

Ramanujan-type congruences for overpartitions modulo 16

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Abstract Let $\bar{p}(n)$ denote the number of overpartitions of n . Recently, Fortin-Jacob-Mathieu and Hirschhorn-Sellers independently obtained 2-, 3- and 4-dissections of the generating function for $\bar{p}(n)$ and derived a number of congruences for $\bar{p}(n)$ modulo 4, 8 and 64 including $\bar{p}(8n+7) \equiv 0 \pmod{64}$, for $n \geq 0$. In this paper, we give a 16-dissection of the generating function for $\bar{p}(n)$ modulo 16 and show that $\bar{p}(16n+14) \equiv 0 \pmod{16}$ for $n \geq 0$. Moreover, using the 2-adic expansion of the generating function for $\bar{p}(n)$ according to Mahlburg, we obtain that $\bar{p}(\ell^2 n + r\ell) \equiv 0 \pmod{16}$, where $n \geq 0$, $\ell \equiv -1 \pmod{8}$ is an odd prime and r is a positive integer with $\ell \nmid r$. In particular, for $\ell = 7$ and $n \geq 0$, we get $\bar{p}(49n+7) \equiv 0 \pmod{16}$ and $\bar{p}(49n+14) \equiv 0 \pmod{16}$. We also find four congruence relations: $\bar{p}(4n) \equiv (-1)^n \bar{p}(n) \pmod{16}$ for $n \geq 0$, $\bar{p}(4n) \equiv (-1)^n \bar{p}(n) \pmod{32}$ where n is not a square of an odd positive integer, $\bar{p}(4n) \equiv (-1)^n \bar{p}(n) \pmod{64}$ for $n \not\equiv 1, 2, 5 \pmod{8}$ and $\bar{p}(4n) \equiv (-1)^n \bar{p}(n) \pmod{128}$ for $n \equiv 0 \pmod{4}$.

Keywords Overpartition · Ramanujan-type congruence · 2-Adic expansion · Dissection formula

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1 Introduction

The objective of this paper is to derive Ramanujan-type congruences for overpartitions modulo 16, 32, 64 and 128 by constructing a 16-dissection of the generating function for overpartitions modulo 16 and applying the 2-adic expansion according to Mahlburg [14].

Recall that an overpartition of a nonnegative integer n is a partition of n where the first occurrence of each distinct part may be overlined. We denote the number of overpartitions of n by $\bar{p}(n)$. For example, there are 8 overpartitions of 3:

$$3, \bar{3}, 2 + 1, \bar{2} + 1, 2 + \bar{1}, \bar{2} + \bar{1}, 1 + 1 + 1, \bar{1} + 1 + 1.$$

Overpartitions arise in combinatorics [5], q -series [4], symmetric functions [2], representation theory [9], mathematical physics [6] and number theory [12, 13], where they are

also called standard MacMahon diagrams, joint partitions, jagged partitions or dotted partitions.

As noted by Corteel and Lovejoy [5], the generating function of $\bar{p}(n)$ is given by

$$\sum_{n \geq 0} \bar{p}(n)q^n = \frac{(-q; q)_\infty}{(q; q)_\infty}, \quad (1.1)$$

where $|q| < 1$ and

$$(a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n).$$

The generating function (1.1) can be written in terms of Ramanujan's theta function $\phi(q)$:

$$\sum_{n \geq 0} \bar{p}(n)q^n = \frac{1}{\phi(-q)}, \quad (1.2)$$

where

$$\phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}.$$

Mahlburg [14] showed that the generating function of $\bar{p}(n)$ has the following 2-adic expansion:

$$\sum_{n \geq 0} \bar{p}(n)q^n = 1 + \sum_{k=1}^{\infty} 2^k \sum_{n=1}^{\infty} (-1)^{n+k} c_k(n)q^n, \quad (1.3)$$

where $c_k(n)$ denotes the number of representations of n as a sum of k squares of positive integers. Employing the above 2-adic expansion (1.3), Mahlburg [14] showed that $\bar{p}(n) \equiv 0 \pmod{64}$ for a set of integers of arithmetic density 1. Moreover, Kim [10] showed that $\bar{p}(n) \equiv 0 \pmod{128}$ for a set of integers of arithmetic density 1.

Congruence properties for $\bar{p}(n)$ have been extensively studied, see, for example, [3, 6–8, 10, 11, 14–16]. Fortin et al. [6] and Hirschhorn and Sellers [7] independently obtained 2-, 3- and 4-dissections of the generating function for $\bar{p}(n)$ and derived a number of Ramanujan-type congruences for $\bar{p}(n)$ modulo 4, 8 and 64, such as

$$\bar{p}(5n + 2) \equiv 0 \pmod{4}, \quad (1.4)$$

$$\bar{p}(4n + 3) \equiv 0 \pmod{8},$$

$$\bar{p}(8n + 7) \equiv 0 \pmod{64}. \quad (1.5)$$

Using dissection techniques, Yao and Xia [16] found some congruences for $\bar{p}(n)$ modulo 8, 16 and 32, such as

$$\bar{p}(48n + 26) \equiv 0 \pmod{8},$$

$$\begin{aligned}\bar{p}(24n + 17) &\equiv 0 \pmod{16}, \\ \bar{p}(72n + 69) &\equiv 0 \pmod{32}.\end{aligned}\tag{1.6}$$

Applying the 2-adic expansion (1.3) of the generating function for $\bar{p}(n)$, Kim [11] proved a conjecture of Hirschhorn and Sellers [7], that is, if ℓ is an odd prime and r is a quadratic nonresidue modulo ℓ , then

$$\bar{p}(\ell n + r) \equiv \begin{cases} 0 \pmod{8} & \text{if } \ell \equiv \pm 1 \pmod{8}, \\ 0 \pmod{4} & \text{if } \ell \equiv \pm 3 \pmod{8}. \end{cases}$$

Moreover, Kim obtained the following congruence:

$$\bar{p}(n) \equiv 0 \pmod{8},\tag{1.7}$$

where n is neither a square nor twice a square.

It should be noted that Kim's congruences (1.7) contain certain Ramanujan-type congruences for $\bar{p}(n)$ modulo 8. Here are some special cases of (1.7). The detailed proofs are omitted. For example, we get the following Ramanujan-type congruences for $\bar{p}(n)$ modulo 8. For $n \geq 0$, we have

$$\begin{aligned}\bar{p}(8n + 5) &\equiv 0 \pmod{8}, \\ \bar{p}(8n + 6) &\equiv 0 \pmod{8}, \\ \bar{p}(12n + 10) &\equiv 0 \pmod{8}, \\ \bar{p}(16n + 10) &\equiv 0 \pmod{8}, \\ \bar{p}(20n + 6) &\equiv 0 \pmod{8}, \\ \bar{p}(20n + 14) &\equiv 0 \pmod{8}.\end{aligned}\tag{1.8}$$

Moreover, as consequences of (1.7), we obtain three infinite families of Ramanujan-type congruences. Let n be a nonnegative integer and ℓ be an odd prime. If r is a positive integer with $\ell \nmid r$, then

$$\bar{p}(\ell^2 n + r\ell) \equiv 0 \pmod{8}.$$

If r is an odd positive integer with $\left(\frac{r}{\ell}\right) = -1$, then

$$\bar{p}(2\ell n + r) \equiv 0 \pmod{8},$$

where $\left(\frac{\cdot}{\ell}\right)$ denotes the Legendre symbol. If $\ell \equiv \pm 3 \pmod{8}$ and $\left(\frac{r}{3\ell}\right) = -1$, then

$$\bar{p}(3\ell n + r) \equiv 0 \pmod{8},$$

where $\left(\frac{\cdot}{3\ell}\right)$ denotes the Jacobi symbol.

In this paper, we are mainly concerned with congruences for $\bar{p}(n)$ modulo 16. We first find a 16-dissection of the generating function for $\bar{p}(n)$ modulo 16 and then establish the following congruence.

Theorem 1.1. *For $n \geq 0$, we have*

$$\bar{p}(16n + 14) \equiv 0 \pmod{16}. \quad (1.9)$$

Applying the 2-adic expansion (1.3), we derive the following infinite family of congruences for $\bar{p}(n)$ modulo 16.

Theorem 1.2. *Let n be a nonnegative integer, $\ell \equiv -1 \pmod{8}$ be an odd prime and r be a positive integer with $\ell \nmid r$. Then we have*

$$\bar{p}(\ell^2 n + r\ell) \equiv 0 \pmod{16}. \quad (1.10)$$

For example, when $\ell = 7$, Theorem 1.2 implies that

$$\bar{p}(49n + 7r) \equiv 0 \pmod{16}$$

holds for $n \geq 0$ and $1 \leq r \leq 6$.

The 2-adic expansion (1.3) can also be used to deduce the following congruence relations for $\bar{p}(n)$ modulo 16, 32, 64 and 128.

Theorem 1.3. *For $n \geq 0$, we have*

$$\bar{p}(4n) \equiv (-1)^n \bar{p}(n) \pmod{16}. \quad (1.11)$$

If n is not a square of an odd positive integer, then

$$\bar{p}(4n) \equiv (-1)^n \bar{p}(n) \pmod{32}. \quad (1.12)$$

If $n \not\equiv 1, 2, 5 \pmod{8}$, then

$$\bar{p}(4n) \equiv (-1)^n \bar{p}(n) \pmod{64}. \quad (1.13)$$

If $n \equiv 0 \pmod{4}$, then

$$\bar{p}(4n) \equiv (-1)^n \bar{p}(n) \pmod{128}. \quad (1.14)$$

Applying the above congruence relations to the congruences (1.5), (1.6) and (1.9), we obtain the following congruences for $n, k \geq 0$:

$$\bar{p}(4^k(16n + 14)) \equiv 0 \pmod{16},$$

$$\bar{p}(4^k(72n + 69)) \equiv 0 \pmod{32},$$

$$\bar{p}(4^k(8n + 7)) \equiv 0 \pmod{64}.$$

2 Proof of Theorem 1.1

In this section, we obtain a 16-dissection of the generating function for $\bar{p}(n)$ modulo 16, which implies Theorem 1.1.

Recall that Ramanujan's theta functions $\phi(q)$ and $\psi(q)$ are given by

$$\begin{aligned}\phi(q) &= \sum_{n=-\infty}^{\infty} q^{n^2}, \\ \psi(q) &= \sum_{n=0}^{\infty} q^{\frac{n^2+n}{2}} = \sum_{n=-\infty}^{\infty} q^{2n^2+n}.\end{aligned}\tag{2.1}$$

From (2.1), it is easy to check that

$$\psi(q) = \psi_1(q^2) + q\psi_2(q^2),\tag{2.2}$$

where

$$\psi_1(q) = \sum_{n=-\infty}^{\infty} q^{4n^2+n}$$

and

$$\psi_2(q) = \sum_{n=-\infty}^{\infty} q^{4n^2-3n}.$$

Theorem 2.1. *We have a 16-dissection of the generating function for $\bar{p}(n)$ modulo 16, namely,*

$$\begin{aligned}\sum_{n \geq 0} \bar{p}(n)q^n &\equiv \frac{\phi^{12}}{\phi(-q^{16})^{16}} \left(\phi^3 + 2q(\phi^2\psi_1 + 4q^{16}\psi^2\psi_2) + 4q^2(\phi\psi_1^2 + q^{16}\phi\psi_2^2) \right. \\ &\quad + 8q^3(\psi_1^3 + q^{16}\psi_1\psi_2^2) + 14q^4\phi^2\psi + 8q^5\phi\psi\psi_1 \\ &\quad + 8q^6(\psi\psi_1^2 + q^{16}\psi\psi_2^2) + 4q^8\phi\psi^2 + 2q^9(\phi^2\psi_2 + 4\psi^2\psi_1) \\ &\quad + 8q^{10}\phi\psi_1\psi_2 + 8q^{11}(\psi_1^2\psi_2 + q^{16}\psi_2^3) + 8q^{12}\psi^3 \\ &\quad \left. + 8q^{13}\phi\psi\psi_2 \right) \pmod{16},\end{aligned}\tag{2.3}$$

where ϕ , ψ , ψ_1 and ψ_2 denote $\phi(q^{16})$, $\psi(q^{32})$, $\psi_1(q^{16})$ and $\psi_2(q^{16})$, respectively.

To prove the above formula, we need the following relations:

$$\phi(q) = \phi(q^4) + 2q\psi(q^8),\tag{2.4}$$

$$\phi(q)^2 = \phi(q^2)^2 + 4q\psi(q^4)^2,\tag{2.5}$$

$$\phi(q)\phi(-q) = \phi(-q^2)^2, \quad (2.6)$$

see Berndt [1, p. 40, Entry 25].

Proof of Theorem 2.1 We claim that

$$\frac{1}{\phi(q)} = \frac{\phi(-q)\phi(q^2)^2\phi(q^4)^4\phi(q^8)^8}{\phi(-q^{16})^{16}}. \quad (2.7)$$

Let $\alpha = e^{\frac{\pi i}{4}}$ and $\beta = e^{\frac{3\pi i}{4}}$. Using (2.6), we find that

$$\begin{aligned} \frac{1}{\phi(q)} &= \frac{\phi(-q)\phi(iq)\phi(-iq)\phi(\alpha q)\phi(-\alpha q)\phi(\beta q)\phi(-\beta q)}{\phi(q)\phi(-q)\phi(iq)\phi(-iq)\phi(\alpha q)\phi(-\alpha q)\phi(\beta q)\phi(-\beta q)} \\ &= \frac{\phi(-q)\phi(q^2)^2\phi(-iq^2)^2\phi(iq^2)^2}{\phi(-q^2)^2\phi(q^2)^2\phi(-iq^2)^2\phi(iq^2)^2} \\ &= \frac{\phi(-q)\phi(q^2)^2\phi(q^4)^4}{\phi(-q^8)^8} \\ &= \frac{\phi(-q)\phi(q^2)^2\phi(q^4)^4\phi(q^8)^8}{\phi(-q^8)^8\phi(q^8)^8} \\ &= \frac{\phi(-q)\phi(q^2)^2\phi(q^4)^4\phi(q^8)^8}{\phi(-q^{16})^{16}}. \end{aligned}$$

Therefore, the generating function for $\bar{p}(n)$ can be written as

$$\sum_{n \geq 0} \bar{p}(n)q^n = \frac{1}{\phi(-q)} = \frac{\phi(q)\phi(q^2)^2\phi(q^4)^4\phi(q^8)^8}{\phi(-q^{16})^{16}}. \quad (2.8)$$

Using (2.2) and (2.4), we obtain that

$$\begin{aligned} \phi(q) &= \phi(q^4) + 2q\psi(q^8) \\ &= \left(\phi(q^{16}) + 2q^4\psi(q^{32})\right) + 2q\left(\psi_1(q^{16}) + q^8\psi_2(q^{16})\right) \\ &= \phi + 2q\psi_1 + 2q^4\psi + 2q^9\psi_2, \end{aligned} \quad (2.9)$$

Applying (2.2), (2.4) and (2.5), we deduce that

$$\begin{aligned} \phi(q^2)^2 &= \phi(q^4)^2 + 4q^2\psi(q^8)^2 \\ &= \left(\phi(q^{16}) + 2q^4\psi(q^{32})\right)^2 + 4q^2\left(\psi_1(q^{16}) + q^8\psi_2(q^{16})\right)^2 \\ &= \phi^2 + 4q^2(\psi_1^2 + q^{16}\psi_2^2) + 4q^4\phi\psi + 4q^8\psi^2 + 8q^{10}\psi_1\psi_2. \end{aligned} \quad (2.10)$$

Similarly, it can be shown that

$$\phi(q^4)^4 = \left(\phi(q^{16}) + 2q^4\psi(q^{32})\right)^4$$

$$= \left(\phi^4 + 16q^{16}\psi^4 \right) + 8q^4\phi^3\psi + 24q^8\phi^2\psi^2 + 32q^{12}\phi\psi^3 \quad (2.11)$$

and

$$\begin{aligned} \phi(q^8)^8 &= \left(\phi(q^{16})^2 + 4q^8\psi(q^{32})^2 \right)^4 \\ &= \left(\phi^8 + 96q^{16}\phi^4\psi^4 + 256q^{32}\psi^8 \right) + 16q^8\phi^2\psi^2 \left(\phi^4 + 16q^{16}\psi^4 \right). \end{aligned} \quad (2.12)$$

Plugging (2.9)—(2.12) into (2.8) and taking modulo 16, we are led to (2.3). This completes the proof. \blacksquare

Notice that the 16-dissection of the generating function for $\bar{p}(n)$ modulo 16 given in Theorem 2.1 contains no terms of powers of q congruent to 7, 14 and 15 modulo 16. So we deduce that $\bar{p}(16n + 14) \equiv 0 \pmod{16}$, $\bar{p}(16n + 7) \equiv 0 \pmod{16}$ and $\bar{p}(16n + 15) \equiv 0 \pmod{16}$, in which the latter two are special cases of (1.5), that is, $\bar{p}(8n+7) \equiv 0 \pmod{64}$.

3 Proof of Theorem 1.2

In this section, we give a proof of Theorem 1.2 by using the 2-adic expansion (1.3) of the generating function for $\bar{p}(n)$. Recall that Theorem 1.2 says that

$$\bar{p}(\ell^2 n + r\ell) \equiv 0 \pmod{16}, \quad (3.1)$$

where $n \geq 0$, $\ell \equiv -1 \pmod{8}$ is an odd prime and r is a positive integer with $\ell \nmid r$.

Proof of Theorem 1.2 By the 2-adic expansion (1.3), we see that for $n \geq 1$,

$$\bar{p}(n) \equiv 2(-1)^{n+1}c_1(n) + 4(-1)^n c_2(n) + 8(-1)^{n+1}c_3(n) \pmod{16}.$$

Thus, to prove congruence (3.1), it suffices to show that

$$c_1(\ell^2 n + r\ell) = 0, \quad (3.2)$$

$$c_2(\ell^2 n + r\ell) \equiv 0 \pmod{4}, \quad (3.3)$$

$$c_3(\ell^2 n + r\ell) \equiv 0 \pmod{2}, \quad (3.4)$$

where n, ℓ and r are given in (3.1).

First, we consider (3.2). Assume to the contrary that there exists a positive integer a such that $\ell^2 n + r\ell = a^2$. It follows that $\ell \mid a^2$. Since ℓ is a prime, we have $\ell^2 \mid a^2$ and $\ell \mid r$, contradicting the assumption that $\ell \nmid r$. This proves (3.2).

Next, to prove (3.3), it suffices to show that the following equation has no positive integer solution in x and y :

$$\ell^2 n + r\ell = x^2 + y^2. \quad (3.5)$$

Otherwise, assume that (a, b) is a positive integer solution of (3.5). Let $d = \gcd(a, b)$, $a = da_1$ and $b = db_1$. Then we have

$$\ell^2 n + r\ell = d^2(a_1^2 + b_1^2),$$

which implies that $\ell \mid d$ or $\ell \mid (a_1^2 + b_1^2)$. If $\ell \mid d$, then $\ell^2 \mid d^2$, and hence $\ell \mid r$, which is a contradiction. If $\ell \mid (a_1^2 + b_1^2)$, namely,

$$a_1^2 + b_1^2 \equiv 0 \pmod{\ell}, \quad (3.6)$$

since $\gcd(a_1, b_1) = 1$, we have $\ell \nmid a_1$ and $\ell \nmid b_1$, so that (3.6) can be written as

$$\frac{a_1^2}{b_1^2} \equiv -1 \pmod{\ell},$$

that is, $\left(\frac{-1}{\ell}\right) = 1$, contradicting the assumption $\ell \equiv -1 \pmod{8}$. Hence (3.3) is proved.

As for (3.4), it suffices to show that the following equation has an even number of positive integer solutions in x, y and z :

$$\ell^2 n + r\ell = x^2 + y^2 + z^2. \quad (3.7)$$

Suppose that (a, b, c) is a positive integer solution of (3.7). We consider the following three cases:

Case 1: $a = b = c$. In this case, (3.7) becomes

$$\ell^2 n + r\ell = 3a^2. \quad (3.8)$$

Since $\ell \neq 3$ and $\ell \nmid r$, it is clear that (3.8) has no positive integer solution.

Case 2: There are exactly two equal numbers among a, b and c . Without loss of generality, we assume that $a = c$, then (3.7) becomes

$$\ell^2 n + r\ell = 2a^2 + b^2. \quad (3.9)$$

Using the argument concerning (3.5), we deduce that (3.9) has no positive integer solution.

Case 3: a, b and c are distinct. If there exists a solution (a, b, c) , then any permutation of (a, b, c) is also a solution of (3.7). Thus the number of solutions of (3.7) is even.

In view of the above three cases, we conclude that (3.7) has an even number of positive integer solutions, and hence the proof is complete. \blacksquare

4 Proof of Theorem 1.3

In this section, we prove Theorem 1.3 by applying the 2-adic expansion (1.3) of the generating function for $\bar{p}(n)$.

Proof of Theorem 1.3 From the 2-adic expansion (1.3), we see that for $n \geq 1$ and $k \geq 0$,

$$(-1)^n \bar{p}(n) \equiv -2c_1(n) + 2^2 c_2(n) + \cdots + (-1)^k 2^k c_k(n) \pmod{2^{k+1}}. \quad (4.1)$$

Replacing n by $4n$ in (4.1), we get

$$\bar{p}(4n) \equiv -2c_1(4n) + 2^2 c_2(4n) + \cdots + (-1)^k 2^k c_k(4n) \pmod{2^{k+1}}. \quad (4.2)$$

By the definition of $c_k(n)$, it is easy to check that for $n \geq 0$,

$$c_1(n) = c_1(4n), \quad c_2(n) = c_2(4n), \quad c_3(n) = c_3(4n).$$

Thus (4.2) becomes

$$\bar{p}(4n) \equiv -2c_1(n) + 2^2 c_2(n) + \cdots + (-1)^k 2^k c_k(4n) \pmod{2^{k+1}}. \quad (4.3)$$

Comparing (4.1) and (4.3), we find that for $n \geq 0$,

$$\begin{aligned} \bar{p}(4n) &\equiv (-1)^n \bar{p}(n) + 2^4 (c_4(4n) - c_4(n)) \\ &\quad + \cdots + (-1)^k 2^k (c_k(4n) - c_k(n)) \pmod{2^{k+1}}. \end{aligned} \quad (4.4)$$

When $k = 3$, it follows from (4.4) that for $n \geq 0$,

$$\bar{p}(4n) \equiv (-1)^n \bar{p}(n) \pmod{16}.$$

Setting $k = 4$ in (4.4), we get

$$\bar{p}(4n) \equiv (-1)^n \bar{p}(n) + 16(c_4(4n) - c_4(n)) \pmod{32}. \quad (4.5)$$

We claim that

$$c_4(4n) - c_4(n) \equiv 0 \pmod{2}, \quad (4.6)$$

where n is not a square of an odd positive integer.

Observe that the following equation has an even number of positive integer solutions in (x, y, z, w) such that x, y, z and w are odd:

$$4n = x^2 + y^2 + z^2 + w^2. \quad (4.7)$$

Assume that (a, b, c, d) is a positive integer solution of (4.7), where a, b, c and d are odd. Clearly, any permutation of (a, b, c, d) is also a solution of (4.7). If there are at least two different numbers among a, b, c and d , then the number of such solutions of equation (4.7) is even. Otherwise, we consider the case $a = b = c = d$. In this case, we get $n = a^2$, which contradicts the assumption that n is not a square of an odd integer. This proves (4.6). Thus, it follows from (4.5) that

$$\bar{p}(4n) \equiv (-1)^n \bar{p}(n) \pmod{32},$$

where n is not a square of an odd positive integer.

To prove (1.13), setting $k = 5$ in (4.4), we find that for $n \geq 0$,

$$\bar{p}(4n) \equiv (-1)^n \bar{p}(n) + 16(c_4(4n) - c_4(n)) - 32(c_5(4n) - c_5(n)) \pmod{64}. \quad (4.8)$$

We claim that for $n \not\equiv 1, 2, 5 \pmod{8}$,

$$c_4(4n) - c_4(n) \equiv 0 \pmod{4}, \quad (4.9)$$

$$c_5(4n) - c_5(n) \equiv 0 \pmod{2}. \quad (4.10)$$

To prove (4.9), we need to show that the number of odd positive integer solutions of the equation

$$4n = x^2 + y^2 + z^2 + w^2 \quad (4.11)$$

is a multiple of 4. Assume that (a, b, c, d) is such a solution of equation (4.11). If $a = b = c = d$, we get $n = a^2$, which contradicts the assumption $n \not\equiv 1 \pmod{8}$. If a, b, c and d are of the pattern $a = b, c = d$, but $a \neq c$, regardless of the order of a, b, c, d , then we get $2n = a^2 + c^2$. It contradicts the assumption that $n \not\equiv 1, 5 \pmod{8}$. For the other cases, the number of odd positive solutions of (4.11) is a multiple of 4. This proves (4.9).

Congruence (4.10) can be proved by showing that the following equation has an even number of solutions in (x, y, z, w, v) :

$$4n = x^2 + y^2 + z^2 + w^2 + v^2, \quad (4.12)$$

where x, y, z, w, v are not all even. Assume that (a, b, c, d, e) is such a solution. If a, b, c, d and e are of the pattern $a = c = d = e$, but $a \neq b$, regardless of the order of a, b, c, d, e , then equation (4.12) becomes

$$4n = 4a^2 + b^2. \quad (4.13)$$

Hence b is even. Since a, b, c, d, e are not all even, a must be odd. Setting $b = 2r$ in (4.13), we deduce that

$$n = a^2 + r^2. \quad (4.14)$$

But this is impossible, since $n \not\equiv 1, 2, 5 \pmod{8}$. For the other cases, the number of the solutions of equation (4.12) is even. Thus we obtain (4.10).

Plugging (4.9) and (4.10) into (4.8), we deduce that for $n \not\equiv 1, 2, 5 \pmod{8}$,

$$\bar{p}(4n) \equiv (-1)^n \bar{p}(n) \pmod{64}.$$

So (1.13) is proved.

Setting $k = 6$ in (4.4), we obtain that

$$\bar{p}(4n) \equiv (-1)^n \bar{p}(n) + 16(c_4(4n) - c_4(n))$$

$$-32(c_5(4n) - c_5(n)) + 64(c_6(4n) - c_6(n)) \pmod{128}. \quad (4.15)$$

Using arguments similar to the proofs of congruences (4.9) and (4.10), we find that for $n \equiv 0 \pmod{4}$,

$$c_4(4n) - c_4(n) \equiv 0 \pmod{8},$$

$$c_5(4n) - c_5(n) \equiv 0 \pmod{4},$$

$$c_6(4n) - c_6(n) \equiv 0 \pmod{2}.$$

It follows from (4.15) that for $n \equiv 0 \pmod{4}$,

$$\bar{p}(4n) \equiv (-1)^n \bar{p}(n) \pmod{128}.$$

This completes the proof. ■

We remark that congruence (1.11) modulo 16 contains the following congruences modulo 4 and 8:

$$\bar{p}(4n) \equiv \bar{p}(n) \pmod{4},$$

$$\bar{p}(4n) \equiv (-1)^n \bar{p}(n) \pmod{8},$$

which can be used to generate more congruences of $\bar{p}(n)$ modulo 4 and 8. For example, from (1.4) and (1.8), we obtain that for $k, n \geq 0$, $\bar{p}(4^k(5n+2)) \equiv 0 \pmod{4}$ and $\bar{p}(4^k(8n+5)) \equiv 0 \pmod{8}$.

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