Ornstein-Uhlenbeck Type Processes on Wasserstein Space*

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Abstract

The Wasserstein space $\mathscr{P}_2$ consists of square integrable probability measures on $\mathbb{R}^d$ and is equipped with the intrinsic Riemannian structure. By using stochastic analysis on the tangent space, we construct the Ornstein-Uhlenbeck (O-U) process on $\mathscr{P}_2$ whose generator is formulated as the intrinsic Laplacian with a drift. This process satisfies the log-Sobolev inequality and has L^2-compact Markov semigroup. Due to the important role played by O-U process in Malliavin calculus on the Wiener space, this measure-valued process should be a fundamental model to develop stochastic analysis on the Wasserstein space. Perturbations of the O-U process is also studied.

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1 Introduction

Let $\mathscr{P}_2$ be the space of all probability measures on $\mathbb{R}^d$ having finite second moments. It is a Polish space under the quadratic Wasserstein distance

$$\mathbb{W}_2(\mu,\nu) := \inf_{\pi\in\mathscr{C}(\mu,\nu)} \bigg(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2\pi(\mathrm{d}x,\mathrm{d}y)\bigg)^{\frac{1}{2}}, \quad \mu,\nu\in\mathscr{P}_2,$$

where $\mathscr{C}(\mu,\nu)$ is the set of all couplings of μ and ν. This space has been equipped with a natural Riemannian structure and becomes an infinite-dimensional Riemannian manifold. We

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will take the Riemannian structure induced by the intrinsic derivative introduced by Albeverio, Kondratiev and Röckner [4], which is consistent with Otto's structure [28] defined for probability measures having smooth and positive density functions, see Remark 2.1 below for details. The Wasserstein space is a fundamental research object in the theory of optimal transport and related analysis, see [3, 37] and reference within.

To study measure-valued diffusion processes, local Dirichlet forms have been constructed by establishing the integration by parts formula of derivatives in measure with respect to a reference distribution Ξ on the space of Radon measures, see [26, 29, 12, 35, 33] and references therein. Moreover, functional inequalities have been derived for measure-valued processes, see [16, 17, 18, 19, 31, 40, 42]. A key point in the construction of Dirichlet form is to establish the integration by parts formula of the reference measure for derivatives in measure. The stationary distribution Ξ in these references is chosen as either the entropic type measures supported on the space of singular distributions without discrete part, or the Dirichlet/Gamma type measures concentrated on the space of discrete distributions, which have reasonable backgrounds from physics and biological genetics. Along a different direction, a Rademacher type theorem is established in [11] for a class of reference measures satisfying the integration by parts formula for the intrinsic derivative.

Corresponding to Dean-Kawasaki type SPDEs, local Dirichlet forms have been constructed on the Wasserstein sub-space over the real line induced by increasing functions (see [23, 24] and references within), and it is proved in [25] that the associated diffusion process is given by the empirical measure of independent particle systems.

By solving a conditional distribution dependent SDE, [41] constructed a diffusion process on $\mathscr{P}_2$ with generator given by second-order differential operator in intrinsic derivative, and establish the Feynman-Kac formula for the underline measure-valued PDE. Since the SDE is driven by finite-dimensional Brownian motion, the measure-valued diffusion process is highly degenerate. Note that the measure-valued diffusion processes constructed in [41] extends that generated by the partial Laplacian investigated in [9]. See also [14] for an extension to the Wasserstein space over a compact Riemannian manifold.

In this paper, we construct and study the Ornstein-Uhlenbeck (O-U) process on the Wesserstein space $\mathscr{P}_2$, whose stationary distribution is a fully supported Gaussian measure, and the generator is the Laplacian with a drift, where the Laplacian is induced by the Riemannian structure and is hence crucial in geometric analysis. In view of the fundamental role played by Ornstein-Uhlenbeck process in Malliavin calculus on the Wiener space, the present study should be crucial for developing stochastic analysis on the Wasserstein space.

Recall that the Brownian motion on a d-dimensional Riemannian manifold can be constructed by using the flat Brownian motion on the tangent space $\mathbb{R}^d$. In the same spirit, we will introduce the tangent space on $\mathscr{P}_2$ which is a separable Hilbert space, then recall the Gaussian measure and Ornstein-Uhlenbeck process on the Hilbert space, and finally construct the corresponding objects on $\mathscr{P}_2$ as projections from the tangent space. The Ornstein-Uhlenbeck process we construct on $\mathscr{P}_2$ shares nice properties of the original process on the tangent space: it satisfies the log-Sobolev inequality and the generator has purely discrete spectrum.

The remainder of the paper is organized as follows. In Section 2, we recall the Riemannian structure induced by the intrinsic derivative due to [4], and calculate the Laplacian operator $\Delta_{\mathscr{P}_2}$. In Section 3, we construct the Gaussian measure $N_{\mu_0,Q}$ determined by a reference measure $\mu_0 \in \mathscr{P}_2$ together with an unbounded positive definite linear operator Q on the tangent space $T_{\mu_0} := L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu_0)$, and study the corresponding O-U process on $\mathscr{P}_2$. In Section 4, we formulate the generator of the O-U process as

$$Lf(\mu) = \Delta_{\mathscr{P}_2} f(\mu) - \langle b(\mu), Df(\mu)\rangle_{T_\mu},$$

where $T_\mu := L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu)$ is the tangent space at μ, and $b(\mu) \in T_\mu$ is induced by the linear operator Q on T_{μ_0}. This formulation is consistent with that of the O-U process on a separable Hilbert space. Finally, in Section 5 we study symmetric diffusion processes on $\mathscr{P}_2$ as perturbations of the O-U process.

2 Riemannian structure on the Wasserstein space

By using the gradient flow of density functions arising from Monge's optimal transport, Otto [28] constructed the Riemannian structure on $\mathscr{P}_2^{ac}$, the space of measures in $\mathscr{P}_2$ having strictly positive smooth density functions with respect to the Lebesgue measure, see also [37, Chapter 13]. Under Otto's structure, the tangent space at $\mu \in \mathscr{P}_2^{ac}$ is the $L^2(\mu)$-closure of $\{\nabla f : f \in C_b^\infty(\mathbb{R}^d)\}$. The Ricci curvature was calculated in [27], while the Levi-Civita connection and parallel displacement have been studied in [13].

In this paper, we adopt the Riemannian structure of $\mathscr{P}_2$ induced by the intrinsic derivative introduced in [4] (see [30, Appendix]). This structure fits well to the Gâteaux derivative in infinite-dimensional analysis, and it works to the space of general Radon measures.

The intrinsic derivative describes the variance in distribution induced by the motion of underlying particle system, while the extrinsic or linear functional derivatives refer to the birth and death of particles, see [8, Section 5.3] and [32] for the relation of different derivatives in measures.

2.1 Intrinsic derivative

We will simply denote $\mu(f) = \int f d\mu$ for a measure μ and a function $f \in L^1(\mu)$. For any $\mu \in \mathscr{P}_2$ and measurable $\phi : \mathbb{R}^d \to \mathbb{R}^d$, let $\mu \circ \phi^{-1}$ be the image of μ under ϕ, i.e.

$$(\mu \circ \phi^{-1})(A) := \mu(\phi^{-1}(A))$$

for measurable sets $A \subset \mathbb{R}^d$. It is easy to see that $\mu \circ \phi^{-1} \in \mathscr{P}_2$ if and only if

$$\phi \in T_\mu := L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu),$$

where $L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu)$ is the space of measurable maps ϕ from $\mathbb{R}^d$ to $\mathbb{R}^d$ with

$$\|\phi\|_{L^2(\mu)} := (\mu(|\phi|^2))^{\frac{1}{2}} < \infty.$$

So, it is natural to take T_μ as the tangent space at μ, which is a separable Hilbert space with inner product

$$\langle \phi_1, \phi_2 \rangle_{T_\mu} := \mu\big(\langle \phi_1, \phi_2 \rangle\big) = \int_{\mathbb{R}^d} \langle \phi_1, \phi_2 \rangle \mathrm{d}\mu, \quad \phi_1, \phi_2 \in T_\mu. \tag{2.1}$$

Let $id \in T_\mu$ be the identity map, i.e. $id(x) = x$.

Definition 2.1. Let $f \in C(\mathscr{P}_2)$, the class of continuous functions on $\mathscr{P}_2$.

(1) We call f intrinsically differentiable, if for any $\mu \in \mathscr{P}_2$,

$$T_\mu \ni \phi \mapsto D_\phi f(\mu) := \lim_{\varepsilon \downarrow 0} \frac{f(\mu \circ (id + \varepsilon\phi)^{-1}) - f(\mu)}{\varepsilon} \in \mathbb{R}$$

is a bounded linear functional. In this case, the intrinsic derivative $Df(\mu)$ is the unique element in T_μ such that

$$\langle Df(\mu), \phi \rangle_{T_\mu} := \mu(\langle \phi, Df(\mu) \rangle) = D_\phi f(\mu), \quad \phi \in T_\mu.$$

(2) We write $f \in C^1(\mathscr{P}_2)$, if f is intrinsically differentiable such that $Df(\mu)(x)$ has a version jointly continuous in $(\mu, x) \in \mathscr{P}_2 \times \mathbb{R}^d$. We denote $f \in C_b^1(\mathscr{P}_2)$ if moreover f and Df are bounded.

(3) We denote $f \in C^2(\mathscr{P}_2)$, if $f \in C^1(\mu)$, the continuous version $Df(\mu)(x)$ is intrinsically differentiable in μ and differentiable in x, such that

$$D^2 f(\mu)(x, y) := D\{Df(\cdot)(x)\}(\mu)(y), \quad \nabla Df(\mu)(x) := \nabla\{Df(\mu)(\cdot)\}(x)$$

have versions jointly continuous in all arguments. We write $f \in C_b^2(\mathscr{P}_2)$ if moreover $f, Df, D^2 f$ and ∇Df are bounded.

When $f \in C^1(\mathscr{P})$, we automatically take $Df(\mu)(x)$ to be the jointly continuous version of Df, which is unique. Indeed, by the continuity, $Df(\mu)(\cdot)$ is unique for each $\mu \in \mathscr{P}_2$ with full support, so that it is unique for all $\mu \in \mathscr{P}_2$ since the set of fully supported measures is dense in $\mathscr{P}_2$. Under the Riemannian metric given by (2.1), the space $\mathscr{P}_2$ becomes an infinite-dimensional Riemannian manifold.

To make calculus on $\mathscr{P}_2$, we introduce the displacement of the tangent space. For any $\phi \in T_\mu$, consider the displacement of measures along ϕ from μ:

$$[0, \infty) \ni s \mapsto \mu \circ (id + s\phi)^{-1} \in \mathscr{P}_2.$$

Then the tangent space is shifted as

$$T_{\mu \circ (id + s\phi)^{-1}} = T_\mu \circ (id + s\phi)^{-1} := \big\{ h \circ (id + s\phi)^{-1} : \ h \in T_\mu \big\}, \quad s \geq 0, \tag{2.2}$$

where $h \circ (id + s\phi)^{-1} \in T_{\mu \circ (id + s\phi)^{-1}}$ is uniquely determined by

$$\big\langle h \circ (id + s\phi)^{-1}, \psi \big\rangle_{T_{\mu \circ (id + s\phi)^{-1}}} := \big\langle h, \psi \circ (id + s\phi) \big\rangle_{T_\mu}, \quad \psi \in T_{\mu \circ (id + s\phi)^{-1}}, \tag{2.3}$$

where $\psi\circ(id+s\phi)\in T_\mu$ is due to

$$
\begin{aligned}
\|\psi\circ(id+s\phi)\|_{T_\mu}^2 &= \mu(|\psi\circ(id+s\phi)|^2)\\
&=\{\mu\circ(id+s\phi)^{-1}\}(|\psi|^2)<\infty,\qquad \psi\in T_{\mu\circ(id+s\phi)^{-1}}.
\end{aligned}\tag{2.4}
$$

Obviously, $T_{\mu\circ(id+s\phi)^{-1}}\supset T_\mu\circ(id+s\phi)^{-1}$. On the other hand, for any $\psi\in T_{\mu\circ(id+s\phi)^{-1}}$, (2.4) implies $\tilde\psi:=\psi\circ(id+s\phi)\in T_\mu$ and

$$\psi=\tilde\psi\circ(id+s\phi)^{-1}\in T_\mu\circ(id+s\phi)^{-1}.$$

Therefore, (2.2) holds.

The following result implies that a function $f\in C_b^1(\mathscr{P}_2)$ is L-differentiable, i.e. it is intrinsically differentiable and

$$\lim_{\|\phi\|_{L^2(\mu)}\downarrow 0}\frac{|f(\mu\circ(id+\phi)^{-1})-f(\mu)-D_\phi f(\mu)|}{\|\phi\|_{L^2(\mu)}}=0.\tag{2.5}$$

In this case, the intrinsic derivative is also called the L-derivative, which coincides with Lions' derivative introduced in [7].

Proposition 2.1. *Let $f\in C^1(\mathscr{P}_2)$ such that for any $\mu\in\mathscr{P}_2$,*

$$\lim_{N\to\infty}\limsup_{\|\phi\|_{L^2(\mu)}\downarrow 0}\int_{\mathbb{R}^d}\Big(\big|Df(\mu\circ(id+\phi)^{-1})\big|^2-N\Big)^+\mathrm{d}\mu=0,\tag{2.6}$$

then (2.5) *holds, i.e. f is L-differentiable.*

Proof. Let $\mu\in\mathscr{P}_2$ and $\phi\in T_\mu$. By (2.2) we have $\phi\circ(id+s\phi)^{-1}\in T_{\mu\circ(id+s\phi)^{-1}}$ for $s\in[0,1]$, and

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}f(\mu\circ(id+s\phi)^{-1})=\lim_{\varepsilon\downarrow 0}\frac{f(\mu\circ(id+(s+\varepsilon)\phi)^{-1})-f(\mu\circ(id+s\phi)^{-1})}{\varepsilon}\\
&=\lim_{\varepsilon\downarrow 0}\frac{f\big(\{\mu\circ(id+s\phi)^{-1}\}\circ(id+\varepsilon\phi\circ(id+s\phi)^{-1})^{-1}\big)-f(\mu\circ(id+s\phi)^{-1})}{\varepsilon}\\
&=D_{\phi\circ(id+s\phi)^{-1}}f(\mu\circ(id+s\phi)^{-1})=\mu\big(\langle\phi,\{Df(\mu\circ(id+s\phi)^{-1})\}\circ(id+s\phi)\rangle\big).
\end{aligned}
$$

Combining this with $f\in C^1(\mathscr{P}_2)$, we arrive at

$$
\begin{aligned}
&\limsup_{\|\phi\|_{L^2(\mu)}\downarrow 0}\frac{|f(\mu\circ(id+\phi)^{-1})-f(\mu)-D_\phi f(\mu)|}{\|\phi\|_{L^2(\mu)}}\\
&=\limsup_{\|\phi\|_{L^2(\mu)}\downarrow 0}\frac{|\int_0^1\frac{\mathrm{d}}{\mathrm{d}s}f(\mu\circ(id+s\phi)^{-1})\mathrm{d}s-D_\phi f(\mu)|}{\|\phi\|_{L^2(\mu)}}\\
&\le\limsup_{\|\phi\|_{L^2(\mu)}\downarrow 0}\int_0^1\frac{1}{\|\phi\|_{L^2(\mu)}}\Big|\mu\big(\langle\phi,\{Df(\mu\circ(id+s\phi)^{-1})\}\circ(id+s\phi)-Df(\mu)\rangle\big)\Big|\mathrm{d}s\\
&\le\limsup_{\|\phi\|_{L^2(\mu)}\downarrow 0}\int_0^1\big\|\{Df(\mu\circ(id+s\phi)^{-1})\}\circ(id+s\phi)-Df(\mu)\big\|_{L^2(\mu)}\mathrm{d}s=0,
\end{aligned}
$$

where the last step follows from the continuity of Df, (2.6) and the dominated convergence theorem. □

We are ready to introduce the chain rule for the intrinsic derivative in the distribution of random variables. Let $\mathscr{L}_\xi$ be the law of a random variable under a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. The following result follows from Proposition 2.1 and [5, Theorem 2.1(2)] for $p = 2$, see also [22, Lemma A.2] for an earlier result.

Proposition 2.2. *Let $(\xi_\varepsilon)_{\varepsilon\in[0,1]}$ be a family of $\mathbb{R}^d$-valued random variables on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, such that $\mu_\varepsilon := \mathscr{L}_{\xi_\varepsilon} \in \mathscr{P}_2$ and*

$$\dot{\xi}_0 := \lim_{s\downarrow 0} \frac{\xi_s - \xi_0}{s}$$

exists in $L^2(\mathbb{P})$. Then for any $f \in C^1(\mathscr{P}_2)$ such that

$$|Df(\mu)(x)| \le c(1 + |x|), \quad x \in \mathbb{R}^d, \quad \mu \in \mathscr{P}_2, \mathbb{W}_2(\mu, \mu_0) \le 1$$

holds for some constant $c > 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} f(\mathscr{L}_{\xi_s}) := \lim_{\varepsilon\downarrow 0} \frac{f(\mathscr{L}_{\xi_s}) - f(\mathscr{L}_{\xi_0})}{s} = \mathbb{E}\big[\langle Df(\mu_0)(\xi_0), \dot{\xi}_0\rangle\big].$$

Remark 2.1. Below we explain that the above Riemannian structure is consistent to that introduced in [28], and $\mathbb{W}_2$ is the intrinsic distance.

(1) We first recall the geodesic on $\mathscr{P}_2$ given by optimal coupling. Let $\mu_0 \in \mathscr{P}_2$ be absolutely continuous with respect to the Lebesgue measure. Then for any $\mu_1, \mu_2 \in \mathscr{P}_2$, there exists an optimal coupling $(h_1, h_2) \in T_{\mu_0} \times T_{\mu_0}$ such that

$$\mu_i = \mu_0 \circ h_i^{-1} (i = 1, 2), \quad \mathbb{W}_2(\mu_1, \mu_2)^2 = \mu_0(|h_1 - h_2|^2),$$

so that

$$\nu_t := \mu_0 \circ (th_1 + (1 - t)h_2)^{-1}, \quad t \in [0, 1]$$

is the geodesic linking μ_1 and μ_2, i.e.

$$\nu_0 = \mu_2, \quad \nu_1 = \mu_1, \quad \mathbb{W}_2(\nu_s, \nu_t) = |t - s|\mathbb{W}_2(\mu_1, \mu_2) \text{ for } t, s \in [0, 1].$$

Indeed, by the definition of $\mathbb{W}_2$, we have

$$\begin{aligned}\mathbb{W}_2(\nu_s, \nu_t) &\le \sqrt{\mu_0(|th_1 + (1 - t)h_2 - sh_1 - (1 - s)h_2|^2)}\\ &\le |t - s|\sqrt{\mu_0(|h_1 - h_2|^2)} = (t - s)\mathbb{W}_2(\mu_1, \mu_2), \quad 0 \le s \le t \le 1,\end{aligned}$$

which together with the triangle inequality implies

$$\mathbb{W}_2(\mu_1, \mu_2) \le \mathbb{W}_2(\nu_0, \nu_s) + \mathbb{W}_2(\nu_s, \nu_t) + \mathbb{W}_2(\nu_t, \nu_1) \le \mathbb{W}_2(\mu_1, \mu_2), \quad 0 \le s \le t \le 1,$$

and hence, $\mathbb{W}_2(\nu_s, \nu_t) = |t - s|\mathbb{W}_2(\mu_1, \mu_2)$ for all $t, s \in [0, 1]$.

(2) To compare the intrinsic derivative with Otto's derivative, let $\mu \in \mathscr{P}_2^{ac}$ as required by the later. Then for any $\psi \in C_0^\infty(\mathbb{R}^d)$, we have $h := \nabla\psi \in T_\mu$ and for small $s > 0$,

$$[0, s] \ni r \mapsto \mu \circ (id + rh)^{-1}$$

is geodesic, so that in the framework of Otto, the directional derivative of a function f at μ along h is given by

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big|_{r=0} f(\mu \circ (id + rh)^{-1}) := \lim_{r \downarrow 0} \frac{f(\mu \circ (id + rh)^{-1}) - f(\mu)}{r},$$

which coincides with the intrinsic directional derivative $D_h f(\mu)$. So, these two Riemannian structures are essentially the same, but the one given by intrinsic derivative is more complete as it works on the whole space $\mathscr{P}_2$.

In general, it seems more reasonable to take the tangent space as $L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu)$ rather than the $L^2(\mu)$-closure of $\{\nabla\psi : \psi \in C_0^\infty(\mathbb{R}^d)\}$, since the optimal map between singular measures may be not of gradient type.

(3) To see that $\mathbb{W}_2$ is the intrinsic distance, let $f \in C_b^1(\mathscr{P}_2)$. By Proposition 2.2 for the reference probability μ_0 such that T_{μ_0} is the class of square integrable random variables, we have

$$\begin{aligned} f(\mu_1) - f(\mu_2) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f(\mathscr{L}_{th_1+(1-t)h_2})\mathrm{d}t \\ &= \int_0^1 \big\langle \{Df(\nu_t)\} \circ (th_1 + (1-t)h_2), h_1 - h_2 \big\rangle_{T_{\mu_0}} \mathrm{d}t. \end{aligned}$$

Therefore, when $\|Df(\nu_t)\|_{T_{\nu_t}} = \big|\{Df(\nu_t)\} \circ (th_1 + (1-t)h_2)\big\|_{T_{\mu_0}} \le 1$, we have

$$|f(\mu_1) - f(\mu_2)| \le \|h_1 - h_1\|_{T_{\mu_0}} = \mathbb{W}_2(\mu_1, \mu_2),$$

while for $Df(\nu_t) = \frac{h_1 - h_2}{|h_1 - h_2|}$ being the constant vector field along the geodesic ν_t, there holds

$$|f(\mu_1) - f(\mu_2)| = \mathbb{W}_2(\mu_1, \mu_2).$$

This exactly fits to the relation of the gradient operator and the intrinsic distance for finite-dimensional Riemannian manifolds.

2.2 Laplacian

Recall that on a d-dimensional Riemannian manifold, the Laplacian is defined as the trace of the second-order derivative ∇^2 (i.e. Hessian operator). Below we define the Laplacian on $\mathscr{P}_2$ in the same way.

Let $\mu \in \mathscr{P}_2$ and let $\{\phi_m\}_{m \ge 1}$ be an ONB (orthonormal basis) of $T_\mu := L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu)$. Note that the number of $\{\phi_m\}$ is finite if and only if μ has finite support. We write $f \in \mathscr{D}(\Delta_\mu)$

if $f \in C_b^2(\mathscr{P}_2)$ such that

$$(2.7)\quad \begin{aligned}\Delta_{\mathscr{P}_2} f(\mu) &:= \mathrm{tr}|_{T_\mu}\{D^2 f(\mu)\} = \sum_{m\geq 1} D^2_{\phi_m} f(\mu) = \sum_{m\geq 1} D_{\phi_m}\{D_{\phi_m} f\}(\mu)\\ &= \sum_{m\geq 1} \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \{D_{\phi_m\circ(id+\varepsilon\phi_m)^{-1}} f\}(\mu\circ(id+\varepsilon\phi_m)^{-1})\end{aligned}$$

exists. We have the following formulation of $\Delta_{\mathscr{P}_2}$.

Proposition 2.3. *For any $\mu \in \mathscr{P}_2$ and $f \in \mathscr{D}(\Delta_\mu)$,*

$$\begin{aligned}\Delta_{\mathscr{P}_2} f(\mu) = \sum_{m\geq 1} \int_{\mathbb{R}^d\times\mathbb{R}^d} \Big\{&\langle D^2 f(\mu)(x,y), h_m(x)\otimes h_m(y)\rangle_{HS}\\ &+ \langle \nabla D f(\mu)(x), \phi_m(x)\otimes\phi_m(x)\rangle_{HS}\Big\}\mu(\mathrm{d}x)\mu(\mathrm{d}y),\end{aligned}$$

where $\langle\cdot,\cdot\rangle_{HS}$ is the Hilbert-Schmidt inner product for matrices, and the right-hand side does not depend on the choice of the ONB $\{\phi\}_{m\geq1}$. Consequently, for any $f \in C_b^2(\mathscr{P}_2)$ and $\mu \in \mathscr{P}_2$, we have $f \in \mathscr{D}(\Delta_\mu)$ if and only if the following series exists:

$$\mathrm{tr}\{\nabla D f(\mu)\} := \sum_{m\geq 1}\int_{\mathbb{R}^d} \langle \nabla D f(\mu)(x), \phi_m(x)\otimes\phi_m(x)\rangle_{HS}\,\mu(\mathrm{d}x).$$

Proof. Noting that

$$\begin{aligned}&\{D_{\phi_m\circ(id+\varepsilon\phi_m)^{-1}} f\}(\mu\circ(id+\varepsilon\phi_m)^{-1})\\ &= \{\mu\circ(id+\varepsilon\phi_m)^{-1}\}\big(\langle \phi_m\circ(id+\varepsilon\phi_m)^{-1}, Df(\mu\circ(id+\varepsilon\phi_m)^{-1})\rangle\big)\\ &= \mu\big(\langle\phi_m, Df(\mu\circ(id+\varepsilon\phi_m)^{-1})(id+\varepsilon\phi_m)\rangle\big),\end{aligned}$$

we deduce from (2.7) that

$$\begin{aligned}D^2_{\phi_m} f(\mu) &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \mu\big(\langle\phi_m, Df(\mu\circ(id+\varepsilon\phi_m)^{-1})(id+\varepsilon\phi_m)\rangle\big)\\ &= \int_{\mathbb{R}^d\times\mathbb{R}^d} \langle D^2 f(\mu)(x,y), \phi_m(x)\otimes\phi_m(y)\rangle_{HS}\,\mu(\mathrm{d}x)\mu(\mathrm{d}y)\\ &\quad + \int_{\mathbb{R}^d} \langle \nabla D f(\mu)(x), \phi_m(x)\otimes\phi_m(x)\rangle_{HS}\,\mu(\mathrm{d}x),\end{aligned}$$

where the first term comes from the derivative $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}|_{\varepsilon=0} Df(\mu\circ(id+\varepsilon\phi_m)^{-1})$ by Proposition 2.2 for $\xi_\varepsilon := id+\varepsilon\phi_m$, and the other term follows from the derivative $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}|_{\varepsilon=0} Df(\mu)(id+\varepsilon\phi_m)$. Therefore, the desired formula holds.

Next, it is easy to see that $\Delta_{\mathscr{P}_2} f(\mu)$ does not depend on the choice of the ONB $\{\phi_m\}_{m\geq1}$. In deed, for another ONB $\{\tilde\phi_m\}_{m\geq1}$ in T_μ, we have

$$\tilde\phi_m = \sum_{l\geq 1} \langle\tilde\phi_m, \phi_l\rangle_{T_\mu}\phi_l, \quad m\geq 1,$$

$$\sum_{m\geq 1}\langle\tilde\phi_m,\phi_l\rangle_{T_\mu}\langle\tilde\phi_m,\phi_k\rangle_{T_\mu}=1_{\{l=k\}},\quad k,l\geq 1.$$

So,

$$\begin{aligned}
&\sum_{m\geq 1}\Big\{\langle D^2f(\mu)(x,y),\tilde\phi_m(x)\otimes\tilde\phi_m(y)\rangle_{HS}+\langle\nabla Df(\mu)(x),\tilde\phi_m(x)\otimes\tilde\phi_m(x)\rangle_{HS}\Big\}\\
&=\sum_{k,l,m\geq 1}\langle\tilde\phi_m,\phi_k\rangle_{T_\mu}\langle\tilde\phi_m,\phi_l\rangle_{T_\mu}\\
&\qquad\qquad\times\Big\{\langle D^2f(\mu)(x,y),\phi_k(x)\otimes\phi_l(y)\rangle_{HS}+\langle\nabla Df(\mu)(x),\phi_k(x)\otimes\phi_l(x)\rangle_{HS}\Big\}\\
&=\sum_{k,l\geq 1}1_{\{k=l\}}\Big\{\langle D^2f(\mu)(x,y),\phi_k(x)\otimes\phi_l(y)\rangle_{HS}+\langle\nabla Df(\mu)(x),\phi_k(x)\otimes\phi_l(x)\rangle_{HS}\Big\}\\
&=\sum_{m=1}\Big\{\langle D^2f(\mu)(x,y),\phi_m(x)\otimes\phi_m(y)\rangle_{HS}+\langle\nabla Df(\mu)(x),\phi_m(x)\otimes\phi_m(x)\rangle_{HS}\Big\}.
\end{aligned}$$

Finally, let $\{e_i\}_{1\leq i\leq d}$ be the standard ONB in $\mathbb{R}^d$. For any $f\in C_b^2(\mathscr{P}_2)$, by the Schwarz inequality we obtain

$$\begin{aligned}
&\sum_{m\geq 1}\left|\int_{\mathbb{R}^d\times\mathbb{R}^d}\langle D^2f(\mu)(x,y),h_m(x)\otimes h_m(y)\rangle_{HS}\,\mu(\mathrm{d}x)\mu(\mathrm{d}y)\right|\\
&=\sum_{m\geq 1}\left|\sum_{i,j=1}^d\int_{\mathbb{R}^d\times\mathbb{R}^d}\{D^2f(\mu)(x,y)\}_{ij}\langle e_i,h_m(x)\rangle\langle e_j,h_m(y)\rangle\,\mu(\mathrm{d}x)\mu(\mathrm{d}y)\right|\\
&\leq\|D^2f\|_\infty\sum_{i,j=1}^d\left(\sum_{m\geq 1}\langle e_i,\phi_m\rangle_{T_\mu}^2\right)^{\frac12}\left(\sum_{m\geq 1}\langle e_j,\phi_m\rangle_{T_\mu}^2\right)^{\frac12}\\
&=\|D^2f\|_\infty\sum_{i,j=1}^d\|e_i\|_{L^2(\mu)}\|e_j\|_{L^2(\mu)}=d^2\|D^2f(\mu)\|_\infty<\infty.
\end{aligned}$$

Therefore, $f\in\mathscr{D}(\Delta_{\mathscr{P}_2})$ if and only if $\mathrm{tr}\{\nabla Df(\mu)\}$ exists.

□

Remark 2.2. We present some comments on $\mathscr{D}(\Delta_\mu)$.

(1) If μ has finite support, then T_μ is finite-dimensional so that $C_b^2(\mathscr{P}_2)\subset\mathscr{D}(\Delta_\mu)$.

(2) Let $f_i(\mu):=\int_{\mathbb{R}^d}x_i\mu(\mathrm{d}x), 1\leq i\leq d$. Then for any $\mu\in\mathscr{P}_2$ and $g\in C_b^2(\mathbb{R}^d)$,

$$f:=g(f_1,\cdots,f_d)\in\mathscr{D}(\Delta_\mu).$$

Indeed, this case we have

$$Df(\mu)=(\nabla g)(f_1(\mu),\cdots,f_d(\mu)),$$

$$D^2 f(\mu) = (\nabla^2 g)(f_1(\mu), \cdots, f_d(\mu)),$$

so that $\nabla Df = 0$ and for the standard ONB $\{e_i\}_{1\le i\le d}$ in $\mathbb{R}^d$,

$$\begin{aligned}\Delta_{\mathscr{P}_2} f(\mu) &= \sum_{m\ge1}\sum_{i,j=1}^d (\partial_i\partial_j g)(f_1(\mu), \cdots, f_d(\mu)) \int_{\mathbb{R}^d\times\mathbb{R}^d} \phi_m^i(x)\phi_m^j(y)\mu(\mathrm{d}x)\mu(\mathrm{d}y)\\ &= \sum_{i,j=1}^d (\partial_i\partial_j g)(f_1(\mu), \cdots, f_d(\mu)) \sum_{m=1}^d \mu(\langle\phi_m, e_i\rangle)\mu(\langle\phi_m, e_j\rangle)\\ &= \sum_{i,j=1}^d (\partial_i\partial_j g)(f_1(\mu), \cdots, f_d(\mu))\mu(\langle e_i, e_j\rangle)\\ &= (\Delta g)(f_1(\mu), \cdots, f_d(\mu)), \quad \mu\in\mathscr{P}_2.\end{aligned}$$

(3) In general, cylindrical functions of type

$$f(\mu) := g(\mu(f_1), \cdots, \mu(f_n)), \quad n\ge 1, f_i\in C_b^2(\mathbb{R}^d), g\in C_b^2(\mathbb{R}^n)$$

may not in $\mathscr{D}(\Delta_\mu)$, although they are in $C_b^2(\mathscr{P}_2)$.

3 Ornstein-Uhlenbeck process on Wasserstein space

As already explained in Introduction based on the construction of Brownian motion on a Riemannian manifold, we will start from the O-U process on the tangent space T_{μ_0} for a reference measure $\mu_0\in\mathscr{P}_2$.

To make sure that any $\mu\in\mathscr{P}_2$ is the distribution of some $h\in T_{\mu_0}$ under the probability μ_0, i.e. $\mu=\mu_0\circ h^{-1}$, we assume that μ_0 is absolutely continuous with respect to the Lebesgue measure. In this case, for any $\mu\in\mathscr{P}_2$, there exists a unique $h\in T_{\mu_0}$ such that

$$\Psi(h) := \mu_0\circ h^{-1} = \mu, \qquad \mathbb{W}_2(\mu_0,\mu)^2 = \mu_0(|id-h|^2).$$

This h is called the optimal map as solution of the Monge problem for $\mathbb{W}_2$, see [37, Theorem 10.41] or [3]. The map $\Psi: T_{\mu_0}\to\mathscr{P}_2$ is a Lipschitz surjection, i.e. $\Psi(T_{\mu_0})=\mathscr{P}_2$ and

$$\mathbb{W}_2(\Psi(h),\Psi(\tilde h)) \le \mu_0(|h-\tilde h|^2)^{\frac12} = \|h-\tilde h\|_{T_{\mu_0}}, \qquad h,\tilde h\in T_{\mu_0}.$$

In the following, we first introduce some facts for the O-U process on the Hilbert space T_{μ_0}, then construct the corresponding one on $\mathscr{P}_2$.

3.1 O-U process on tangent space

Let Q be a positive definite self-adjoint operator in T_{μ_0} with eigenvalues $\{q_n\}_{n\ge1}$ and eigenbasis $\{h_n\}_{n\ge1}$ such that $q_n\uparrow\infty$ as $n\uparrow\infty$ and

$$\sum_{n=1}^\infty q_n^{-1} < \infty.$$

Then the centred Gaussian measure on T_{μ_0} with covariance Q^{-1} is given by

$$G_Q(\mathrm{d}h) := \prod_{n=1}^{\infty} \Big(\frac{q_n}{2\pi}\Big)^{\frac{1}{2}} \exp\Big[-\frac{q_n\langle h, h_n\rangle_{T_{\mu_0}}^2}{2}\Big]\mathrm{d}\langle h, h_n\rangle_{T_{\mu_0}}$$

under the coordinates $\{\langle h, h_n\rangle_{T_\mu}\}_{n\geq 1}$ referring to the expansion

$$h = \sum_{n=1}^{\infty} \langle h, h_n\rangle_{T_{\mu_0}} h_n, \quad h \in T_{\mu_0}.$$

The associated O-U process can be constructed as (see [10, (5.2.9) or (6.2.1)])

$$h_t = \mathrm{e}^{-Qt} h_0 + \sqrt{2}\int_0^t \mathrm{e}^{-Q(t-s)}\mathrm{d}W_s, \quad t \geq 0, \tag{3.1}$$

where W_t is the cylindrical Brownian motion on T_{μ_0}, i.e.

$$W_t = \sum_{n=1}^{\infty} B_t^n h_n, \quad t \geq 0$$

for independent one-dimensional Brownian motions $\{B_t^n\}_{n\geq 1}$.

Let $(\tilde{L}, \mathscr{D}(\tilde{L}))$ be generator of the O-U process, which is a negative definite self-adjoint operator in $L^2(G_Q)$. The domain $\mathscr{D}(\tilde{L})$ includes the class of cylindrical functions $\mathscr{F}C_b^2(T_{\mu_0})$ consisting of

$$h \mapsto \tilde{f}(h) := F(\langle h, h_1\rangle_{T_{\mu_0}}, \cdots, \langle h, h_n\rangle_{T_{\mu_0}}), \quad n \geq 1, F \in C_b^2(\mathbb{R}^n),$$

and for such a function,

$$\tilde{L}\tilde{f}(h) = \Delta\tilde{f}(h) - \langle Q\nabla f(h), h\rangle_{T_{\mu_0}} = \sum_{i=1}^{n} \big(\partial_i^2 F - q_n\partial_i F\big)(F)(\langle h, h_1\rangle_{T_{\mu_0}}, \cdots, \langle h, h_n\rangle_{T_{\mu_0}}),$$

where Δ and ∇ are the Laplacian and gradient operators on T_{μ_0} respectively. Moreover, the integration by parts formula yields

$$\tilde{\mathscr{E}}(\tilde{f}, \tilde{g}) := \int_{T_{\mu_0}} \langle \nabla\tilde{f}, \nabla\tilde{g}\rangle_{T_{\mu_0}} \mathrm{d}G_Q = -\int_{T_{\mu_0}} (\tilde{f}\tilde{L}\tilde{g})\mathrm{d}G_Q, \quad \tilde{f}, \tilde{g} \in \mathscr{F}C_b^2(T_{\mu_0}). \tag{3.2}$$

Consequently, $(\tilde{\mathscr{E}}, \mathscr{F}C_b^2(T_{\mu_0}))$ is closable in $L^2(G_Q)$ and the closure $(\tilde{\mathscr{E}}, \mathscr{D}(\tilde{\mathscr{E}}))$ is a symmetric conservative Dirichlet form. Moreover, it satisfies the log-Sobolev inequality (see [20, 21])

$$G_Q(\tilde{f}^2 \log \tilde{f}^2) \leq \frac{2}{q_1}\tilde{\mathscr{E}}(\tilde{f}, \tilde{f}), \quad \tilde{f} \in \mathscr{D}(\tilde{\mathscr{E}}),\ G_Q(\tilde{f}^2) = 1. \tag{3.3}$$

The generator has purely discrete spectrum, i.e. the essential spectrum of $\tilde{L}$ is empty. Indeed, consider the following one-dimensional O-U operators $\{L_i\}_{i\geq 1}$:

$$L_i\varphi(r) = \varphi''(r) - q_i r\varphi'(r), \quad r \in \mathbb{R}.$$

It is well known that each $-L_i$ has purely discrete spectrum consisting of simple eigenvalues

$$\sigma(-L_i) = \{\lambda_{i,k} : k \ge 0\},$$

where $\lambda_{i,0} = 0, \lambda_{i,1} = q_i$ and $\lambda_{i,k} \uparrow \infty$ with linear growth as $k \uparrow \infty$. Since $q_i \uparrow \infty$ as $i \uparrow \infty$ and L is the independent sum of these operators, i.e.

$$Lf(h) = \sum_{i=1}^{\infty} L_i f_{i,h}(\langle h, h_i\rangle_{T_{\mu_0}}), \quad f_{i,h}(r) := f(h - \langle h, h_i\rangle_{T_{\mu_0}} h_i + r h_i),$$

the spectrum of $-L$ is purely discrete with eigenvalues

$$\sum_{i=1}^{n} \lambda_{i,k_i}, \quad n \ge 1, k_i \ge 0.$$

According to the spectral theory, the pure discreteness of the spectrum for $\tilde{L}$ is equivalent to the compactness in $L^2(G_Q)$ of the associated Markov semigroup $\tilde{P}_t := \mathrm{e}^{\tilde{L}t}$ for $t > 0$, they are also equivalent to the compactness in $L^2(G_Q)$ of the set

$$\big\{\tilde{f} \in \mathscr{D}(\tilde{\mathscr{E}}) : \ \tilde{\mathscr{E}}_1(\tilde{f}) := \tilde{\mathscr{E}}(f, f) + G_Q(\tilde{f}^2) \le 1\big\}.$$

Let $C_b^1(T_{\mu_0})$ be the class of all bounded functions on T_{μ_0} with bounded and continuous Fréchet derivative. By an approximation argument, see the proof of Lemma 5.2 below for $F = 0$, we have $\mathscr{D}(\tilde{\mathscr{E}}) \supset C_b^1(T_{\mu_0})$ and (3.2) implies

$$\tilde{\mathscr{E}}(\tilde{f}, \tilde{g}) = G_Q\big(\langle \nabla \tilde{f}, \nabla \tilde{g}\rangle_{T_{\mu_0}}\big) := \int_{T_{\mu_0}} \langle \nabla \tilde{f}, \nabla \tilde{g}\rangle_{T_{\mu_0}} \mathrm{d}G_Q, \quad \tilde{f}, \tilde{g} \in C_b^1(T_{\mu_0}).$$

3.2 O-U process on $\mathscr{P}_2$

We first introduce the Gaussian measure and the corresponding O-U Dirichlet form on $\mathscr{P}_2$.

Definition 3.1. Let $\Psi : T_{\mu_0} \to \mathscr{P}_2, \Psi(h) := \mu_0 \circ h^{-1}$.

(1) $N_{\mu_0,Q} := G_Q \circ \Psi^{-1}$ is called the Gaussian measure on $\mathscr{P}_2$ with parameter (μ_0, Q).

(2) Define the following O-U bilinear form on $L^2(N_{\mu_0,Q})$:

$$\mathscr{D}(\mathscr{E}) := \big\{f \in L^2(N_{\mu_0,Q}) : \ f \circ \Psi \in \mathscr{D}(\tilde{\mathscr{E}})\big\},$$
$$\mathscr{E}(f, g) := \tilde{\mathscr{E}}(f \circ \Psi, g \circ \Psi), \qquad f, g \in \mathscr{D}(\mathscr{E}).$$

It is easy to see that $L^2(N_{\mu_0,Q})$ consists of measurable functions f on $\mathscr{P}_2$ such that $f \circ \Psi \in L^2(G_Q)$, so that

$$L^2(N_{\mu_0,Q}) = \big\{\mu \mapsto G_Q(\tilde{f}|\Psi = \mu) : \ \tilde{f} \in L^2(G_Q)\big\},$$

where $G_Q(\cdot|\Psi)$ is the conditional expectation of G_Q given Ψ, and

$$G_Q(\tilde{f}|\Psi = \mu) := G_Q(\tilde{f}|\Psi)\big|_{\Psi=\mu}.$$

It is easy to see that $N_{\mu_0,Q}$ is shift-invariant in the following sense.

Proposition 3.1. *Let $\tilde{h} \in T_{\mu_0}$ be a homeomorphism on $\mathbb{R}^d$. Then $N_{\mu_0,Q} = N_{\mu_0\circ\tilde{h}^{-1},Q\circ\tilde{h}^{-1}}$ for $Q \circ \tilde{h}^{-1}$ being the linear operator on $T_{\mu_0\circ\tilde{h}^{-1}}$ determined by*

$$\{Q \circ \tilde{h}^{-1}\}\tilde{h}_n := q_n \tilde{h}_n, \quad n \geq 1,$$

where $\{\tilde{h}_n\}_{n\geq 1} := \{h_n \circ \tilde{h}^{-1}\}_{n\geq 1}$ is an ONB of $T_{\mu_0\circ\tilde{h}^{-1}}$.

We have the following result for the O-U process on $\mathscr{P}_2$.

Theorem 3.2. *Let $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ be defined above. Then the following assertions hold.*

(1) *$(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is a conservative symmetric Dirichlet form on $L^2(N_{\mu_0,Q})$ with $\mathscr{D}(\mathscr{E}) \supset C_b^1(\mathscr{P}_2)$ and*

$$\mathscr{E}(f,g) = \int_{\mathscr{P}_2} \langle Df(\mu), Dg(\mu)\rangle_{T_\mu} N_{\mu_0,Q}(\mathrm{d}\mu), \quad f,g \in C_b^1(\mathscr{P}_2). \tag{3.4}$$

Moreover, the following log-Sobolev inequality holds:

$$N_{\mu_0,Q}(f^2 \log f^2) \leq \frac{2}{q_1}\mathscr{E}(f,f), \quad f \in \mathscr{D}(\mathscr{E}),\ N_{\mu_0,Q}(f^2) = 1. \tag{3.5}$$

(2) *The generator $(L, \mathscr{D}(L))$ of $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ has discrete spectrum, and satisfies*

$$\begin{aligned}&\mathscr{D}(L) \supset \tilde{\mathscr{D}}(L) := \big\{f \in L^2(N_{\mu_0,Q}):\ f\circ\Psi \in \mathscr{D}(\tilde{L})\big\},\\ &Lf(\mu) = G_Q\big(\tilde{L}(f\circ\Psi)\big|\Psi = \mu\big) := G_Q\big(\tilde{L}(f\circ\Psi)\big|\Psi\big)\big|_{\Psi=\mu}, \quad f \in \tilde{\mathscr{D}}(L).\end{aligned}$$

(3) *Let P_t be the associated Markov semigroup of $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$. Then P_t is compact in $L^2(N_{\mu_0,Q})$ for any $t > 0$, P_t converges exponentially to $N_{\mu_0,Q}$ in entropy:*

$$N_{\mu_0,Q}((P_tf)\log P_tf) \leq \mathrm{e}^{-2q_1t} N_{\mu_0,Q}(f\log f), \quad t \geq 0,\ 0 \leq f,\ N_{\mu_0,Q}(f) = 1, \tag{3.6}$$

and it is hypercontractive:

$$\begin{aligned}&\|P_t\|_{L^p(N_{\mu_0,Q})\to L^{p_t}(N_{\mu_0,Q})} := \sup_{\|f\|_{L^p(N_{\mu_0,Q})}\leq 1} \|P_tf\|_{L^{p_t}(N_{\mu_0,Q})} \leq 1,\\ &t > 0,\ p > 1,\ p_t := 1 + (p-1)\mathrm{e}^{2q_1t}.\end{aligned} \tag{3.7}$$

Proof. (1) We first prove that $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is closed and (3.4) holds, so that $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is a symmetric conservative Dirichlet form in $L^2(N_{\mu_0,Q})$.

Let

$$\mathscr{E}_1(f) := \mathscr{E}(f,f) + \|f\|^2_{L^2(N_{\mu_0,Q})}, \quad \tilde{\mathscr{E}}_1(\tilde{f}) := \tilde{\mathscr{E}}(\tilde{f},\tilde{f}) + \|\tilde{f}\|^2_{L^2(G_Q)}.$$

Let $\{f_n\}_{n\geq 1} \subset \mathscr{D}(\mathscr{E})$ such that

$$\lim_{m,n\to\infty} \mathscr{E}_1(f_n - f_m) = 0.$$

Then $f := \lim_{n\to\infty} f_n$ exists in $L^2(N_{\mu_0,Q})$ and by definition, $\{f_n \circ \Psi\}_{n\geq 1} \subset \mathscr{D}(\tilde{\mathscr{E}})$ with

$$\lim_{m,n\to\infty} \tilde{\mathscr{E}}_1(f_n \circ \Psi - f_m \circ \Psi) = 0.$$

Thus, the closed property of $(\tilde{\mathscr{E}}, \mathscr{D}(\tilde{\mathscr{E}}))$ implies

$$f \circ \Psi = \lim_{n\to\infty} f_n \circ \Psi \in \mathscr{D}(\tilde{\mathscr{E}}),$$

so that $f \in \mathscr{D}(\mathscr{E})$ by definition. Thus, $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is closed.

On the other hand, let $f \in C_b^1(\mathscr{P}_2)$. By Proposition 2.2 for the reference probability μ_0, we see that for any h,

$$\nabla_\phi (f \circ \Psi)(h) := \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} (f \circ \Psi)(h + s\phi) = \langle \phi, (Df)(\Psi(h))(h)\rangle_{T_{\mu_0}}, \quad \phi \in T_{\mu_0},$$

so that

$$\nabla (f \circ \Psi)(h) = (Df)(\Psi(h)) \circ h, \qquad f \in C_b^1(\mathscr{P}_2),\ h \in T_{\mu_0}. \tag{3.8}$$

Hence, $f \in C_b^1(\mathscr{P}_2)$ implies $f \circ \Psi \in C_b^1(T_{\mu_0}) \subset \mathscr{D}(\tilde{\mathscr{E}})$, so that $f \in \mathscr{D}(\mathscr{E})$ by definition.

It remains to verify (3.4), which together with (3.3) implies (3.5). By (3.8) and $N_{\mu_0,Q} = G_Q \circ \Psi^{-1}$,

$$\begin{aligned}
\mathscr{E}(f,g) &:= \tilde{\mathscr{E}}(f \circ \Psi, g \circ \Psi) = \int_{T_{\mu_0}} \langle \nabla (f \circ \Psi), \nabla (g \circ \Psi)\rangle_{T_{\mu_0}} \mathrm{d}G_Q \\
&= \int_{T_{\mu_0}} \mu_0\big(\langle \{Df(\Psi(h))\} \circ h, \{Dg(\Psi(h))\} \circ h\rangle\big) G_Q(\mathrm{d}h) \\
&= \int_{T_{\mu_0}} \{\Psi(h)\}\big(\langle Df(\Psi(h)), Dg(\Psi(h))\rangle\big) G_Q(\mathrm{d}h) \\
&= \int_{\mathscr{P}_2} \mu\big(\langle Df(\mu), Dg(\mu)\rangle\big) N_{\mu_0,Q}(\mathrm{d}\mu) \\
&= \int_{\mathscr{P}_2} \langle Df(\mu), Dg(\mu)\rangle_{T_\mu} N_{\mu_0,Q}(\mathrm{d}\mu), \quad f,g \in C_b^1(\mathscr{P}_2).
\end{aligned}$$

Therefore, (3.4) holds.

(2) Since $(\tilde{L}, \mathscr{D}(\tilde{L}))$ has purely discrete spectrum, the set

$$\big\{\tilde{f} \in \mathscr{D}(\tilde{\mathscr{E}}) :\ \tilde{\mathscr{E}}_1(\tilde{f}) := \tilde{\mathscr{E}}(\tilde{f}, \tilde{f}) + \|\tilde{f}\|^2_{L^2(G_Q)} \leq 1\big\}$$

is compact in $L^2(G_Q)$. By the definitions of $N_{\mu_0,Q}$ and $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$, this implies that the set

$$\big\{f \in \mathscr{D}(\mathscr{E}) :\ \mathscr{E}_1(f) := \mathscr{E}(f,f) + \|f\|^2_{L^2(N_{\mu_0,Q})} \leq 1\big\}$$

is compact in $L^2(N_{\mu_0,Q})$. So, L has purely discrete spectrum.

Next, let $f \in L^2(N_{\mu_0,Q})$ such that $f\circ\Psi \in \mathscr{D}(\tilde{L})$. By the definition of $(\mathscr{E},\mathscr{D}(\mathscr{E}))$, we obtain

$$\int_{\mathscr{P}_2} g(\mu) G_Q\big(\tilde{L}(f\circ\Psi)\big|\Psi=\mu\big) N_{\mu_0,Q}(\mathrm{d}\mu) = \int_{T_{\mu_0}} (g\circ\Psi)\tilde{L}(f\circ\Psi)\mathrm{d}G_Q$$
$$= -\tilde{\mathscr{E}}(g\circ\Psi, f\circ\Psi) = -\mathscr{E}(f,g), \quad g\in\mathscr{D}(\mathscr{E}).$$

Thus, $f\in\mathscr{D}(L)$ and $Lf(\mu) = G_Q(\tilde{L}(f\circ\Psi)|\Psi=\mu)$.

(3) The log-Sobolev inequality (3.5) is equivalent to each of (3.6) and (3.7), see [21]. Moreover, by the spectral theory, since L has purely discrete spectrum, P_t is compact in $L^2(N_{\mu_0,Q})$ for $t>0$. □

Remark 3.1. In general, $\mathscr{D}(L)\neq\tilde{\mathscr{D}}(L)$. For any $f\in\mathscr{D}(L)$, we have

$$\tilde{\mathscr{E}}(g\circ\Psi, f\circ\psi) = \mathscr{E}(g,f) = -N_{\nu_0,Q}(g(Lf)) = G_Q((g\circ\Psi)(Lf)\circ\Psi), \ \ g\in\mathscr{D}(\mathscr{E}).$$

If $\{g\circ\Psi : g\in\mathscr{D}(\mathscr{E})\}$ is dense in $\mathscr{D}(\tilde{\mathscr{E}})$, this would imply that $f\circ\Psi\in\mathscr{D}(\tilde{L})$ and $\tilde{L}(f\circ\Psi) = (Lf)\circ\Psi$. If so, the O-U process on $\mathscr{P}_2$ stating at ν_0 could be constructed as $\nu_t = \Psi(h_t)$ for h_t in (3.1) with $\nu_0 = \Psi(h_0)$. However, in general this is not true, as there might be different h_0 satisfying $\nu_0=\Psi(h_0)$ and the corresponding $\Psi(h_t)$ may have different distributions. In the next section, we formulate L as Laplacian with a drift on $\mathscr{P}_2$.

4 Generator as Laplacian with drift

We shall introduce a subclass of $\mathscr{D}(L)$, such that for functions in this class the generator is formulated as

$$Lf(\mu) = \Delta_{\mathscr{P}_2} f(\mu) - \langle b(\mu), Df(\mu)\rangle_{T_\mu}$$

for a drift $(b,\mathscr{D}(b))$:

$$b: \mathscr{P}_2\supset\mathscr{D}(b)\ni\mu\mapsto b(\mu)\in T_\mu.$$

This is compatible with the finite-dimensional case where the O-U process has a generator of type

$$L_0 f(x) = \Delta f(x) - (Ax)\cdot\nabla f(x)$$

for a positive definite $d\times d$-matrix A.

Definition 4.1. Let $\mathscr{D}$ be the space of functions $f\in C_b^2(\mathscr{P}_2)$ such that for G_Q-a.e. h,

$$\{Df(\mu_0\circ h^{-1})\}\circ h\in\mathscr{D}(Q), \qquad f\in\mathscr{D}(\Delta_{\mu_0\circ h^{-1}}),$$

and

$$\int_{T_{\mu_0}} \Big|\Delta_{\mathscr{P}_2} f(\mu_0\circ h^{-1}) - \big\langle h, Q[\{Df(\mu_0\circ h^{-1})\}\circ h]\big\rangle_{T_{\mu_0}}\Big|^2 G_Q(\mathrm{d}h) < \infty.$$

Theorem 4.1. *We have* $\mathscr{D} \subset \mathscr{D}(L)$ *and for* $U_Q^f(h) := \big\langle h, Q[\{Df(\mu_0 \circ h^{-1})\} \circ h]\big\rangle_{T_{\mu_0}}$,

$$Lf(\mu) = \Delta_{\mathscr{P}_2} f(\mu) - G_Q(U_Q^f | \Psi = \mu), \quad f \in \mathscr{D}. \tag{4.1}$$

Formally, we may write $Lf(\mu) = \Delta_{\mathscr{P}_2} f(\mu) - \langle b(\mu), Df(\mu)\rangle_{T_\mu}$ *where the drift is given by*

$$b(\mu) = G_Q\big(\{Qh\} \circ h^{-1} \big| \Psi(h) = \mu\big).$$

Proof. (a) Let $h \in T_{\mu_0}$ and $\mu := \mu_0 \circ h^{-1}$. Recall that for any $\tilde{h} \in T_{\mu_0}$, $\tilde{h} \circ h^{-1} \in T_\mu$ is determined by

$$\langle \tilde{h} \circ h^{-1}, \phi\rangle_{T_\mu} = \langle \tilde{h}, \phi \circ h\rangle_{T_{\mu_0}}, \quad \phi \in T_\mu. \tag{4.2}$$

Then by Proposition 2.2 for the reference probability $\mathbb{P} = \mu_0$, we obtain

$$\begin{aligned} \nabla_{\tilde{h}}(f \circ \Psi)(h) &:= \lim_{\varepsilon \downarrow 0} \frac{(f \circ \Psi)(h + \varepsilon \tilde{h}) - (f \circ \Psi)(h)}{\varepsilon} \\ &= \lim_{\varepsilon \downarrow 0} \frac{f(\mu_0 \circ (h + \varepsilon \tilde{h})^{-1}) - f(\mu_0 \circ h^{-1})}{\varepsilon} = \int_{\mathbb{R}^d} \langle Df(\mu)(h), \tilde{h}\rangle \mathrm{d}\mu_0 \\ &= \langle Df(\mu), \tilde{h} \circ h^{-1}\rangle_{T_\mu} = D_{\tilde{h} \circ h^{-1}} f(\mu), \quad \tilde{h}, h \in T_{\mu_0}, \quad \mu = \mu_0 \circ h^{-1}. \end{aligned} \tag{4.3}$$

(b) By definition, for any $f \in \mathscr{D}$, Lf given in (4.1) is a well-defined function in $L^2(N_{\mu_0,Q})$. It suffices to prove the integration by parts formula

$$\mathscr{E}(f, g) = -\int_{\mathscr{P}_2} g(\mu) Lf(\mu) N_{\mu_0,Q}(\mathrm{d}\mu), \quad g \in C_b^1(\mathscr{P}_2). \tag{4.4}$$

Simply denote $\tilde{f} = f \circ \Psi$ and $\tilde{g} = g \circ \Psi$. By (3.4) and the integration by parts formula for G_Q, we obtain

$$\begin{aligned} \mathscr{E}(f, g) &= \int_{T_{\mu_0}} \langle \nabla \tilde{f}, \nabla \tilde{g}\rangle \mathrm{d}G_Q = \sum_{n=1}^\infty \int_{T_{\mu_0}} \big(\nabla_{h_n} \tilde{f}\big)(h) \big(\nabla_{h_n} \tilde{g}\big)(h) G_Q(\mathrm{d}h) \\ &= \sum_{n=1}^\infty \int_{T_{\mu_0}} \Big[\nabla_{h_n}\{(\nabla_{h_n} \tilde{f})\tilde{g}\}(h) - \tilde{g}(h) \nabla_{h_n} \nabla_{h_n} \tilde{f}(h)\Big] G_Q(\mathrm{d}h) \\ &= \sum_{n=1}^\infty \int_{T_{\mu_0}} \tilde{g}(h) \Big[q_n \langle h, h_n\rangle_{T_{\mu_0}} \big(\nabla_{h_n} \tilde{f}\big)(h) - \nabla_{h_n} \nabla_{h_n} \tilde{f}(h)\Big] G_Q(\mathrm{d}h). \end{aligned} \tag{4.5}$$

By (4.3),

$$\begin{aligned} &\sum_{n=1}^\infty q_n \langle h, h_n\rangle_{T_{\mu_0}} \big(\nabla_{h_n} \tilde{f}\big)(h) = \sum_{n=1}^\infty q_n \langle h, h_n\rangle_{T_{\mu_0}} \langle h_n, \nabla \tilde{f}(h)\rangle_{T_{\mu_0}} \\ &= \sum_{n=1}^\infty q_n \langle h, h_n\rangle_{T_{\mu_0}} \langle h_n, \{Df(\mu_0 \circ h^{-1}\} \circ h)\rangle_{T_{\mu_0}} = \langle h, Q[\{Df(\mu_0 \circ h^{-1})\} \circ h]\rangle_{T_{\mu_0}}, \end{aligned}$$

so that

$$\begin{aligned}(4.6)\qquad &\sum_{n=1}^\infty \int_{T_{\mu_0}} \tilde g(h)\ q_n\langle h, h_n\rangle_{T_{\mu_0}} \langle h_n, \nabla \tilde f(h)\rangle_{T_{\mu_0}} G_Q(\mathrm{d}h)\\ &= \int_{\mathscr{P}_2} g(\mu) G_Q\Big(\big\langle h, Q[\{Df(\mu_0\circ h^{-1})\}\circ h]\big\rangle_{T_{\mu_0}}\Big|\Psi(h)=\mu\Big) N_{\mu_0,Q}(\mathrm{d}\mu).\end{aligned}$$

By Proposition 2.2, (4.3) also implies

$$\begin{aligned}(4.7)\qquad &\nabla_{h_n}\nabla_{h_n}\tilde f(h) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \mu_0\big(\langle (Df)(\mu_0\circ(h+\varepsilon h_n)^{-1})(h+\varepsilon h_n), h_n\rangle\big)\\ &= \int_{\mathbb{R}^d\times\mathbb{R}^d} \big\langle (D^2 f)(\mu_0\circ h^{-1})(h(x),h(y)), h_n(x)\otimes h_n(y)\big\rangle_{HS} \mu_0(\mathrm{d}x)\mu_0(\mathrm{d}y)\\ &\quad + \int_{\mathbb{R}^d} \big\langle (\nabla Df)(\mu_0\circ h^{-1})(h(x)), h_n(x)\otimes h_n(x)\big\rangle_{HS}\mu_0(\mathrm{d}x)\\ &= I_1(n)+I_2(n),\end{aligned}$$

where $\mu=\mu_0\circ h^{-1}$.

$$\begin{aligned}I_1(n) &:= \int_{\mathbb{R}^d\times\mathbb{R}^d} \big\langle (D^2 f)(\mu_0\circ h^{-1})(x,y), (h_n\circ h^{-1})(x)\otimes (h_n\circ h^{-1})(y)\big\rangle_{HS}\mu(\mathrm{d}x)\mu(\mathrm{d}y),\\ I_2(n) &:= \int_{\mathbb{R}^d} \big\langle (\nabla Df)(\mu_0\circ h^{-1})(x), (h_n\circ h^{-1})(x)\otimes (h_n\circ h^{-1})(x)\big\rangle_{HS}\mu(\mathrm{d}x).\end{aligned}$$

Let $\{\phi_m\}_{m\ge1}$ be an ONB of $T_\mu := L^2(\mathbb{R}^d\to\mathbb{R}^d,\mu)$. By (4.2),

$$h_n\circ h^{-1} = \sum_{m\ge1}\langle h_n\circ h^{-1},\phi_m\rangle_{T_\mu}\phi_m = \sum_{m\ge1}\langle h_n, \phi_m\circ h\rangle_{T_{\mu_0}}\phi_m,$$

so that

$$\begin{aligned}\sum_{n=1}^\infty I_1(n) &= \sum_{m,l\ge1}\sum_{n=1}^\infty \mu_0(\langle h_n,\phi_m\circ h\rangle)\mu_0(\langle h_n,\phi_l\circ h\rangle)\\ &\qquad\cdot\int_{\mathbb{R}^d\times\mathbb{R}^d}\big\langle (D^2f)(\mu)(x,y),\phi_m(x)\otimes\phi_l(y)\big\rangle_{HS}\mu(\mathrm{d}x)\mu(\mathrm{d}y)\\ &= \sum_{m,l\ge1}\mu_0(\langle\phi_m\circ h,\phi_l\circ h\rangle)\int_{\mathbb{R}^d\times\mathbb{R}^d}\big\langle (D^2f)(\mu)(x,y),\phi_m(x)\otimes\phi_l(y)\big\rangle_{HS}\mu(\mathrm{d}x)\mu(\mathrm{d}y)\\ &= \sum_{m,l\ge1}\mu(\langle\phi_m,\phi_l\rangle)\int_{\mathbb{R}^d\times\mathbb{R}^d}\big\langle (D^2f)(\mu)(x,y),\phi_m(x)\otimes\phi_l(y)\big\rangle_{HS}\mu(\mathrm{d}x)\mu(\mathrm{d}y)\\ &= \sum_{m\ge1}\int_{\mathbb{R}^d\times\mathbb{R}^d}\big\langle (D^2f)(\mu)(x,y),\phi_m(x)\otimes\phi_m(y)\big\rangle_{HS}\mu(\mathrm{d}x)\mu(\mathrm{d}y).\end{aligned}$$

Similarly,

$$\sum_{n=1}^\infty I_2(n) = \sum_{m,l\ge1}\sum_{n=1}^\infty \mu_0(\langle h_n,\phi_m\circ h\rangle)\mu_0(\langle h_n,\phi_l\circ h\rangle)$$

$$\cdot \int_{\mathbb{R}^d} \big\langle (\nabla D f)(\mu)(x), \phi_m(x) \otimes \phi_l(x) \big\rangle_{HS} \mu(\mathrm{d}x)$$

$$= \sum_{m\geq 1} \int_{\mathbb{R}^d} \big\langle (\nabla D f)(\mu)(x), \phi_m(x) \otimes \phi_m(x) \big\rangle_{HS} \mu(\mathrm{d}x).$$

These together with (4.7) and Proposition 2.3 yield

$$\sum_{n=1}^{\infty} \nabla_{h_n} \nabla_{h_n} \tilde{f}(h) = \Delta_{\mathscr{P}_2} f(\mu_0 \circ h^{-1}).$$

Combining this with (4.5) and (4.6), we prove (4.1).

□

5 Perturbation of the O-U process

Let V be a measurable function on $\mathscr{P}_2$ such that

$$N^V_{\mu_0,Q}(\mathrm{d}\mu) := \mathrm{e}^{V(\mu)} N_{\mu_0,Q}(\mathrm{d}\mu)$$

is a probability measure on $\mathscr{P}_2$. We consider the pre-Dirichlet form

$$\mathscr{E}^V(f,g) := \int_{\mathscr{P}_2} \langle Df(\mu), Dg(\mu)\rangle_{T_\mu} N^V_{\mu_0,Q}(\mathrm{d}\mu), \quad f,g \in C^1_b(\mathscr{P}_2).$$

If this form is closable in $L^2(N^V_{\mu_0,Q})$, then its closure $(\mathscr{E}^V, \mathscr{D}(\mathscr{E}^V))$ is a symmetric conservative Dirichlet form, whose generator can be formally written as

$$L^V f(\mu) = Lf(\mu) + \langle DV(\mu), Df(\mu)\rangle_{T_\mu}.$$

We call the associated Markov process a perturbation of the O-U process.

A simple situation is that V is bounded. In this case, the closability of $(\mathscr{E}^V, C^1_b(\mathscr{P}_2))$ follows from that of $(\mathscr{E}, C^1_b(\mathscr{P}_2))$, and (3.3) implies the log-Sobolev inequality (see [15])

$$N^V_{\mu_0,Q}(f^2 \log f^2) \leq \frac{2}{q_1} \mathrm{e}^{\sup V - \inf V} \mathscr{E}^V(f,f), \quad f \in \mathscr{D}(\mathscr{E}^V),\ N^V_{\mu_0,Q}(f^2) = 1.$$

Consequently, the associate Markov semigroup P^V_t is hypercontractive and exponentially convergent in entropy. Moreover, the compactness of $\{f \in \mathscr{D}(\mathscr{E}) : \mathscr{E}_1(f) \leq 1\}$ in $L^2(N_{\mu_0,Q})$ implies that of $\{f \in \mathscr{D}(\mathscr{E}^V) : \mathscr{E}^V_1(f) \leq 1\}$ in $L^2(N^V_{\mu_0,Q})$, so that the generator L^V has empty essential spectrum and P^V_t is compact in $L^2(N^V_{\mu_0,Q})$ for $t > 0$. In the following, we intend to extend these to unbounded perturbation V.

In the framework of local Dirichlet forms, unbounded perturbations have been studied in many papers, where the key points are to prove the closability of the pre-Dirichlet form and to see which properties of the original Dirichlet form can be kept under the perturbation, see for instance [2, 6, 34] and references within. However, in all of related references one needs an algebra of bounded measurable functions $\mathscr{A} \subset \mathscr{D}(L)$ which is dense in $\mathscr{D}(L)$ such that the square field is given by

$$\Gamma(f,g) = \frac{1}{2}\big\{L(fg) - fLg - gLf\big\}, \quad f,g \in \mathscr{A}. \tag{5.1}$$

In the present situation, the square field reads

$$\Gamma(f,g)(\mu) = \langle Df(\mu), Dg(\mu)\rangle_{T_\mu}, \qquad \mu \in \mathscr{P}_2,\ f,g \in C_b^1(\mathscr{P}_2).$$

But we do not have explicit choice of the algebra $\mathscr{A}$ such that (5.1) holds, since from Theorem 4.1 we can not confirm that cylindrical functions of type

$$\mu \mapsto \Phi(\mu(f_1), \cdots, \mu(f_n)), \quad n \ge 1, \Phi \in C_b^\infty(\mathbb{R}^n), f_i \in C_b^\infty(\mathbb{R}^d)$$

are included in $\mathscr{D}(L)$. Therefore, we again come back to the tangent space T_{μ_0} by considering the following probability measure on T_{μ_0}:

$$G_Q^V(\mathrm{d}h) := \mathrm{e}^{(V\circ\Psi)(h)} G_Q(\mathrm{d}h),$$

and the corresponding bilinear form

$$\tilde{\mathscr{E}}^V(\tilde f, \tilde g) := \int_{T_{\mu_0}} \langle \nabla \tilde f, \nabla \tilde g\rangle_{T_{\mu_0}} \mathrm{d}G_Q^V, \quad \tilde f, \tilde g \in C_b^1(T_{\mu_0}).$$

By studying properties of $\tilde{\mathscr{E}}^V$, we obtain the following result under assumption

(*A*) $V \in C^1(\mathscr{P}_2)$ such that $\mathrm{d}N^V_{\mu_0,Q} := \mathrm{e}^V \mathrm{d}N_{\mu_0,Q}$ is a probability measure on $\mathscr{P}_2$, and there exists $p > 1$ such that

$$\int_{\mathscr{P}_2} \Big\{\|DV(\mu)\|_{T_\mu} \mathrm{e}^{V(\mu)^+} + \|DV(\mu)\|^p_{T_\mu}\Big\} N_{\mu_0,Q}(\mathrm{d}\mu) < \infty.$$

Theorem 5.1. *Assume (A). Then the following assertions hold.*

(1) *$(\mathscr{E}^V, C_b^1(\mathscr{P}_2))$ is closable in $L^2(N^V_{\mu_0,Q})$, and the closure $(\mathscr{E}^V, \mathscr{D}(\mathscr{E}^V))$ is a symmetric conservative Dirichlet form.*

(2) *If there exists $\lambda > \frac{1}{2q_1}$ such that $N_{\mu_0,Q}(\mathrm{e}^{\lambda\|DV\|^2}) < \infty$, where $\|DV\|(\mu) := \|DV(\mu)\|_{T_\mu}$, then the associated Markov semigroup P_t^V is compact in $L^2(N^F_{\mu_0,Q})$ for $t > 0$.*

(2) *If there exists $\varepsilon > 0$ such that*

$$\int_{\mathscr{P}_2} \Big(\mathrm{e}^{\frac{1+\varepsilon}{2q_1}\|DV\|^2} + \mathrm{e}^{V^+ + \varepsilon V^-}\Big) \mathrm{d}N_{\mu_0,Q} < \infty, \tag{5.2}$$

then there exists a constant $c > 0$ such that

$$N^V_{\mu_0,Q}(f^2 \log f^2) \le c\mathscr{E}^V(f,f), \quad f \in \mathscr{D}(\mathscr{E}^V),\ N^V_{\mu_0,Q}(f^2) = 1. \tag{5.3}$$

Consequently, for any $t > 0$,

$$\|P_t^V\|_{L^p(N^V_{\mu_0,Q}) \to L^{p_t}(N^V_{\mu_0,Q})} \le 1, \quad p > 1,\ p_t = 1 + (p-1)\mathrm{e}^{4t/c},$$
$$N^V_{\mu_0,Q}\big((P_t^V f)\log P_t^V f\big) \le \mathrm{e}^{-4t/c} N^V_{\mu_0,Q}(f \log f), \quad f \ge 0,\ N^V_{\mu_0,Q}(f) = 1.$$

Noting that $N_{\mu_0,Q} = G_Q \circ \Psi^{-1}$, $\mathrm{d}G_Q^V := \mathrm{e}^{V\circ\Psi}\mathrm{d}G_Q$ is a probability measure on T_{μ_0}, and

$$\mathscr{E}^V(f,g) = \int_{T_{\mu_0}} \langle \nabla\tilde f, \nabla\tilde g\rangle_{T_{\mu_0}} \mathrm{d}G_Q^V, \quad \tilde f, \tilde g \in C_b^1(T_{\mu_0}),$$

according to the proof of Theorem 3.2, Theorem 5.1 is a consequence of the following Lemma 5.2 for $F = V \circ \Psi$, where the condition $\|\nabla F\|_{T_{\mu_0}} \mathrm{e}^{F^+} + \|\nabla F\|_{T_{\mu_0}}^p \in L^1(G_Q)$ for some $p > 1$ is much weaker than $\mathrm{e}^{\|\nabla F\|_{T_{\mu_0}}^2 + |F|} \in \cap_{p>1} L^p(G_Q)$ used in [2, Proposition 3.2]. We will use the dimension-free Harnack inequality and Bismut formula for $\tilde P_t$ to prove the closability under this weaker condition. Moreover, the condition (5.4) for the log-Sobolev inequality is slightly better than that in [1, Lemma 4.1] where $F^+ + \varepsilon F^-$ is replaced by $(1+\varepsilon)|F|$.

Lemma 5.2. *Let $F \in C^1(T_{\mu_0})$ such that $G_Q^F(\mathrm{d}h) := \mathrm{e}^{F(h)} G_Q(\mathrm{d}h)$ is a probability measure on T_{μ_0} and $\|\nabla F\|_{T_{\mu_0}} \mathrm{e}^{F^+} + \|\nabla F\|_{T_{\mu_0}}^p \in L^1(G_Q)$ for some constant $p > 1$. Then:*

(1) *The bilinear form*

$$\tilde{\mathscr{E}}^F(\tilde f, \tilde g) := \int_{T_{\mu_0}} \langle \nabla\tilde f, \nabla\tilde g\rangle_{T_{\mu_0}} \mathrm{d}G_Q^F, \quad \tilde f, \tilde g \in \mathscr{F}C_b^1(T_{\mu_0})$$

is closable in $L^2(G_Q^F)$, and the closure $(\tilde{\mathscr{E}}^F, \mathscr{D}(\tilde{\mathscr{E}}^F))$ is a symmetric conservative Dirichlet form. Moreover, $\mathscr{D}(\tilde{\mathscr{E}}^F) \supset C_b^1(T_{\mu_0})$.

(2) *If there exists $\lambda > \frac{1}{2q_1}$ such that $G_Q(\mathrm{e}^{\lambda\|\nabla F\|_{T_{\mu_0}}^2}) < \infty$, then the associated Markov semigroup $\tilde P_t^F$ is compact in $L^2(G_Q^F)$ for $t > 0$.*

(3) *If there exists $\varepsilon > 0$ such that*

$$\int_{T_{\mu_0}} \mathrm{e}^{\frac{1+\varepsilon}{2q_1}\|\nabla F\|_{T_{\mu_0}}^2 + F^+ + \varepsilon F^-} \mathrm{d}G_Q < \infty, \tag{5.4}$$

then there exists a constant $c > 0$ such that

$$G_Q^F(\tilde f^2 \log \tilde f^2) \le c\tilde{\mathscr{E}}^F(\tilde f, \tilde f), \quad \tilde f \in \mathscr{D}(\tilde{\mathscr{E}}^F),\ G_Q^F(\tilde f^2) = 1. \tag{5.5}$$

Proof. (1) We will establish the integration by parts formula

$$\tilde{\mathscr{E}}^F(\tilde f, \tilde g) = -\int_{T_{\mu_0}} \tilde g\big(\tilde L^V \tilde f\big)\mathrm{d}G_Q^F, \quad \tilde f, \tilde g \in \mathscr{F}C_b^2(T_{\mu_0}) \tag{5.6}$$

for $\tilde L^V \tilde f := \tilde L \tilde f + \langle \nabla F, \nabla \tilde f\rangle_{T_{\mu_0}}$, so that $(\tilde{\mathscr{E}}^F, \mathscr{F}C_b^2(T_{\mu_0}))$ is closable. Since a function in $\mathscr{F}C_b^1$ can be approximated by functions in $\mathscr{F}C_b^2(T_{\mu_0})$ under the C_b^1-norm, this also implies that $(\tilde{\mathscr{E}}^V, \mathscr{F}C_b^1(T_{\mu_0}))$ is closable.

To this end, we make approximations of F. Let $\varphi \in C^\infty(\mathbb{R})$ such that $\varphi(r) = r$ for $|r| \le 1$, $1 \ge \varphi' \ge 0$ and $\varphi(r) = 0$ for $|r| \ge 2$. For any $m, n \ge 1$, let

$$F_m := m\varphi(F/m), \quad F_{m,n} := \tilde P_{\frac{1}{n}} F_m.$$

We have

$$F_m \in C_b(T_{\mu_0}) \cap C^1(T_{\mu_0}), \quad \|\nabla F_m\| \le \|\nabla F\|. \tag{5.7}$$

Since $\tilde P_t f(h_0) = \mathbb{E}[f(h_t)]$ for h_t in (3.1), by [38, Theorem 3.2.1 and Theorem 3.2.2] for $A = -Q, b = 0$ and $\sigma(t) = \sqrt{2}$, we have the Harnack inequality

$$(\tilde P_t \tilde f(h+v))^p \le (\tilde P_t \tilde f^p(h)) \mathrm{e}^{\frac{p}{2(p-1)}\|v\|^2_{T_{\mu_0}}}, \quad \tilde f \ge 0 p > 1, h, v \in T_{\mu_0}, \tag{5.8}$$

and the Bismut formula

$$\nabla \tilde P_t \tilde f(h_0) = \frac{\sqrt{2}}{t} \mathbb{E} \int_0^t \tilde f(h_t) \mathrm{e}^{-Qs} \mathrm{d}W_s, \quad t > 0, \tilde f \in \mathscr{B}_b(T_{\mu_0}). \tag{5.9}$$

By (5.9), we see that $F_{m,n} \in C_b^1(T_{\mu_0})$, and (5.8) together with (5.7) and $Q \ge 0$ implies

$$\begin{aligned}
&|\tilde P_t F_m(h_0 + \varepsilon v) - \tilde P_t F_m(h_0)| \le \mathbb{E}|F_m(\varepsilon \mathrm{e}^{-Qt} v + h_t) - F_m(h_t)| \\
&\le \mathbb{E} \int_0^\varepsilon \|\nabla F_m\|_{T_{\mu_0}} (r \mathrm{e}^{-Qt} v + h_t) \mathrm{d}r \\
&\le \int_0^\varepsilon (\tilde P_t \|\nabla F_m\|^p_{T_{\mu_0}})^{\frac{1}{p}} (h_0) \mathrm{e}^{\frac{r^2|v|^2}{2(p-1)}} \mathrm{d}r \\
&\le \varepsilon (\tilde P_t \|\nabla F\|^p_{T_{\mu_0}})^{\frac{1}{p}} (h_0) \mathrm{e}^{\frac{\varepsilon^2 |v|^2}{2(p-1)}}, \quad \varepsilon > 0, h_0, v \in T_{\mu_0}.
\end{aligned}$$

By letting $\varepsilon \downarrow 0$ we derive

$$\|\nabla F_{m,n}\|_{T_{\mu_0}} = \|\tilde P_{\frac{1}{n}} F_m\|_{T_{\mu_0}} \le (\tilde P_{\frac{1}{n}} \|\nabla F\|^p_{T_{\mu_0}})^{\frac{1}{p}}. \tag{5.10}$$

Since $F_{m,n} \in C_b^1(T_{\mu_0})$, the integration by parts formula for G_Q yields

$$\begin{aligned}
&\int_{T_{\mu_0}} \langle \nabla \tilde f, \nabla \tilde g \rangle_{T_{\mu_0}} \mathrm{e}^{F_{m,n}} \mathrm{d}G_Q = \int_{T_{\mu_0}} \Big\{ \nabla_{\nabla \tilde f} (\mathrm{e}^{F_{m,n}} \tilde g) - \mathrm{e}^{F_{m,n}} \tilde g \nabla_{\nabla \tilde f} F_{m,n} \Big\} \mathrm{d}G_Q \\
&= - \int_{T_{\mu_0}} \mathrm{e}^{F_{m,n}} \tilde g \big\{ \tilde L \tilde f + \nabla_{\nabla F_{m,n}} \tilde f \big\} \mathrm{d}G_Q \\
&= - \int_{T_{\mu_0}} \Big\{ \tilde g (\tilde L + \nabla_{\nabla F_{m,n}}) \tilde f \Big\} \mathrm{e}^{F_{m,n}} \mathrm{d}G_Q, \quad \tilde f, \tilde g \in \mathscr{F}C_b^2(T_{\mu_0}).
\end{aligned}$$

Noting that (5.10) implies

$$\Big| \tilde g (\tilde L + \nabla_{\nabla F_{m,n}}) \tilde f \Big| \mathrm{e}^{F_{m,n}} \le c_m (1 + \tilde P_{\frac{1}{n}} \|\nabla F\|^p_{T_{\mu_0}})^{\frac{1}{p}}, \quad n \ge 1$$

for some constant $c_m > 0$, which are bounded in $L^p(G_Q)$ since

$$G_Q\big(\tilde P_{\frac{1}{n}} \|\nabla F\|^p_{T_{\mu_0}}\big) = G_Q(\|\nabla F\|^p_{T_{\mu_0}}) < \infty,$$

by the dominated convergence theorem we may let $n \to \infty$ to derive

$$\int_{T_{\mu_0}} \langle \nabla \tilde f, \nabla \tilde g \rangle_{T_{\mu_0}} \mathrm{e}^{F_m} \mathrm{d}G_Q = - \int_{T_{\mu_0}} \Big\{ \tilde g (\tilde L + \nabla_{\nabla F_m}) \tilde f \Big\} \mathrm{e}^{F_m} \mathrm{d}G_Q.$$

Since (5.7) and $\mathrm{e}^F + \|\nabla F\|_{T_{\mu_0}} \mathrm{e}^{F^+} \in L^1(G_Q)$ implies

$$\big|\tilde g(\tilde L + \nabla_{\nabla F_m})\tilde f\big|\mathrm{e}^{F_m} \le c\big(1+\|\nabla F\|_{T_{\mu_0}}\big)\mathrm{e}^{F^+} \in L^1(G_Q),$$

by using the dominated convergence theorem again, we may let $m\to\infty$ to get (5.6).

Next, let $\tilde f \in C_b^1(T_{\mu_0})$, and for any $n\ge 1$ let

$$\tilde f_n := \tilde f\circ \pi_n,\quad \pi_n h := \sum_{i=1}^n \langle h, h_i\rangle_{T_{\mu_0}} h_i.$$

Then $\{\tilde f_n\}_{n\ge1}\subset \mathscr{F}C_b^1(T_{\mu_0})\subset \mathscr{D}(\tilde{\mathscr{E}}^F)$, and

$$\begin{aligned}
&\lim_{n\to\infty}\int_{T_{\mu_0}} \big\{|\tilde f_n - f|^2 + \|\nabla(\tilde f_n-\tilde f)\|_{T_{\mu_0}}^2\big\}\mathrm{d}G_Q^V\\
&= \lim_{n\to\infty}\int_{T_{\mu_0}} \Big\{\sum_{l=1}^n \big\{|(\nabla_{h_l}\tilde f)\circ\pi_n - \nabla_{h_l}\tilde f|^2 + \sum_{l=n+1}^\infty |\nabla_{h_l}\tilde f|^2\Big\}\mathrm{d}G_Q^F\\
&\le \lim_{n\to\infty}\int_{T_{\mu_0}} \|(\nabla\tilde f)\circ\pi_n - \nabla\tilde f\|_{T_{\mu_0}}^2 + \sum_{l=n+1}^\infty |\nabla_{h_l}\tilde f|^2\Big\}\mathrm{d}G_Q^F = 0,
\end{aligned}$$

where the last step follows from $\nabla\tilde f\in C_b(T_{\mu_0})$ and the dominated convergence theorem. So, $\tilde f\in\mathscr{D}(\tilde{\mathscr{E}}^F)$.

(2) It suffices to prove that the set

$$B_1^F := \big\{\tilde f\in C_b^1(T_{\mu_0}):\ \tilde{\mathscr{E}}_1^F(\tilde f) := \tilde{\mathscr{E}}^F(\tilde f,\tilde f) + G_Q^F(\tilde f^2)\le 1\big\}$$

is relatively compact in $L^2(G_Q^F)$. By the chain rule and Young's inequality, for any $\varepsilon>0$ and $f\in B_1^F$, we have

$$\begin{aligned}
&\tilde{\mathscr{E}}_1(\tilde f\mathrm{e}^{\frac F2},\tilde f\mathrm{e}^{\frac F2}) \le G_Q^F(\tilde f^2) + (1+\varepsilon^{-1})\tilde{\mathscr{E}}^F(\tilde f,\tilde f) + \frac{1+\varepsilon}{4}G_Q(\tilde f^2\mathrm{e}^F\|\nabla F\|_{T_{\mu_0}}^2)\\
&\le 1+\varepsilon^{-1} + \frac{1+\varepsilon}{4\lambda}G_Q\bigg(\tilde f^2\mathrm{e}^F\log\frac{\tilde f^2\mathrm{e}^F}{G_Q(\tilde f^2\mathrm{e}^F)}\bigg) + \frac{1+\varepsilon}{4\lambda}G_Q(\tilde f^2\mathrm{e}^F)\log G_Q\big(\mathrm{e}^{\lambda\|\nabla F\|_{T_{\mu_0}}^2}\big).
\end{aligned}$$

Since $G_Q(\|\nabla(\tilde f\mathrm{e}^{\frac F2})\|_{T_{\mu_0}}^2)\le \tilde{\mathscr{E}}_1(\tilde f\mathrm{e}^{\frac F2},\tilde f\mathrm{e}^{\frac F2})$ and $G_Q(\tilde f^2\mathrm{e}^F) = G_Q^F(\tilde f^2)\le 1$, by combining this with (3.3) we derive

$$\tilde{\mathscr{E}}_1(\tilde f\mathrm{e}^{\frac F2},\tilde f\mathrm{e}^{\frac F2}) \le 1+\varepsilon^{-1} + \frac{1+\varepsilon}{2q_1\lambda}\tilde{\mathscr{E}}_1(\tilde f\mathrm{e}^{\frac F2},\tilde f\mathrm{e}^{\frac F2}) + \frac{1+\varepsilon}{4\lambda}\log G_Q\big(\mathrm{e}^{\lambda\|\nabla F\|_{T_{\mu_0}}^2}\big).$$

Since $\lambda > \frac{1}{2q_1}$, we may take small $\varepsilon>0$ such that $\frac{1+\varepsilon}{2q_1\lambda}<1$, so that this estimate and $G_Q\big(\mathrm{e}^{\lambda\|\nabla F\|_{T_{\mu_0}}^2}\big)<\infty$ yield

$$\tilde{\mathscr{E}}_1(\tilde f\mathrm{e}^{\frac F2},\tilde f\mathrm{e}^{\frac F2})\le C,\quad \tilde f\in B_1^F$$

for some constant $C>0$. Since $\tilde L$ has empty essential spectrum, this implies that the set

$$\{\tilde f\mathrm{e}^{\frac F2}: \tilde f\in B_1^F\}$$

is relatively compact in $L^2(G_Q)$, equivalently, B_1^F is relatively compact in $L^2(G_Q^F)$.

(3) The proof of (5.5) is similar to that of [1, Lemma 4.1], but we make a more careful estimate by separating F^+ and F^-. Let $\tilde f \in C_b^1(T_{\mu_0})$ such that $G_Q^F(\tilde f^2) = 1$. By (3.3) and Young's inequality, we obtain

$$\begin{aligned}
&G_Q^F(\tilde f^2 \log \tilde f^2) = G_Q\big(\tilde f^2 \mathrm{e}^F \log(\tilde f^2 \mathrm{e}^F)\big) - G_Q(\tilde f^2 F \mathrm{e}^F)\\
&\le \frac{2}{q_1} G_Q\big(\|\nabla (f \mathrm{e}^{\frac{F}{2}})\|_{T_{\mu_0}}^2\big) + G_Q(\tilde f^2 F^- \mathrm{e}^F)\\
&\le \frac{2(1+r_1^{-1})}{q_1} \tilde{\mathscr{E}}^F(\tilde f, \tilde f) + G_Q^F\bigg(\tilde f^2\Big[\frac{1+r_1}{2q_1}\|\nabla F\|_{T_{\mu_0}}^2 + F^-\Big]\bigg)\\
&\le \frac{2(1+r_1^{-1})}{q_1} \tilde{\mathscr{E}}^F(\tilde f, \tilde f) + r_2 G_Q^F(\tilde f^2 \log \tilde f^2) + r_2 \log G_Q^F\big(\mathrm{e}^{\frac{1+r_1}{2r_2q_1}}\|\nabla F\|_{T_{\mu_0}}^2 + \frac{1}{r_2}F^-\big)\\
&= \frac{2(1+r_1^{-1})}{q_1} \tilde{\mathscr{E}}^F(\tilde f, \tilde f) + r_2 G_Q^F(\tilde f^2 \log \tilde f^2) + r_2 \log G_Q\Big(\mathrm{e}^{\frac{1+r_1}{2r_2q_1}\|\nabla F\|_{T_{\mu_0}}^2 + F^+ + \frac{1-r_2}{r_2}F^-}\Big)
\end{aligned}$$

for any $r_1, r_2 \in (0,1)$. By taking r_1 small enough and r_2 close enough to r_2 such that

$$\frac{1-r_2}{r_2} \vee \Big(\frac{1+r_1}{r_2} - 1\Big) \le \varepsilon,$$

we deduce from this and (5.4) that the defective log-Sobolev inequality

$$G_Q^F(\tilde f^2 \log \tilde f^2) \le c_1 \mathscr{E}^V(\tilde f, \tilde f) + c_2, \quad \tilde f \in \mathscr{D}(\tilde{\mathscr{E}}^F), G_Q^F(\tilde f^2) = 1$$

holds for some constants $c_1, c_2 > 0$. Since the Dirichlet form is irreducible, according to [39, Corollary 1.3] for $\phi(p) = 2 - p$, this implies the log-Sobolev inequality (5.5) for some constant $c > 0$. □

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