infty C_\Omega\right\}\frac{|\Omega|}{|\partial\Omega|}\int_\Omega|\nabla\tilde{f}|^2d\lambda_\Omega.$$

Thus we have

$$Ent_{\lambda_{\partial\Omega}}(f^2)\le\left(\inf_{p\in[2,\frac{2d-2}{d-2}]}\frac{p}{p-2}\frac{1}{e}\left(\left(\frac{|\Omega|}{|\partial\Omega|}|\nabla\rho|_\infty p\right)^{2/p}C_{2(p-1),2}^{2(p-1)/p}+\left(\frac{|\Omega|}{|\partial\Omega|}|(\Delta\rho)^-|_\infty\right)^{2/p}C_{p,2}^2\right)\right.$$
$$\left.+\left\{2|\nabla\rho|_\infty C_\Omega^{1/2}+|(\Delta\rho)^-|_\infty C_\Omega\right\}\frac{|\Omega|}{|\partial\Omega|}\right)\int_\Omega|\nabla\tilde{f}|^2d\lambda_\Omega$$
$$=\left(\inf_{p\in[2,\frac{2d-2}{d-2}]}\frac{p}{p-2}\frac{1}{e}\left(\left(\frac{|\Omega|}{|\partial\Omega|}|\nabla\rho|_\infty p\right)^{2/p}C_{2(p-1),2}^{2(p-1)/p}+\left(\frac{|\Omega|}{|\partial\Omega|}|(\Delta\rho)^-|_\infty\right)^{2/p}C_{p,2}^2\right)\right.$$
$$\left.+\left\{2|\nabla\rho|_\infty C_\Omega^{1/2}+|(\Delta\rho)^-|_\infty C_\Omega\right\}\frac{|\Omega|}{|\partial\Omega|}\right)\int_\Omega|\nabla f|^2d\lambda_\Omega.$$

□

Combining this with Lemma 2.2 again results in an explicit upper bound for $L_{\partial\Omega,\Omega}$.

**Proposition 4.4.** *Assume that $d \geq 3$. Let $k_2 \in \mathbb{R}$ such that $sect \leq k_2$ and $\gamma_2 \in \mathbb{R}$ such that $\Pi \leq \gamma_2 id$. Then equation* (10) *in Proposition 4.1 is fulfilled with*

$$L_{\partial\Omega,\Omega} = \begin{cases} \inf_{p\in[2,\frac{2d-2}{d-2}]} \frac{p}{p-2}\frac{1}{e}\Bigg[\left(\frac{|\Omega|}{|\partial\Omega|}p\right)^{2/p} C_{2(p-1),2}^{2(p-1)/p} \\ +\left(\frac{|\Omega|}{|\partial\Omega|}\inf_{t_1\in(0,h_2^{-1}(0))}\sup_{t\in(0,t_1)}\left(\left((d-1)\frac{h_2'}{h_2}(t)\left(1-\frac{t}{t_1}\right)-\frac{1}{t_1}\right)^{-}\right)^{2/p} C_{p,2}^2\Bigg] \\ +\left\{2C_\Omega^{1/2}+\inf_{t_1\in(0,h_2^{-1}(0))}\sup_{t\in(0,t_1)}\left((d-1)\frac{h_2'}{h_2}(t)\left(1-\frac{t}{t_1}\right)-\frac{1}{t_1}\right)^{-}C_\Omega\right\}\frac{|\Omega|}{|\partial\Omega|}, & \text{if } k_2 \geq -\gamma_2^2 \\ \inf_{p\in[2,\frac{2d-2}{d-2}]} \frac{p}{p-2}\frac{1}{e}\left[\left(\frac{|\Omega|}{|\partial\Omega|}p\right)^{2/p} C_{2(p-1),2}^{2(p-1)/p}+\left(\frac{|\Omega|}{|\partial\Omega|}\max(\gamma_2(d-1),0)\right)^{2/p}C_{p,2}^2\right] \\ +\left\{2C_\Omega^{1/2}+\max(\gamma_2(d-1),0)C_\Omega\right\}\frac{|\Omega|}{|\partial\Omega|}, & \text{if } k_2 < -\gamma_2^2. \end{cases}$$

As in section 2 instead of Brownian motion with sticky reflecting boundary diffusion the above results may as well be used to give upper bounds for Brownian motion with sticky reflection from the boundary (but without boundary diffusion). The Logarithmic Sobolev inequality in this setting is

$$Ent_{\lambda_\alpha} \leq \hat{L}_\alpha \hat{\mathcal{E}}_\alpha(f)\ \forall f \in C^1(\Omega),$$

where

$$\hat{\mathcal{E}}_\alpha(f) = \alpha \int_\Omega |\nabla f|^2 d\lambda_\Omega,\ f \in C^1(\Omega).$$

**Proposition 4.5.** *Assume there exist constants $K_{\partial\Omega,\Omega}, L_{\partial\Omega,\Omega}, K_1$ such that for any $f \in C^1(\Omega)$*

$$Var_{\lambda_{\partial\Omega}} f \leq K_{\partial\Omega,\Omega}\int_\Omega |\nabla f|^2 d\lambda_\Omega,$$

$$Ent_{\lambda_{\partial\Omega}}(f^2) \leq L_{\partial\Omega,\Omega}\int_\Omega |\nabla f|^2 d\lambda_\Omega,$$

$$\left(\int_\Omega f d\lambda_\Omega - \int_{\partial\Omega} f d\lambda_{\partial\Omega}\right)^2 \leq K_1 \int_\Omega |\nabla f|^2 d\lambda_\Omega,$$

*then it holds for any $\alpha \in (0,1)$*

$$\hat{L}_\alpha \leq \left(L_\Omega + \frac{(1-\alpha)}{\alpha}L_{\partial\Omega,\Omega} + \frac{(1-\alpha)(log(\alpha)-log(1-\alpha))}{2\alpha-1}(C_\Omega + K_{\partial\Omega,\Omega}+K_1)\right).$$

*Proof.* As above we use a decomposition of the entropy with respect to the mixture of two measures as well as an optimal logarithmic Sobolev inequality for Bernoulli measures as described in [**?**][section 4]:

$$\begin{aligned} Ent_{\lambda_\alpha}(f^2) \leq & \alpha Ent_{\lambda_\Omega}(f^2) + (1-\alpha)Ent_{\lambda_{\partial\Omega}}(f^2) \\ & + \frac{\alpha(1-\alpha)(log(\alpha)-log(1-\alpha))}{2\alpha-1}\left(Var_{\lambda_\Omega}(f)+Var_{\lambda_{\partial\Omega}}(f)+(\mathbb{E}_{\lambda_\Omega}(f)-\mathbb{E}_{\lambda_{\partial\Omega}}(f))^2\right) \\ \leq & \left(L_\Omega + \frac{(1-\alpha)}{\alpha}L_{\partial\Omega,\Omega} + \frac{(1-\alpha)(log(\alpha)-log(1-\alpha))}{2\alpha-1}(C_\Omega+K_{\partial\Omega,\Omega}+K_1)\right)\alpha\int_\Omega|\nabla f|^2 d\lambda_\Omega. \end{aligned}$$

□

Of course again the interpolation approach from above is not of any use any more in this setting.

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