HESSIAN ESTIMATES FOR DIRICHLET AND NEUMANN EIGENFUNCTIONS OF LAPLACIAN

LI-JUAN CHENG[1]*, ANTON THALMAIER[2], FENG-YU WANG[3]

[1] *School of Mathematics, Hangzhou Normal University*
Hangzhou 311123, The People's Republic of China

[2] *Department of Mathematics, University of Luxembourg, Maison du Nombre,*
L-4364 Esch-sur-Alzette, Luxembourg

[3] *Center for Applied Mathematics, Tianjin University,*
Tianjin 300072, People's Republic of China

Abstract. By methods of stochastic analysis on Riemannian manifolds, we develop two approaches to determine an explicit constant $c(D)$ for an n-dimensional compact manifold D with boundary such that

$$\frac{\lambda}{n}\|\phi\|_\infty \le \|\operatorname{Hess}\phi\|_\infty \le c(D)\lambda\|\phi\|_\infty$$

holds for any Dirichlet eigenfunction ϕ of $-\Delta$ with eigenvalue λ. Our results provide the sharp Hessian estimate $\|\operatorname{Hess}\phi\|_\infty \lesssim \lambda^{\frac{n+3}{4}}$. Corresponding Hessian estimates for Neumann eigenfunctions are derived in the second part of the paper.

1. Introduction

Let D be an n-dimensional compact Riemannian manifold with boundary ∂D. We write $(\phi,\lambda)\in \operatorname{Eig}(\Delta)$ if ϕ is a Dirichlet eigenfunction of $-\Delta$ on D with eigenvalue $\lambda>0$, i.e. $-\Delta\phi=\lambda\phi$. We always assume eigenfunctions ϕ to be normalized in $L^2(D)$ such that $\|\phi\|_{L^2}=1$. According to [15], there exist two positive constants $c_1(D)$ and $c_2(D)$ such that

$$c_1(D)\sqrt{\lambda}\,\|\phi\|_\infty \leqslant \|\nabla\phi\|_\infty \leqslant c_2(D)\sqrt{\lambda}\,\|\phi\|_\infty,\quad (\phi,\lambda)\in\operatorname{Eig}(\Delta), \tag{1.1}$$

where we write $\|\nabla\phi\|_\infty := \||\nabla\phi|\|_\infty$ for simplicity. An analogous statement for Neumann eigenfunctions has been derived by Hu, Shi and Xui [8]. Subsequently, by methods of stochastic analysis on Riemannian manifolds, Arnaudon, Thalmaier and Wang [2] determined explicit constants $c_1(D)$ and $c_2(D)$ in (1.1) for Dirichlet and Neumann eigenfunctions. From this, together with the uniform estimate of ϕ (see [7, 6, 11]),

$$\|\phi\|_\infty \le c_D\lambda^{\frac{n-1}{4}}$$

for some positive constant c_D, the optimal uniform bound of the gradient writes as

$$\|\nabla\phi\|_\infty \lesssim \lambda^{\frac{n+1}{4}}.$$

E-mail address: lijuan.cheng@hznu.edu.cn, anton.thalmaier@uni.lu, wangfy@tju.edu.cn.
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*Corresponding author.
♯The author contributed equally to this work.

Results of this type have been used to study gradient estimates for unit spectral projection operators and to give a new proof of Hörmander's multiplier theorem, see [23, 24, 25].

Concerning higher order estimates of eigenfunctions, not much is known. Very recently, Steinerberger [16] studied Laplacian eigenfunctions of $-\Delta$ with Dirichlet boundary conditions on bounded domains $\Omega \subset \mathbb{R}^n$ with smooth boundary and proved a sharp Hessian estimate for the eigenfunctions which reads as

$$\|\operatorname{Hess}\phi\|_\infty \lesssim \lambda^{\frac{n+3}{4}}$$

where

$$\|\operatorname{Hess}\phi\|_\infty := \sup\{|\operatorname{Hess}\phi(v,v)|(x) : x \in \mathbb{R}^n,\ v \in \mathbb{R}^n,\ |v| = 1\}.$$

It is a natural question under which geometric assumptions such estimates extend to compact manifolds (with boundary). Following the lines of [2], one may ask the question how for the Hessian to derive explicit numerical constants $C_1(D)$ and $C_2(D)$ such that

$$C_1(D)\lambda\|\phi\|_\infty \leqslant \|\operatorname{Hess}\phi\|_\infty \leqslant C_2(D)\lambda\|\phi\|_\infty, \quad (\phi,\lambda) \in \operatorname{Eig}(\Delta). \tag{1.2}$$

Note that for eigenfunctions of the Laplacian, one trivially has

$$|\operatorname{Hess}\phi| \geq \frac{1}{n}|\Delta\phi| = \frac{\lambda}{n}|\phi|,$$

and hence there is always the obvious lower bound

$$\frac{\|\operatorname{Hess}\phi\|_\infty}{\|\phi\|_\infty} \geq \frac{\lambda}{n}.$$

For this reason, we shall concentrate in the sequel on upper bounds for $\|\operatorname{Hess}\phi\|_\infty/\|\phi\|_\infty$.

In [2] a derivative formula for Dirichlet eigenfunctions has been given from where an upper bound for the gradient of the eigenfunction could be derived directly. Let us briefly describe this method. Assume that X_t is a Brownian motion on $D\setminus\partial D$ with generator $\frac{1}{2}\Delta$, and write $X_t(x)$ to indicate the starting point $X_0 = x$. Then $X.(x)$ is defined up to the first hitting time $\tau_D = \inf\{t > 0 \colon X_t(x) \in \partial D\}$ of the boundary. For $x \in \partial D$ we use the convention that $X.(x)$ is defined with lifetime $\tau_D \equiv 0$; in this case the subsequent statements usually hold automatically.

Suppose that $Q_t \colon T_xD \to T_{X_t(x)}D$ is defined by

$$\mathrm{D}Q_t = -\frac{1}{2}\operatorname{Ric}^\sharp(Q_t)\,dt, \quad Q_0 = \mathrm{id},$$

where $\mathrm{D} := /\!/_t\, d\, /\!/_t^{-1}$ with $/\!/_t := /\!/_{0,t} \colon T_xD \to T_{X_t(x)}D$ parallel transport along $X(x)$ and $\operatorname{Ric}^\sharp(v)(w) = \operatorname{Ric}(v,w)$ for $v, w \in TM$. Suppose that $(\phi,\lambda) \in \operatorname{Eig}(\Delta)$. Then, for $v \in T_xM$ and any $k \in C_b^1([0,\infty);\mathbb{R})$, i.e., k bounded with bounded derivative, the process

$$k(t)\,\mathrm{e}^{\lambda t/2}\langle\nabla\phi(X_t), Q_t(v)\rangle - \mathrm{e}^{\lambda t/2}\phi(X_t)\int_0^t\langle\dot{k}(s)Q_sv, /\!/_s dB_s\rangle, \quad t \leq \tau_D$$

is a martingale. From this, by taking expectation, a formula involving $\nabla\phi$ can be obtained which allows to derive an upper bound for $|\nabla\phi|$ on D by estimating $|\nabla\phi|$ on the boundary ∂D and carefully choosing the function k. Along this circle of ideas, we aim at establishing a similar formula for the Hessian of an eigenfunction ϕ.

In view of the fact that $P_t\phi = \mathrm{e}^{-\lambda t/2}\phi$ where P_t is the semigroup generated by $\frac{1}{2}\Delta$, we focus first on martingales which are appropriate for attaining uniform Hessian estimates of eigenfunctions. Let us start with some background on Bismut type formulas for the second-order derivatives of heat semigroups. A second-order differential formula for the heat semigroup P_t was first obtained by Elworthy and Li [5, 12] for a non-compact manifold, however with restrictions on the curvature of the manifold. An intrinsic formula for $\operatorname{Hess}P_tf$ has been given by Stroock [17] for a compact Riemannian manifold, and a localized version of such a formula was obtained in [1, 3] adopting martingale arguments. For the Hessian of the Feynman-Kac semigroup of an operator $\Delta + V$ with a potential function V on manifolds, we refer the reader to see [13, 14, 18].

For a complete Riemannian manifold M without boundary, an appropriate version of a Bismut-type Hessian formula gives the following estimate (see [3, Corollary 4.3] and Lemma 2.2, or Corollary 3.2 with $\sigma_1 = \sigma_2 = 0$):

$$\|\mathrm{Hess}\, P_t f\|_\infty \le \left(K_1\sqrt{t} + \frac{K_2 t}{2}\right)\mathrm{e}^{K_0 t}\,\|f\|_\infty + \frac{2}{t}\,\mathrm{e}^{K_0 t}\,\|f\|_\infty$$

where

$$\begin{aligned} K_0 &:= \sup\left\{-\mathrm{Ric}(v,v)\colon\, y\in M,\ v\in T_yM,\ |v|=1\right\};\\ K_1 &:= \sup\{|R|(y)\colon\, y\in M\};\\ K_2 &:= \sup\left\{|(\mathbf{d}^*R+\nabla\mathrm{Ric})^\sharp(v,w)|(y)\colon\, y\in M,\ v,w\in T_yM,\ |v|=|w|=1\right\}\end{aligned} \tag{1.3}$$

and

$$|R|(y) := \sup\left\{\sqrt{\sum_{i,j=1}^n R(e_i,v,w,e_j)^2(y)} : |v|\le 1, |w|\le 1\right\}$$

for an orthonormal base $\{e_i\}_{i=1}^n$ of T_yM.

Thus if $f=\phi$ and $(\phi,\lambda)\in\mathrm{Eig}(\Delta)$, then

$$\|\mathrm{Hess}\,\phi\|_\infty \le \left(K_1\sqrt{t} + \frac{K_2 t}{2}\right)\mathrm{e}^{(K_0+\lambda/2)t}\,\|\phi\|_\infty + \frac{2\,\mathrm{e}^{(K_0+\lambda/2)t}}{t}\,\|\phi\|_\infty$$

for any $t>0$. Letting $t=\frac{1}{K_0+\lambda/2}$ then yields the estimate

$$\frac{\|\mathrm{Hess}\,\phi\|_\infty}{\|\phi\|_\infty} \le \left(K_1\sqrt{\frac{2}{2K_0+\lambda}} + \frac{K_2}{2K_0+\lambda}\right)\mathrm{e} + (\lambda+2K_0)\,\mathrm{e}\,.$$

To carry over such results to (compact) manifold D with boundary, the influence of the boundary has to be studied. In this paper, we shall discuss a martingale approach to the Hessian of Dirichlet eigenfunctions. This is based on the construction of a suitable martingale which builds a relation between $\mathrm{Hess}\,\phi$ and $\mathbf{d}\phi$ and then to estimate $C_2(D)$ in (1.2) by finding explicit constants C_1, C_2 and C_3 such that

$$\|\mathrm{Hess}\,\phi\|_\infty \le C_1\|\mathrm{Hess}\,\phi\|_{\partial D,\infty} + C_2\|\nabla\phi\|_{\partial D,\infty} + C_3\|\nabla\phi\|_\infty \tag{1.4}$$

where $\|\mathrm{Hess}\,\phi\|_{\partial D,\infty} := \sup_{x\in\partial D}|\mathrm{Hess}\,\phi|(x)$ and $\|\nabla\phi\|_{\partial D,\infty} := \sup_{x\in\partial D}|\nabla\phi|(x)$. The final estimate for $|\mathrm{Hess}\,\phi|$ is then received by combining the last inequality with estimate (1.1) in [2].

Let us start with the general principle behind the construction of the relevant martingale. Let $k\in C_b^1([0,\infty);\mathbb{R})$ and define an operator-valued process $W_t^k\colon T_xD\otimes T_xD\to T_{X_t(x)}D$ as solution to the following covariant Itô equation

$$dW_t^k(v,w) = R(/\!/_t dB_t, Q_t(k(t)v))Q_t(w) - \frac{1}{2}(\mathbf{d}^*R+\nabla\mathrm{Ric})^\sharp(Q_t(k(t)v),Q_t(w))\,dt - \frac{1}{2}\mathrm{Ric}^\sharp(W_t^k(v,w))\,dt,$$

with initial condition $W_0^k(v,w)=0$. Here the operator $\mathbf{d}^*R$ is defined by $\mathbf{d}^*R(v_1,v_2) := -\mathrm{tr}\,\nabla.R(\cdot,v_1)v_2$ and thus satisfies

$$\langle \mathbf{d}^*R(v_1,v_2),v_3\rangle = \langle(\nabla_{v_3}\mathrm{Ric}^\sharp)(v_1),v_2\rangle - \langle(\nabla_{v_2}\mathrm{Ric}^\sharp)(v_3),v_1\rangle$$

for all $v_1,v_2,v_3\in T_xD$ and $x\in D$. Then the process

$$\begin{aligned} M_t := {}&\mathrm{e}^{\lambda t/2}\,\mathrm{Hess}\,\phi(Q_t(k(t)v),Q_t(v)) + \mathrm{e}^{\lambda t/2}\,\mathbf{d}\phi(W_t^k(v,v))\\ &- \mathrm{e}^{\lambda t/2}\,\mathbf{d}\phi(Q_t(v))\int_0^t\langle Q_s(\dot k(s)v), /\!/_s dB_s\rangle \end{aligned} \tag{1.5}$$

is a martingale on $[0,\tau_D]$ in the sense that $(M_{t\wedge\tau_D})_{t\ge0}$ is a globally defined martingale where $\tau_D = \inf\{t>0 : X_t(x)\in\partial D\}$ denotes the first hitting time of $X.(x)$ of the boundary ∂D. The martingale property of (1.5) then allows to establish an inequality of the type (1.4) by equating the expectations at time 0 and at time $t\wedge\tau_D$. This approach then requires to estimate the boundary values of $|\mathbf{d}\phi|$ and $|\mathrm{Hess}\,\phi|$, in order to obtain the wanted upper bound for $\|\mathrm{Hess}\,\phi\|_\infty$. To this end, we establish the required estimates in Lemmas 2.4-2.5 by using the information on the second fundamental form II and the second derivative of N, where for $X,Y\in T_x\partial D$ and $x\in\partial D$, the second fundamental form is defined by

$$\mathrm{II}(X,Y)=-\langle\nabla_X N,\ Y\rangle.$$

Note that we observe that the N is canonically extended to vector fields on a neighbourhood of the boundary. Let

$$\ell(t):=\ell_{k,\sigma}(t):=\begin{cases}\cos\sqrt{k}t-\frac{\sigma}{\sqrt{k}}\sin\sqrt{k}t, & k>0,\\ 1-\sigma t, & k=0,\\ \cosh\sqrt{-k}t-\frac{\sigma}{\sqrt{-k}}\sinh\sqrt{-k}t, & k<0,\end{cases}\tag{1.6}$$

We state now the first main result of this paper.

Theorem 1.1. *Let D be a compact Riemannian manifold with boundary ∂D. Assume that $|\mathrm{Ric}|\le K_0$, $|R|\le K_1$ and $|\mathbf{d}^*R+\nabla\mathrm{Ric}|\le K_2$ on D. Moreover, assume that N is the extended vector field of the normal vector field to the neighbour area $\partial_{r_0}D:=\{x:\rho_{\partial D}(x)\le r_0\}$ such that $\rho_{\partial D}$ is smooth, $|\mathrm{Sect}|\le k$, $|\nabla N|\le\sigma$ and $|\Delta^{(1)}N|\le\beta$ on $\partial_{r_0}D$ for some $r_0>0$. Then*

$$\frac{\|\mathrm{Hess}\,\phi\|_\infty}{\|\phi\|_\infty}\le C_\lambda(D)\lambda,$$

where if $\sqrt{\lambda+K_0}\ge 2A$, then

$$\begin{aligned}C_\lambda(D)=&\frac{2(n-1)\mathrm{e}\sigma A}{\lambda}+\frac{2\,\mathrm{e}(\mathrm{e}^{nr_1/2}+2)}{\lambda}\sqrt{\lambda+K_0}\max\left\{\sqrt{\lambda+2K_0+\frac{n\sigma}{r_1}+\sigma^2},\,2\,\mathrm{e}^{nr_1/2}(\sigma+\frac{3}{r_2})\right\}\\&+\frac{2\,\mathrm{e}}{\lambda}\left[\mathrm{e}^{nr_1/2}\left(\beta+\frac{6(\sigma\vee\sqrt{k})}{r_2}+\frac{6}{r_2^2}+K_0+\lambda\right)+K_1\right]\\&+\frac{K_2\,\mathrm{e}}{\lambda\sqrt{\lambda+2K_0+\frac{n\sigma}{r_1}+\sigma^2}};\end{aligned}$$

if $\sqrt{\lambda+K_0}<2A$, then

$$\begin{aligned}C_\lambda(D)=&\frac{2(n-1)\sigma\mathrm{e}A}{\lambda}+2\,\mathrm{e}(\mathrm{e}^{nr_1/2}+2)\max\left\{\sqrt{\lambda+2K_0+\frac{n\sigma}{r_1}+\sigma^2},\,2\,\mathrm{e}^{nr_1/2}(\sigma+\frac{3}{r_2})\right\}\left(\frac{A}{\lambda}+\frac{\lambda+K_0}{4A\lambda}\right)\\&+2\,\mathrm{e}\frac{\mathrm{e}^{nr_1/2}\left(\beta+\frac{6(\sigma\vee\sqrt{k})}{r_2}+\frac{6}{r_2}+K_0+\lambda\right)+K_1}{\sqrt{\lambda+2K_0+\sigma(\frac{n}{r_1}+\sigma)}}\left(\frac{A}{\lambda}+\frac{\lambda+K_0}{4A\lambda}\right)\\&+\frac{K_2\,\mathrm{e}}{\lambda+2K_0+\sigma(\frac{n}{r_1}+\sigma)}\left(\frac{A}{\lambda}+\frac{\lambda+K_0}{4A\lambda}\right),\end{aligned}$$

where

$$A=2(\sqrt{k}\wedge\sigma)+\frac{\sqrt{2(\lambda+K_0)}}{\sqrt{\pi}}\exp\left(-\frac{k\wedge\sigma^2}{2\lambda}\right);$$

$$r_1=r_0\wedge\ell^{-1}(0);\quad r_2=r_0\wedge\ell^{-1}(\frac{1}{2}).$$

Remark 1.2. It is easy to see that both $C_\lambda(D)$ is decreasing in λ, and hence $C_\lambda(D)\le C_{\lambda_1}(D)$ where λ_1 is the first Dirichlet eigenvalue of $-\Delta$ which gives

$$\frac{\|\mathrm{Hess}\,\phi\|_\infty}{\|\phi\|_\infty}\le C_{\lambda_1}(D)\lambda.$$

Considering now Hessian estimate for Neumann eigenfunctions, we denote by $\mathrm{Eig}_N(\Delta)$ the set of non-trivial eigenpairs (ϕ,λ) for the Neumann eigenproblem, i.e., ϕ is non-constant, $\Delta\phi = -\lambda\phi$ with $N\phi|_{\partial D} = 0$ for the unit inward normal vector field N of ∂D. Along the previous idea, a big different is that we do not consider the process before hitting the boundary ∂D. Thus when constructing the suitable martingales, the boundary behaviour of the process should be considered at first. Here, we will use the reflecting Brownian motion as the base process to consider this question. Due to our very recent work on Bismut-type Hessian formula for Neumann semigroup, we have the following formula linking $\mathrm{Hess}\,P_t f$ and $\mathbf{d}f$ directly:

$$\mathrm{Hess}\,P_t f(v,v) = \mathbb{E}\left[-\mathbf{d}f(\tilde{Q}_t(v))\int_0^t \langle \tilde{Q}_s(\dot{k}(s)v), //_s dB_s\rangle + \mathbf{d}f(\tilde{W}_t^k(v,v))\right],$$

where $\tilde{Q}$ and $\tilde{W}^k$ are defined in (3.1) and (3.2) in Section 3. By observing the fact that $P_t\phi = \mathrm{e}^{-\frac{1}{2}\lambda t}$ and estimating $\tilde{Q}_\cdot$ and $\tilde{W}_\cdot$ carefully under suitable curvature condition, we obtain the following theorem gives an upper estimate for $\mathrm{Hess}\,\phi$ of the type (1.2) with an explicit constant $C_2(D)$.

Theorem 1.3. *Let D be an n-dimensional compact Riemannian manifold with boundary ∂D. Assume that* $\mathrm{Ric} \geq -K_0$, $|R| \leq K_1$ *and* $|\mathbf{d}^*R + \nabla\mathrm{Ric}| \leq K_2$ *on D, and that* $-\sigma_1 \leq \mathrm{II} \leq \sigma$ *and* $|\nabla^2 N - R(N)| \leq \sigma_2$ *on the boundary ∂D. If there exists a positive constant r_0 such that on $\partial_{r_0}D := \{x \in D : \rho_{\partial D}(x) \leq r_0\}$ the distance function $\rho_{\partial D}$ to the boundary ∂D is smooth and* $\mathrm{Sect} \leq k$*. Then for any non-trivial $(\phi,\lambda) \in \mathrm{Eig}_N(\Delta)$,*

$$\frac{\|\mathrm{Hess}\ \phi\|_\infty}{\|\phi\|_\infty} \leq C_{N,\lambda}(D)\lambda$$

where

$$C_{N,\lambda}(D) = \left(1 + \frac{K_1 + 2K_0 + 2\sigma_1^+(\frac{n}{r_1} + 2\sigma_1^+)}{\lambda} + \frac{K_2 + 2\sigma_2(\frac{n}{r_1} + 2\sigma_1^+)}{\lambda\sqrt{2\lambda + 4K_0 + 4\sigma_1(\frac{n}{r_1} + 2\sigma_1^+)}}\right)\mathrm{e}^{\frac{3}{2}\sigma_1^+ n r_1 + 1}$$
$$+ \frac{\sigma_2 n r_1}{2\lambda}\sqrt{2\lambda + 4K_0 + 4\sigma_1(\frac{n}{r_1} + 2\sigma_1^+)}\,\mathrm{e}^{\frac{3}{2}\sigma_1^+ n r_1 + 1},$$

for $r_1 = r_0 \wedge \ell^{-1}(0)$. Denoting by λ_1 the first Neumann eigenvalue of $-\Delta$, then

$$\frac{\|\mathrm{Hess}\,\phi\|_\infty}{\|\phi\|_\infty} \leq C_{N,\lambda_1}(D)\,\lambda.$$

The remainder of the paper is organized as follows. In Section 2 we first show for Dirichlet eigenfunctions

$$\|\mathrm{Hess}\,\phi\|_\infty/\|\phi\|_\infty \leq C_\lambda(D)\lambda \tag{1.7}$$

by verifying that the process (1.5) is a martingale, in combination with boundary estimates for $|\mathrm{Hess}\,\phi|$. Section 3 deals with Neumann eigenfunctions where we give a proof of Theorem 1.3 by using Bismut type Hessian formulae for the Neumann semigroup and an estimate of the local time.

2. Hessian estimates of Dirichlet eigenfunctions

This section is dedicated to the the approach described in the Introduction. In fact, the proof of Theorem 1.1 is also divided into two steps by first showing Theorem 2.8 with some testing functions, which will be constructed in Section 4. We start by constructing the fundamental martingale which will be the basis for our method.

Theorem 2.1. *On a compact Riemannian manifold D with boundary ∂D, let $X_\cdot(x)$ be a Brownian motion starting from $x \in D$ and denote by $\tau_D = \inf\{t \geq 0\colon X_t(x) \in \partial D\}$ its first hitting time of ∂D.*

Define Q_t and W_t^k as above where $k \in C_b^1([0,\infty);\mathbb{R})$. Then, for $(\phi,\lambda) \in \mathrm{Eig}_N(\Delta)$ and $v \in T_xD$, the process

$$
\begin{aligned}
&\mathrm{e}^{\lambda t/2}\,\mathrm{Hess}\,\phi\big(Q_t(k(t)v), Q_t(v)\big) + \mathrm{e}^{\lambda t/2}\,\mathbf{d}\phi(W_t^k(v,v)) \\
&\quad - \mathrm{e}^{\lambda t/2}\,\mathbf{d}\phi(Q_t(v)) \int_0^t \langle Q_s(\dot{k}(s)v), //_s dB_s\rangle
\end{aligned}
\tag{2.1}
$$

is a martingale on $[0,\tau_D]$.

Proof. Due to the compactness of D it is sufficient to check that (2.1) is a local martingale on $[0,\tau_D)$. Fixing a time $T>0$, for $v \in T_xD$, we let

$$N_t(v,v) = \mathrm{Hess}\,P_{T-t}\phi(Q_t(v),Q_t(v)) + (\mathbf{d}P_{T-t}\phi)(W_t(v,v)), \quad t \le T\wedge\tau_D,$$

where

$$W_t(v,v) = Q_t \int_0^t Q_r^{-1} R(//_r dB_r, Q_r(v))Q_r(v) - \frac{1}{2} Q_t \int_0^t Q_r^{-1}(\mathbf{d}^*R + \nabla\mathrm{Ric})^\sharp(Q_r(v),Q_r(v))\,dr.$$

Then $N_t(v,v)$ is a local martingale, see for instance the proof of [19, Lemma 2.7] with potential $V\equiv 0$. Since $(\phi,\lambda)\in\mathrm{Eig}(\Delta)$, we know that $P_{T-t}\phi(X_t) = \mathrm{e}^{-\lambda(T-t)/2}\,\phi(X_t)$ and thus

$$\mathrm{e}^{\lambda t/2}\,\mathrm{Hess}\,\phi(Q_t(v),Q_t(v)) + \mathrm{e}^{\lambda t/2}(\mathbf{d}\phi)(W_t(v,v))$$

is also a local martingale. Furthermore, consider

$$N_t^k(v,v) := \mathrm{e}^{\lambda t/2}\,\mathrm{Hess}\,\phi(Q_t(k(t)v),Q_t(v)) + (\mathrm{e}^{\lambda t/2}\,\mathbf{d}\phi)(W_t^k(v,v)).$$

According to the definition of $W_t^k(v,v)$, resp. $W_t(v,v)$, and in view of the fact that $N_t(v,v)$ is a local martingale, it is easy to see that

$$\mathrm{e}^{\lambda t/2}\,\mathrm{Hess}\,\phi(Q_t(k(t)v),Q_t(v)) + (\mathrm{e}^{\lambda t/2}\,\mathbf{d}\phi)(W_t^k(v,v)) - \int_0^t \mathrm{e}^{\lambda s/2}\,\mathrm{Hess}\,\phi(Q_s(\dot{k}(s)v),Q_s(v))\,ds$$

is a local martingale as well. From the formula

$$\mathrm{e}^{\lambda t/2}\,\mathbf{d}\phi(Q_t(v)) = \mathbf{d}\phi(v) + \int_0^t \mathrm{e}^{\lambda s/2}(\mathrm{Hess}\,\phi)(//_s dB_s, Q_s(v))$$

it follows that

$$\int_0^t \mathrm{e}^{\lambda s/2}(\mathrm{Hess}\,\phi)(Q_s(\dot{k}(s)v),Q_s(v))\,ds - \mathrm{e}^{\lambda t/2}\,\mathbf{d}\phi(Q_t(v)) \int_0^t \langle Q_s(\dot{k}(s)v), //_s dB_s\rangle \tag{2.2}$$

is a local martingale. We conclude that

$$(\mathrm{e}^{\lambda t/2}\,\mathrm{Hess}\,\phi)(Q_t(k(t)v),Q_t(v)) + (\mathrm{e}^{\lambda t/2}\,\mathbf{d}\phi)(W_t^k(v,v)) - \mathrm{e}^{\lambda t/2}\,\mathbf{d}\phi(Q_t(v)) \int_0^t \langle Q_s(\dot{k}(s)v), //_s dB_s\rangle$$

is a local martingale. □

We shall use the following estimate to proceed with the Hessian formula for ϕ.

Lemma 2.2. *Assume that* $\mathrm{Ric} \ge -K_0$, $|R| \le K_1$ *and* $|\mathbf{d}^*R + \nabla\mathrm{Ric}| \le K_2$ *on D for non-negative constants K_0, K_1 and K_2. Let $k \in C_b^1([0,\infty);\mathbb{R})$. For $t \ge 0$ and $\delta > 0$, it holds*

$$|Q_t| \le \mathrm{e}^{K_0 t/2} \quad \textit{and} \tag{2.3}$$

$$\mathbb{E}\Big[\big|W_t^k(v,\dot{k}(t)v)\big|\mathbb{1}_{\{t\le\tau_D\}}\Big] \le \Big(K_1\sqrt{t} + \frac{K_2}{2}t\Big)\mathrm{e}^{K_0 t}\,\dot{k}(t), \tag{2.4}$$

where K_0, K_1 and K_2 are defined as in (1.3).

Proof. The first inequality follows from the lower Ricci curvature bound condition and the definition of Q_t. According to the definition of W_t^k, it is easy to see that

$$\begin{aligned} W_t^k(v,v) =& Q_t \int_0^t Q_s^{-1} R(//_s dB_s, Q_s(k(s)v))Q_s(v) \\ & - \frac{1}{2} Q_t \int_0^t Q_s^{-1} (\mathbf{d}^* R + \nabla \mathrm{Ric})^\sharp (Q_s(k(s)v), Q_s(v))\, ds. \end{aligned}$$

Then we have

$$\begin{aligned} \mathbb{E}\Big(|W_t^k(v,v)| \mathbb{1}_{\{t\le\tau_D\}}\Big) \le& \mathbb{E}\bigg[\mathbb{1}_{\{t\le\tau_D\}} \Big| Q_t \int_0^t Q_s^{-1} R(//_s dB_s, Q_s(k(s)v))Q_s(v) \Big|\bigg] \\ & + \frac{1}{2} \mathbb{E}\bigg[\mathbb{1}_{\{t\le\tau_D\}} \Big| Q_t \int_0^t Q_s^{-1} (\mathbf{d}^* R + \nabla \mathrm{Ric})(Q_s(k(s)v), Q_s(v))\, ds \Big|\bigg] \\ \le& \, \mathrm{e}^{\frac{K_0 t}{2}} \mathbb{E}\bigg[\mathbb{1}_{\{t\le\tau_D\}} \Big| \mathrm{e}^{-\frac{K_0 t}{2}} Q_t \int_0^t Q_s^{-1} R(//_s dB_s, Q_s(k(s)v))Q_s(v) \Big|^2\bigg]^{1/2} \\ & + \frac{K_1}{2} \mathbb{E}\bigg[\mathbb{1}_{\{t\le\tau_D\}} \Big| \mathrm{e}^{\frac{1}{2}K_0 t} \int_0^t \mathrm{e}^{\frac{1}{2}K_0 s}\, ds \Big|\bigg]. \end{aligned} \tag{2.5}$$

Moreover,

$$\begin{aligned} & d \Big| \mathrm{e}^{-\frac{1}{2}K_0 t} Q_t \int_0^t Q_s^{-1} R(//_s dB_s, Q_s(k(s)v))Q_s(v) \Big|^2 \\ &= 2\, \mathrm{e}^{-K_0 t} \Big\langle R(//_t dB_t, Q_t(k(t)v))Q_t(v), Q_t \int_0^t Q_s^{-1} R(//_s dB_s, Q_s(k(s)v))Q_s(v) \Big\rangle \\ &\quad + \mathrm{e}^{-K_0 t} \Big| R^{\sharp,\sharp}(Q_t(k(t)v), Q_t(v)) \Big|_{\mathrm{HS}}^2 dt \\ &\quad - \mathrm{e}^{-K_0 t} \mathrm{Ric}\bigg(Q_t \int_0^t Q_s^{-1} R(//_s dB_s, Q_s(k(s)v))Q_s(v), Q_t \int_0^t Q_s^{-1} R(//_s dB_s, Q_s(k(s)v))Q_s(v) \bigg) dt \\ &\quad - K_0\, \mathrm{e}^{-K_0 t} \Big| Q_t \int_0^t Q_s^{-1} R(//_s dB_s, Q_s(k(s)v))Q_s(v) \Big|^2 dt \\ &\overset{m}{\le} \mathrm{e}^{-K_0 t} \Big| R^{\sharp,\sharp}(Q_t(k(t)v), Q_t(v)) \Big|_{\mathrm{HS}}^2 dt \le K_2^2\, \mathrm{e}^{-K_0 t} |Q_t|^4\, dt \le K_2^2\, \mathrm{e}^{K_0 t}\, dt, \qquad t \le \tau_D. \end{aligned}$$

Combining this with (2.5), we have

$$\mathbb{E}\Big(|W_t^k(v,v)| \mathbb{1}_{\{t\le\tau_D\}}\Big) \le K_2\, \mathrm{e}^{\frac{1}{2}K_0 t} \bigg(\int_0^t \mathrm{e}^{K_0 s}\, ds \bigg)^{1/2} + \frac{K_1}{2} \mathrm{e}^{K_0 t}\, t.$$

We then complete the proof. □

By the results above, the following Hessian formula for eigenfunctions ϕ is obtained.

Theorem 2.3. *Let D be a compact Riemannian manifold with boundary ∂D. Let $X_\cdot(x)$ be a Brownian motion starting from $x \in D$ and τ_D be its first hitting time of ∂D. Suppose that k is a non-negative function in $C_b^1([0,\infty);\mathbb{R})$ such that $k(0) = 1$. Then for $(\phi, \lambda) \in \mathrm{Eig}(\Delta)$, $t \ge 0$ and $v \in T_x D$,*

$$\begin{aligned} (\mathrm{Hess}\,\phi)(v,v) =& \mathbb{E}^x \Big[\mathrm{e}^{(t\wedge\tau_D)\lambda/2} (\mathrm{Hess}\,\phi)(Q_{t\wedge\tau_D}(k(t\wedge\tau_D)v), Q_{t\wedge\tau_D}(v)) + \mathrm{e}^{(t\wedge\tau_D)\lambda/2} (\mathbf{d}\phi)(W_{t\wedge\tau_D}^k(v,v)) \Big] \\ & - \mathbb{E}^x \bigg[\mathrm{e}^{(t\wedge\tau_D)\lambda/2}\, \mathbf{d}\phi(Q_{t\wedge\tau_D}(v)) \int_0^{t\wedge\tau_D} \langle Q_s(\dot{k}(s)v), //_s dB_s \rangle \bigg]. \end{aligned} \tag{2.6}$$

Proof. The claim follows by taking expectation of the martingale (2.1) at time 0 and $t \wedge \tau_D$. Recall that $|Q_t| \le \mathrm{e}^{K_0 t/2}$. For $x \in \partial D$ formula (2.6) is obviously tautological since $\tau_D \equiv 0$. □

To get Hessian estimates from Theorem 2.3 requires estimates of $\mathrm{Hess}\,\phi$ on the boundary ∂D. To this end, we first note the following observation. Since $\phi = 0$ on the boundary ∂D, then $\mathbf{d}\phi = N(\phi)N$.

Lemma 2.4. *For $x \in \partial D$, let $H(x)$ be the mean curvature of the boundary. Then*

$$N^2(\phi)(x) = -H(x)N(\phi)(x), \quad x \in \partial D.$$

Proof. For $x \in \partial D$, we have

$$\begin{aligned} 0 &= \lambda\phi(x) = \Delta\phi(x) \\ &= \mathrm{div}(\mathbf{d}\phi)(x) = \mathrm{div}(N(\phi)N)(x) \\ &= \langle \nabla N(\phi), N\rangle(x) + N(\phi)\mathrm{div}(N)(x). \end{aligned}$$

Taking into account that $\mathrm{div}(N)(x) = H(x)$, the proof is completed. □

The following lemma is taken from [2, Proposition 2.5] and allows to estimate the values of $|\nabla\phi|$ on the boundary.

Lemma 2.5. *Let $\alpha \in \mathbb{R}$ such that*

$$\frac{1}{2}\Delta\rho_{\partial D} \le \alpha. \tag{2.7}$$

Then for any $t > 0$,

$$\|\nabla\phi\|_{\partial D,\infty} = \|N(\phi)\|_{\partial D,\infty} \le \|\phi\|_\infty \, \mathrm{e}^{\lambda t/2} f(t,\alpha), \tag{2.8}$$

where

$$f(t,\alpha) = 2\alpha^+ + \frac{\sqrt{2}}{\sqrt{\pi t}}\,\mathrm{e}^{-\frac{\alpha^2 t}{2}}. \tag{2.9}$$

Remark 2.6. With constants $K_0, \theta > 0$ such that $\mathrm{Ric} \ge -K_0$ on D and $H \ge -\theta$ on the boundary ∂D, where $H(x)$ is the mean curvature of D at $x \in D$, let

$$\alpha = \frac{1}{2}\max\left\{\theta, \sqrt{(n-1)K_0}\right\}.$$

Then estimate (2.7) holds for such α.

Lemmas 2.4 and 2.5 allow to derive an estimate of $|\mathrm{Hess}\,\phi|$ on the boundary ∂D.

Lemma 2.7. *Assume that* $\mathrm{Ric} \ge -K_0$ *and N is the extended vector field of the normal vector field to the neighbour area $\partial_{r_0}D := \{x : \rho_{\partial D}(x) \le r_0\}$ such that $|\nabla N| \le \sigma$ and $|\Delta^{(1)}N| \le \beta$ on $\partial_{r_0}D$. Then for $x \in \partial D$,*

$$\begin{aligned} \Big|\mathrm{Hess}(\phi)\Big|(x) \le& (n-1)\sigma\|N(\phi)\|_{\partial D,\infty} \\ &+ \|h\|_\infty \,\mathrm{e}^{\frac{1}{2}(K_0+\sigma K_{h,\sigma})t}\left(\frac{1}{\sqrt{t}} + \sqrt{t}(\beta + \|\Delta\psi\|_\infty + K_0 + \lambda)\right)\|\nabla\phi\|_\infty \\ &+ \|h\|_\infty \,\mathrm{e}^{\frac{1}{2}(K_0+\sigma K_{h,\sigma})t}\sqrt{t}(\sigma + \|\nabla\psi\|_\infty)\|\mathrm{Hess}\,\phi\|_\infty, \end{aligned}$$

where $h \in C^\infty(D)$ such that $h \ge 1$ and $N\log h \ge 1$ and

$$K_{h,\sigma} = \sup\{-\Delta\log h + \sigma|\nabla\log h|^2\},$$

and $\psi \in C^2(D)$ is a cut-off function satisfying $\psi|_{\partial D} = 1$ and $\psi(x) = 0$ for $\rho_\partial(x) \ge r_0$.

Proof. Given $x \in \partial D$, let $\{X_i\}_{1\le i\le n}$ be an orthonormal basis of T_xD with $X_1 = N$. Then

$$\begin{aligned} |\mathrm{Hess}(\phi)(X_i,X_j)| &= |\nabla\mathbf{d}\phi(X_i,X_j)| = |\langle\nabla_{X_i}\nabla\phi, X_j\rangle| \\ &= |X_i\langle\nabla\phi, X_j\rangle - \langle\nabla\phi, \nabla_{X_i}X_j\rangle|. \end{aligned}$$

From the condition $|\nabla N| \le \sigma$ on $\partial_{r_0}D$, we know that $|\mathrm{II}| \le \sigma$. If $X_i, X_j \in T_x\partial D$, i.e. $i, j \ne 1$, then $\langle\nabla\phi, X_j\rangle|_{\partial D} = 0$ and

$$|\mathrm{Hess}(\phi)(X_i,X_j)| = |-N(\phi)\langle N, \nabla_{X_i}X_j\rangle| \le \sigma|N(\phi)|. \tag{2.10}$$

If $X_i = X_j = N$, i.e. $i = j = 1$, then $\nabla_N N|_{\partial D} = 0$ and

$$|\operatorname{Hess}(\phi)(N,N)| = |N^2(\phi)| \le (n-1)\sigma|N(\phi)|. \tag{2.11}$$

If $X_j \in T_x\partial D$ and $X_i = N$ (i.e. $j \ne 1$ and $i = 1$), then

$$|\operatorname{Hess}(\phi)(X_j,N)|(x) = |NX_j(\phi)|(x).$$

Let X_t be the reflecting Brownian motion and $P_t^N f(x) = \mathbb{E}^x[f(X_t)]$ for $f \in \mathcal{B}_b(D)$ the Neumann semigroup. We have known that for $N(P_\delta^N f)|_{\partial D} = 0$ for $\delta > 0$. Then according to Kolmogorov equation,

$$P_\delta^N(N(\phi))(x) = P_{t+\delta}^N(N(\phi))(x) + \frac{1}{2}\int_0^t P_s^N(\Delta P_\delta^N(N(\phi))(x)\,ds.$$

Set $\partial_{r_0}D := \{x : \rho(x,\partial D) \le r_0\}$ and $\psi \in C^2(\partial_{r_0}D)$ such that $\psi|_{\partial D} = 1$ and $\psi = 0$ outside $\partial_{r_0}D$. Taking derivative on both sides of the above equation yields

$$X_iP_\delta^N(\psi N(\phi))(x) = X_iP_{t+\delta}^N(\psi N(\phi))(x) + \frac{1}{2}\int_0^t X_iP_s^N\Delta P_\delta^N(\psi N(\phi))(x)\,ds.$$

It has been known from [22] that

$$|\nabla P_t^N f| \le \frac{1}{\sqrt{t}}\,\mathrm{e}^{\frac{1}{2}K_0t}\,\mathbb{E}^x[\mathrm{e}^{\frac{1}{2}\sigma l_t}]\|f\|_\infty,$$

where l_t is the local time supported on ∂D. By [22, Corollary 3.2.8.] or Lemma 3.3 presented in the next section, one has

$$\mathbb{E}^x[\mathrm{e}^{\frac{1}{2}\sigma l_t}] \le \|h\|_\infty^\sigma \exp\Big(\frac{\sigma}{2}K_{h,\sigma}t\Big),$$

where $h \in C^\infty(D)$ such that $h \ge 1$ and $N\log h \ge 1$ and

$$K_{h,\sigma} = \sup\{-\Delta\log h + \sigma|\nabla\log h|^2\}.$$

We then conclude that

$$\begin{aligned}|X_iP_\delta^N(\psi N(\phi))|(x) &\le \|h\|_\infty\,\mathrm{e}^{\frac{1}{2}(K_0+\sigma K_{h,\sigma})t}\left[\frac{1}{\sqrt{t}}\|\psi N(\phi)\|_\infty + \sqrt{t}\|\Delta P_\delta^N(\psi N(\phi))\|_\infty\right]\\ &\le \|h\|_\infty\,\mathrm{e}^{\frac{1}{2}(K_0+\sigma K_{h,\sigma})t}\left[\frac{1}{\sqrt{t}}\|\nabla\phi\|_\infty + \sqrt{t}\|\Delta P_\delta^N(\psi N(\phi))\|_\infty\right].\end{aligned} \tag{2.12}$$

Since for $0 < s \le \delta$,

$$d[\Delta P_{\delta-s}^N(\psi N(\phi))(X_s)] = \frac{1}{2}N\Big(\Delta P_{\delta-s}^N(\psi N(\phi))(X_s)\Big)(X_s)\,dl_s,$$

then there exists constants $c_1, c_2 > 0$ such that

$$\begin{aligned}|\Delta P_\delta^N(\psi N(\phi))|(x) &\le |P_\delta^N[\Delta(\psi N(\phi))]|(x) + \frac{1}{2}\mathbb{E}^x\left[\int_0^\delta (\Delta P_{\delta-s}^N(\psi N(\phi)))(X_s)\,dl_s\right]\\ &\le \|\Delta(\psi N(\phi))\|_\infty + \frac{c_1}{2}\mathbb{E}^x l_\delta \le \|\Delta(\psi N(\phi))\|_\infty + \frac{c_2}{2}\sqrt{\delta}.\end{aligned} \tag{2.13}$$

Letting δ go to 0 yields

$$|\Delta P_\delta^N(\psi N(\phi))|(x) \le \|\Delta(\psi N(\phi))\|_\infty.$$

Moreover,

$$\begin{aligned}d//_t^{-1}\nabla P_{\delta-t}^N(\psi N(\phi))(X_t) &= //_t^{-1}\nabla_{//_tdB_t}P_{\delta-t}^N(\psi N(\phi))(X_t) + \frac{1}{2}//_t^{-1}\mathrm{Ric}^\sharp(\nabla P_{\delta-t}^N(\psi N(\phi)))(X_t)\,dt\\ &\quad + \frac{1}{2}//_t^{-1}\nabla_N\nabla P_{\delta-t}^N(\psi N(\phi))(X_t)\,dl_t,\end{aligned}$$

which implies

$$
\begin{aligned}
&|\langle \nabla P^N_\delta(\psi N(\phi)), X_i\rangle(x) - \mathbb{E}^x\langle //_\delta^{-1}\nabla(\psi N(\phi))(X_\delta), X_i\rangle| \\
&\le \frac{K_0}{2}\int_0^\delta \mathbb{E}^x|\nabla P^N_{\delta-s}(\psi N(\phi))|(X_s)\,ds + \frac{\sigma}{2}\mathbb{E}^x\int_0^\delta |\nabla P^N_{\delta-s}(\psi N(\phi))|(X_s)\,dl_s.
\end{aligned}
$$

Again using Itô's formula for $//_s\nabla(\psi N(\phi))(X_s)$, we further have

$$
\begin{aligned}
&|\langle \nabla P^N_\delta(\psi N(\phi)), X_i\rangle(x) - \langle \nabla(\psi N(\phi)), X_i\rangle(x)| \\
&\le |\langle \nabla P^N_\delta(\psi N(\phi)), X_i\rangle(x) - \mathbb{E}^x\langle //_\delta^{-1}\nabla(\psi N(\phi))(X_\delta), X_i\rangle| \\
&\quad + |\mathbb{E}^x\langle //_\delta^{-1}\nabla(\psi N(\phi))(X_\delta), X_i\rangle - \langle \nabla(\psi N(\phi)), X_i\rangle(x)| \\
&\le \frac{K_0}{2}\int_0^\delta \mathbb{E}^x|\nabla P^N_{\delta-s}(\psi N(\phi))|(X_s)\,ds + \frac{\sigma}{2}\mathbb{E}^x\int_0^\delta |\nabla P^N_{\delta-s}(\psi N(\phi))|(X_s)\,dl_s \\
&\quad + \int_0^\delta \mathbb{E}^x|\Delta^{(1)}\nabla(\psi N(\phi))|(X_s)\,ds + \mathbb{E}^x\int_0^\delta |\nabla_N\nabla(\psi N(\phi))|(X_s)\,dl_s.
\end{aligned}
$$

Letting δ go to 0 we obtain

$$
\lim_{\delta\to 0}|X_iP^N_\delta(\psi N(\phi))|(x) = |X_i(\psi N(\phi))|(x).
$$

Together this with (2.12) and (2.13), we have

$$
|X_i(\psi N(\phi))|(x) \le \|h\|_\infty\, \mathrm{e}^{\frac{1}{2}(K_0+\sigma K_{h,\sigma})t}\left[\frac{1}{\sqrt{t}}\|\nabla\phi\|_\infty + \sqrt{t}\|\Delta(\psi N(\phi))\|_\infty\right].
$$

By Weitzenböck's formula, we calculate that

$$
\begin{aligned}
\Delta(\psi N(\phi)) &= \Delta\psi N(\phi) + \psi\langle\Delta^{(1)}N, \nabla\phi\rangle + \psi\langle N, \Delta^{(1)}\nabla\phi\rangle + \langle\nabla\psi, \nabla N(\phi)\rangle \\
&= \Delta\psi N(\phi) + \psi\langle\Delta^{(1)}N, \nabla\phi\rangle + \psi\langle N, \nabla\Delta^{(1)}\phi\rangle + \psi\langle\nabla N, \operatorname{Hess}\phi\rangle + \langle\nabla\psi, \nabla N(\phi)\rangle \\
&= \Delta\psi N(\phi) + \psi\langle\Delta^{(1)}N, \nabla\phi\rangle + \operatorname{Ric}(\psi N, \nabla\phi) - \lambda\psi\langle N, \nabla\phi\rangle + \psi\langle\nabla N, \operatorname{Hess}\phi\rangle + \langle\nabla\psi, \nabla N(\phi)\rangle,
\end{aligned}
$$

and for $x \in \partial D$,

$$
X_i(\psi N(\phi))(x) = X_i(\psi)(x)N(\phi)(x) + \psi(x)X_iN(\phi)(x) = X_iN(\phi)(x).
$$

We finally conclude that

$$
\begin{aligned}
|X_iN(\phi)|(x) \le{}& \|h\|_\infty\, \mathrm{e}^{\frac{1}{2}(K_0+\sigma K_{h,\sigma})t}\left(\frac{1}{\sqrt{t}} + \sqrt{t}(\|\Delta^{(1)}N\|_{\partial_{r_0}D,\infty} + \|\Delta\psi\|_\infty + K_0 + \lambda)\right)\|\nabla\phi\|_\infty \\
&+ \|h\|_\infty\, \mathrm{e}^{\frac{1}{2}(K_0+\sigma K_{h,\sigma})t}\sqrt{t}(\|\nabla N\|_{\partial_{r_0}D,\infty} + \|\nabla\psi\|_\infty)\|\operatorname{Hess}\phi\|_\infty.
\end{aligned}
$$

We then complete the proof by combining the above estimate with (2.10) and (2.11).

□

Combining the estimates in Lemmas 2.5 and 2.7 with Theorem 2.3, we are now in a position to prove our first main result.

Theorem 2.8. *Let D be a compact Riemannian manifold with boundary ∂D. Assume that $|\mathrm{Ric}| \le K_0$, $|R| \le K_1$ and $|\mathbf{d}^*R + \nabla\mathrm{Ric}| \le K_2$ on D. Moreover, assume that N is the extended vector field of the normal vector field to the neighbour area $\partial_{r_0}D := \{x : \rho_{\partial D}(x) \le r_0\}$ such that $|\nabla N| \le \sigma$ and $|\Delta^{(1)}N| \le \beta$ on $\partial_{r_0}D$. Let $\alpha \in \mathbb{R}$ such that*

$$
\frac{1}{2}\Delta\rho_{\partial D} \le \alpha.
$$

For $h \in C^\infty(D)$ with $\min_D h = 1$ and $N\log h|_{\partial D} \ge 1$ and some cut-off function $\psi \in C^2(D)$ satisfying $\psi|_{\partial D} = 1$ and $\psi(x) = 0$ for $\rho_\partial(x) \ge r_0$,

- *if* $\sqrt{\lambda+K_0} \geq 2f(\frac{1}{\lambda+K_0},\alpha)$, *then*

$$\begin{aligned}\frac{\|\mathrm{Hess}\,\phi\|}{\|\phi\|_\infty} \leq & 2(n-1)\,\mathrm{e}\,\sigma f(\frac{1}{\lambda+K_0},\alpha)\\ &+2\,\mathrm{e}(\|h\|_\infty+2)\sqrt{\lambda+K_0}\max\Big\{\sqrt{\lambda+2K_0+\sigma K_{h,\sigma}},\,2\|h\|_\infty(\sigma+\|\nabla\psi\|_\infty)\Big\}\\ &+2\,\mathrm{e}\,[\|h\|_\infty(\beta+\|\Delta\psi\|_\infty+K_0+\lambda)+K_1]\\ &+\frac{K_2\,\mathrm{e}}{\sqrt{\lambda+2K_0+\sigma K_{h,\sigma}}};\end{aligned}$$

- *if* $\sqrt{\lambda+K_0} < 2f(\frac{1}{\lambda+K_0},\alpha)$, *then*

$$\begin{aligned}\frac{\|\mathrm{Hess}\,\phi\|}{\|\phi\|_\infty} \leq & \,2(n-1)\sigma\,\mathrm{e}\,f(\frac{1}{\lambda+K_0},\alpha)\\ &+2\,\mathrm{e}(\|h\|_\infty+2)\max\Big\{\sqrt{\lambda+2K_0+\sigma K_{h,\sigma}},\,2\|h\|_\infty(\sigma+\|\nabla\psi\|_\infty)\Big\}\left(f(\frac{1}{\lambda+K_0},\alpha)+\frac{\lambda+K_0}{4f(\frac{1}{\lambda+K_0},\alpha)}\right)\\ &+2\sqrt{\mathrm{e}}\frac{\|h\|_\infty(\beta+\|\Delta\psi\|_\infty+K_0+\lambda)+K_1}{\sqrt{\lambda+2K_0+\sigma K_{h,\sigma}}}\left(f(\frac{1}{\lambda+K_0},\alpha)+\frac{\lambda+K_0}{4f(\frac{1}{\lambda+K_0},\alpha)}\right)\\ &+\frac{K_2\sqrt{\mathrm{e}}}{\lambda+2K_0+\sigma K_{h,\sigma}}\left(f(\frac{1}{\lambda+K_0},\alpha)+\frac{\lambda+K_0}{4f(\frac{1}{\lambda+K_0},\alpha)}\right),\end{aligned}$$

where $K_{h,\alpha} := \sup_D\{-\Delta\log h+\alpha|\nabla\log h|^2\}$ *with* α *a positive constant.*

Proof. According to the formula (2.6) we have

$$\begin{aligned}|\mathrm{Hess}\,\phi(v,v)| = & \,\mathbb{E}\Big[\mathrm{e}^{\lambda(t\wedge\tau_D)/2}\,\mathrm{Hess}\,\phi(Q_{t\wedge\tau_D}(k(t\wedge\tau_D)v),Q_{t\wedge\tau_D}(v))\Big]\\ &+\mathbb{E}\Big[\mathrm{e}^{\lambda(t\wedge\tau_D)/2}\,\mathbf{d}\phi(W^k_{t\wedge\tau_D}(v,v))\Big]\\ &-\mathbb{E}\left[\mathrm{e}^{\lambda(t\wedge\tau_D)/2}\,\mathbf{d}\phi(Q_{t\wedge\tau_D}(v))\int_0^{t\wedge\tau_D}\langle Q_s(\dot{k}(s)v),//_s dB_s\rangle\right].\end{aligned}$$

Taking $k(s)=(t-s)/t$ for $s\in[0,t]$ in the equation yields

$$\begin{aligned}&|\mathrm{Hess}\,\phi(v,v)|\\ &\leq (n-1)\sigma\mathbb{E}\left[\mathbb{1}_{\{\tau_D\leq t\}}\,\mathrm{e}^{(\frac{\lambda}{2}+K_0)\tau_D}\,\frac{t-\tau_D}{t}\right]\|N(\phi)\|_{\partial D,\infty}\\ &\quad+\|h\|_\infty\mathbb{E}\left[\mathbb{1}_{\{\tau_D\leq t\}}\,\mathrm{e}^{(\frac{\lambda}{2}+K_0)\tau_D}\,\frac{t-\tau_D}{t}\,\mathrm{e}^{\frac12(K_0+\sigma K_{h,\sigma})(t-\tau_D)}\left(\frac{1}{\sqrt{t-\tau_D}}+\sqrt{t-\tau_D}(\|\Delta^{(1)}N\|_{\partial_{r_0}D,\infty}+\|\Delta\psi\|_\infty+K_0+\lambda)\right)\right]\|\mathbf{d}\phi\|_\infty\\ &\quad+\|h\|_\infty\mathbb{E}\left[\mathbb{1}_{\{\tau_D\leq t\}}\,\mathrm{e}^{(\frac{\lambda}{2}+K_0)\tau_D}\,\frac{t-\tau_D}{t}\,\mathrm{e}^{\frac12(K_0+\sigma K_{h,\sigma})(t-\tau_D)}\,\sqrt{t-\tau_D}(\|\nabla N\|_{\partial_{r_0}D,\infty}+\|\nabla\psi\|_\infty)\right]\|\mathrm{Hess}\,\phi\|_\infty\\ &\quad+\|\mathbf{d}\phi\|_\infty\left(K_1\sqrt{t}+\frac{K_2}{2}t\right)\mathrm{e}^{\left(\frac12\lambda+K_0\right)t}\\ &\quad+2\|\mathbf{d}\phi\|_\infty\frac{\mathrm{e}^{\left(\frac12\lambda+K_0\right)t}}{\sqrt{t}}.\end{aligned}\tag{2.14}$$

Combining this with the fact that

$$\frac{t-\tau_D}{t}\frac{1}{\sqrt{t-\tau_D}}\leq\frac{\sqrt{t-\tau_D}}{t}\leq\frac{1}{\sqrt{t}}$$

and then substituting back into (2.14) and using (2.8) by letting $t = \frac{1}{\lambda+K_0}$ we obtain

$$
\begin{aligned}
&|\operatorname{Hess}\phi(v,v)| \\
&\le (n-1)\sigma\, \mathrm{e}^{(\frac{\lambda}{2}+K_0)t}\ \sqrt{\mathrm{e}} f(\frac{1}{\lambda+K_0},\alpha)\|\phi\|_\infty \\
&\quad + \|h\|_\infty\, \mathrm{e}^{(\frac{\lambda}{2}+K_0+\frac{\sigma K_{h,\sigma}}{2})t}\left(\frac{1}{\sqrt{t}} + \sqrt{t}(\|\Delta^{(1)}N\|_{\partial_{r_0}D,\infty} + \beta + K_0 + \lambda)\right)\|\mathbf{d}\phi\|_\infty \\
&\quad + \|h\|_\infty\, \mathrm{e}^{(\frac{\lambda}{2}+K_0+\frac{\sigma K_{h,\sigma}}{2})t}\ \sqrt{t}(\sigma + \|\nabla\psi\|_\infty)\|\operatorname{Hess}\phi\|_\infty \\
&\quad + \left(\frac{2}{\sqrt{t}} + K_1\sqrt{t} + \frac{K_2}{2}t\right)\mathrm{e}^{(\frac{1}{2}\lambda+K_0)t}\|\mathbf{d}\phi\|_\infty.
\end{aligned}
\tag{2.15}
$$

Now let $t = t_0 := \frac{1}{\max\{\lambda+2K_0+\sigma K_{h,\sigma},\, 4\|h\|_\infty^2(\sigma+\|\nabla\psi\|_\infty)^2\}}$. Then

$$
\|h\|_\infty\, \mathrm{e}^{(\frac{\lambda}{2}+K_0+\frac{\sigma K_{h,\sigma}}{2})t_0}\ \sqrt{t_0}(\|\nabla N\|_{\partial_{r_0}D,\infty} + \|\nabla\psi\|_\infty)\|\operatorname{Hess}\phi\|_\infty \le \frac{1}{2}\|\operatorname{Hess}\phi\|_\infty
$$

and

$$
\begin{aligned}
|\operatorname{Hess}\phi(v,v)| &\le 2(n-1)\sigma\, \mathrm{e}\, f(\frac{1}{\lambda+K_0},\alpha)\|\phi\|_\infty \\
&\quad + 2\sqrt{\mathrm{e}}(\|h\|_\infty+2)\max\left\{\sqrt{\lambda+2K_0+\sigma K_{h,\sigma}},\, 2\|h\|_\infty(\sigma+\|\nabla\psi\|_\infty)\right\}\|\mathbf{d}\phi\|_\infty \\
&\quad + 2\sqrt{\mathrm{e}}\frac{\|h\|_\infty(\beta+\|\Delta\psi\|_\infty+K_0+\lambda)+K_1}{\max\{\sqrt{\lambda+2K_0+\sigma K_{h,\sigma}},\, 2\|h\|_\infty(\sigma+\|\nabla\psi\|_\infty)\}}\|\mathbf{d}\phi\|_\infty \\
&\quad + K_2\sqrt{\mathrm{e}}\frac{1}{\max\{\lambda+2K_0+\sigma K_{h,\sigma},\, 4\|h\|_\infty^2(\sigma+\|\nabla\psi\|_\infty)^2\}}\|\mathbf{d}\phi\|_\infty \\
&\le 2(n-1)\sigma\, \mathrm{e}\, f(\frac{1}{\lambda+K_0},\alpha)\|\phi\|_\infty \\
&\quad + 2\sqrt{\mathrm{e}}(\|h\|_\infty+2)\max\left\{\sqrt{\lambda+2K_0+\sigma K_{h,\sigma}},\, 2\|h\|_\infty(\sigma+\|\nabla\psi\|_\infty)\right\}\|\mathbf{d}\phi\|_\infty \\
&\quad + 2\sqrt{\mathrm{e}}\frac{\|h\|_\infty(\beta+\|\Delta\psi\|_\infty+K_0+\lambda)+K_1}{\sqrt{\lambda+2K_0+\sigma K_{h,\sigma}}}\|\mathbf{d}\phi\|_\infty \\
&\quad + \frac{K_2\sqrt{\mathrm{e}}}{\lambda+2K_0+\sigma K_{h,\sigma}}\|\mathbf{d}\phi\|_\infty.
\end{aligned}
\tag{2.16}
$$

It is known from Arnaudon, Thalmaier and Wang [2] that

$$
\frac{\|\mathbf{d}\phi\|_\infty}{\|\phi\|_\infty} \le \begin{cases} \sqrt{\mathrm{e}(\lambda+K_0)}, & \text{if } \sqrt{\lambda+K_0} \ge 2f(\frac{1}{\lambda+K_0},\alpha); \\ \sqrt{\mathrm{e}}\left(f(\frac{1}{\lambda+K_0},\alpha) + \frac{\lambda+K_0}{4f(\frac{1}{\lambda+K_0},\alpha)}\right), & \text{if } \sqrt{\lambda+K_0} < 2f(\frac{1}{\lambda+K_0},\alpha), \end{cases}
$$

Combining this with (2.16) implies that

- if $\sqrt{\lambda+K_0} \ge 2f(\frac{1}{\lambda+K_0},\alpha)$, then

$$
\begin{aligned}
\frac{\|\operatorname{Hess}\phi\|}{\|\phi\|_\infty} \le\, & 2(n-1)\,\mathrm{e}\,\sigma f(\frac{1}{\lambda+K_0},\alpha) \\
& + 2\mathrm{e}(\|h\|_\infty+2)\sqrt{\lambda+K_0}\max\left\{\sqrt{\lambda+2K_0+\sigma K_{h,\sigma}},\, 2\|h\|_\infty(\sigma+\|\nabla\psi\|_\infty)\right\} \\
& + 2\mathrm{e}\left[\|h\|_\infty(\beta+\|\Delta\psi\|_\infty+K_0+\lambda)+K_1\right] \\
& + \frac{K_2\,\mathrm{e}}{\sqrt{\lambda+2K_0+\sigma K_{h,\sigma}}};
\end{aligned}
\tag{2.17}
$$

- if $\sqrt{\lambda+K_0} < 2f(\frac{1}{\lambda+K_0},\alpha)$, then

$$\begin{aligned}\frac{\|\mathrm{Hess}\,\phi\|}{\|\phi\|_\infty} &\le 2(n-1)\sigma\,\mathrm{e}\,f(\frac{1}{\lambda+K_0},\alpha)\\
&\quad + 2\,\mathrm{e}(\|h\|_\infty+2)\max\Big\{\sqrt{\lambda+2K_0+\sigma K_{h,\sigma}},\,2\|h\|_\infty(\sigma+\|\nabla\psi\|_\infty)\Big\}\left(f(\frac{1}{\lambda+K_0},\alpha)+\frac{\lambda+K_0}{4f(\frac{1}{\lambda+K_0},\alpha)}\right)\\
&\quad + 2\sqrt{\mathrm{e}}\frac{\|h\|_\infty(\beta+\|\Delta\psi\|_\infty+K_0+\lambda)+K_1}{\sqrt{\lambda+2K_0+\sigma K_{h,\sigma}}}\left(f(\frac{1}{\lambda+K_0},\alpha)+\frac{\lambda+K_0}{4f(\frac{1}{\lambda+K_0},\alpha)}\right)\\
&\quad + \frac{K_2\sqrt{\mathrm{e}}}{\lambda+2K_0+\sigma K_{h,\sigma}}\left(f\left(\frac{1}{\lambda+K_0},\alpha\right)+\frac{\lambda+K_0}{4f(\frac{1}{\lambda+K_0},\alpha)}\right). \qquad (2.18)\end{aligned}$$

□

3. Hessian estimates on Neumann eigenfunctions of Laplacian

We also use a stochastic approach to prove Theorem 1.3. Let us first introduce the Hessian formulas for the Neumann semigroups, which are established in [4] very recently. The reflecting Brownian motion on D with generator $\frac{1}{2}\Delta$ satisfies the SDE

$$dX_t = //_t \circ dB_t^x + \frac{1}{2}N(X_t)\,dl_t, \quad X_0 = x,$$

where B_t^x is a standard Brownian motion on the Euclidean space $T_xD \cong \mathbb{R}^n$. We write again $X_t = X_t(x)$ to indicate the starting point $x \in D$ (which may be on the boundary ∂D). Here $//_t : T_xD \to T_{X_t(x)}D$ denotes the ∇-parallel transport along $X_t(x)$ and l_t the local time of $X_t(x)$ supported on ∂D. Note that the reflecting Brownian motion $X_t(x)$ is defined for all $t \ge 0$.

Suppose that $\tilde{Q}_t\colon T_xD \to T_{X_t(x)}D$ satisfies

$$\mathrm{D}\tilde{Q}_t = -\frac{1}{2}\mathrm{Ric}^\sharp(\tilde{Q}_t)\,dt + \frac{1}{2}(\nabla N)^\sharp(\tilde{Q}_t)\,dl_t, \quad \tilde{Q}_0 = \mathrm{id}. \qquad (3.1)$$

For $k \in C_b^1([0,\infty);\mathbb{R})$ define an operator-valued process $\tilde{W}_t^k\colon T_xD\otimes T_xD \to T_{X_t(x)}D$ as solution to the following covariant Itô equation

$$\begin{aligned}\mathrm{D}\tilde{W}_t^k(v,w) &= R(//_t\,dB_t, \tilde{Q}_t(k(t)v))\tilde{Q}_t(w)\\
&\quad -\frac{1}{2}(\mathbf{d}^*R+\nabla\mathrm{Ric})^\sharp(\tilde{Q}_t(k(t)v),\tilde{Q}_t(w))\,dt\\
&\quad -\frac{1}{2}(\nabla^2N - R(N))^\sharp(\tilde{Q}_t(k(t)v),\tilde{Q}_t(w))\,dl_t\\
&\quad -\frac{1}{2}\mathrm{Ric}^\sharp(\tilde{W}_t^k(v,w))\,dt + \frac{1}{2}(\nabla N)^\sharp(\tilde{W}_t^k(v,w))\,dl_t, \qquad (3.2)\end{aligned}$$

with initial condition $\tilde{W}_0^k(v,w) = 0$.

Theorem 3.1 ([4])**.** *Let D be a compact Riemannian manifold with boundary ∂D. Let $X(x)$ be the reflecting Brownian motion on D with starting point x (possibly on the boundary) and denote by $P_tf(x) = \mathbb{E}[f(X_t(x)]$ the corresponding Neumann semigroup acting on $f \in \mathcal{B}_b(D)$. Then, for $v \in T_xD$, $t \ge 0$ and $k \in C_b^1([0,\infty);\mathbb{R})$,*

$$\mathrm{Hess}\,P_tf(v,v) = \mathbb{E}\left[-df(\tilde{Q}_t(v))\int_0^t \langle \tilde{Q}_s(\dot{k}(s)v), //_s dB_s\rangle + df(\tilde{W}_t^k(v,v))\right].$$

By estimating $\tilde{W}^k$ and $\tilde{Q}$ explicitly, we can get pointwise bounds for the Hessian of Neumann eigenfunctions.

Proof. By [4, Theorem 3.5] the Hessian of the semigroup can be estimated as

$$
\begin{aligned}
|\operatorname{Hess} P_t f|(x) \le & \left(K_1 + \frac{K_2}{2}\sqrt{t} + \frac{2}{t}\right) e^{K_0 t}\, \mathbb{E}[e^{\sigma_1 l_t}]\|f\|_\infty \\
& + \frac{\sigma_2}{2\sqrt{t}} e^{K_0 t}\, \mathbb{E}\left[e^{\sigma_1 l_t}\right]^{1/2} \left(\mathbb{E}\left[\int_0^t e^{\frac{1}{2}\sigma_1 l_s}\, dl_s\right]^2\right)^{1/2} \|f\|_\infty,
\end{aligned}
$$

We complete the proof by observing that $P_t\phi = e^{-\lambda t/2}\phi$. □

Corollary 3.2. *We keep the assumptions of Theorem 3.1. Assume that* $\operatorname{Ric} \ge -K_0$, $|R| \le K_1$ *and* $|\mathbf{d}^* R + \nabla \operatorname{Ric}| \le K_2$ *on D, and* $\mathrm{II} \ge -\sigma_1$, $|\nabla^2 N + R(N)| < \sigma_2$ *on the boundary* ∂D. *Then, for* $(\phi, \lambda) \in \operatorname{Eig}_N(D)$,

$$
\begin{aligned}
|\operatorname{Hess}\phi|(x) \le & e^{(\frac{1}{2}\lambda + K_0)t}\, \mathbb{E}[e^{\sigma_1^+ l_t}] \left(\frac{1}{\sqrt{t}} + K_1\sqrt{t} + \frac{K_2}{2} t\right) \|\mathbf{d}\phi\|_\infty \\
& + \frac{\sigma_2}{2} e^{(K_0 + \frac{\lambda}{2})t}\, \mathbb{E}\left(e^{\frac{1}{2}\sigma_1^+ l_t} \int_0^t e^{\frac{1}{2}\sigma_1^+ l_s}\, dl_s\right) \|\mathbf{d}\phi\|_\infty.
\end{aligned}
$$

Proof. By [4, Corollary 3.9] the Hessian of the semigroup can be estimated as

$$
\begin{aligned}
|\operatorname{Hess} P_t f| \le & \left(K_1\sqrt{t} + \frac{K_2}{2} t + \frac{1}{\sqrt{t}}\right) \mathbb{E}\left[e^{\sigma_1^+ l_t}\right] e^{K_0 t} \|\nabla f\|_\infty \\
& + \frac{\sigma_2}{2} \mathbb{E}\left(e^{\frac{1}{2}\sigma_1^+ l_t} \int_0^t e^{\frac{1}{2}\sigma_1^+ l_s}\, dl_s\right) e^{K_0 t} \|\nabla f\|_\infty.
\end{aligned}
$$

We complete the proof by observing that $P_t\phi = e^{-\lambda t/2}\phi$. □

Now we turn to the problem of estimating $\mathbb{E}[e^{\alpha l_t/2}]$ for $\alpha > 0$ by introducing a specific class of function h.

Lemma 3.3. *Suppose that* $h \in C^\infty(D)$ *such that* $h \ge 1$ *and* $N \log h \ge 1$. *For* $\alpha > 0$ *let*

$$
K_{h,\alpha} = \sup\{-\Delta \log h + \alpha |\nabla \log h|^2\}.
$$

Then

$$
\mathbb{E}[e^{\alpha l_t/2}] \le \|h\|_\infty^\alpha \exp\left(\frac{\alpha}{2} K_{h,\alpha} t\right).
$$

Proof. By the Itô formula we have

$$
\begin{aligned}
dh^{-\alpha}(X_t) &= \langle \nabla h^{-\alpha}(X_t), //_t\, dB_t\rangle + \frac{1}{2}\Delta h^{-\alpha}(X_t)\, dt + \frac{1}{2} N h^{-\alpha}(X_t)\, dl_t \\
&\le \langle \nabla h^{-\alpha}(X_t), //_t\, dB_t\rangle - \alpha h^{-\alpha}(X_t)\left(-\frac{1}{2} K_{h,\alpha}\, dt + \frac{1}{2} N \log h(X_t)\, dl_t\right).
\end{aligned}
$$

Hence,

$$
M_t := h^{-\alpha}(X_t) \exp\left(-\frac{\alpha}{2} K_{h,\alpha} t + \frac{\alpha}{2}\int_0^t N \log h(X_s)\, dl_s\right)
$$

is a local submartingale. Therefore, by Fatou's lemma and taking into account that $h \ge 1$, we get

$$
\begin{aligned}
& \mathbb{E}\left[h^{-\alpha}(X_t) \exp\left(-\frac{\alpha}{2} K_{h,\alpha} t + \frac{\alpha}{2}\int_0^t N \log h(X_s)\, dl_s\right)\right] \\
& \le \mathbb{E}\left[h^{-\alpha}(X_{t\wedge\tau_D}) \exp\left(-\frac{\alpha}{2} K_{h,\alpha}(t \wedge \tau_D) + \frac{\alpha}{2}\int_0^{t\wedge\tau_D} N \log h(X_s)\, dl_s\right)\right] \\
& \le h^{-\alpha}(x) \le 1.
\end{aligned}
$$

Since $N\log h(x)\geq 1$ we conclude that

$$\mathbb{E}\left[\exp\left(\frac{\alpha}{2}l_t\right)\right]\leq \mathbb{E}\left[\exp\left(\frac{\alpha}{2}\int_0^t N\log h(X_s)\,dl_s\right)\right]\leq \|h\|_\infty^{\alpha}\exp\left(\frac{\alpha}{2}K_{h,\alpha}t\right). \qquad \square$$

Combining Theorem 3.2 and Lemma 3.3, we are now in a position to prove Theorem 1.3.

Theorem 3.4. *Let D be an n-dimensional compact Riemannian manifold with boundary ∂D. Assume that* $\mathrm{Ric}\geq -K_0$, $|R|\leq K_1$ *and* $|\mathbf{d}^*R+\nabla\mathrm{Ric}|\leq K_2$ *on D, and that* $\mathrm{II}\geq -\sigma_1$ *and* $|\nabla^2 N-R(N)|\leq\sigma_2$ *on the boundary ∂D. For $h\in C^\infty(D)$ with* $\min_D h=1$ *and* $N\log h|_{\partial D}\geq 1$, *let* $K_{h,\alpha}:=\sup_D\{-\Delta\log h+\alpha|\nabla\log h|^2\}$ *with α a positive constant. Then for any non-trivial $(\phi,\lambda)\in\mathrm{Eig}_N(\Delta)$,*

$$\frac{\|\mathrm{Hess}\,\phi\|_\infty}{\|\phi\|_\infty}\leq C_{N,\lambda}(D)\lambda$$

where

$$C_{N,\lambda}(D)=\mathrm{e}\left(1+\frac{K_1+2K_0^++2\sigma_1^+K_{h,2\sigma_1^+}}{\lambda}+\frac{K_2+2\sigma_2K_{h,2\sigma_1^+}}{\lambda\sqrt{2\lambda+4K_0^++4\sigma_1K_{h,2\sigma_1^+}}}\right)\|h\|_\infty^{3\sigma_1^+}$$
$$+\frac{\sigma_2\,\mathrm{e}}{\lambda}\sqrt{2\lambda+4K_0^++4\sigma_1^+K_{h,2\sigma_1^+}}\|h\|_\infty^{3\sigma_1^+}\ln\|h\|_\infty;$$

Proof. By Lemma 3.3, we have

$$\mathbb{E}[\mathrm{e}^{\sigma_1 l_t}]\leq\mathbb{E}[\mathrm{e}^{\sigma_1^+ l_t}]\leq\|h\|_\infty^{2\sigma_1^+}\exp\left(\sigma_1^+K_{h,2\sigma_1^+}t\right),$$

and

$$\mathbb{E}[\mathrm{e}^{\sigma_1 l_t}]\leq\|h\|_\infty^{2\sigma_1^+}\exp\left(\sigma_1^+K_{h,2\sigma_1^+}t\right).$$

Moreover, we observe that

$$\begin{aligned}
\mathbb{E}\left[\mathrm{e}^{\frac12\sigma_1 l_t}\int_0^t\mathrm{e}^{\frac12\sigma_1 l_s}\,dl_s\right]&\leq\frac{2(\mathbb{E}[\mathrm{e}^{(\sigma_1^++\varepsilon)l_t}]-1)}{\sigma_1^++\varepsilon}\\
&\leq\frac{2}{\sigma_1^++\varepsilon}\left(\|h\|_\infty^{2(\sigma_1^++\varepsilon)}\exp\left((\sigma_1^++\varepsilon)K_{h,2(\sigma_1^++\varepsilon)}t\right)-1\right)\\
&\leq\frac{2}{\sigma_1^++\varepsilon}\left(\|h\|_\infty^{2(\sigma_1^++\varepsilon)}\exp\left((\sigma_1^++\varepsilon)K_{h,(\sigma_1^++\varepsilon)}t\right)-1\right)\\
&\leq\frac{2}{\sigma_1^++\varepsilon}\left(\|h\|_\infty^{2(\sigma_1^++\varepsilon)}-1\right)+\frac{2}{\sigma_1^++\varepsilon}\|h\|_\infty^{2(\sigma_1^++\varepsilon)}\left[\exp\left((\sigma_1^++\varepsilon)K_{h,2(\sigma_1^++\varepsilon)}t\right)-1\right]\\
&\leq 4\|h\|_\infty^{2(\sigma_1^++\varepsilon)}\ln\|h\|_\infty+2\|h\|_\infty^{2(\sigma_1^++\varepsilon)}\exp\left((\sigma_1^++\varepsilon)K_{h,2(\sigma_1^++\varepsilon)}t\right)K_{h,2(\sigma_1^++\varepsilon)}t.
\end{aligned}$$

Letting ε tend to 0, we arrive at

$$\mathbb{E}\left[\mathrm{e}^{\frac12\sigma_1 l_t}\int_0^t\mathrm{e}^{\frac12\sigma_1 l_s}\,dl_s\right]\leq 4\|h\|_\infty^{2\sigma_1^+}\ln\|h\|_\infty+2\|h\|_\infty^{2\sigma_1^+}\exp\left(\sigma_1^+K_{h,2\sigma_1^+}t\right)K_{h,2\sigma_1^+}t.$$

Therefore, combining this with Theorem 3.2, we obtain

$$\begin{aligned}
\frac{\|\mathrm{Hess}\,\phi\|_\infty}{\|\mathbf{d}\phi\|_\infty}&\leq\mathrm{e}^{(\frac12\lambda+K_0)t}\left(\frac{1}{\sqrt{t}}+K_1\sqrt{t}+\frac{K_2}{2}t\right)\|h\|_\infty^{2\sigma_1^+}\exp\left(\sigma_1^+K_{h,2\sigma_1^+}t\right)\\
&\quad+\sigma_2\,\mathrm{e}^{(\frac12\lambda+K_0)t}\left[2\ln\|h\|_\infty+K_{h,\sigma_1^+}t\right]\|h\|_\infty^{2\sigma_1^+}\exp\left(\sigma_1^+K_{h,2\sigma_1^+}t\right)\\
&\leq\mathrm{e}^{(\frac12\lambda+K_0)t}\left(\frac{1}{\sqrt{t}}+K_1\sqrt{t}+\frac{K_2}{2}t\right)\|h\|_\infty^{2\sigma_1^+}\exp\left(\sigma_1^+K_{h,2\sigma_1^+}t\right)\\
&\quad+\sigma_2\,\mathrm{e}^{(\frac12\lambda+K_0)t}\left[2\ln\|h\|_\infty+K_{h,\sigma_1^+}t\right]\|h\|_\infty^{2\sigma_1^+}\exp\left(\sigma_1^+K_{h,2\sigma_1^+}t\right).
\end{aligned}$$

Let $t = \left(\lambda + 2K_0 + 2\sigma_1^+ K_{h,2\sigma_1^+}\right)^{-1}$. Then we get

$$\frac{\|\operatorname{Hess}\phi\|_\infty}{\|\mathbf{d}\phi\|_\infty} \le \left(\frac{K_1}{\sqrt{\lambda + 2K_0 + 2\sigma_1^+ K_{h,2\sigma_1^+}}} + \sqrt{\lambda + 2K_0 + 2\sigma_1^+ K_{h,2\sigma_1^+}} \right.$$
$$\left. + \frac{K_2 + 2\sigma_2 K_{h,\sigma_1^+}}{2(\lambda + 2K_0 + 2\sigma_1^+ K_{h,2\sigma_1^+})} + 2\sigma_2 \ln \|h\|_\infty \right) \|h\|_\infty^{2\sigma_1^+} \sqrt{\mathrm{e}}.$$

On the other hand, from [2], it has already been shown that

$$\frac{\|\mathbf{d}\phi\|_\infty}{\|\phi\|_\infty} \le \frac{1}{\sqrt{t}} \mathbb{E}[\mathrm{e}^{\sigma_1 l_t}]^{1/2} \mathrm{e}^{\frac{1}{2}(K_0+\lambda)t} \le \frac{1}{\sqrt{t}} \|h\|_\infty^{\sigma_1^+} \exp\left(\frac{1}{2}(\lambda + \sigma_1^+ K_{h,2\sigma_1^+} + K_0)t\right).$$

Let $t = \left(\lambda + K_0 + \sigma_1^+ K_{h,2\sigma_1^+}\right)^{-1}$. Then we get

$$\frac{\|\mathbf{d}\phi\|_\infty}{\|\phi\|_\infty} \le \sqrt{\lambda + K_0 + \sigma_1^+ K_{h,2\sigma_1^+}} \|h\|_\infty^{\sigma_1^+} \sqrt{\mathrm{e}}.$$

We then conclude that

$$\frac{\|\operatorname{Hess}\phi\|_\infty}{\|\phi\|_\infty} \le \left(\lambda + K_1 + 2K_0 + 2\sigma_1^+ K_{h,2\sigma_1^+} + \frac{K_2 + 2\sigma_2 K_{h,\sigma_1^+}}{2\sqrt{\lambda + 2K_0 + 2\sigma_1^+ K_{h,2\sigma_1^+}}} \right.$$
$$\left. + 2\sigma_2 \ln \|h\|_\infty \sqrt{\lambda + K_0 + \sigma_1^+ K_{h,2\sigma_1^+}} \right) \|h\|_\infty^{3\sigma_1^+} \mathrm{e}.$$

□

4. Construction of functions h and ψ

In Section 4 we explain that if more refined geometric information about the boundary is available (for instance as **Condition (B)** below), then following F.-Y. Wang's construction of the function h (see [21, p.1436] or [22, Theorem 3.2.9]), we can derive explicit upper bounds for $\|h\|_\infty$ and the constant $K_{h,2(\sigma_1^+ + \delta)}$ in Theorem 3.4. See Theorem 1.3 above for a precise formulation of the result. Let us first formulate a more refined condition on the boundary.

Condition (B) There exists a non-negative constant σ such that $\mathrm{II} \le \sigma$ and a positive constant r_0 such that on $\partial_{r_0} D := \{x \in D : \rho_{\partial D}(x) \le r_0\}$ the distance function $\rho_{\partial D}$ to the boundary ∂D is smooth.

Furthermore let k be a positive constant such that $\mathrm{Sect} \le k$ which exists by compactness of D. Using Condition (B), F.-Y. Wang constructed a function $h \in \mathscr{D}$ (see [21, p.1436] or [22, Theorem 3.2.9] for the notation and result). Modifying his construction one defines

$$\log h(x) = \frac{1}{\Lambda_0} \int_0^{\rho_\partial(x)} (\ell(s) - \ell(r_1))^{1-n} \, ds \int_{s \wedge r_1}^{r_1} (\ell(u) - \ell(r_1))^{n-1} \, du \tag{4.1}$$

where ℓ is defined in (1.6), $r_1 := r_0 \wedge \ell^{-1}(0)$ and

$$\Lambda_0 := (1 - \ell(r_1))^{1-n} \int_0^{r_1} (\ell(s) - \ell(r_1))^{n-1} \, ds.$$

Then from the proof of [20, Theorem 1.1], we get:

$$K_{h,\alpha} \le K_\alpha := \frac{n}{r_1} + \alpha \quad \text{and} \quad \|h\|_\infty \le \mathrm{e}^{\frac{1}{2} n r_1}. \tag{4.2}$$

Lemma 4.1. *Let $\sigma, k \in \mathbb{R}$ be non-negative constants such that $|\mathrm{II}| \le \sigma$ and $|\mathrm{Sect}| \le k$ on $\partial_{r_0} D := \{x : \rho_{\partial D}(x) \le r_0\}$ for some $r_0 > 0$. Then*

$$-2(\sigma \vee \sqrt{k}) \le \Delta \rho_\partial(x) \le \sigma \vee \sqrt{k}, \quad \rho_\partial(x) \le r_2,$$

where

$$r_2 := r_0 \wedge \ell^{-1}(\frac{1}{2}). \tag{4.3}$$

Then there exists $\psi \in C^2(D)$ *satisfying* $\psi|_{\partial D} = 0$ *and* $\psi(x)|_{\rho_\partial(x)\geq r_0} = 0$ *such that*

$$\|\nabla\psi\|_\infty \leq \frac{3}{r_2}; \qquad \|\Delta\psi\|_\infty \leq \frac{6(\sigma \vee \sqrt{k})}{r_2} + \frac{6}{r_2^2}.$$

Proof. Using the comparison theorem for $\Delta\rho_\partial$, we have the following estimates due to Kasue [9, 10],

$$\frac{\ell'_{k,\sigma}(\rho_\partial(x))}{\ell_{k,\sigma}(\rho_\partial(x))} \leq \Delta\rho_\partial(x) \leq \frac{\ell'_{-k,-\sigma}(\rho_\partial(x))}{\ell_{-k,-\sigma}(\rho_\partial(x))}, \quad \rho_\partial(x) \leq r_0 \wedge \ell_{k,\sigma}^{-1}(0).$$

It is easy to have for $k, \sigma \geq 0$,

$$\Delta\rho_\partial(x) \leq \sigma \vee \sqrt{k}.$$

For $\rho_\partial(x) \leq r_0 \wedge \ell_{k,\sigma}^{-1}(\frac{1}{2})$,

$$\Delta\rho_\partial(x) \geq \frac{\ell_{k,\sigma}(\rho_\partial(x))}{\ell_{k,\sigma}(\rho_\partial(x))} \geq 2\ell_{k,\sigma}(\rho_\partial(x)) \geq -2(\sigma \vee \sqrt{k}).$$

Let

$$\psi(x) = \begin{cases} \left(\frac{r_2 - \rho_{\partial D}(x)}{r_2}\right)^3, & 0 \leq \rho_{\partial D}(x) \leq r_2; \\ 0, & \rho_{\partial D}(x) > r_2, \end{cases}. \tag{4.4}$$

where $r_2 = r_0 \wedge \ell^{-1}(\frac{1}{2})$ and

$$\ell^{-1}(\frac{1}{2}) = \left[\left(\arcsin\left(\sqrt{k/(k+\sigma^2)}\right) - \arcsin\left(\sqrt{k/2(k+\sigma^2)}\right)\right)/\sqrt{k}\right], \quad \text{for } k > 0.$$

Then we have

$$\|\nabla\psi\|_\infty \leq \frac{3}{r_2}, \quad \text{and} \quad \|\Delta\psi\|_\infty \leq \frac{6(\sigma \vee \sqrt{k})}{r_2} + \frac{6}{r_2^2}. \tag{4.5}$$

□

Proof of Theorem 1.1. Substituting the upper bound $\alpha = \sqrt{k} \vee \sigma$ from $\Delta\rho_\partial(x) \leq \sqrt{k} \vee \sigma$ for $f(t, \alpha)$ into inequalities (2.17) and (2.18). Then using h and ψ defined in (4.1) and (4.4) and substituting the estimates (4.2) and (4.5), we replace

$$K_{h,\alpha}, \ \|h\|_\infty, \ \|\nabla\psi\|_\infty, \ \|\Delta\psi\|_\infty$$

by

$$\frac{n}{r_1} + \alpha, \ e^{nr_1/2}, \ \frac{3}{r_2}, \ \frac{6(\sigma \vee \sqrt{k})}{r_2} + \frac{6}{r_2^2}$$

respectively. We then complete the proof of inequality (1.7) □

Proof of Theorem 1.3. From the conditions we see that Condition (**B**) is satisfied. Then, the Hessian estimate of Neumann eigenfunctions in Theorem 3.4 remain valid by substituting the h defined in (4.1). Then under replacing

$$K_{h,\alpha} \ \text{and} \ \|h\|_\infty$$

by

$$K_\alpha := \frac{n}{r_1} + \alpha \ \text{and} \ e^{nr_1/2}$$

respectively, the conclusion is just listed in Theorem 1.3. □

Conflict of Interest and Ethics Statements. On behalf of all authors, the corresponding author Li-Juan Chen declares that there is no conflict of interest. Data sharing is not applicable to this article as no datasets were created or analysed in this study.

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