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WASSERSTEIN CONVERGENCE FOR CONDITIONAL EMPIRICAL MEASURES OF SUBORDINATED DIRICHLET DIFFUSIONS ON RIEMANNIAN MANIFOLDS

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Abstract. In this paper, we investigate the rate of convergence on the quadratic Wasserstein distance between conditional empirical measures associated with subordinated Dirichlet diffusion processes on a connected compact Riemannian manifold with boundary and the quasi-ergodic distribution. We obtain the sharp rate of convergence for any initial distribution and even prove the precise limit for a large class of initial distributions. This proof is based on the PDE method developed by L. Ambrosio et al. in [3] and the arguments on diffusion cases by F.-Y. Wang in [27].

1. **Introduction.** In this section, we first introduce the framework and then present the main results. Some related literatures are discussed subsequently.

Let M be a d-dimensional connected compact Riemannian manifold with smooth boundary ∂M and $\mathcal{P}$ be the set of all Borel probability measures on M. Let $U \in C^2(M)$ such that $\mu(\mathrm{d}x) = \mathrm{e}^{U(x)}\mathrm{d}x$ belongs to $\mathcal{P}$, where $\mathrm{d}x$ is the volume measure on M. Denote $\mathcal{L} = \Delta + \nabla U$, where Δ and ∇ stand for the Laplace–Beltrami operator and the gradient operator on M, respectively. Let $(X_t)_{t\geq 0}$ be the diffusion process corresponding to $\mathcal{L}$ with hitting time

$$\tau := \inf\{t \geq 0 : X_t \in \partial M\}.$$

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and let ϕ_m, λ_m, $m \in \mathbb{N}_0$, be Dirichlet eigenfunctions and Dirichlet eigenvalues of the operator $-\mathcal{L}$ in $L^2(\mu)$ respectively (see Section 2 for details). Set $\mu_0 := \phi_0^2\mu$, which clearly belongs to $\mathcal{P}$.

We are interested in the asymptotic behavior of conditional empirical measures of a large class of Markov processes subordinated to $(X_t)_{t\geq 0}$. For this purpose, we should recall some basics on Bernstein functions and subordinated processes;

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see e.g. [6, 21, 4] for a systematic study. A function $B \in C([0,\infty);[0,\infty)) \cap C^\infty((0,\infty);[0,\infty))$ is called a Bernstein function if, for each $k \in \mathbb{N}$,

$$(-1)^{k-1}\frac{\mathrm{d}^k}{\mathrm{d}t^k}B(t) \geq 0, \quad t > 0.$$

We will use the following class of Bernstein functions (see [33]), i.e.,

$$\mathbf{B} := \big\{B:\ B \text{ is a Bernstein function with } B(0)=0,\ B'(0)>0\big\}.$$

Let $B \in \mathbf{B}$, and let $(S_t^B)_{t\geq 0}$ be the unique subordinator corresponding to B, i.e., an increasing stochastic process with stationary, independent increments, taking values in $[0,\infty)$ and $S_0^B = 0$ such that B is the Laplace exponent of $(S_t^B)_{t\geq 0}$, i.e.,

$$\mathbb{E}\mathrm{e}^{-\lambda S_t^B} = \mathrm{e}^{-tB(\lambda)}, \quad t, \lambda \geq 0. \tag{1.1}$$

Let $(X_t^B)_{t\geq 0}$ be the Markov process on M generated by $-B(-\mathcal{L})$. It is well known that $(X_t^B)_{t\geq 0}$ can be constructed as the time-changed process of $(X_t)_{t\geq 0}$ by $(S_t^B)_{t\geq 0}$; more precisely,

$$X_t^B = X_{S_t^B \wedge \tau}, \quad t \geq 0,$$

where $(S_t^B)_{t\geq 0}$ is the subordinator introduced above, independent of $(X_t)_{t\geq 0}$. We call $(X_t^B)_{t\geq 0}$ the Dirichlet diffusion process subordinated to $(X_t)_{t\geq 0}$ or B-subordinated Dirichlet diffusion process[1].

Our main results are based on the following class of Bernstein function. For any $\alpha \in [0,1]$, let

$$\mathbf{B}^\alpha := \left\{B \in \mathbf{B}:\ \liminf_{\lambda\to\infty} \lambda^{-\alpha}B(\lambda) > 0\right\}.$$

From the table in [21, Chapter 16], we can find many examples belonging to $\mathbf{B}^\alpha$ such as λ^α, $\alpha \in (0,1]$ (algebraic type), $\sqrt{\lambda}(1-e^{-2\sqrt{\lambda}})$ (exponential type), $\frac{\lambda(\lambda+1)}{(\lambda+2)\log(\lambda+2)}$ (logarithmic type), etc. Let $B \in \mathbf{B}^\alpha$ for some $\alpha \in [0,1]$. Set

$$\sigma_\tau^B := \inf\{t \geq 0:\ S_t^B > \tau\},$$

which can be regarded as the hitting time of the B-subordinated Dirichlet diffusion process $(X_t^B)_{t\geq 0}$ at the boundary ∂M. Denote $\mathring{M}$ as the interior of M i.e., $\mathring{M} := M \setminus \partial M$. Then we can verify that μ_0 is the unique quasi-ergodic distribution of the B-subordinated Dirichlet diffusion process $(X_t^B)_{t\geq 0}$ for $B \in \mathbf{B}^\alpha$, $\alpha \in (0,1]$, i.e., for every $\nu \in \mathcal{P}$ supported on $\mathring{M}$,

$$\lim_{t\to\infty} \mathbb{E}^\nu\Big[\frac{1}{t}\int_0^t f(X_s^B)\,\mathrm{d}s\Big|\sigma_\tau^B > t\Big] = \int_M f\,\mathrm{d}\mu_0, \quad f \in \mathcal{B}_b(M),$$

where $\mathbb{E}^\nu$ denotes the expectation for the process $(X_t^B)_{t>0}$ with initial distribution $\nu \in \mathcal{P}$; see Appendix for a proof. This implies that the family of conditional empirical measures $(\mu_t^{B,\nu})_{t>0}$, i.e.,

$$\mu_t^{B,\nu} := \mathbb{E}^\nu\Big(\frac{1}{t}\int_0^t \delta_{X_s^B}\,\mathrm{d}s\Big|\sigma_\tau^B > t\Big), \quad t > 0,$$

converges weakly to μ_0 as $t \to \infty$.

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[1] According to the literature, it seems more appropriate to call $(X_t^B)_{t\geq 0}$ the subordinate killed diffusion process w.r.t. B. See e.g. [22] for studies on subordinate killed processes.

Let ρ be the geodesic distance on M. The quadratic Wasserstein distance $\mathbb{W}_2$ is defined as

$$\mathbb{W}_2(\mu_1,\mu_2)=\inf_{\pi\in\mathcal{C}(\mu_1,\mu_2)}\left(\int_{M\times M}\rho(x,y)^2\,\pi(\mathrm{d}x,\mathrm{d}y)\right)^{\frac{1}{2}},\quad \mu_1,\mu_2\in\mathcal{P},$$

where $\mathcal{C}(\mu_1,\mu_2)$ is the set of all Borel probability measures on the product space $M\times M$ with respective marginal distributions μ_1 and μ_2. According to [24, Theorem 7.12], we have that

$$\mathbb{W}_2(\mu_t^{B,\nu},\mu_0)\to 0,\quad t\to\infty.$$

So, it is interesting and significant to estimate the rate of convergence on $\mathbb{W}_2(\mu_t^{B,\nu}, \mu_0)$ as t tends to infinity.

In order to avoid the situation that $\mathbb{P}^\nu(\sigma_\tau^B>t)=0$ for some $\nu\in\mathcal{P}$, we should consider the conditional empirical measure $\mu_t^{B,\nu}$ with $\nu\in\mathcal{P}_0$, where

$$\mathcal{P}_0:=\{\nu\in\mathcal{P}:\ \nu(\mathring{M})>0\}.$$

For convenience, for every $B\in\mathbf{B}$ and every $\nu\in\mathcal{P}_0$, we set

$$I:=\frac{1}{\{\mu(\phi_0)\nu(\phi_0)\}^2}\sum_{m=1}^\infty\frac{\{\nu(\phi_0)\mu(\phi_m)+\mu(\phi_0)\nu(\phi_m)\}^2}{(\lambda_m-\lambda_0)[B(\lambda_m)-B(\lambda_0)]^2}.$$

Now we present the first main result of this paper, which contains the rate of convergence for any initial distribution from $\mathcal{P}_0$.

Theorem 1.1. Let $\alpha\in(0,1]$ and $B\in\mathbf{B}^\alpha$. Then

$$\limsup_{t\to\infty}\{t^2\mathbb{W}_2(\mu_t^{B,\nu},\mu_0)^2\}\le 4I\in(0,\infty],\tag{1.2}$$

and moreover, I is finite in either of the following cases:

(1) $d\le 2(1+2\alpha)$,
(2) $d>2(1+2\alpha)$ and $\nu=h\mu$ with $h\in L^{2d/(d+2+4\alpha)}(\mu)$.

Some further remarks are in order.

Remark 1.2. (a) In particular, if we choose $B(r)=r$, $r\ge 0$, then we are in the framework of [27]. Compared with the upper bound in [27], the rate of convergence in (1.2) is sharp, although there is an extra factor 4 which comes from the application of inequality (3.1). It seems that the original idea used in [27, Section 3] to prove the upper bound is not applicable to the present non-local setting due to the difficulty in employing (4.2) instead of (3.1). Nevertheless, the precise limit is obtained in Theorem 1.3 below for a large class of initial distributions.

(b) Let $\mathfrak{m}_t^B=\frac{1}{t}\int_0^t\delta_{X_s^B}\,\mathrm{d}s$ for any $t>0$. In the recent work [17], the rate of convergence on conditional expectations of the quadratic Wasserstein distance between $\mathfrak{m}_t^B$ and μ_0 is studied, which in particularly shows that for any $\nu\in\mathcal{P}_0$, $\mathbb{E}^\nu[\mathbb{W}_2(\mathfrak{m}_t^B,\mu_0)^2|t<\sigma_\tau^B]$ decays with rate t^{-1} as $t\to\infty$, slower than the one t^{-2} in Theorem 1.1.

The next main result contains the exact limit for a large class of initial distributions.

Theorem 1.3. Let $\alpha\in(0,1]$ and $B\in\mathbf{B}^\alpha$. Then, for any $\nu\in\mathcal{P}_0$ such that $\nu=h\mu$ with $h\phi_0^{-1}\in L^p(\mu_0)$ for some $p\in(p_0,\infty]$,

$$\lim_{t\to\infty}\{t^2\mathbb{W}_2(\mu_t^{B,\nu},\mu_0)^2\}=I,\tag{1.3}$$

where

$$p_0 := \max\Big\{\frac{6(d+2)}{d+2+12\alpha}, \frac{3}{2}\Big\}.$$

Remark 1.4. (1) It is clear that the limit in (1.3) belongs to $(0,\infty)$, which follows from Theorem 1.1. The main novel contribution is the introduction of a new regularization procedure for the proof of Theorem 1.3, different from [27], which leads us to apply (4.2) successfully. However, this approach does not seem to be effective without additional regularity on the initial distribution $\nu \in \mathcal{P}_0$. Although we do not know how to get rid of the extra assumption on the initial distribution at the moment, we believe that (1.3) holds for all $\nu \in \mathcal{P}_0$ at least for α close enough to 1.

(2) It is interesting to notice that, due to our approach, we do not need any upper control on the Bernstein function for the lower bound in Theorem 1.3, which is different from the recent results [17, Theorem 1.3(2)] and [33, Theorem 1.1(2)].

Recently, besides [27, 17] mentioned above, large time asymptotic behaviors of empirical measures associated with (subordinated) diffusion processes to the reference measure under Wasserstein distances on Riemannian manifolds have been investigated in a series of papers. (i) Let M be a compact Riemannian manifold with ∂M empty or convex. Uniformly in $x \in M$, the precise limit of $t\mathbb{E}^x[\mathbb{W}_2(\mu_t,\mu)^2]$ as $t \to \infty$ and sharp rates of convergence on $\mathbb{E}^x[\mathbb{W}_2(\mu_t,\mu)^2]$ for large t are obtained in [34], where μ_t is the empirical measure associated with the given (reflecting) diffusion process (when $\partial M \neq \emptyset$) and μ is the invariant measure. Furthermore, these results are successfully generalized to subordinated diffusion processes by the second named author joint with F.-Y. Wang in a more recent paper [33]. See related studies on empirical measures in the quadratic Wasserstein distance under the conditional expectation in [28], where the rate of convergence turns out to be quite different from [27]. (ii) Let M be a noncompact Riemannian manifold with ∂M empty or convex. Sharp rates of convergence on $\mathbb{E}^\mu[\mathbb{W}_2(\mu_t,\mu)^2]$ for large t are obtained in [29], where μ_t and μ are similar as the ones in (i). The results are generalized to a large class of subordinated processes more recently in [16] by the same authors of the present paper. (iii) Further studies on this subject can be found in [30] for stochastic partial differential equations, in [13] for the fractional Brownian motion on flat torus and its subordinated case [18, 35] for weighted empirical measures of symmetric diffusion processes on compact Riemannian manifolds without boundary, [31] for subordinated non-symmetric diffusion processes, and [32] for general ergodic Markov processes. Last but not the least, being a classic research subject with a wild range of applications, the study on asymptotic behaviors of empirical measures associated with i.i.d. random variables to the reference measure under Wasserstein distances, particularly on quantifying the rate of convergence, has received considerable attention over years; see e.g. the papers [3, 5, 14, 15, 12, 11, 1, 36, 19], and the book [23] as well as references therein for many deep results.

The remainder of the paper is laid out as follows. In Section 2, we recall some known results and present some useful properties needed for the later sections. The proof of Theorem 1.1 and Theorem 1.3 are presented in Sections 3 and 4 respectively. An appendix on quasi-ergodic distributions is included.

2. **Preparations.**

In this section, we briefly recall some well known facts on the Dirichlet diffusion semigroup and the Dirichlet heat kernel, which are mainly borrowed from [27, Section 2]; see e.g. [26, 7] for more details. Then we deduce some useful properties on the subordinated Dirichlet diffusion semigroup and introduce necessary notations.

It is well known that the spectrum of the operator $-\mathcal{L}$ is discrete, whose eigenvalues λ_k, $k \in \mathbb{N}_0$, are nonnegative and listed in an ascending order counting multiplicities, and the corresponding eigenfunctions ϕ_k, $k \in \mathbb{N}_0$, satisfying the Dirichlet boundary condition, form a complete orthonormal system in the function space $L^2(\mu)$. We may assume that $\phi_0 > 0$ in $\mathring{M}$ since ϕ_0 does not change the sign in $\mathring{M}$. It is also well known that $\lambda_0 > 0$,

$$\alpha_0^{-1} k^{2/d} \le \lambda_k - \lambda_0 \le \alpha_0 k^{2/d}, \quad \|\phi_k\|_\infty \le \alpha_0 \sqrt{k}, \qquad k \in \mathbb{N}, \tag{2.1}$$

for some constant $\alpha_0 > 1$, and

$$\|\phi_0^{-1}\|_{L^p(\mu_0)} < \infty, \quad p \in [1, 3). \tag{2.2}$$

Here and in the sequel, we denote the supremum norm by $\|\cdot\|_\infty$.

Let p_t^D and P_t^D be the Dirichlet heat kernel and the Dirichlet diffusion semigroup corresponding to $\mathcal{L}$, respectively. It is well known that p_t^D has the following spectral representation, i.e.,

$$p_t^D(x, y) = \sum_{m=0}^{\infty} \mathrm{e}^{-\lambda_m t} \phi_m(x) \phi_m(y), \quad t > 0,\, x, y \in M. \tag{2.3}$$

Then, we can use (2.3) to express the Dirichlet diffusion semigroup as

$$\begin{aligned} P_t^D f(x) &:= \mathbb{E}^x[f(X_t) 1_{\{t<\tau\}}] = \int_M p_t^D(x, y) f(y)\, \mu(\mathrm{d}y) \\ &= \sum_{m=0}^{\infty} \mathrm{e}^{-\lambda_m t} \mu(\phi_m f) \phi_m(x), \quad t > 0,\, x \in M,\, f \in L^2(\mu). \end{aligned} \tag{2.4}$$

Moreover, there exists a constant $c > 0$ such that

$$\begin{aligned} \|P_t^D\|_{L^p(\mu) \to L^q(\mu)} &:= \sup_{\|f\|_{L^p(\mu)} \le 1} \|P_t^D f\|_{L^q(\mu)} \\ &\le c \mathrm{e}^{-\lambda_0 t} (1 \wedge t)^{-\frac{d(q-p)}{2pq}}, \quad t > 0,\, 1 \le p \le q \le \infty. \end{aligned} \tag{2.5}$$

Since the Dirichlet diffusion operator $\mathcal{L}$ is non-symmetric w.r.t. μ_0, we can not use [3, Proposition 2.3] directly. One important technique to overcome the difficulty is by employing Doob's transform and considering

$$\mathcal{L}_0 = \mathcal{L} + 2\nabla \log \phi_0.$$

Then $\mathcal{L}_0$ is a non-positive self-adjoint operator in $L^2(\mu_0)$, and the corresponding semigroup, defined by $P_t^0 := \mathrm{e}^{t\mathcal{L}_0}$ in the sense of functional analysis, satisfies

$$P_t^0 f = \mathrm{e}^{\lambda_0 t} \phi_0^{-1} P_t^D (f \phi_0), \quad t \ge 0,\, f \in L^2(\mu_0). \tag{2.6}$$

Moreover, μ_0 is the invariant measure of P_t^0 since P_t^0 is conservative (i.e., $P_t^0 1 = 1$ for every $t \ge 0$) and symmetric w.r.t. μ_0. By taking $f = \phi_0^{-1}\phi_k$ in (2.6) and noting that $P_t^D \phi_k = \mathrm{e}^{-\lambda_k t} \phi_k$ for every $k \in \mathbb{N}_0$, we clearly see that

$$\begin{aligned} &P_t^0(\phi_k \phi_0^{-1}) = \mathrm{e}^{-(\lambda_k - \lambda_0)t} \phi_k \phi_0^{-1}, \quad k \in \mathbb{N}_0,\, t \ge 0, \\ &\mathcal{L}_0(\phi_k \phi_0^{-1}) = -(\lambda_k - \lambda_0) \phi_k \phi_0^{-1}, \quad k \in \mathbb{N}_0, \end{aligned} \tag{2.7}$$

and hence, $\{\phi_0^{-1}\phi_m\}_{m\in\mathbb{N}_0}$ is an eigenbasis of $-\mathcal{L}_0$ in $L^2(\mu_0)$. Thus, by (2.4) and (2.6),

$$P_t^0 f = \sum_{m=0}^{\infty} \mu_0(f\phi_m\phi_0^{-1})\mathrm{e}^{-(\lambda_m-\lambda_0)t}\phi_m\phi_0^{-1}, \quad f \in L^2(\mu_0),\, t \geq 0, \tag{2.8}$$

and the heat kernel of P_t^0 w.r.t. μ_0, denoted by p_t^0, can be represented by

$$p_t^0(x,y) = \sum_{m=0}^{\infty} \mathrm{e}^{-(\lambda_m-\lambda_0)t}(\phi_m\phi_0^{-1})(x)(\phi_m\phi_0^{-1})(y), \quad x,y \in M,\, t > 0. \tag{2.9}$$

By the intrinsic ultra-contractivity (introduced first in [10]), we can find a constant $\alpha_1 \geq 1$ such that

$$\begin{aligned}\|P_t^0 - \mu_0\|_{L^1(\mu_0)\to L^\infty(\mu_0)} &:= \sup_{\|f\|_{L^1(\mu_0)}\leq 1} \|P_t^0 f - \mu_0(f)\|_{L^\infty(\mu_0)} \\ &\leq \frac{\alpha_1 \mathrm{e}^{-(\lambda_1-\lambda_0)t}}{(1\wedge t)^{(d+2)/2}}, \quad t > 0,\end{aligned} \tag{2.10}$$

which along with the semigroup property and the contractivity of P_t^0 in $L^p(\mu)$ implies that, there exists a constant $\alpha_2 \geq 1$ such that

$$\begin{aligned}\|P_t^0 - \mu_0\|_{L^p(\mu_0)\to L^p(\mu_0)} &:= \sup_{\|f\|_{L^p(\mu_0)}\leq 1} \|P_t^0 f - \mu_0(f)\|_{L^p(\mu_0)} \\ &\leq \alpha_2 \mathrm{e}^{-(\lambda_1-\lambda_0)t}, \quad t \geq 0,\, \infty \geq p \geq 1.\end{aligned} \tag{2.11}$$

Combining the Riesz–Thorin interpolation theorem (see e.g. [9, page 3]) with (2.10) and (2.11), we obtain that

$$\|P_t^0 - \mu_0\|_{L^p(\mu_0)\to L^q(\mu_0)} \leq \alpha_3 \mathrm{e}^{-(\lambda_1-\lambda_0)t}\{1\wedge t\}^{-\frac{(d+2)(q-p)}{2pq}}, \quad t > 0,\, \infty \geq q \geq p \geq 1, \tag{2.12}$$

for some constant $\alpha_3 > 0$. Thus, (2.12) and (2.1) lead to that, there exists a constant $\alpha_4 > 0$ such that

$$\|\phi_k\phi_0^{-1}\|_\infty \leq \alpha_4 k^{\frac{d+2}{2d}}, \quad k \in \mathbb{N}. \tag{2.13}$$

Moreover, by [27, Lemma 2.4], we have

$$\|\nabla(\phi_k\phi_0^{-1})\|_\infty \leq \alpha_5 k^{\frac{d+4}{2d}}, \quad k \in \mathbb{N}, \tag{2.14}$$

for some constant $\alpha_5 > 0$.

We now turn to the non-local situation. Let $B \in \mathbf{B}$, $t > 0$, and let $p_t^{D,B}$ and $P_t^{D,B}$ be the subordinated Dirichlet heat kernel and the subordinated Dirichlet diffusion semigroup associated with the B-subordinated Dirichlet diffusion process $(X_t^B)_{t\geq 0}$, respectively. By (1.1), (2.3) and (2.4), one has that

$$p_t^{D,B}(x,y) = \sum_{m=0}^{\infty} \mathrm{e}^{-B(\lambda_m)t}\phi_m(x)\phi_m(y), \quad x,y \in M, \tag{2.15}$$

and

$$\begin{aligned}P_t^{D,B} f(x) &:= \mathbb{E}^x[f(X_t^B)1_{\{t<\sigma_\tau^B\}}] = \int_M p_t^{D,B}(x,y)f(y)\,\mu(\mathrm{d}y) \\ &= \sum_{m=0}^{\infty} \mathrm{e}^{-B(\lambda_m)t}\mu(\phi_m f)\phi_m(x), \quad x \in M,\, f \in L^2(\mu).\end{aligned} \tag{2.16}$$

By (2.6), we immediately obtain that

$$P_t^D f = \mathrm{e}^{-\lambda_0 t}\phi_0 P_t^0(f\phi_0^{-1}), \quad f \in L^2(\mu_0).$$

Hence, the semigroup $P_t^{D,B}$ can be written as

$$\begin{aligned} P_t^{D,B} f &= \int_0^\infty P_s^D f\, \mathbb{P}(S_t^B \in \mathrm{d}s) \\ &= \int_0^\infty \mathrm{e}^{-\lambda_0 s}\phi_0 P_s^0(f\phi_0^{-1})\, \mathbb{P}(S_t^B \in \mathrm{d}s), \quad f \in L^2(\mu_0), \end{aligned} \tag{2.17}$$

where $\mathbb{P}(S_t^B \in \cdot)$ is the distribution of the subordinator S_t^B.

We also need the next useful facts. Let $\alpha \in [0,1]$ and $B \in \mathbf{B}^\alpha$. It is easy to see that, there exist constants $a, c > 0$ and $b \geq 0$ such that

$$B(r) \geq c(r^\alpha \wedge r) \geq ar^\alpha - b, \quad r \geq 0; \tag{2.18}$$

see also [33, (3.12)] and [16, page 17]. Moreover, according to (2.18) (which particularly implies that $\lim_{r\to\infty} B(r) = \infty$), we have for every $r_0 \geq 0$,

$$\lim_{r\to\infty} \frac{B(r - r_0)}{B(r) - B(r_0)} = 1. \tag{2.19}$$

Together with (2.18) and (2.19), applying (2.16) and (2.1), we get a constant $C > 0$ such that

$$\begin{aligned} &|\mathrm{e}^{B(\lambda_0)t}\mathbb{P}^\nu(t < \sigma_\tau^B) - \mu(\phi_0)\nu(\phi_0)| = |\mathrm{e}^{B(\lambda_0)t}\nu(P_t^{D,B}1) - \mu(\phi_0)\nu(\phi_0)| \\ &\leq \sum_{m=1}^\infty \mathrm{e}^{-[B(\lambda_m)-B(\lambda_0)]t}|\mu(\phi_m)\nu(\phi_m)| \\ &\leq \mathrm{e}^{-[B(\lambda_1)-B(\lambda_0)]t/2} \sum_{m=1}^\infty \mathrm{e}^{-[B(\lambda_m)-B(\lambda_0)]t/2}\|\phi_m\|_\infty^2 \\ &\leq C\mathrm{e}^{-[B(\lambda_1)-B(\lambda_0)]t/2}, \quad t \geq 1,\, \nu \in \mathcal{P}_0, \end{aligned} \tag{2.20}$$

which clearly implies that

$$\lim_{t\to\infty}\{\mathrm{e}^{B(\lambda_0)t}\mathbb{P}^\nu(t < \sigma_\tau^B)\} = \mu(\phi_0)\nu(\phi_0), \quad \nu \in \mathcal{P}_0. \tag{2.21}$$

The following notation is used frequently in the sequel. Let $\nu \in \mathcal{P}_0$ and $t > 0$. Define

$$\eta_t^\nu = \int_M \phi_0(x) p_t^0(x, \cdot)\, \nu(\mathrm{d}x),$$

which is obviously non-negative. Then, by (2.9), we have the spectral representation of η_t^ν as follows:

$$\eta_t^\nu = \nu(\phi_0) + \sum_{m=1}^\infty \nu(\phi_m)\mathrm{e}^{-(\lambda_m-\lambda_0)t}\phi_m\phi_0^{-1} \geq 0. \tag{2.22}$$

Let $\mathcal{B}_+(M)$ (resp. $\mathcal{B}_b(M)$) be the class of non-negative (resp. bounded) measurable functions on M, and set $\mathcal{B}_1(M) := \{f \in \mathcal{B}_b(M) : \|f\|_\infty \leq 1\}$. Denote the standard gamma function as $\Gamma(\cdot)$. For any $a, b \in \mathbb{R} \cup \{\infty, -\infty\}$, $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$; in particular, $a \vee 0 =: a^+$.

Throughout the following Sections 3 and 4, we always assume that $\alpha \in (0,1]$ and $B \in \mathbf{B}^\alpha$ unless explicitly stated otherwise.

3. **Proofs of Theorem 1.1.** In this section, we aim to prove Theorem 1.1. Recall that $\mu_0 = \phi_0^2\mu$. One of the key steps to reach this target is based on the following inequality:

$$\mathbb{W}_2(f\mu_0,\mu_0)^2 \le 4\int_M |\nabla(-\mathcal{L}_0)^{-1}(f-1)|^2\,\mathrm{d}\mu_0,\quad f\ge 0,\ \mu_0(f)=1; \tag{3.1}$$

see [14, Theorem 2] and see also [20, 3, 29] for related results.

Let $\nu \in \mathcal{P}_0$. In order to employ (3.1) to estimate $\mathbb{W}_2(\mu_t^{B,\nu},\mu_0)$, we should first calculate the Radon–Nikodym derivative $\frac{\mathrm{d}\mu_t^{B,\nu}}{\mathrm{d}\mu_0}$, which is presented in the next lemma. The main tools to reach this are the Markov property and the spectral representation of the subordinated Dirichlet diffusion semigroup $(P_t^{D,B})_{t\ge 0}$.

Lemma 3.1. Let $\nu\in\mathcal{P}_0$ and $t>0$. Then

$$\frac{\mathrm{d}\mu_t^{B,\nu}}{\mathrm{d}\mu_0} = \rho_t^{B,\nu}+1,$$

where

$$\rho_t^{B,\nu} := \tilde\rho_t^{B,\nu} + \frac{1}{t\mathbb{E}^\nu[1_{\{t<\sigma_\tau^B\}}]}\int_0^t \xi_s\,\mathrm{d}s - A_t,$$

and

$$\begin{aligned}
\tilde\rho_t^{B,\nu} &:= \frac{1}{t\mathbb{E}^\nu[1_{\{t<\sigma_\tau^B\}}]}\sum_{m=1}^\infty \frac{\big[\mu(\phi_0)\nu(\phi_m)+\nu(\phi_0)\mu(\phi_m)\big]\mathrm{e}^{-B(\lambda_0)t}}{B(\lambda_m)-B(\lambda_0)}\phi_m\phi_0^{-1},\\
A_t &:= \frac{1}{t\mathbb{E}^\nu[1_{\{t<\sigma_\tau^B\}}]}\sum_{m=1}^\infty \frac{\big[\mu(\phi_0)\nu(\phi_m)+\nu(\phi_0)\mu(\phi_m)\big]\mathrm{e}^{-B(\lambda_m)t}}{B(\lambda_m)-B(\lambda_0)}\phi_m\phi_0^{-1},\\
\xi_s &:= \left(\sum_{m=1}^\infty \mathrm{e}^{-B(\lambda_m)s}\nu(\phi_m)\phi_m\phi_0^{-1}\right)\left(\sum_{n=1}^\infty \mathrm{e}^{-B(\lambda_n)(t-s)}\mu(\phi_n)\phi_n\phi_0^{-1}\right)\\
&\quad -\sum_{m=1}^\infty \mathrm{e}^{-B(\lambda_m)t}\mu(\phi_m)\nu(\phi_m),\quad 0<s\le t.
\end{aligned} \tag{3.2}$$

Proof. Let $t\ge s>0$ and $f\in\mathcal{B}_+(M)$. By the Markov property,

$$\begin{aligned}
\int_M f\,\mathrm{d}\mathbb{E}^\nu[\delta_{X_s^B}1_{\{t<\sigma_\tau^B\}}] &= \mathbb{E}^\nu\big[f(X_s^B)1_{\{s<\sigma_\tau^B\}}\mathbb{E}^{X_s^B}(1_{\{t-s<\sigma_\tau^B\}})\big]\\
&= \mathbb{E}^\nu\big[f(X_s^B)1_{\{s<\sigma_\tau^B\}}(P_{t-s}^{D,B}1)(X_s^B)\big]\\
&= \nu\big(P_s^{D,B}\{fP_{t-s}^{D,B}1\}\big).
\end{aligned} \tag{3.3}$$

By (2.16), we have

$$P_s^{D,B}\{fP_{t-s}^{D,B}1\}(x) = \sum_{m=0}^\infty \mathrm{e}^{-B(\lambda_m)s}\mu(\phi_m fP_{t-s}^{D,B}1)\phi_m(x),\quad x\in M.$$

Applying (2.16) again, we derive that

$$\begin{aligned}
\mu(\phi_m fP_{t-s}^{D,B}1) &= \int_M \phi_m(y)f(y)\left(\sum_{n=0}^\infty \mathrm{e}^{-B(\lambda_n)(t-s)}\mu(\phi_n)\phi_n(y)\right)\mu(\mathrm{d}y)\\
&= \sum_{n=0}^\infty \mathrm{e}^{-B(\lambda_n)(t-s)}\mu(\phi_n)\int_M \phi_m(y)\phi_n(y)f(y)\,\mu(\mathrm{d}y)
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \mathrm{e}^{-B(\lambda_n)(t-s)} \mu(\phi_n) \int_M (\phi_m \phi_0^{-1})(y)(\phi_n \phi_0^{-1})(y) f(y)\, \mu_0(\mathrm{d}y).$$

Hence

$$\int_M f \,\mathrm{d}\mathbb{E}^\nu[\delta_{X_s^B} 1_{\{t<\sigma_\tau^B\}}] \tag{3.4}$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \mathrm{e}^{-B(\lambda_m)s}\mathrm{e}^{-B(\lambda_n)(t-s)} \mu(\phi_n)\nu(\phi_m) \int_M (\phi_m \phi_0^{-1})(y)(\phi_n \phi_0^{-1})(y) f(y)\, \mu_0(\mathrm{d}y).$$

According to(3.4), we deduce that the Radon–Nikodym derivative of $\mathbb{E}^\nu[\delta_{X_s^B} 1_{\{t<\sigma_\tau^B\}}]$ w.r.t. μ_0 can be written as

$$\begin{aligned}
&\frac{\mathrm{d}\mathbb{E}^\nu[\delta_{X_s^B} 1_{\{t<\sigma_\tau^B\}}]}{\mathrm{d}\mu_0}\\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \mathrm{e}^{-B(\lambda_m)s}\mathrm{e}^{-B(\lambda_n)(t-s)} \mu(\phi_n)\nu(\phi_m)(\phi_m \phi_0^{-1})(\phi_n \phi_0^{-1})\\
&= \left(\sum_{m=0}^{\infty} \mathrm{e}^{-B(\lambda_m)s}\nu(\phi_m)\phi_m\phi_0^{-1}\right)\left(\sum_{n=0}^{\infty} \mathrm{e}^{-B(\lambda_n)(t-s)}\mu(\phi_n)\phi_n\phi_0^{-1}\right)\\
&= \mathrm{e}^{-B(\lambda_0)t}\mu(\phi_0)\nu(\phi_0) + \mathrm{e}^{-B(\lambda_0)s}\nu(\phi_0)\sum_{m=1}^{\infty} \mathrm{e}^{-B(\lambda_m)(t-s)}\mu(\phi_m)\phi_m\phi_0^{-1}\\
&\quad + \mathrm{e}^{-B(\lambda_0)(t-s)}\mu(\phi_0)\sum_{m=1}^{\infty} \mathrm{e}^{-B(\lambda_m)s}\nu(\phi_m)\phi_m\phi_0^{-1}\\
&\quad + \left(\sum_{m=1}^{\infty} \mathrm{e}^{-B(\lambda_m)s}\nu(\phi_m)\phi_m\phi_0^{-1}\right)\left(\sum_{n=1}^{\infty} \mathrm{e}^{-B(\lambda_n)(t-s)}\mu(\phi_n)\phi_n\phi_0^{-1}\right).
\end{aligned}\tag{3.5}$$

Applying (2.16) again, we get

$$\mathbb{E}^\nu[1_{\{t<\sigma_\tau^B\}}] = \mathrm{e}^{-B(\lambda_0)t}\mu(\phi_0)\nu(\phi_0) + \sum_{m=1}^{\infty} \mathrm{e}^{-B(\lambda_m)t}\mu(\phi_m)\nu(\phi_m).$$

Combining this with (3.5), we have

$$\begin{aligned}
&\frac{\mathrm{d}\mathbb{E}^\nu[\delta_{X_s^B} 1_{\{t<\sigma_\tau^B\}}]}{\mathrm{d}\mu_0} - \mathbb{E}^\nu[1_{\{t<\sigma_\tau^B\}}]\\
&= \mathrm{e}^{-B(\lambda_0)s}\nu(\phi_0)\sum_{m=1}^{\infty} \mathrm{e}^{-B(\lambda_m)(t-s)}\mu(\phi_m)\phi_m\phi_0^{-1}\\
&\quad + \mathrm{e}^{-B(\lambda_0)(t-s)}\mu(\phi_0)\sum_{m=1}^{\infty} \mathrm{e}^{-B(\lambda_m)s}\nu(\phi_m)\phi_m\phi_0^{-1}\\
&\quad + \left(\sum_{m=1}^{\infty} \mathrm{e}^{-B(\lambda_m)s}\nu(\phi_m)\phi_m\phi_0^{-1}\right)\left(\sum_{n=1}^{\infty} \mathrm{e}^{-B(\lambda_n)(t-s)}\mu(\phi_n)\phi_n\phi_0^{-1}\right)\\
&\quad - \sum_{m=1}^{\infty} \mathrm{e}^{-B(\lambda_m)t}\mu(\phi_m)\nu(\phi_m).
\end{aligned}$$

Noting that

$$\begin{aligned}\mathrm{I} &:= \int_0^t \mathrm{e}^{-B(\lambda_0)s}\nu(\phi_0)\sum_{m=1}^{\infty}\mathrm{e}^{-B(\lambda_m)(t-s)}\mu(\phi_m)\phi_m\phi_0^{-1}\,\mathrm{d}s\\ &= \sum_{m=1}^{\infty}\frac{\nu(\phi_0)\mu(\phi_m)(\mathrm{e}^{-B(\lambda_0)t}-\mathrm{e}^{-B(\lambda_m)t})}{B(\lambda_m)-B(\lambda_0)}\phi_m\phi_0^{-1},\end{aligned}$$

and

$$\begin{aligned}\mathrm{II} &:= \int_0^t \mathrm{e}^{-B(\lambda_0)(t-s)}\mu(\phi_0)\sum_{m=1}^{\infty}\mathrm{e}^{-B(\lambda_m)s}\nu(\phi_m)\phi_m\phi_0^{-1}\,\mathrm{d}s\\ &= \sum_{m=1}^{\infty}\frac{\mu(\phi_0)\nu(\phi_m)(\mathrm{e}^{-B(\lambda_0)t}-\mathrm{e}^{-B(\lambda_m)t})}{B(\lambda_m)-B(\lambda_0)}\phi_m\phi_0^{-1},\end{aligned}$$

we have

$$\begin{aligned}\mathrm{I}+\mathrm{II} &= \sum_{m=1}^{\infty}\frac{\big[\mu(\phi_0)\nu(\phi_m)+\nu(\phi_0)\mu(\phi_m)\big]\mathrm{e}^{-B(\lambda_0)t}}{B(\lambda_m)-B(\lambda_0)}\phi_m\phi_0^{-1}\\ &\quad -\sum_{m=1}^{\infty}\frac{\big[\mu(\phi_0)\nu(\phi_m)+\nu(\phi_0)\mu(\phi_m)\big]\mathrm{e}^{-B(\lambda_m)t}}{B(\lambda_m)-B(\lambda_0)}\phi_m\phi_0^{-1}.\end{aligned}$$

Therefore, we can write

$$\begin{aligned}\frac{\mathrm{d}\mu_t^{B,\nu}}{\mathrm{d}\mu_0} &= \frac{1}{t\mathbb{E}^{\nu}[1_{\{t<\sigma_\tau^B\}}]}\int_0^t\frac{\mathrm{d}\mathbb{E}^{\nu}[\delta_{X_s^B}1_{\{t<\sigma_\tau^B\}}]}{\mathrm{d}\mu_0}\,\mathrm{d}s\\ &= \frac{1}{t\mathbb{E}^{\nu}[1_{\{t<\sigma_\tau^B\}}]}\int_0^t\Big(\frac{\mathrm{d}\mathbb{E}^{\nu}[\delta_{X_s^B}1_{\{t<\sigma_\tau^B\}}]}{\mathrm{d}\mu_0}-\mathbb{E}^{\nu}[1_{\{t<\sigma_\tau^B\}}]\Big)\,\mathrm{d}s+1\\ &= \frac{1}{t\mathbb{E}^{\nu}[1_{\{t<\sigma_\tau^B\}}]}\left(\mathrm{I}+\mathrm{II}+\int_0^t\xi_s\,\mathrm{d}s\right)+1\\ &= \tilde{\rho}_t^{B,\nu}-A_t+\frac{1}{t\mathbb{E}^{\nu}[1_{\{t<\sigma_\tau^B\}}]}\int_0^t\xi_s\,\mathrm{d}s+1,\end{aligned}$$

where $\tilde{\rho}_t^{B,\nu}$, A_t and ξ_s are explicitly expressed in (3.2). □

Indeed, we have the following useful integral representation of ξ_s.

Remark 3.2. Let $\nu\in\mathcal{P}_0$ and $t>0$. Then

$$\begin{aligned}\xi_s &= \int_0^{\infty}\int_0^{\infty}\mathrm{e}^{-\lambda_0(k+l)}[\eta_l^{\nu}-\nu(\phi_0)][P_k^0\phi_0^{-1}-\mu(\phi_0)]\,\mathbb{P}(S_{t-s}^B\in\mathrm{d}k)\mathbb{P}(S_s^B\in\mathrm{d}l)\\ &\quad -\int_0^{\infty}\mathrm{e}^{-\lambda_0 l}\nu\big(\phi_0\{P_l^0\phi_0^{-1}-\mu(\phi_0)\}\big)\,\mathbb{P}(S_t^B\in\mathrm{d}l),\quad 0<s\le t.\end{aligned}\tag{3.6}$$

Proof. Indeed, according to (2.8),

$$P_t^0(\phi_0^{-1})=\mu(\phi_0)+\sum_{m=1}^{\infty}\mu(\phi_m)\mathrm{e}^{-(\lambda_m-\lambda_0)t}\phi_m\phi_0^{-1},$$

which together with (2.22) and (1.1) imply that

$$\int_0^{\infty}\int_0^{\infty}\mathrm{e}^{-\lambda_0(k+l)}[\eta_l^{\nu}-\nu(\phi_0)][P_k^0\phi_0^{-1}-\mu(\phi_0)]\,\mathbb{P}(S_{t-s}^B\in\mathrm{d}k)\mathbb{P}(S_s^B\in\mathrm{d}l)$$

$$= \left(\sum_{m=1}^{\infty} \mathrm{e}^{-B(\lambda_m)s} \nu(\phi_m)\phi_m\phi_0^{-1} \right) \left(\sum_{n=1}^{\infty} \mathrm{e}^{-B(\lambda_n)(t-s)} \mu(\phi_n)\phi_n\phi_0^{-1} \right), \quad 0 < s \le t,$$

and

$$\int_0^\infty \mathrm{e}^{-\lambda_0 l} \nu\big(\phi_0\{P_l^0\phi_0^{-1} - \mu(\phi_0)\}\big)\, \mathbb{P}(S_t^B \in \mathrm{d}l) = \sum_{m=1}^{\infty} \mathrm{e}^{-B(\lambda_m)t} \mu(\phi_m)\nu(\phi_m).$$

Thus, we immediately obtain (3.6) from the definition of ξ_s in (3.2). □

With Lemma 3.1 and inequality (3.1) in hand, we establish (1.2) for particular initial distributions in the next proposition.

Proposition 3.3. Assume that $B \in \mathbf{B}^\alpha$ for some $\alpha \in (0, 1]$. Then, for every $\nu \in \mathcal{P}_0$ satisfying that $\nu = h\mu$ and $\|h\phi_0^{-1}\|_\infty < \infty$,

$$\limsup_{t\to\infty} \{t^2 \mathbb{W}_2(\mu_t^{B,\nu}, \mu_0)^2\} \le 4I.$$

Proof. Since $\frac{\mathrm{d}\mu_t^{B,\nu}}{\mathrm{d}\mu_0} = 1 + \rho_t^{B,\nu}$, applying (3.1), we have

$$\mathbb{W}_2(\mu_t^{B,\nu}, \mu_0)^2 \le 4 \int_M |\nabla(-\mathcal{L}_0)^{-1}\rho_t^{B,\nu}|^2 \,\mathrm{d}\mu_0, \quad t > 0.$$

According to Lemma 3.1 and the triangle inequality of $\|\cdot\|_{L^2(\mu_0)}$, we deduce that for any $\delta > 0$,

$$\begin{aligned} \mathbb{W}_2(\mu_t^{B,\nu}, \mu_0)^2 \le{}& 4(1+\delta) \int_M |\nabla(-\mathcal{L}_0)^{-1}\tilde{\rho}_t^{B,\nu}|^2 \,\mathrm{d}\mu_0 \\ &+ 8(1+\delta^{-1}) \int_M |\nabla(-\mathcal{L}_0)^{-1}A_t|^2 \,\mathrm{d}\mu_0 \\ &+ 8(1+\delta^{-1}) \int_M \left| \nabla(-\mathcal{L}_0)^{-1} \frac{1}{t\mathbb{E}^\nu[1_{\{t<\sigma_\tau^B\}}]} \int_0^t \xi_s \,\mathrm{d}s \right|^2 \mathrm{d}\mu_0. \end{aligned} \tag{3.7}$$

Since $-\mathcal{L}_0(\phi_m\phi_0^{-1}) = (\lambda_m - \lambda_0)\phi_m\phi_0^{-1}$ and $\|\phi_m\phi_0^{-1}\|_{L^2(\mu_0)} = 1$ for every $m \in \mathbb{N}$, by the integration-by-parts formula, for any $m \in \mathbb{N}$

$$\int_M |\nabla(-\mathcal{L}_0)^{-1}(\phi_m\phi_0^{-1})|^2 \,\mathrm{d}\mu_0 = \int_M \phi_m\phi_0^{-1}(-\mathcal{L}_0)^{-1}(\phi_m\phi_0^{-1}) \,\mathrm{d}\mu_0 = \frac{1}{\lambda_m - \lambda_0}.$$

Recalling the definition of $\tilde{\rho}_t^{B,\nu}$ and A_t in (3.2), it is easy to see that, for every $t > 0$,

$$\int_M |\nabla(-\mathcal{L}_0)^{-1}\tilde{\rho}_t^{B,\nu}|^2 \,\mathrm{d}\mu_0 = \frac{\mathrm{e}^{-2B(\lambda_0)t}}{(t\mathbb{E}^\nu[1_{\{t<\sigma_\tau^B\}}])^2} \sum_{m=1}^{\infty} \frac{[\mu(\phi_0)\nu(\phi_m) + \nu(\phi_0)\mu(\phi_m)]^2}{(\lambda_m - \lambda_0)[B(\lambda_m) - B(\lambda_0)]^2}, \tag{3.8}$$

and

$$\begin{aligned} &\int_M |\nabla(-\mathcal{L}_0)^{-1}A_t|^2 \,\mathrm{d}\mu_0 \\ &= \frac{1}{(t\mathbb{E}^\nu[1_{\{t<\sigma_\tau^B\}}])^2} \sum_{m=1}^{\infty} \frac{[\mu(\phi_0)\nu(\phi_m) + \nu(\phi_0)\mu(\phi_m)]^2 \mathrm{e}^{-2B(\lambda_m)t}}{(\lambda_m - \lambda_0)[B(\lambda_m) - B(\lambda_0)]^2}. \end{aligned} \tag{3.9}$$

By the definition of ξ_s in (3.2), since $\{\phi_m\phi_0^{-1}\}_{m\in\mathbb{N}_0}$ is an orthonormal basis in $L^2(\mu_0)$, we have

$$\mu_0(\xi_s) = \mu_0\Big(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \mathrm{e}^{-B(\lambda_m)s} \mathrm{e}^{-B(\lambda_n)(t-s)} \nu(\phi_m)\mu(\phi_n)\phi_m\phi_0^{-1}\phi_n\phi_0^{-1} \Big)$$

$$
\begin{aligned}
&- \sum_{m=1}^{\infty} \mathrm{e}^{-B(\lambda_m)t}\mu(\phi_m)\nu(\phi_m) \\
&= \sum_{m=1}^{\infty} \mathrm{e}^{-B(\lambda_m)s}\mathrm{e}^{-B(\lambda_m)(t-s)}\mu(\phi_m)\nu(\phi_m) - \sum_{m=1}^{\infty} \mathrm{e}^{-B(\lambda_m)t}\mu(\phi_m)\nu(\phi_m) \\
&= 0, \quad 0 < s \le t. \qquad (3.10)
\end{aligned}
$$

By the fact that $(-\mathcal{L}_0)^{-\frac{1}{2}} = \frac{2}{\sqrt{\pi}} \int_0^\infty P^0_{s^2}\,\mathrm{d}s$ and Minkowski's inequality, we obtain that

$$
\begin{aligned}
&\int_M \left| \nabla(-\mathcal{L}_0)^{-1} \frac{1}{t\mathbb{E}^\nu[1_{\{t<\sigma_\tau^B\}}]} \int_0^t \xi_s\,\mathrm{d}s \right|^2 \mathrm{d}\mu_0 \\
&= \frac{1}{(t\mathbb{E}^\nu[1_{\{t<\sigma_\tau^B\}}])^2} \int_M \left| \frac{2}{\sqrt{\pi}} \int_0^\infty \int_0^t P^0_{r^2}\xi_s\,\mathrm{d}s\mathrm{d}r \right|^2 \mathrm{d}\mu_0 \\
&\le \frac{4}{\pi(t\mathbb{E}^\nu[1_{\{t<\sigma_\tau^B\}}])^2} \left(\int_0^\infty \int_0^t \|P^0_{r^2}\xi_s\|_{L^2(\mu_0)}\,\mathrm{d}s\mathrm{d}r \right)^2, \quad t > 0. \qquad (3.11)
\end{aligned}
$$

Recalling the integral representation of ξ_s in (3.6), we have

$$
\begin{aligned}
&\|\xi_s\|_{L^2(\mu_0)} \\
&\le \int_0^\infty \int_0^\infty \mathrm{e}^{-\lambda_0(k+l)} \|[\eta_l^\nu - \nu(\phi_0)][P^0_k\phi_0^{-1} - \mu(\phi_0)]\|_{L^2(\mu_0)}\,\mathbb{P}(S^B_{t-s} \in \mathrm{d}k)\mathbb{P}(S^B_s \in \mathrm{d}l) \\
&+ \int_0^\infty \mathrm{e}^{-\lambda_0 l} |\nu(\phi_0\{P^0_l\phi_0^{-1} - \mu(\phi_0)\})|\,\mathbb{P}(S^B_t \in \mathrm{d}l), \quad 0 < s \le t. \qquad (3.12)
\end{aligned}
$$

Since $\nu = h\mu \in \mathcal{P}_0$, $\|\phi_0^{-1}\|_{L^2(\mu_0)} = 1$, $\eta_s^\nu = P^0_s(h\phi_0^{-1})$ for any $s > 0$, and $\|h\phi_0^{-1}\|_{L^2(\mu_0)} \le \|h\phi_0^{-1}\|_\infty$, (2.11) yields that for every $k, l > 0$,

$$
\begin{aligned}
&\|[\eta_l^\nu - \nu(\phi_0)][P^0_k\phi_0^{-1} - \mu(\phi_0)]\|_{L^2(\mu_0)} \\
&\le \|P^0_l(h\phi_0^{-1}) - \mu_0(h\phi_0^{-1})\|_\infty \|P^0_k\phi_0^{-1} - \mu(\phi_0)\|_{L^2(\mu_0)} \\
&\le \mathrm{e}^{-(\lambda_1-\lambda_0)l}\|h\phi_0^{-1}\|_\infty \mathrm{e}^{-(\lambda_1-\lambda_0)k}\|\phi_0^{-1}\|_{L^2(\mu_0)} \\
&= \|h\phi_0^{-1}\|_\infty \mathrm{e}^{-(\lambda_1-\lambda_0)(k+l)}, \qquad (3.13)
\end{aligned}
$$

and

$$
\begin{aligned}
|\nu(\phi_0\{\mu(\phi_0) - P^0_l(\phi_0^{-1})\})| &\le \|h\phi_0^{-1}\|_{L^2(\mu_0)}\|P^0_l\phi_0^{-1} - \mu(\phi_0)\|_{L^2(\mu_0)} \\
&\le \|h\phi_0^{-1}\|_\infty \mathrm{e}^{-(\lambda_1-\lambda_0)l}. \qquad (3.14)
\end{aligned}
$$

Hence, by (1.1), (3.12), (3.13) and (3.14), we have

$$
\begin{aligned}
\|\xi_s\|_{L^2(\mu_0)} &\le \|h\phi_0^{-1}\|_\infty \int_0^\infty \int_0^\infty \mathrm{e}^{-\lambda_0(k+l)}\mathrm{e}^{-(\lambda_1-\lambda_0)(k+l)}\,\mathbb{P}(S^B_{t-s} \in \mathrm{d}k)\mathbb{P}(S^B_s \in \mathrm{d}l) \\
&\quad + \|h\phi_0^{-1}\|_\infty \int_0^\infty \mathrm{e}^{-\lambda_0 l}\mathrm{e}^{-(\lambda_1-\lambda_0)l}\,\mathbb{P}(S^B_t \in \mathrm{d}l) \\
&= 2\|h\phi_0^{-1}\|_\infty \mathrm{e}^{-B(\lambda_1)t}, \quad s > 0.
\end{aligned}
$$

Then we can apply (2.11) to get that

$$
\|P^0_{r^2}\xi_s\|_{L^2(\mu_0)} \le 2\alpha_2\|h\phi_0^{-1}\|_\infty \mathrm{e}^{-B(\lambda_1)t}\mathrm{e}^{-(\lambda_1-\lambda_0)r^2}, \quad r > 0, \qquad (3.15)
$$

where α_2 is the same constant in (2.11). Thus, according to (3.15), (2.20) and (3.11), there exist some constants $c_1, t_0 > 0$ such that

$$\begin{aligned}&\int_M \left|\nabla(-\mathcal{L}_0)^{-1}\frac{1}{t\mathbb{E}^\nu[1_{\{t<\sigma_\tau^B\}}]}\int_0^t \xi_s\,\mathrm{d}s\right|^2 \mathrm{d}\mu_0\\&\le c_1\|h\phi_0^{-1}\|_\infty^2 \mathrm{e}^{-2[B(\lambda_1)-B(\lambda_0)]t}, \quad t\ge t_0. \end{aligned}\tag{3.16}$$

Therefore, by (3.7), (3.8), (3.9) and (3.16), we find constants $c_2, t_0 > 0$ such that

$$\begin{aligned}t^2\mathbb{W}_2(\mu_t^{B,\nu},\mu_0)^2 \le{}& \frac{4(1+\delta)\mathrm{e}^{-2B(\lambda_0)t}}{(\mathbb{E}^\nu[1_{\{t<\sigma_\tau^B\}}])^2}\sum_{m=1}^\infty \frac{[\mu(\phi_0)\nu(\phi_m)+\nu(\phi_0)\mu(\phi_m)]^2}{(\lambda_m-\lambda_0)[B(\lambda_m)-B(\lambda_0)]^2}\\&+c_2(1+\delta^{-1})\sum_{m=1}^\infty \frac{[\mu(\phi_0)\nu(\phi_m)+\nu(\phi_0)\mu(\phi_m)]^2}{(\lambda_m-\lambda_0)[B(\lambda_m)-B(\lambda_0)]^2}\mathrm{e}^{-2[B(\lambda_m)-B(\lambda_0)]t}\\&+c_2(1+\delta^{-1})\|h\phi_0^{-1}\|_\infty^2\mathrm{e}^{-2[B(\lambda_1)-B(\lambda_0)]t}, \quad t\ge t_0.\end{aligned}\tag{3.17}$$

Since $\|h\phi_0^{-1}\|_\infty < \infty$, by (2.19) and (2.21), letting $t\to\infty$ first and then $\delta\to 0$, we arrive at

$$\limsup_{t\to\infty}\{t^2\mathbb{W}_2(\mu_t^{B,\nu},\mu_0)^2\} \le \frac{4}{[\mu(\phi_0)\nu(\phi_0)]^2}\sum_{m=1}^\infty \frac{[\mu(\phi_0)\nu(\phi_m)+\nu(\phi_0)\mu(\phi_m)]^2}{(\lambda_m-\lambda_0)[B(\lambda_m)-B(\lambda_0)]^2}.$$

We finish the proof. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. The proof consists of two parts. Let $\alpha\in(0,1]$ and $\nu\in\mathcal{P}_0$.

PART 1. In this part, we aim to prove (1.2), which is divided into three steps. Assume that $B\in\mathbf{B}^\alpha$.

Step (i). Let $t\ge s\ge\varepsilon>0$. By the Markov property, the definition of η_t^ν, (2.17) and Fubini's theorem, we have, for any $f\in\mathcal{B}_b(M)$,

$$\begin{aligned}&\mathbb{E}^\nu[f(X_s^B)1_{\{t<\sigma_\tau^B\}}]\\&=\mathbb{E}^\nu\big(1_{\{\varepsilon<\sigma_\tau^B\}}\mathbb{E}^{X_\varepsilon^B}[f(X_{s-\varepsilon}^B)1_{\{t-\varepsilon<\sigma_\tau^B\}}]\big)=\int_M P_\varepsilon^{D,B}\psi(x)\,\nu(\mathrm{d}x)\\&=\int_M\int_0^\infty \mathrm{e}^{-\lambda_0 k}\phi_0(x)P_k^0(\psi\phi_0^{-1})(x)\,\mathbb{P}(S_\varepsilon^B\in\mathrm{d}k)\,\nu(\mathrm{d}x)\\&=\int_0^\infty \mathrm{e}^{-\lambda_0 k}\int_M(\eta_k^\nu\phi_0)(y)\psi(y)\,\mu(\mathrm{d}y)\mathbb{P}(S_\varepsilon^B\in\mathrm{d}k)\\&=\int_0^\infty \mathrm{e}^{-\lambda_0 k}\int_M(\eta_k^\nu\phi_0)(y)\mathbb{E}^y[f(X_{s-\varepsilon}^B)1_{\{t-\varepsilon<\sigma_\tau^B\}}]\,\mu(\mathrm{d}y)\mathbb{P}(S_\varepsilon^B\in\mathrm{d}k),\end{aligned}$$

where we set $\psi(\cdot)=\mathbb{E}^\cdot[f(X_{s-\varepsilon}^B)1_{\{t-\varepsilon<\sigma_\tau^B\}}]$ for convenience. Taking $f=1$ in this equality, we immediately obtain

$$\mathbb{P}^\nu(t<\sigma_\tau^B)=\int_0^\infty\int_M \mathrm{e}^{-\lambda_0 k}(\eta_k^\nu\phi_0)(y)\,\mathbb{P}^y(t-\varepsilon<\sigma_\tau^B)\,\mu(\mathrm{d}y)\mathbb{P}(S_\varepsilon^B\in\mathrm{d}k).$$

Letting

$$\nu_\varepsilon=\frac{\int_0^\infty \mathrm{e}^{-\lambda_0 k}\eta_k^\nu\phi_0\,\mathbb{P}(S_\varepsilon^B\in\mathrm{d}k)\mu}{\int_0^\infty \mathrm{e}^{-\lambda_0 k}\mu(\eta_k^\nu\phi_0)\,\mathbb{P}(S_\varepsilon^B\in\mathrm{d}k)}=:h_\varepsilon\mu,$$

by the Markov property again, we have

$$\mathbb{E}^\nu[f(X_s^B)|t<\sigma_\tau^B]=\frac{\mathbb{E}^{\nu_\varepsilon}[f(X_{s-\varepsilon}^B)1_{\{t-\varepsilon<\sigma_\tau^B\}}]}{\mathbb{P}^{\nu_\varepsilon}(t-\varepsilon<\sigma_\tau^B)}=\mathbb{E}^{\nu_\varepsilon}[f(X_{s-\varepsilon}^B)|t-\varepsilon<\sigma_\tau^B].$$

Thus,

$$\hat\mu_{r,\varepsilon}^{B,\nu}:=\frac{1}{r-\varepsilon}\int_\varepsilon^r\mathbb{E}^\nu(\delta_{X_s^B}|r<\sigma_\tau^B)\,\mathrm{d}s=\mu_{r-\varepsilon}^{B,\nu_\varepsilon},\quad r>\varepsilon. \tag{3.18}$$

Noting that for any $k>0$,

$$\mu(\eta_k^\nu\phi_0)=\nu(\phi_0P_k^0\phi_0^{-1})\ge\nu(\phi_0)\|\phi_0\|_\infty^{-1}=:\gamma\in(0,1],$$

we obtain

$$\int_0^\infty\mathrm{e}^{-\lambda_0k}\mu(\eta_k^\nu\phi_0)\,\mathbb{P}(S_\varepsilon^B\in\mathrm{d}k)\ge\gamma\mathrm{e}^{-B(\lambda_0)\varepsilon},$$

where we applied (1.1). Then, by (1.1) and (2.18),

$$\begin{aligned}\mathbb{E}[(1\wedge S_\varepsilon^B)^{-\frac{d+2}{2}}]&\le1+\mathbb{E}\left[(S_\varepsilon^B)^{-\frac{d+2}{2}}\right]\\&=1+\frac{1}{\Gamma(\frac{d+2}{2})}\int_0^\infty t^{\frac{d+2}{2}-1}\mathrm{e}^{-\varepsilon B(t)}\mathrm{d}t\\&\le1+\frac{1}{\Gamma(\frac{d+2}{2})}\int_0^\infty t^{\frac{d+2}{2}-1}\mathrm{e}^{\varepsilon(b-at^\alpha)}\mathrm{d}t\\&\le c\left(1+\varepsilon^{-\frac{d+2}{2\alpha}}\right),\quad\varepsilon>0,\end{aligned}\tag{3.19}$$

for some constant $c>0$. Hence, by the intrinsic ultra-contractivity (2.10), there exists a constant $c_1>0$ such that, for every $y\in M$,

$$\begin{aligned}|(h_\varepsilon\phi_0^{-1})(y)|&=\frac{\int_0^\infty\mathrm{e}^{-\lambda_0k}\eta_k^\nu(y)\,\mathbb{P}(S_\varepsilon^B\in\mathrm{d}k)}{\int_0^\infty\mathrm{e}^{-\lambda_0k}\mu(\eta_k^\nu\phi_0)\,\mathbb{P}(S_\varepsilon^B\in\mathrm{d}k)}\\&\le\gamma^{-1}\mathrm{e}^{B(\lambda_0)\varepsilon}\int_0^\infty\mathrm{e}^{-\lambda_0k}\int_Mp_k^0(x,y)\phi_0(x)\,\nu(\mathrm{d}x)\mathbb{P}(S_\varepsilon^B\in\mathrm{d}k)\\&\le\gamma^{-1}\mathrm{e}^{B(\lambda_0)}\|\phi_0\|_\infty\int_0^\infty\mathrm{e}^{-\lambda_0k}\|p_k^0\|_{L^\infty(\mu_0\times\mu_0)}\,\mathbb{P}(S_\varepsilon^B\in\mathrm{d}k)\\&\le c_1\varepsilon^{-\frac{d+2}{2\alpha}},\quad\varepsilon\in(0,1),\end{aligned}$$

and hence,

$$\|h_\varepsilon\phi_0^{-1}\|_\infty\le c_1\varepsilon^{-\frac{d+2}{2\alpha}},\quad\varepsilon\in(0,1). \tag{3.20}$$

Thus, (3.17) and (3.20) imply that, there exist constants $c_2>0$ and $t_0\ge1$ such that, for every $\delta>0$ and every $\varepsilon\in(0,1)$,

$$\begin{aligned}&t^2\mathbb{W}_2(\hat\mu_{t,\varepsilon}^{B,\nu},\mu_0)^2\\&=t^2\mathbb{W}_2(\mu_{t-\varepsilon}^{B,\nu_\varepsilon},\mu_0)^2\\&\le\frac{4(1+\delta)\mathrm{e}^{-2B(\lambda_0)t}}{(\mathbb{E}^{\nu_\varepsilon}[1_{\{t<\sigma_\tau^B\}}])^2}\sum_{m=1}^\infty\frac{[\mu(\phi_0)\nu_\varepsilon(\phi_m)+\nu_\varepsilon(\phi_0)\mu(\phi_m)]^2}{(\lambda_m-\lambda_0)[B(\lambda_m)-B(\lambda_0)]^2}\\&\quad+c_2(1+\delta^{-1})\sum_{m=1}^\infty\frac{[\mu(\phi_0)\nu_\varepsilon(\phi_m)+\nu_\varepsilon(\phi_0)\mu(\phi_m)]^2}{(\lambda_m-\lambda_0)[B(\lambda_m)-B(\lambda_0)]^2}\mathrm{e}^{-2[B(\lambda_m)-B(\lambda_0)](t-\varepsilon)}\\&\quad+c_2(1+\delta^{-1})\varepsilon^{-\frac{d+2}{\alpha}}\mathrm{e}^{-2[B(\lambda_1)-B(\lambda_0)](t-\varepsilon)},\quad t\ge t_0,\end{aligned}\tag{3.21}$$

Step (ii). Let

$$I_\varepsilon := \frac{1}{\{\mu(\phi_0)\nu_\varepsilon(\phi_0)\}^2}\sum_{m=1}^\infty \frac{\{\nu_\varepsilon(\phi_0)\mu(\phi_m)+\mu(\phi_0)\nu_\varepsilon(\phi_m)\}^2}{(\lambda_m-\lambda_0)[B(\lambda_m)-B(\lambda_0)]^2},\quad \varepsilon>0.$$

Next, we prove that

$$\lim_{\varepsilon\to 0^+} I_\varepsilon = I.$$

By Fubini's theorem, (2.6), (2.7) and (2.22), for every $k>0$, it is direct to verify that

$$\mu(\eta_k^\nu\phi_0)=\nu(\phi_0P_k^0\phi_0^{-1})=\mathrm{e}^{\lambda_0 k}\nu(P_k^D 1),$$
$$\mu(\eta_k^\nu\phi_0\phi_m)=\nu\big(\phi_0P_k^0(\phi_m\phi_0^{-1})\big)=\mathrm{e}^{-(\lambda_m-\lambda_0)k}\nu(\phi_m),\quad m\in\mathbb{N}_0.$$

Let $\varepsilon\in(0,1)$. Then

$$\nu_\varepsilon(\phi_m)=\frac{\int_0^\infty \mathrm{e}^{-\lambda_0 k}\mu(\eta_k^\nu\phi_0\phi_m)\,\mathbb{P}(S_\varepsilon^B\in\mathrm{d}k)}{\int_0^\infty \mathrm{e}^{-\lambda_0 k}\mu(\eta_k^\nu\phi_0)\,\mathbb{P}(S_\varepsilon^B\in\mathrm{d}k)}=\frac{\mathrm{e}^{-B(\lambda_m)\varepsilon}\nu(\phi_m)}{\nu(P_\varepsilon^{D,B}1)},\quad m\in\mathbb{N}_0.$$

It is easy to see that $\lim_{\varepsilon\to0^+}\nu_\varepsilon(\phi_m)=\nu(\phi_m)$ for each $m\in\mathbb{N}_0$ since $\nu(P_\varepsilon^{D,B}1)=\mathbb{P}^\nu(\varepsilon<\sigma_\tau^B)\to 1$ as $\varepsilon\to0^+$. By (2.17) and (1.1), we have

$$\mathrm{e}^{B(\lambda_m)\varepsilon}\nu(P_\varepsilon^{D,B}1)\ge\mathrm{e}^{B(\lambda_0)\varepsilon}\nu\Big(\int_0^\infty \mathrm{e}^{-\lambda_0 s}\phi_0P_s^0\phi_0^{-1}\,\mathbb{P}(S_\varepsilon^B\in\mathrm{d}s)\Big)\ge\gamma\in(0,1],\ m\in\mathbb{N}_0,$$

which together with $\nu(P_\varepsilon^{D,B}1)\le1$ implies that

$$\mathrm{e}^{-B(\lambda_m)\varepsilon}|\nu(\phi_m)|\le|\nu_\varepsilon(\phi_m)|\le\gamma^{-1}|\nu(\phi_m)|,\quad m\in\mathbb{N}_0. \tag{3.22}$$

Since $\{\phi_m\}_{m\in\mathbb{N}_0}$ is an orthonormal sequence in $L^2(\mu)$, by Bessel's inequality, it is clear that

$$\sum_{m=1}^\infty \mu(\phi_m)^2\le1. \tag{3.23}$$

On the one hand, if $I<\infty$, by (3.22) and (3.23), applying the dominated convergence theorem, we have $\lim_{\varepsilon\to0^+}I_\varepsilon=I$. On the other hand, if $I=\infty$, which is equivalent to

$$\sum_{m=1}^\infty\frac{\nu(\phi_m)^2}{(\lambda_m-\lambda_0)[B(\lambda_m)-B(\lambda_0)]^2}=\infty.$$

then, by (3.22) and the monotone convergence theorem, we have

$$\begin{aligned}&\liminf_{\varepsilon\to0^+}\sum_{m=1}^\infty\frac{\nu_\varepsilon(\phi_m)^2}{(\lambda_m-\lambda_0)[B(\lambda_m)-B(\lambda_0)]^2}\\ &\ge\liminf_{\varepsilon\to0^+}\sum_{m=1}^\infty\frac{\mathrm{e}^{-2B(\lambda_m)\varepsilon}\nu(\phi_m)^2}{(\lambda_m-\lambda_0)[B(\lambda_m)-B(\lambda_0)]^2}=\infty.\end{aligned}$$

Combining this with (3.23) and $\nu_\varepsilon(\phi_0)\to\nu(\phi_0)$ as $\varepsilon\to0^+$, we have

$$\begin{aligned}\liminf_{\varepsilon\to0^+}I_\varepsilon&=\frac{1}{\{\mu(\phi_0)\nu(\phi_0)\}^2}\liminf_{\varepsilon\to0^+}\sum_{m=1}^\infty\frac{\{\nu_\varepsilon(\phi_0)\mu(\phi_m)+\mu(\phi_0)\nu_\varepsilon(\phi_m)\}^2}{(\lambda_m-\lambda_0)[B(\lambda_m)-B(\lambda_0)]^2}\\ &\ge\frac{1}{\{\mu(\phi_0)\nu(\phi_0)\}^2}\liminf_{\varepsilon\to0^+}\sum_{m=1}^\infty\frac{\{\mu(\phi_0)\nu_\varepsilon(\phi_m)\}^2-2\|\phi_0\|_\infty^2\mu(\phi_m)^2}{2(\lambda_m-\lambda_0)[B(\lambda_m)-B(\lambda_0)]^2}=\infty.\end{aligned}$$

Thus,

$$\lim_{\varepsilon\to0^+}I_\varepsilon=I. \tag{3.24}$$

Step (iii). Taking $\varepsilon = t^{-2}$ in (3.18), by (2.21), (3.21) and (3.24), we obtain

$$\lim_{\delta\to 0}\limsup_{t\to\infty}\{t^2\mathbb{W}_2(\hat{\mu}^{B,\nu}_{t,t^{-2}},\mu_0)^2\} \le 4I. \tag{3.25}$$

It is easy to verify that

$$\|\hat{\mu}^{B,\nu}_{t,\varepsilon} - \mu^{B,\nu}_t\|_{\mathrm{var}} \le \int_\varepsilon^t \left(\frac{1}{t-\varepsilon} - \frac{1}{t}\right) \mathrm{d}s + \frac{1}{t}\int_0^\varepsilon \mathrm{d}s \le 2\varepsilon t^{-1}, \quad \varepsilon \in (0,t),$$

where $\|\mu - \nu\|_{\mathrm{var}} = \sup_{f\in\mathcal{B}_1(M)} |\mu(f) - \nu(f)|$ for any $\mu,\nu \in \mathcal{P}$ by definition. Then, for any $t > 1$,

$$\mathbb{W}_2(\mu^{B,\nu}_t, \hat{\mu}^{B,\nu}_{t,t^{-2}})^2 \le \frac{1}{2}D^2\|\hat{\mu}^{B,\nu}_{t,t^{-2}} - \mu^{B,\nu}_t\|_{\mathrm{var}} \le D^2 t^{-3}.$$

Combining this with the triangle inequality of $\mathbb{W}_2$, we arrive at

$$t^2\mathbb{W}_2(\mu^{B,\nu}_t,\mu_0)^2 \le (1+\delta^{-1})D^2t^{-1} + (1+\delta)t^2\mathbb{W}_2(\hat{\mu}^{B,\nu}_{t,t^{-2}},\mu_0)^2, \quad \delta > 0.$$

Letting $t\to\infty$ first and then $\delta \to 0^+$, by (3.25), we finally complete the proof of (1.2).

<u>Part 2</u>. In this part, we prove the remaining assertions of Theorem 1.1. Assume that $B \in \mathbf{B}^\alpha$.

It is obvious that $I \ge 0$. The same argument in [27, pages 21-22] shows that $I > 0$. So it is left to us to show that $I < \infty$ under the condition (1) or (2). Set

$$J = \sum_{m=1}^\infty \frac{\nu(\phi_m)^2}{\lambda_m^{1+2\alpha}}.$$

Recall that $\sum_{m=1}^\infty \mu(\phi_m)^2 \le 1$. Then, since $B\in\mathbf{B}^\alpha$, by (2.18) and (2.19), it is easy to see that $I \le bJ$, for some constant $b > 0$. Hence it remains to prove that if the condition (1) or (2) holds then $J < \infty$.

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of probability density functions w.r.t. μ such that the sequence of probability measures $(f_n\mu)_{n\in\mathbb{N}}$ converges weakly to ν. Then

$$\begin{aligned}\|(-\mathcal{L})^{-\frac{1+2\alpha}{2}} f_n\|^2_{L^2(\mu)} &= \sum_{m=0}^\infty \Big(\int_M (-\mathcal{L})^{-\frac{1+2\alpha}{2}} f_n\phi_m \,\mathrm{d}\mu\Big)^2 \\ &= \sum_{m=0}^\infty \Big(\frac{1}{\lambda_m^{(1+2\alpha)/2}}\int_M f_n\phi_m\,\mathrm{d}\mu\Big)^2 = \sum_{m=0}^\infty \frac{\mu(f_n\phi_m)^2}{\lambda_m^{1+2\alpha}}. \end{aligned} \tag{3.26}$$

(i) Let $d > 2+4\alpha$. Assume that $\nu = h\mu$ with $h\in L^q(\mu)$. By (2.5) and the fact that $(P^D_t)_{t>0}$ is contractive in both $L^1(\mu)$ and $L^\infty(\mu)$, applying [25, Theorem II.2.7], i.e., for any $p\in(1,\infty)$ and $\beta \in (0,d/p)$,

$$\|(-\mathcal{L})^{-\beta/2} f\|_{L^{dp/(d-\beta p)}(\mu)} \le c_3\|f\|_{L^p(\mu)}, \quad f\in L^p(\mu),$$

for some constant $c_3 > 0$, we have

$$\|(-\mathcal{L})^{-\frac{1+2\alpha}{2}} f_n\|_{L^2(\mu)} \le c_4\|f_n\|_{L^q(\mu)}, \tag{3.27}$$

for some constant $c_4 > 0$, where $q := \frac{2d}{d+2+4\alpha}$. Then by (3.26) and (3.27) with f_n replaced by h, we have

$$J = \sum_{m=1}^\infty \frac{\mu(h\phi_m)^2}{\lambda_m^{1+2\alpha}} \le \|(-\mathcal{L})^{-\frac{1+2\alpha}{2}} h\|^2_{L^2(\mu)} \le c_4^2\|h\|^2_{L^q(\mu)} < \infty.$$

(ii) Let $d \le 2+4\alpha$. By (2.5) and (3.26), we can find a constant $c_5>0$ such that

$$\begin{aligned} J &\le \liminf_{n\to\infty} \sum_{m=1}^{\infty} \frac{\mu(f_n\phi_m)^2}{\lambda_m^{1+2\alpha}} \le \liminf_{n\to\infty} \|(-\mathcal{L})^{-\frac{1+2\alpha}{2}} f_n\|_{L^2(\mu)}^2 \\ &\le c_5 \liminf_{n\to\infty} \int_0^\infty \left\| P^D_{t^{\frac{2}{1+2\alpha}}} \right\|_{L^1(\mu)\to L^2(\mu)} \|f_n\|_{L^1(\mu)}^2 \,\mathrm{d}t < \infty. \end{aligned}$$

Therefore, the proof is completed. □

4. **Proofs of Theorem 1.3.** This section is divided into two parts. In the first part, we present the proof for the upper bound in Theorem 1.3, and in the second part, we prove the lower bound in Theorem 1.3.

We begin by introducing some frequently used notations. Let $t,\beta>0$ and $\nu\in\mathcal{P}_0$. Recall the definition of $\rho_t^{B,\nu}$ and $\tilde\rho_t^{B,\nu}$ in Lemma 3.1. Set

$$\mu_{t,\beta}^{B,\nu} := (1+\rho_{t,\beta}^{B,\nu})\mu_0, \quad \rho_{t,\beta}^{B,\nu} := P^0_{t^{-\beta}}\rho_t^{B,\nu}.$$

Note that $\mu_{t,\beta}^{B,\nu}$ should be regarded as the regularized version of $\mu_t^{B,\nu}$. Similarly, set

$$\tilde\mu_{t,\beta}^{B,\nu} := (1+\tilde\rho_{t,\beta}^{B,\nu})\mu_0, \quad \tilde\rho_{t,\beta}^{B,\nu} := P^0_{t^{-\beta}}\tilde\rho_t^{B,\nu}.$$

We should point out that our regularization procedure is different from [27], where in the particular case when B is the identity map, F.-Y. Wang proved that $\{(1+\tilde\rho_t^{B,\nu})\mu\}_{t>t_0}$ is a family of probability measures for some big enough t_0 and then he employed it to approximate the conditional empirical measure $\mu_t^{B,\nu}$. However, this approach seems invalid in our non-local setting.

4.1. **Upper bounds.** Since $\mu_t^{B,\nu}$ is a probability measure, it is easy to see that $\mu_{t,\beta}^{B,\nu}\in\mathcal{P}_0$. From Lemma 4.1, for every $\beta\in(0,\frac{2}{2d-2\alpha+1})$, there exists a constant $t_0>0$ such that $\tilde\mu_{t,\beta}^{B,\nu}$ is a probability measure for every $t\ge t_0$. Applying the triangle inequality of $\mathbb{W}_2$, we have

$$\begin{aligned} \mathbb{W}_2(\mu_t^{B,\nu},\mu_0)^2 &\le (1+\delta)\mathbb{W}_2(\tilde\mu_{t,\beta}^{B,\nu},\mu_0)^2 + (1+\delta^{-1})\mathbb{W}_2(\tilde\mu_{t,\beta}^{B,\nu},\mu_t^{B,\nu})^2 \\ &\le (1+\delta)\mathbb{W}_2(\tilde\mu_{t,\beta}^{B,\nu},\mu_0)^2 + 2(1+\delta^{-1})\mathbb{W}_2(\tilde\mu_{t,\beta}^{B,\nu},\mu_{t,\beta}^{B,\nu})^2 \\ &\quad + 2(1+\delta^{-1})\mathbb{W}_2(\mu_{t,\beta}^{B,\nu},\mu_t^{B,\nu})^2, \quad t\ge t_0,\ \delta>0. \end{aligned} \tag{4.1}$$

Clearly, in order to get the upper bound of $\mathbb{W}_2(\mu_t^{B,\nu},\mu_0)$, we need to estimate the three terms in the right hand side of (4.1). The term $\mathbb{W}_2(\tilde\mu_{t,\beta}^{B,\nu},\mu_0)$ should be regarded as the dominant term, and the others as error terms.

To bound $\mathbb{W}_2(\tilde\mu_{t,\beta}^{B,\nu},\mu_0)$ from above, the crucial tool is the following inequality: for any probability density functions f_0 and f_1 w.r.t. μ_0,

$$\mathbb{W}_2(f_0\mu_0,f_1\mu_0)^2 \le \int_M \frac{|\nabla(-\mathcal{L}_0)^{-1}(f_0-f_1)|^2}{\mathcal{M}(f_0,f_1)}\,\mathrm{d}\mu_0, \tag{4.2}$$

where the function $\mathcal{M}:[0,\infty)\times[0,\infty)\to[0,\infty)$ is defined as

$$\mathcal{M}(a,b) = \begin{cases} \frac{a-b}{\log a-\log b}1_{\{a\wedge b>0\}}, & a\ne b, \\ \frac{1}{a}1_{\{a>0\}}, & \text{otherwise}. \end{cases}$$

Refer to [3, Proposition 2.3] for (4.2). To bound $\mathbb{W}_2(\tilde\mu_{t,\beta}^{B,\nu},\mu_{t,\beta}^{B,\nu})$, we use the total variation norm since M is compact; see Lemma 4.5. As for the estimation of $\mathbb{W}_2(\mu_{t,\beta}^{B,\nu},\mu_t^{B,\nu})$, we apply inequality (4.7) introduced later.

In order to apply (4.2), the following estimate on $\tilde{\rho}_{t,\beta}^{B,\nu}$ is important.

Lemma 4.1. For every $\beta > 0$, there exist constants $c, t_0 > 0$ such that

$$\|\tilde{\rho}_{t,\beta}^{B,\nu}\|_\infty \leq \frac{c}{\nu(\phi_0)} t^{\frac{(2d-2\alpha+1)\beta}{2}-1}, \quad t \geq t_0,\, \nu \in \mathcal{P}_0. \tag{4.3}$$

Moreover, if $\beta \in (0, \frac{2}{2d-2\alpha+1})$, then $\tilde{\mu}_{t,\beta}^{B,\nu}$ is a probability measure for every $t \geq t_0$ and every $\nu \in \mathcal{P}_0$.

Proof. Combining (2.20) with (2.1), (2.7), (2.13), (2.18), (2.19) and (3.2), we can find some constants $c_1, c_2, c_3, t_0 > 0$ such that

$$\begin{aligned}
\|\tilde{\rho}_{t,\beta}^{B,\nu}\|_\infty &= \frac{1}{t\mathbb{E}^\nu[1_{\{t<\sigma_\tau^B\}}]} \left\| \sum_{m=1}^\infty \frac{[\mu(\phi_0)\nu(\phi_m) + \nu(\phi_0)\mu(\phi_m)]\mathrm{e}^{-B(\lambda_0)t}}{[B(\lambda_m) - B(\lambda_0)]\mathrm{e}^{(\lambda_m-\lambda_0)t^{-\beta}}} \phi_m \phi_0^{-1} \right\|_\infty \\
&\leq \frac{c_1}{t\nu(\phi_0)} \sum_{m=1}^\infty \frac{\|\phi_m\|_\infty \|\phi_m\phi_0^{-1}\|_\infty}{B(\lambda_m) - B(\lambda_0)} \mathrm{e}^{-(\lambda_m-\lambda_0)t^{-\beta}} \\
&\leq \frac{c_2}{t\nu(\phi_0)} t^{\frac{(2d-2\alpha+1)\beta}{2}} \int_0^\infty u^{\frac{d-2\alpha+1}{d}} \mathrm{e}^{-u^{\frac{2}{d}}}\, \mathrm{d}u \\
&\leq \frac{c_3}{\nu(\phi_0)} t^{\frac{(2d-2\alpha+1)\beta}{2}-1}, \quad t \geq t_0,
\end{aligned}$$

which proves (4.3).

Now let $\beta \in (0, \frac{2}{2d-2\alpha+1})$. It is clear that (4.3) implies that there exists a constant $t_1 > 0$, such that for any $t \geq t_1$, $\|\tilde{\rho}_{t,\beta}^{B,\nu}\|_\infty \leq \frac{1}{2}$. Hence, $1 + \tilde{\rho}_{t,\beta}^{B,\nu} \geq \frac{1}{2}$ and $\mu_0(1 + |\tilde{\rho}_{t,\beta}^{B,\nu}|) \leq \frac{3}{2}$ for every $t \geq t_1$. Noting that $\mu_0(\phi_m\phi_0^{-1}) = \mu(\phi_m\phi_0) = 0$ for every $m \in \mathbb{N}$, we easily see that $\mu_0(1 + \tilde{\rho}_{t,\beta}^{B,\nu}) = 1$ for any $t > 0$. Thus, $\tilde{\mu}_{t,\beta}^{B,\nu}$ is a probability measure for every $t \geq t_1$. □

Remark 4.2. In the particular case when $B(r) = r$ for every $r \geq 0$, the pointwise lower bound of $\tilde{\rho}_t^{B,\nu}$ is obtained in [27, Lemma 3.2]. However, it seems that the original method of proof does not work in the general setting of Lemma 4.1. That is why we introduce $\tilde{\rho}_{t,\beta}^{B,\nu}$, the regularized version of $\tilde{\rho}_t^{B,\nu}$, and establish (4.3). Indeed, the pointwise lower bound in (4.3) is enough for our purpose.

In the next lemma, we give an upper bound estimate on $\mathbb{W}_2(\tilde{\mu}_{t,\beta}^{B,\nu}, \mu_0)^2$.

Lemma 4.3. For every $\beta \in (0, \frac{2}{2d-2\alpha+1})$, there exist constants $c, t_0 > 0$ such that

$$t^2\mathbb{W}_2(\tilde{\mu}_{t,\beta}^{B,\nu}, \mu_0)^2 \leq \frac{1+ct^{\frac{(2d-2\alpha+1)\beta}{2}-1}}{\{\mu(\phi_0)\nu(\phi_0)\}^2} \sum_{m=1}^\infty \frac{[\mu(\phi_0)\nu(\phi_m)+\nu(\phi_0)\mu(\phi_m)]^2}{(\lambda_m-\lambda_0)[B(\lambda_m)-B(\lambda_0)]^2} \mathrm{e}^{-2(\lambda_m-\lambda_0)t^{-\beta}}$$

for any $t \geq t_0$ and any $\nu \in \mathcal{P}_0$.

Proof. The proof is a direct application of Lemma 4.1 and inequality (4.2). By Lemma 4.1, for every $\beta \in (0, \frac{2}{2d-2\alpha+1})$, there exist constants $c, t_0 > 0$ such that $\tilde{\mu}_{t,\beta}^{B,\nu} \in \mathcal{P}_0$ for every $t \geq t_0$, and

$$\mathcal{M}(1 + \tilde{\rho}_{t,\beta}^{B,\nu}, 1) \geq 1 \wedge (1 + \tilde{\rho}_{t,\beta}^{B,\nu}) \geq \frac{1}{1 + ct^{\frac{(2d-2\alpha+1)\beta}{2}-1}}, \quad t \geq t_0.$$

So, (4.2) implies that

$$\mathbb{W}_2(\tilde{\mu}_{t,\beta}^{B,\nu}, \mu_0)^2 \leq \left(1 + ct^{\frac{(2d-2\alpha+1)\beta}{2}-1}\right) \mu_0\big(|\nabla(-\mathcal{L}_0)^{-1}\tilde{\rho}_{t,\beta}^{B,\nu}|^2\big), \quad t \geq t_0. \tag{4.4}$$

Next, (2.7), (3.2) and the integration-by-parts formula yield that

$$t^2\mu_0(|\nabla(-\mathcal{L}_0)^{-1}\tilde\rho_{t,\beta}^{B,\nu}|^2)$$
$$= \frac{1}{(\mathbb{E}^\nu[1_{\{t<\sigma_\tau^B\}}])^2}\sum_{m=1}^\infty \frac{[\mu(\phi_0)\nu(\phi_m)+\nu(\phi_0)\mu(\phi_m)]^2\mathrm{e}^{-2B(\lambda_0)t}}{(\lambda_m-\lambda_0)[B(\lambda_m)-B(\lambda_0)]^2}\mathrm{e}^{-2(\lambda_m-\lambda_0)t^{-\beta}}.$$

Combining this with (2.20) and (4.4), we finish the proof. □

In order to use the total variation $\|\mu_{t,\beta}^{B,\nu}-\tilde\mu_{t,\beta}^{B,\nu}\|_{\mathrm{var}}$ to bound $\mathbb{W}_2(\tilde\mu_{t,\beta}^{B,\nu},\mu_{t,\beta}^{B,\nu})$, we need the following lemma.

Lemma 4.4. For every $p\in(3/2,\infty]$, there exist constants $c,t_0>0$ such that, for every $\nu\in\mathcal{P}_0$ with $\nu=h\mu$ and every $t\ge t_0$,

$$\mu_0(|\rho_t^{B,\nu}-\tilde\rho_t^{B,\nu}|)\le c\|h\phi_0^{-1}\|_{L^p(\mu_0)}\mathrm{e}^{-[B(\lambda_1)-B(\lambda_0)]t}.$$

Proof. Let $p\in(3/2,\infty]$ and p' be its conjugate number. Since $\eta_s^\nu=P_s^0(h\phi_0^{-1})$ for any $s>0$ and any $\nu=h\mu\in\mathcal{P}_0$, by Hölder's inequality and (2.11), there exist constants $c_1,c_2>0$ such that

$$\begin{aligned}&\big\|\big(\eta_l^\nu-\nu(\phi_0)\big)P_k^0[\phi_0^{-1}-\mu(\phi_0)]\big\|_{L^1(\mu_0)}\\&\le\|P_l^0(h\phi_0^{-1})-\mu_0(h\phi_0^{-1})\|_{L^p(\mu_0)}\|P_k^0\phi_0^{-1}-\mu_0(\phi_0^{-1})\|_{L^{p'}(\mu_0)}\\&\le c_1\|h\phi_0^{-1}\|_{L^p(\mu_0)}\|\phi_0^{-1}\|_{L^{p'}(\mu_0)}\mathrm{e}^{-(\lambda_1-\lambda_0)(k+l)}\\&\le c_2\|h\phi_0^{-1}\|_{L^p(\mu_0)}\mathrm{e}^{-(\lambda_1-\lambda_0)(k+l)},\quad k,l>0,\end{aligned}$$

where we used (2.2) in the last inequality. Similarly, we find a constant $c_3>0$ such that

$$\begin{aligned}|\nu(\phi_0\{\mu(\phi_0)-P_l^0\phi_0^{-1}\})|&\le\|h\phi_0^{-1}\|_{L^p(\mu_0)}\|P_l^0\phi_0^{-1}-\mu_0(\phi_0^{-1})\|_{L^{p'}(\mu_0)}\\&\le c_3\|h\phi_0^{-1}\|_{L^p(\mu_0)}\mathrm{e}^{-(\lambda_1-\lambda_0)l},\quad l>0.\end{aligned}\tag{4.5}$$

Together with (2.20), we find some constants $c_4,t_0>0$ such that

$$\begin{aligned}&\mu_0\left(\left|\frac{1}{t\mathbb{E}^\nu[1_{\{t<\sigma_\tau^B\}}]}\int_0^t\xi_s\,\mathrm{d}s\right|\right)\\&\le\frac{1}{t\mathbb{E}^\nu[1_{\{t<\sigma_\tau^B\}}]}\int_0^t\|\xi_s\|_{L^1(\mu_0)}\,\mathrm{d}s\\&\le\frac{c_4\mathrm{e}^{B(\lambda_0)t}}{t}\int_0^t\int_0^\infty\int_0^\infty\mathrm{e}^{-\lambda_0(k+l)}\big\|[\eta_l^\nu-\nu(\phi_0)]P_k^0[\phi_0^{-1}-\mu(\phi_0)]\big\|_{L^1(\mu_0)}\\&\qquad\qquad\mathbb{P}(S_{t-s}^B\in\mathrm{d}k)\mathbb{P}(S_s^B\in\mathrm{d}l)\mathrm{d}s\\&\quad+\frac{c_4\mathrm{e}^{B(\lambda_0)t}}{t}\int_0^t\int_0^\infty\mathrm{e}^{-\lambda_0 l}\|\nu(\phi_0\{P_l^0[\phi_0^{-1}-\mu(\phi_0)]\})\|_{L^1(\mu_0)}\,\mathbb{P}(S_t^B\in\mathrm{d}l)\mathrm{d}s\\&\le\frac{c_4\mathrm{e}^{B(\lambda_0)t}}{t}\|h\phi_0^{-1}\|_{L^p(\mu_0)}\int_0^t\int_0^\infty\int_0^\infty\mathrm{e}^{-\lambda_1(k+l)}\,\mathbb{P}(S_{t-s}^B\in\mathrm{d}k)\mathbb{P}(S_s^B\in\mathrm{d}l)\mathrm{d}s\\&\quad+\frac{c_4\mathrm{e}^{B(\lambda_0)t}}{t}\|h\phi_0^{-1}\|_{L^p(\mu_0)}\int_0^t\int_0^\infty\mathrm{e}^{-\lambda_1 l}\,\mathbb{P}(S_t^B\in\mathrm{d}l)\mathrm{d}s\\&=2c_4\|h\phi_0^{-1}\|_{L^p(\mu_0)}\mathrm{e}^{-[B(\lambda_1)-B(\lambda_0)]t},\quad t\ge t_0,\end{aligned}\tag{4.6}$$

where we used (3.6) in the second inequality and (1.1) in the last equality. By (3.2), (2.20), (2.19), (2.18), (2.1) and the fact that $\|\phi_m\phi_0^{-1}\|_{L^2(\mu_0)} = 1$ for every $m \in \mathbb{N}$, we find some constants $c_5, c_6, t_0 > 0$ such that

$$\begin{aligned}\mu_0(|A_t|) &\le \frac{c_5}{t}\sum_{m=1}^{\infty}\frac{\|\phi_m\|_\infty \mathrm{e}^{-[B(\lambda_m)-B(\lambda_0)]t}}{B(\lambda_m)-B(\lambda_0)}\mu_0(|\phi_m\phi_0^{-1}|)\\ &\le c_6\mathrm{e}^{-[B(\lambda_1)-B(\lambda_0)]t},\quad t\ge t_0.\end{aligned}$$

Combining this with (4.6) and

$$\mu_0(|\rho_t^{B,\nu}-\tilde\rho_t^{B,\nu}|) \le \mu_0\left(\left|\frac{1}{t\mathbb{E}^\nu[1_{\{t<\sigma_\tau^B\}}]}\int_0^t \xi_s\,\mathrm{d}s\right|\right) + \mu_0(|A_t|),$$

we complete the proof of the desired result. □

With Lemma 4.4 at our disposal, we may obtain upper bound estimates on $\mathbb{W}_2(\tilde\mu_{t,\beta}^{B,\nu}, \mu_{t,\beta}^{B,\nu})$.

Lemma 4.5. For every $p \in (3/2, \infty]$ and every $\beta \in (0, \frac{2}{2d-2\alpha+1})$, there exist constants $t_0, c > 0$ such that

$$\mathbb{W}_2(\mu_{t,\beta}^{B,\nu}, \tilde\mu_{t,\beta}^{B,\nu})^2 \le c\|h\phi_0^{-1}\|_{L^p(\mu_0)}\mathrm{e}^{-[B(\lambda_1)-B(\lambda_0)]t},\quad t\ge t_0,\ \nu = h\mu\in\mathcal{P}_0.$$

Proof. We use D to denote the diameter of M, i.e., $D = \sup\{\rho(x,y) : x, y \in M\}$, which is obviously finite since M is compact. According to Lemmas 4.1 and 4.4, for every $p \in (3/2, \infty]$ and every $\beta \in (0, \frac{2}{2d-2\alpha+1})$, there exist constants $t_0, c > 0$ such that, $\tilde\mu_{t,\beta}^{B,\nu}$ is a probability measure for any $t \ge t_0$ and $\nu \in \mathcal{P}_0$, and

$$\begin{aligned}\mathbb{W}_2(\mu_{t,\beta}^{B,\nu}, \tilde\mu_{t,\beta}^{B,\nu})^2 &\le \frac12 D^2\|\mu_{t,\beta}^{B,\nu}-\tilde\mu_{t,\beta}^{B,\nu}\|_{\mathrm{var}} = \frac12 D^2\mu_0(|\rho_{t,\beta}^{B,\nu}-\tilde\rho_{t,\beta}^{B,\nu}|)\\ &\le c\|h\phi_0^{-1}\|_{L^p(\mu_0)}\mathrm{e}^{-[B(\lambda_1)-B(\lambda_0)]t},\quad t\ge t_0,\ \nu=h\mu\in\mathcal{P}_0,\end{aligned}$$

where in the first equality we used the fact that

$$\|\mu_{t,\beta}^{B,\nu}-\tilde\mu_{t,\beta}^{B,\nu}\|_{\mathrm{var}} = \left\|\frac{\mathrm{d}\mu_{t,\beta}^{B,\nu}}{\mathrm{d}\mu_0}-\frac{\mathrm{d}\tilde\mu_{t,\beta}^{B,\nu}}{\mathrm{d}\mu_0}\right\|_{L^1(\mu_0)} = \|\rho_{t,\beta}^{B,\nu}-\tilde\rho_{t,\beta}^{B,\nu}\|_{L^1(\mu_0)}.$$ □

Next, we estimate the error term $\mathbb{W}_2(\mu_t^{B,\nu}, \mu_{t,\beta}^{B,\nu})$. We need the following inequality borrowed from [29, Theorem A.1] (see also [2, Corollary 4.4]), i.e., for any probability density functions f_1 and f_2 with respect to μ_0 such that $f_1 \vee f_2 > 0$,

$$\mathbb{W}_2(f_1\mu_0, f_2\mu_0)^2 \le 4\int_M \frac{|\nabla(-\mathcal{L}_0)^{-1}(f_2-f_1)|^2}{f_1}\,\mathrm{d}\mu_0. \tag{4.7}$$

Recall the number p_0 introduced in Theorem 1.3, i.e.,

$$p_0 = \frac{6(d+2)}{d+2+12\alpha}\vee\frac32.$$

Lemma 4.6. Let $\beta \in (0, \frac{2}{2d-2\alpha+1})$ and $p \in (p_0, \infty]$. Then there exist constants $c, t_0 > 0$ such that, for every $t \ge t_0$ and every $\nu = h\mu \in \mathcal{P}_0$ with $h\phi_0^{-1} \in L^p(\mu_0)$,

$$t^2\mathbb{W}_2(\mu_t^{B,\nu}, \mu_{t,\beta}^{B,\nu})^2 \le c\|h\phi_0^{-1}\|_{L^p(\mu_0)}^2 t^{-\beta}.$$

Proof. We divide the proof into five steps.

STEP 1. Since $\mu_0(\xi_s)=0$ for any $s>0$ (see (3.10)) and $p_0\ge 3/2$, by (2.10) and (4.6), we find some constants $c_1,c_2,t_0>0$ such that

$$\begin{aligned}
&\left\|\frac{1}{t\mathbb{E}^\nu[1_{\{t<\sigma_\tau^B\}}]}\int_0^t P^0_{t^{-\beta}}\xi_s\,\mathrm{d}s\right\|_\infty \le \frac{1}{t\mathbb{E}^\nu[1_{\{t<\sigma_\tau^B\}}]}\int_0^t \|P^0_{t^{-\beta}}\xi_s\|_\infty\,\mathrm{d}s\\
&\le \frac{1}{t\mathbb{E}^\nu[1_{\{t<\sigma_\tau^B\}}]}\int_0^t \|P^0_{t^{-\beta}}-\mu_0\|_{L^1(\mu_0)\to L^\infty(\mu_0)}\|\xi_s\|_{L^1(\mu_0)}\,\mathrm{d}s\\
&\le c_1\mathrm{e}^{-(\lambda_1-\lambda_0)t^{-\beta}}\{1\wedge t^{-\beta}\}^{-\frac{d+2}{2}}\frac{1}{t\mathbb{E}^\nu[1_{\{t<\sigma_\tau^B\}}]}\int_0^t\|\xi_s\|_{L^1(\mu_0)}\,\mathrm{d}s\\
&\le c_2\|h\phi_0^{-1}\|_{L^p(\mu_0)}t^{\frac{(d+2)\beta}{2}}\mathrm{e}^{-[B(\lambda_1)-B(\lambda_0)]t},\quad t\ge t_0.
\end{aligned}\tag{4.8}$$

According to (2.7) and the definition of A_t in (3.2), we have

$$P^0_{t^{-\beta}}A_t=\frac{1}{t\mathbb{E}^\nu[1_{\{t<\sigma_\tau^B\}}]}\sum_{m=1}^\infty\frac{[\mu(\phi_0)\nu(\phi_m)+\nu(\phi_0)\mu(\phi_m)]\mathrm{e}^{-B(\lambda_m)t}}{[B(\lambda_m)-B(\lambda_0)]\mathrm{e}^{(\lambda_m-\lambda_0)t^{-\beta}}}\phi_m\phi_0^{-1},\quad t>0.$$

Combining this with (2.20), Lemma 4.1 and $B(\lambda_m)\ge B(\lambda_0)$ for each $m\in\mathbb{N}$, we easily deduce that there exist constants $c_3,t_0>0$ such that

$$\|P^0_{t^{-\beta}}A_t\|_\infty\le\|\tilde\rho^{B,\nu}_{t,\beta}\|_\infty\le c_3t^{\frac{(2d-2\alpha+1)\beta}{2}-1},\quad t\ge t_0.\tag{4.9}$$

By Lemma 3.1, Lemma 4.1, (4.8) and (4.9), we obtain

$$\|\rho^{B,\nu}_{t,\beta}\|_\infty\le c_2\|h\phi_0^{-1}\|_{L^p(\mu_0)}t^{\frac{(d+2)\beta}{2}}\mathrm{e}^{-[B(\lambda_1)-B(\lambda_0)]t}+2c_3t^{\frac{(2d-2\alpha+1)\beta}{2}-1},\quad t\ge t_0.$$

Since $\beta\in(0,\frac{2}{2d-2\alpha+1})$, by Lemma 4.1, we can find a constant $t_1>0$ such that for any $t\ge t_1$, $\|\rho^{B,\nu}_{t,\beta}\|_\infty\le\frac12$. So, $1+\rho^{B,\nu}_{t,\beta}\ge\frac12$, $t\ge t_1$. Hence, by (4.7), we have

$$\mathbb{W}_2(\mu^{B,\nu}_t,\mu^{B,\nu}_{t,\beta})^2\le 8\int_M|\nabla(-\mathcal{L}_0)^{-1}(\rho^{B,\nu}_t-\rho^{B,\nu}_{t,\beta})|^2\,\mathrm{d}\mu_0,\quad t\ge t_1.\tag{4.10}$$

So we need to estimate the right hand side of (4.10).

STEP 2. Let $\epsilon=t^{-\beta}$. By the fact that $(-\mathcal{L}_0)^{-1/2}=a\int_0^\infty P^0_{s^2}\,\mathrm{d}s$ with $a=\frac{2}{\sqrt\pi}$, we have

$$\begin{aligned}
\mathrm{J}&:=\|\nabla(-\mathcal{L}_0)^{-1}(\rho^{B,\nu}_t-\rho^{B,\nu}_{t,\beta})\|_{L^2(\mu_0)}=\|(-\mathcal{L}_0)^{-1/2}(\rho^{B,\nu}_t-\rho^{B,\nu}_{t,\beta})\|_{L^2(\mu_0)}\\
&=a\Big\|\int_0^\infty P^0_{s^2}(\rho^{B,\nu}_t-P^0_\epsilon\rho^{B,\nu}_t)\,\mathrm{d}s\Big\|_{L^2(\mu_0)}\\
&=\frac a2\Big\|\int_0^\infty\frac{1}{\sqrt r}(P^0_r\rho^{B,\nu}_t-P^0_{r+\epsilon}\rho^{B,\nu}_t)\,\mathrm{d}r\Big\|_{L^2(\mu_0)}\\
&=\frac a2\Big\|\int_0^\infty\frac{1}{\sqrt r}P^0_r\rho^{B,\nu}_t\,\mathrm{d}r-\int_\epsilon^\infty\frac{1}{\sqrt{r-\epsilon}}P^0_r\rho^{B,\nu}_t\,\mathrm{d}r\Big\|_{L^2(\mu_0)}\\
&=\frac a2\Big\|\int_\epsilon^\infty\Big(\frac{1}{\sqrt{r-\epsilon}}-\frac{1}{\sqrt r}\Big)P^0_r\rho^{B,\nu}_t\,\mathrm{d}r-\int_0^\epsilon\frac{1}{\sqrt r}P^0_r\rho^{B,\nu}_t\,\mathrm{d}r\Big\|_{L^2(\mu_0)}\\
&\le\frac a2(\mathrm{J}_1+\mathrm{J}_2),
\end{aligned}\tag{4.11}$$

where we applied twice the change-of-variables method, and let

$$\mathrm{J}_1:=\int_\epsilon^\infty\Big(\frac{1}{\sqrt{r-\epsilon}}-\frac{1}{\sqrt r}\Big)\|P^0_r\rho^{B,\nu}_t\|_{L^2(\mu_0)}\,\mathrm{d}r,\quad \mathrm{J}_2:=\int_0^\epsilon\frac{1}{\sqrt r}\|P^0_r\rho^{B,\nu}_t\|_{L^2(\mu_0)}\,\mathrm{d}r.$$

Hence it remains to estimate both J_1 and J_2.

<u>STEP 3</u>. As for the estimate on J_1, we claim that, for every $p \in (p_0, \infty]$, there exist constants $c, t_0 > 0$ such that

$$\begin{aligned}\mathrm{J}_{1,1} &:= \int_\epsilon^\infty \Big(\frac{1}{\sqrt{r-\epsilon}} - \frac{1}{\sqrt{r}}\Big)(\|P_r^0 \tilde\rho_t^{B,\nu}\|_{L^2(\mu_0)} + \|P_r^0 A_t\|_{L^2(\mu_0)})\,\mathrm{d}r \\ &\le c\|h\phi_0^{-1}\|_{L^p(\mu_0)} t^{-(1+\beta/2)}, \quad t \ge t_0,\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}\mathrm{J}_{1,2} &:= \int_\epsilon^\infty \Big(\frac{1}{\sqrt{r-\epsilon}} - \frac{1}{\sqrt{r}}\Big)\frac{1}{t\mathbb{E}^\nu[1_{\{t<\sigma_\tau^B\}}]}\Big\|\int_0^t P_r^0 \xi_s\,\mathrm{d}s\Big\|_{L^2(\mu_0)}\,\mathrm{d}r \\ &\le c\mathrm{e}^{-[B(\lambda_1)-B(\lambda_0)]t}\|h\phi_0^{-1}\|_{L^p(\mu_0)} t^{-\beta/2}, \quad t \ge t_0.\end{aligned} \tag{4.13}$$

<u>*Proof of* (4.12)</u>. By the expression of $\tilde\rho_t^{B,\nu}$, A_t in (3.2) and by (2.7),

$$P_r^0 \tilde\rho_t^{B,\nu} = \frac{\mathrm{e}^{-B(\lambda_0)t}}{t\mathbb{E}^\nu[1_{\{t<\sigma_\tau^B\}}]}\sum_{m=1}^\infty \frac{\mu(\phi_0)\nu(\phi_m)+\nu(\phi_0)\mu(\phi_m)}{B(\lambda_m)-B(\lambda_0)}\mathrm{e}^{-(\lambda_m-\lambda_0)r}\phi_m\phi_0^{-1},$$

and

$$P_r^0 A_t = \frac{1}{t\mathbb{E}^\nu[1_{\{t<\sigma_\tau^B\}}]}\sum_{m=1}^\infty \frac{[\mu(\phi_0)\nu(\phi_m)+\nu(\phi_0)\mu(\phi_m)]\mathrm{e}^{-B(\lambda_m)t}}{B(\lambda_m)-B(\lambda_0)}\mathrm{e}^{-(\lambda_m-\lambda_0)r}\phi_m\phi_0^{-1}.$$

Since $\{\phi_m\phi_0^{-1}\}_{m\in\mathbb{N}_0}$ is an eigenbasis of $\mathcal{L}_0$ in $L^2(\mu_0)$, by (2.20), (2.19) and (2.18), we derive that

$$\begin{aligned}\|P_r^0 A_t\|_{L^2(\mu_0)} &\le \big\|P_r^0\tilde\rho_t^{B,\nu}\big\|_{L^2(\mu_0)} \\ &= \frac{\mathrm{e}^{-B(\lambda_0)t}}{t\mathbb{E}^\nu[1_{\{t<\sigma_\tau^B\}}]}\Big(\sum_{m=1}^\infty \frac{[\mu(\phi_0)\nu(\phi_m)+\nu(\phi_0)\mu(\phi_m)]^2}{[B(\lambda_m)-B(\lambda_0)]^2}\mathrm{e}^{-2(\lambda_m-\lambda_0)r}\Big)^{1/2} \\ &\le \frac{c_4}{t}\Big(\sum_{m=1}^\infty \frac{[\mu(\phi_0)\nu(\phi_m)+\nu(\phi_0)\mu(\phi_m)]^2}{(\lambda_m-\lambda_0)^{2\alpha}}\mathrm{e}^{-2(\lambda_m-\lambda_0)r}\Big)^{1/2}, \quad t \ge t_0,\end{aligned} \tag{4.14}$$

for some constant $c_4 > 0$.

Let $\hat h = \mu(\phi_0)h\phi_0^{-1} + \nu(\phi_0)\phi_0^{-1}$. If $p_0 < p < 2 \vee p_0$, which is equivalent to that $p_0 < p < 2$, then

$$\begin{aligned}\|\hat h\|_{L^p(\mu_0)} &\le \mu(\phi_0)\|h\phi_0^{-1}\|_{L^p(\mu_0)} + \nu(\phi_0)\|\phi_0^{-1}\|_{L^p(\mu_0)} \\ &\le \mu(\phi_0)\|h\phi_0^{-1}\|_{L^p(\mu_0)} + \|h\phi_0^{-1}\|_{L^1(\mu_0)}\|\phi_0^{-1}\|_{L^2(\mu_0)} \\ &\le 2\|h\phi_0^{-1}\|_{L^p(\mu_0)},\end{aligned} \tag{4.15}$$

and if $\infty \ge p \ge 2 \vee p_0$, then

$$\|\hat h\|_{L^2(\mu_0)} \le \|h\phi_0^{-1}\|_{L^2(\mu_0)} + \|h\phi_0^{-1}\|_{L^1(\mu_0)} \le 2\|h\phi_0^{-1}\|_{L^p(\mu_0)}, \tag{4.16}$$

since $\mu(\phi_0) \le 1$ and $\|\phi_0^{-1}\|_{L^2(\mu_0)} = 1$. By (2.8),

$$(P_r^0 - \mu_0)\hat h = \sum_{m=1}^\infty [\mu(\phi_0)\nu(\phi_m)+\nu(\phi_0)\mu(\phi_m)]\mathrm{e}^{-(\lambda_m-\lambda_0)r}\phi_m\phi_0^{-1},$$

which immediately leads to

$$\begin{aligned}&\big\|(-\mathcal{L}_0)^{-\alpha}(P_r^0-\mu_0)\hat h\big\|_{L^2(\mu_0)} \\ &= \Big\|\sum_{m=1}^\infty \frac{\mu(\phi_0)\nu(\phi_m)+\nu(\phi_0)\mu(\phi_m)}{(\lambda_m-\lambda_0)^\alpha}\mathrm{e}^{-(\lambda_m-\lambda_0)r}\phi_m\phi_0^{-1}\Big\|_{L^2(\mu_0)}\end{aligned}$$

$$= \Big(\sum_{m=1}^{\infty} \frac{[\mu(\phi_0)\nu(\phi_m) + \nu(\phi_0)\mu(\phi_m)]^2}{(\lambda_m - \lambda_0)^{2\alpha}} \mathrm{e}^{-2(\lambda_m - \lambda_0)r} \Big)^{1/2}. \tag{4.17}$$

Due to (2.11), (4.14), (4.17), (4.15) (resp. (4.16)) and the fact that $(-\mathcal{L}_0)^{-\alpha} = C\int_0^\infty P^0_{s^{1/\alpha}}\,\mathrm{d}s$ for some constant $C > 0$, we have

$$\begin{aligned}
&\mathrm{J}_{1,1} \\
&\leq \frac{c}{t}\int_\epsilon^\infty \Big(\frac{1}{\sqrt{r-\epsilon}} - \frac{1}{\sqrt{r}}\Big) \int_0^\infty \|(P^0_{r+s^{1/\alpha}} - \mu_0)\hat{h}\|_{L^2(\mu_0)}\,\mathrm{d}s\mathrm{d}r \\
&\leq \frac{c}{t}\int_\epsilon^\infty \Big(\frac{1}{\sqrt{r-\epsilon}} - \frac{1}{\sqrt{r}}\Big) \int_0^\infty \|P^0_{r+s^{1/\alpha}} - \mu_0\|_{L^p(\mu_0)\to L^2(\mu_0)}\|\hat{h}\|_{L^p(\mu_0)}\,\mathrm{d}s\mathrm{d}r \\
&\leq \frac{c}{t}\|h\phi_0^{-1}\|_{L^p(\mu_0)}\int_\epsilon^\infty \Big(\frac{1}{\sqrt{r-\epsilon}} - \frac{1}{\sqrt{r}}\Big)\int_0^\infty \mathrm{e}^{-(\lambda_1-\lambda_0)(r+s^{1/\alpha})} \\
&\qquad\qquad\qquad\qquad \times \big[1\wedge(r+s^{1/\alpha})\big]^{-\frac{(d+2)(2-p)}{4p}}\,\mathrm{d}s\mathrm{d}r \\
&\leq \frac{c}{t}\|h\phi_0^{-1}\|_{L^p(\mu_0)}\int_\epsilon^\infty \Big(\frac{1}{\sqrt{r-\epsilon}} - \frac{1}{\sqrt{r}}\Big)\mathrm{d}r\int_0^\infty \mathrm{e}^{-(\lambda_1-\lambda_0)s^{1/\alpha}}(1\wedge s^{1/\alpha})^{-\frac{(d+2)(2-p)}{4p}}\,\mathrm{d}s \\
&\leq c\|h\phi_0^{-1}\|_{L^p(\mu_0)}t^{-(1+\beta/2)}, \quad t \geq t_0,
\end{aligned}$$

where the positive constant c may vary from line to line, and in the last inequality we used the fact that

$$\int_\epsilon^\infty \Big(\frac{1}{\sqrt{r-\epsilon}} - \frac{1}{\sqrt{r}}\Big)\mathrm{d}r = \frac{2\sqrt{\epsilon}}{\sqrt{\pi}}, \quad \int_0^\infty \mathrm{e}^{-(\lambda_1-\lambda_0)s^{1/\alpha}}(1\wedge s^{1/\alpha})^{-\frac{(d+2)(2-p)}{4p}}\,\mathrm{d}s < \infty,$$

since $p > p_0 > 2(d+2)/(d+2+4\alpha)$. We finish the proof of (4.12).

Proof of (4.13). Suppose that $p_0 < p \leq 6$. (Note that $p_0 < 6$.) By Hölder's inequality, (2.12) and (2.2), there exist constants $c_5, c_6 > 0$ such that, for any $k, l > 0$,

$$\begin{aligned}
&\|[\eta_l^\nu - \nu(\phi_0)][P^0_k\phi_0^{-1} - \mu(\phi_0)]\|_{L^2(\mu_0)} \\
&\leq \|P^0_l(h\phi_0^{-1}) - \mu_0(h\phi_0^{-1})\|_{L^q(\mu_0)}\|P^0_k\phi_0^{-1} - \mu(\phi_0)\|_{L^{\frac{2q}{q-2}}(\mu_0)} \\
&\leq \|P^0_l - \mu_0\|_{L^p(\mu_0)\to L^q(\mu_0)}\|h\phi_0^{-1}\|_{L^p(\mu_0)}\|P^0_k(\phi_0^{-1}) - \mu(\phi_0)\|_{L^{\frac{2q}{q-2}}(\mu_0)} \\
&\leq c_5\mathrm{e}^{-(\lambda_1-\lambda_0)(k+l)}(1\wedge l)^{-\frac{(d+2)(q-p)}{2pq}}\|h\phi_0^{-1}\|_{L^p(\mu_0)}\|\phi_0^{-1}\|_{L^{\frac{2q}{q-2}}(\mu_0)} \\
&\leq c_6\mathrm{e}^{-(\lambda_1-\lambda_0)(k+l)}(1\wedge l)^{-\frac{(d+2)(q-p)}{2pq}}\|h\phi_0^{-1}\|_{L^p(\mu_0)}, \quad q \in (6,\infty].
\end{aligned}$$

By (4.5), for every $p \in (p_0, \infty]$, we find a constant $c_7 > 0$ such that

$$|\nu(\phi_0\{\mu(\phi_0) - P^0_l\phi_0^{-1}\})| \leq c_7\mathrm{e}^{-(\lambda_1-\lambda_0)l}\|h\phi_0^{-1}\|_{L^p(\mu_0)}, \quad l > 0.$$

Hence, according to (1.1) and (3.6), we have

$$\begin{aligned}
&\|\xi_s\|_{L^2(\mu_0)} \\
&\leq \|h\phi_0^{-1}\|_{L^p(\mu_0)}\int_0^\infty\int_0^\infty \mathrm{e}^{-\lambda_0(k+l)}\mathrm{e}^{-(\lambda_1-\lambda_0)(k+l)}(1\wedge l)^{-\frac{(d+2)(q-p)}{2pq}} \\
&\qquad\qquad\qquad \mathbb{P}(S^B_{t-s}\in\mathrm{d}k)\mathbb{P}(S^B_s\in\mathrm{d}l) \\
&\quad + \|h\phi_0^{-1}\|_{L^p(\mu_0)}\int_0^\infty \mathrm{e}^{-\lambda_0 l}\mathrm{e}^{-(\lambda_1-\lambda_0)l}\mathbb{P}(S^B_t\in\mathrm{d}l)
\end{aligned}$$

$$\begin{aligned}
&\leq c\|h\phi_0^{-1}\|_{L^p(\mu_0)}\mathrm{e}^{-B(\lambda_1)(t-s)}\int_0^\infty \mathrm{e}^{-\lambda_1 l}(1\wedge l)^{-\frac{(d+2)(q-p)}{2pq}}\mathbb{P}(S_s^B\in \mathrm{d}l)\\
&\quad + c\|h\phi_0^{-1}\|_{L^p(\mu_0)}\mathrm{e}^{-B(\lambda_1)t}\\
&\leq c\|h\phi_0^{-1}\|_{L^p(\mu_0)}\mathrm{e}^{-B(\lambda_1)t}\Big(1+s^{-\frac{(d+2)(q-p)}{2\alpha pq}}\Big)+c\|h\phi_0^{-1}\|_{L^p(\mu_0)}\mathrm{e}^{-B(\lambda_1)t},
\end{aligned}$$

where the positive constant c may vary from line to line. A similar argument as in (3.15) leads to

$$\|P_r^0\xi_s\|_{L^2(\mu_0)}\leq c_8\|h\phi_0^{-1}\|_{L^p(\mu_0)}\mathrm{e}^{-B(\lambda_1)t}\Big(1+s^{-\frac{(d+2)(q-p)}{2\alpha pq}}\Big),\quad r>0, \tag{4.18}$$

for some constant $c_8>0$.

It is easy to see that

$$0\leq\frac{(d+2)(6-p)}{12\alpha p}<1,\quad p\in(p_0,6].$$

Then, for every $p\in(p_0,6]$, there exists $\bar p\in(6,\infty]$ such that

$$\vartheta:=\frac{(d+2)(\bar p-p)}{2\alpha p\bar p}\in(0,1).$$

Hence, by (4.18) and (2.20), we can find constants $c_9,c_{10},t_0>0$ such that

$$\begin{aligned}
\frac{1}{t\mathbb{E}^\nu[1_{\{t<\sigma_\tau^B\}}]}\Big\|\int_0^t P_r^0\xi_s\,\mathrm{d}s\Big\|_{L^2(\mu_0)}&\leq\frac{1}{t\mathbb{E}^\nu[1_{\{t<\sigma_\tau^B\}}]}\int_0^t\|P_r^0\xi_s\|_{L^2(\mu_0)}\,\mathrm{d}s\\
&\leq\frac{c_9\mathrm{e}^{-B(\lambda_1)t}}{t\mathbb{E}^\nu[1_{\{t<\sigma_\tau^B\}}]}\|h\phi_0^{-1}\|_{L^p(\mu_0)}\int_0^t(1+s^{-\vartheta})\,\mathrm{d}s\\
&\leq c_{10}\mathrm{e}^{-[B(\lambda_1)-B(\lambda_0)]t}\|h\phi_0^{-1}\|_{L^p(\mu_0)},\quad t\geq t_0.
\end{aligned}$$

Thus, for every $p\in(p_0,6]$, there exist constants $t_0,c_{11}>0$ such that

$$\begin{aligned}
\mathrm{J}_{1,2}&\leq c_{10}\mathrm{e}^{-[B(\lambda_1)-B(\lambda_0)]t}\|h\phi_0^{-1}\|_{L^p(\mu_0)}\int_\epsilon^\infty\Big(\frac{1}{\sqrt{r-\epsilon}}-\frac{1}{\sqrt r}\Big)\,\mathrm{d}r\\
&\leq c_{11}\mathrm{e}^{-[B(\lambda_1)-B(\lambda_0)]t}\|h\phi_0^{-1}\|_{L^p(\mu_0)}t^{-\beta/2},\quad t\geq t_0.
\end{aligned}$$

Now suppose that $p\in(6,\infty]$. Then there exists constant $c_{12}>0$ such that, for any $k,l>0$,

$$\begin{aligned}
&\|[\eta_l^\nu-\nu(\phi_0)][P_k^0\phi_0^{-1}-\mu(\phi_0)]\|_{L^2(\mu_0)}\\
&\leq\|P_l^0(h\phi_0^{-1})-\mu_0(h\phi_0^{-1})\|_{L^p(\mu_0)}\|P_k^0\phi_0^{-1}-\mu(\phi_0)\|_{L^{\frac{2p}{p-2}}(\mu_0)}\\
&\leq c_{12}\mathrm{e}^{-(\lambda_1-\lambda_0)(k+l)}\|h\phi_0^{-1}\|_{L^p(\mu_0)},
\end{aligned}$$

and a similar argument also leads to (4.13).

Thus, by the definition of $\rho_t^{B,\nu}$, (4.12) and (4.13) imply that, for every $p\in(p_0,\infty]$, there exist constants $c_{13},t_0>0$ such that

$$\mathrm{J}_1\leq c_{13}\|h\phi_0^{-1}\|_{L^p(\mu_0)}t^{-(1+\beta/2)},\quad t\geq t_0. \tag{4.19}$$

<u>Step 4</u>. To estimate J_2, by an analogous argument for (4.12) and (4.13), we have

$$\begin{aligned}
&\int_0^\epsilon\frac{1}{\sqrt r}\big(\|P_r^0\tilde\rho_t^{B,\nu}\|_{L^2(\mu_0)}+\|P_r^0A_t^{B,\nu}\|_{L^2(\mu_0)}\big)\,\mathrm{d}r\\
&\leq\frac{c}{t}\int_0^\epsilon\frac{1}{\sqrt r}\|(-\mathcal{L}_0)^{-\alpha}(P_r^0-\mu_0)\hat h\|_{L^2(\mu_0)}\,\mathrm{d}r
\end{aligned}$$

$$\le \frac{c}{t}\|h\phi_0^{-1}\|_{L^p(\mu_0)}\int_0^\epsilon \frac{\mathrm{d}r}{\sqrt{r}}\int_0^\infty \mathrm{e}^{-(\lambda_1-\lambda_0)s^{1/\alpha}}(1\wedge s^{1/\alpha})^{-\frac{(d+2)(2-p)}{4p}}\,\mathrm{d}s$$
$$\le c\|h\phi_0^{-1}\|_{L^p(\mu_0)}t^{-(1+\beta/2)}, \quad t\ge t_0. \tag{4.20}$$

and

$$\int_0^\epsilon \frac{1}{\sqrt{r}}\frac{1}{t\mathbb{E}^\nu[1_{\{t<\sigma_\tau^B\}}]}\left\|\int_0^t P_r^0\xi_s\mathrm{d}s\right\|_{L^2(\mu_0)}\mathrm{d}r$$
$$\le c\|h\phi_0^{-1}\|_{L^p(\mu_0)}\mathrm{e}^{-[B(\lambda_1)-B(\lambda_0)]t}\int_0^\epsilon \frac{1}{\sqrt{r}}\,\mathrm{d}r$$
$$\le c\|h\phi_0^{-1}\|_{L^p(\mu_0)}\mathrm{e}^{-[B(\lambda_1)-B(\lambda_0)]t}t^{-\beta/2}, \quad t\ge t_0, \tag{4.21}$$

where the constant $c>0$ may vary from line to line. Thus, there exists a constant $c_{14}>0$, such that

$$\mathrm{J}_2\le c_{14}\|h\phi_0^{-1}\|_{L^p(\mu_0)}t^{-(1+\beta/2)}, \quad t\ge t_0. \tag{4.22}$$

STEP 5. Finally, by (4.10), (4.11), (4.19), (4.22), we complete the proof of Lemma 4.6. □

An alternative proof leads to the following result, which improves the rate of convergence in the case when $\alpha\in(1/2,1]$. However, $\alpha=1/2$ seems critical for the approach employed below. We postpone the proof of Remark 4.7 to the end of this subsection.

Remark 4.7. Assume that $\alpha\in(1/2,1]$ and $B\in\mathbf{B}^\alpha$. Let $\beta\in(0,\frac{2}{2d-2\alpha+1})$ and $p\in(p_0,\infty]$. Then there exist constants $c,t_0>0$ such that, for any $t\ge t_0$ and any $\nu=h\mu\in\mathcal{P}_0$ with $h\phi_0^{-1}\in L^p(\mu_0)$,

$$t^2\mathbb{W}_2(\mu_t^{B,\nu},\mu_{t,\beta}^{B,\nu})^2\le c\|h\phi_0^{-1}\|_{L^p(\mu_0)}^2t^{-2\beta}.$$

The next proposition establishes the upper bound in Theorem 1.3.

Proposition 4.8. Let $\alpha\in(0,1]$, $B\in\mathbf{B}^\alpha$ and $p\in(p_0,\infty]$. Then for any $\nu=h\mu\in\mathcal{P}_0$ with $h\phi_0^{-1}\in L^p(\mu_0)$,

$$\limsup_{t\to\infty}\{t^2\mathbb{W}_2(\mu_t^{B,\nu},\mu_0)^2\}\le\frac{1}{\{\mu(\phi_0)\nu(\phi_0)\}^2}\sum_{m=1}^\infty\frac{[\mu(\phi_0)\nu(\phi_m)+\nu(\phi_0)\mu(\phi_m)]^2}{(\lambda_m-\lambda_0)[B(\lambda_m)-B(\lambda_0)]^2}.$$

Proof. By the triangle inequality of $\mathbb{W}_2$, we see that for any $\beta\in(0,\frac{2}{2d-2\alpha+1})$, there exists some constant $t_0>0$ such that $\tilde\mu_{t,\beta}^{B,\nu}\in\mathcal{P}_0$ for every $t\ge t_0$ and

$$t^2\mathbb{W}_2(\mu_t^{B,\nu},\mu_0)^2\le(1+\delta)t^2\mathbb{W}_2(\tilde\mu_{t,\beta}^{B,\nu},\mu_0)^2+2(1+\delta^{-1})t^2\mathbb{W}_2(\mu_{t,\beta}^{B,\nu},\tilde\mu_{t,\beta}^{B,\nu})^2$$
$$+2(1+\delta^{-1})t^2\mathbb{W}_2(\mu_t^{B,\nu},\mu_{t,\beta}^{B,\nu})^2, \quad t\ge t_0,\ \delta>0.$$

According to this and Lemmas 4.3, 4.5 and 4.6, for any $\beta\in(0,\frac{2}{2d-2\alpha+1})$ and any $p\in(p_0,\infty]$, there exist some constants $t_0,c>0$ such that, for every $t\ge t_0$,

$$t^2\mathbb{W}_2(\mu_t^{B,\nu},\mu_0)^2$$
$$\le(1+\delta)\frac{1+ct^{\frac{(2d-2\alpha+1)\beta}{2}-1}}{\{\mu(\phi_0)\nu(\phi_0)\}^2}\sum_{m=1}^\infty\frac{[\mu(\phi_0)\nu(\phi_m)+\nu(\phi_0)\mu(\phi_m)]^2}{(\lambda_m-\lambda_0)[B(\lambda_m)-B(\lambda_0)]^2}\mathrm{e}^{-2(\lambda_m-\lambda_0)t^{-\beta}}$$
$$+2c(1+\delta^{-1})t^2\|h\phi_0^{-1}\|_{L^p(\mu_0)}\mathrm{e}^{-[B(\lambda_1)-B(\lambda_0)]t}+2c(1+\delta^{-1})\|h\phi_0^{-1}\|_{L^p(\mu_0)}^2t^{-\beta}.$$

Since $\|h\phi_0^{-1}\|_{L^p(\mu_0)} < \infty$, by letting $t \to \infty$ first and then $\delta \to 0$, we have

$$\lim_{t\to\infty}\{t^2\mathbb{W}_2(\mu_t^{B,\nu},\mu_0)^2\} \le \frac{1}{\{\mu(\phi_0)\nu(\phi_0)\}^2}\sum_{m=1}^{\infty}\frac{[\mu(\phi_0)\nu(\phi_m)+\nu(\phi_0)\mu(\phi_m)]^2}{(\lambda_m-\lambda_0)[B(\lambda_m)-B(\lambda_0)]^2}.$$

We finish the proof. □

To end this subsection, we present the proof of Remark 4.7.

Proof of Remark 4.7. We use the same notations as in Lemma 4.6. The positive constant c used below may vary from line to line. By the definition of $\rho_t^{B,\nu}, \rho_{t,\beta}^{B,\nu}$ and (3.2), we have

$$\mathrm{J} = \|\nabla(-\mathcal{L}_0)^{-1}(\rho_t^{B,\nu} - \rho_{t,\beta}^{B,\nu})\|_{L^2(\mu_0)} \le \tilde{\mathrm{J}}_1 + \tilde{\mathrm{J}}_2 + \tilde{\mathrm{J}}_3, \tag{4.23}$$

where

$$\begin{aligned}
\tilde{\mathrm{J}}_1 &:= \|\nabla(-\mathcal{L}_0)^{-1}(\tilde\rho_t^{B,\nu} - \tilde\rho_{t,\beta}^{B,\nu})\|_{L^2(\mu_0)},\\
\tilde{\mathrm{J}}_2 &:= \|\nabla(-\mathcal{L}_0)^{-1}(A_t - P^0_{t^{-\beta}}A_t)\|_{L^2(\mu_0)},\\
\tilde{\mathrm{J}}_3 &:= \frac{1}{t\mathbb{E}^\nu[1_{\{t<\sigma_\tau^B\}}]}\int_0^t \|\nabla(-\mathcal{L}_0)^{-1}(\xi_s - P^0_{t^{-\beta}}(\xi_s))\|_{L^2(\mu_0)}\,\mathrm{d}s.
\end{aligned}$$

So it suffices to estimate the right hand side of (4.23).

(1) Firstly, we estimate $\tilde{\mathrm{J}}_1 + \tilde{\mathrm{J}}_2$. Noting that

$$\tilde\rho_t^{B,\nu} - \tilde\rho_{t,\beta}^{B,\nu} = \tilde\rho_t^{B,\nu} - P^0_{t^{-\beta}}\tilde\rho_t^{B,\nu} = \int_0^{t^{-\beta}}(-\mathcal{L}_0)P_r^0\tilde\rho_t^{B,\nu}\,\mathrm{d}r,$$

we have

$$\begin{aligned}
\tilde{\mathrm{J}}_1 &= \|\nabla(-\mathcal{L}_0)^{-1}(\tilde\rho_t^{B,\nu} - \tilde\rho_{t,\beta}^{B,\nu})\|_{L^2(\mu_0)} = \|(-\mathcal{L}_0)^{-1/2}(\tilde\rho_t^{B,\nu} - \tilde\rho_{t,\beta}^{B,\nu})\|_{L^2(\mu_0)}\\
&= \Big\|\int_0^\epsilon(-\mathcal{L}_0)^{1/2}P_r^0\tilde\rho_t^{B,\nu}\,\mathrm{d}r\Big\|_{L^2(\mu_0)} \le \int_0^\epsilon \big\|(-\mathcal{L}_0)^{1/2}P_r^0\tilde\rho_t^{B,\nu}\big\|_{L^2(\mu_0)}\,\mathrm{d}r.
\end{aligned}\tag{4.24}$$

By the expression of $\tilde\rho_t^{B,\nu}$ in (3.2) and (2.7),

$$(-\mathcal{L}_0)^{1/2}P_r^0\tilde\rho_t^{B,\nu} = a_t\sum_{m=1}^{\infty}\frac{\mu(\phi_0)\nu(\phi_m)+\nu(\phi_0)\mu(\phi_m)}{B(\lambda_m)-B(\lambda_0)}\sqrt{\lambda_m-\lambda_0}\,\mathrm{e}^{-(\lambda_m-\lambda_0)r}\phi_m\phi_0^{-1},$$

where $a_t := \frac{\mathrm{e}^{-B(\lambda_0)t}}{t\mathbb{E}^\nu[1_{\{t<\sigma_\tau^B\}}]}$. Since $(\phi_m\phi_0^{-1})_{m\in\mathbb{N}_0}$ is an eigenbasis of $\mathcal{L}_0$ in $L^2(\mu_0)$, by (2.20), (2.19) and (2.18), we derive that

$$\begin{aligned}
&\big\|(-\mathcal{L}_0)^{1/2}P_r^0\tilde\rho_t^{B,\nu}\big\|_{L^2(\mu_0)}\\
&= a_t\Big(\sum_{m=1}^{\infty}\frac{[\mu(\phi_0)\nu(\phi_m)+\nu(\phi_0)\mu(\phi_m)]^2}{[B(\lambda_m)-B(\lambda_0)]^2}(\lambda_m-\lambda_0)\mathrm{e}^{-2(\lambda_m-\lambda_0)r}\|\phi_m\phi_0^{-1}\|^2_{L^2(\mu_0)}\Big)^{1/2}\\
&\le \frac{c}{t}\Big(\sum_{m=1}^{\infty}\frac{[\mu(\phi_0)\nu(\phi_m)+\nu(\phi_0)\mu(\phi_m)]^2}{(\lambda_m-\lambda_0)^{2\alpha-1}}\mathrm{e}^{-2(\lambda_m-\lambda_0)r}\Big)^{1/2},\quad t\ge t_0,
\end{aligned}\tag{4.25}$$

for some constant $c>0$. By a similar argument as $\tilde{\mathrm{J}}_1$ and (4.25), we have

$$\tilde{\mathrm{J}}_2 \le \int_0^\varepsilon \|(-\mathcal{L}_0)^{\frac12}P_r^0A_t\|_{L^2(\mu_0)}\,\mathrm{d}r, \tag{4.26}$$

and

$$\big\|(-\mathcal{L}_0)^{1/2}P_r^0A_t\big\|_{L^2(\mu_0)}$$

$$= \frac{1}{t\mathbb{E}^\nu[1_{\{t<\sigma_\tau^B\}}]}\Big(\sum_{m=1}^\infty \frac{[\mu(\phi_0)\nu(\phi_m)+\nu(\phi_0)\mu(\phi_m)]^2 \mathrm{e}^{-2B(\lambda_m)t}}{[B(\lambda_m)-B(\lambda_0)]^2\mathrm{e}^{2(\lambda_m-\lambda_0)r}(\lambda_m-\lambda_0)^{-1}}\|\phi_m\phi_0^{-1}\|^2_{L^2(\mu_0)}\Big)^{1/2}$$

$$\le \frac{c}{t}\Big(\sum_{m=1}^\infty \frac{[\mu(\phi_0)\nu(\phi_m)+\nu(\phi_0)\mu(\phi_m)]^2}{(\lambda_m-\lambda_0)^{2\alpha-1}}\mathrm{e}^{-2(\lambda_m-\lambda_0)r}\Big)^{1/2}, \quad t\ge t_0, \tag{4.27}$$

for some constant $c>0$.

Recall that $\hat h=\mu(\phi_0)h\phi_0^{-1}+\nu(\phi_0)\phi_0^{-1}$. Since

$$(P_r^0-\mu_0)\hat h=\sum_{m=1}^\infty[\mu(\phi_0)\nu(\phi_m)+\nu(\phi_0)\mu(\phi_m)]\mathrm{e}^{-(\lambda_m-\lambda_0)r}\phi_m\phi_0^{-1},$$

by (2.7), we immediately have

$$\begin{aligned}&\big\|(-\mathcal{L}_0)^{1/2-\alpha}(P_r^0-\mu_0)\hat h\big\|_{L^2(\mu_0)}\\ &\quad=\Big(\sum_{m=1}^\infty \frac{[\mu(\phi_0)\nu(\phi_m)+\nu(\phi_0)\mu(\phi_m)]^2}{(\lambda_m-\lambda_0)^{2\alpha-1}}\mathrm{e}^{-2(\lambda_m-\lambda_0)r}\Big)^{1/2}.\end{aligned} \tag{4.28}$$

Thus, combining (4.24), (4.25) (4.26), (4.27) and (4.28), we obtain

$$\tilde{\mathrm{J}}_1+\tilde{\mathrm{J}}_2\le\frac{c}{t}\int_0^\epsilon\big\|(-\mathcal{L}_0)^{1/2-\alpha}(P_r^0-\mu_0)\hat h\big\|_{L^2(\mu_0)}\,\mathrm{d}r, \quad t\ge t_0. \tag{4.29}$$

Suppose that $p_0<p<2\vee p_0$. By (2.11), (4.15) and the fact that $(-\mathcal{L}_0)^{1/2-\alpha}=c\int_0^\infty P^0_{s^{2/(2\alpha-1)}}\,\mathrm{d}s$, we have

$$\begin{aligned}&\tilde{\mathrm{J}}_1+\tilde{\mathrm{J}}_2\\ &\le\frac{c}{t}\int_0^\epsilon\Big\|\int_0^\infty(P^0_{r+s^{2/(2\alpha-1)}}-\mu_0)\hat h\,\mathrm{d}s\Big\|_{L^2(\mu_0)}\mathrm{d}r\\ &\le\frac{c}{t}\int_0^\epsilon\int_0^\infty\big\|(P^0_{r+s^{2/(2\alpha-1)}}-\mu_0)\hat h\big\|_{L^2(\mu_0)}\,\mathrm{d}s\mathrm{d}r\\ &\le\frac{c}{t}\int_0^\epsilon\int_0^\infty\|P^0_{r+s^{2/(2\alpha-1)}}-\mu_0\|_{L^p(\mu_0)\to L^2(\mu_0)}\|\hat h\|_{L^p(\mu_0)}\,\mathrm{d}s\mathrm{d}r\\ &\le\frac{c}{t}\|h\phi_0^{-1}\|_{L^p(\mu_0)}\int_0^\epsilon\mathrm{e}^{-(\lambda_1-\lambda_0)r}\mathrm{d}r\int_0^\infty\mathrm{e}^{-(\lambda_1-\lambda_0)s^{2/(2\alpha-1)}}(1\wedge s^{2/(2\alpha-1)})^{-\frac{(d+2)(2-p)}{4p}}\,\mathrm{d}s\\ &\le c\|h\phi_0^{-1}\|_{L^p(\mu_0)}t^{-(1+\beta)}, \quad t\ge t_0,\end{aligned} \tag{4.30}$$

for some constant $c>0$, where we applied the fact that

$$\int_0^\infty\mathrm{e}^{-(\lambda_1-\lambda_0)s^{2/(2\alpha-1)}}(1\wedge s^{2/(2\alpha-1)})^{-\frac{(d+2)(2-p)}{4p}}\mathrm{d}s<\infty,$$

since

$$0\le\frac{2}{2\alpha-1}\frac{(d+2)(2-p)}{4p}<1, \quad p\in(p_0,2],\ \alpha\in(1/2,1].$$

Suppose that $p_0\vee 2\le p\le\infty$. By an analogous argument, we also have (4.30).

(2) Secondly, we turn to estimate $\tilde{\mathrm{J}}_3$. By (4.21) and a similar argument as in (4.11) and (4.13),

$$\begin{aligned}\tilde{\mathrm{J}}_3&\le\frac{c}{t\mathbb{E}^\nu[1_{\{t<\sigma_\tau^B\}}]}\int_0^t\Big\|\int_\epsilon^\infty\Big(\frac{1}{\sqrt{r-\epsilon}}-\frac{1}{\sqrt r}\Big)P_r^0\xi_s\,\mathrm{d}r-\int_0^\epsilon\frac{1}{\sqrt r}P_r^0\xi_s\,\mathrm{d}r\Big\|_{L^2(\mu_0)}\mathrm{d}s\\ &\le\frac{c}{t\mathbb{E}^\nu[1_{\{t<\sigma_\tau^B\}}]}\int_0^t\int_\epsilon^\infty\Big(\frac{1}{\sqrt{r-\epsilon}}-\frac{1}{\sqrt r}\Big)\|P_r^0\xi_s\|_{L^2(\mu_0)}\,\mathrm{d}r\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&+\frac{c}{t\mathbb{E}^\nu[1_{\{t<\sigma_\tau^B\}}]}\int_0^t\int_0^\epsilon \frac{1}{\sqrt{r}}\|P_r^0\xi_s\|_{L^2(\mu_0)}\,\mathrm{d}r\mathrm{d}s\\
&\le c\|h\phi_0^{-1}\|_{L^p(\mu_0)}\mathrm{e}^{-[B(\lambda_1)-B(\lambda_0)]t}\left[\int_\epsilon^\infty\left(\frac{1}{\sqrt{r-\epsilon}}-\frac{1}{\sqrt{r}}\right)\mathrm{d}r+\int_0^\epsilon\frac{1}{\sqrt{r}}\,\mathrm{d}r\right]\\
&\le c\|h\phi_0^{-1}\|_{L^p(\mu_0)}\mathrm{e}^{-[B(\lambda_1)-B(\lambda_0)]t}t^{-\beta/2},\quad t\ge t_0, \qquad (4.31)
\end{aligned}
$$

for some constant $c>0$.

(3) Finally, combining (4.10), (4.23), (4.30) and (4.31) together, we complete the proof. □

4.2. **Lower bounds.** In this subsection, we present the proof of the lower bound in Theorem 1.3. Note that, by Lemma 4.1, for any $\beta\in(0,\frac{2}{2d-2\alpha+1})$, there exists some constant $t_0>0$ such that $\tilde\mu_{t,\beta}^{B,\nu}\in\mathcal{P}_0$ for every $t\ge t_0$ and every $\nu\in\mathcal{P}_0$.

Set $f_{t,\beta}^B:=(-\mathcal{L}_0)^{-1}\tilde\rho_{t,\beta}^{B,\nu}$. The next lemma establishes useful regularity estimates for $f_{t,\beta}^B$ and its gradient.

Lemma 4.9. For any $\alpha\in(0,1]$, $\beta>0$, there exists a constant $c>0$ such that

$$\|f_{t,\beta}^B\|_\infty+\|\mathcal{L}_0f_{t,\beta}^B\|_\infty+\|\nabla f_{t,\beta}^B\|_\infty\le ct^{\frac{(5d+2-4\alpha)\beta}{4}-1},\quad t\ge1,\,\nu\in\mathcal{P}_0.$$

Proof. By (2.7) and (3.2), we have, for every $t\ge1$,

$$
\begin{aligned}
&(-\mathcal{L}_0)f_{t,\beta}^B\\
&=\frac{1}{t\mathbb{E}^\nu[1_{\{t<\sigma_\tau^B\}}]}\sum_{m=1}^\infty\frac{[\mu(\phi_0)\nu(\phi_m)+\nu(\phi_0)\mu(\phi_m)]\mathrm{e}^{-B(\lambda_0)t}}{B(\lambda_m)-B(\lambda_0)}\mathrm{e}^{-(\lambda_m-\lambda_0)t^{-\beta}}\phi_m\phi_0^{-1},
\end{aligned}
$$

and

$$f_{t,\beta}^B=\frac{1}{t\mathbb{E}^\nu[1_{\{t<\sigma_\tau^B\}}]}\sum_{m=1}^\infty\frac{[\mu(\phi_0)\nu(\phi_m)+\nu(\phi_0)\mu(\phi_m)]\mathrm{e}^{-B(\lambda_0)t}}{(\lambda_m-\lambda_0)[B(\lambda_m)-B(\lambda_0)]}\mathrm{e}^{-(\lambda_m-\lambda_0)t^{-\beta}}\phi_m\phi_0^{-1}.$$

Combining these identities with (2.1), (2.13), (2.20), (2.19), (2.18) and the fact that

$$|\mu(\phi_0)\nu(\phi_m)+\nu(\phi_0)\mu(\phi_m)|\le\|\phi_0\|_\infty+\|\phi_m\|_\infty\le c_1m^{\frac12},\quad m\in\mathbb{N},$$

for some constant $c_1>0$, we find constants $c_2,c_3,c_4>0$ such that

$$
\begin{aligned}
t\{\|f_{t,\beta}^B\|_\infty+\|\mathcal{L}_0f_{t,\beta}^B\|_\infty\}&\le c_2\sum_{m=1}^\infty\frac{\mathrm{e}^{-(\lambda_m-\lambda_0)t^{-\beta}}}{B(\lambda_m)-B(\lambda_0)}m^{\frac{d+1}{d}}\\
&\le c_3\int_0^\infty \mathrm{e}^{-\alpha_0^{-1}s^{\frac2d}t^{-\beta}}s^{\frac{d+1-2\alpha}{d}}\,\mathrm{d}s\\
&\le c_4t^{\frac{(2d+1-2\alpha)\beta}{2}},\quad t\ge1,
\end{aligned}
$$

where α_0 is from (2.1). By a similar argument as above, applying (2.14), we deduce that there exist constants $c_5,c_6,c_7>0$ such that

$$
\begin{aligned}
t\|\nabla f_{t,\beta}^B\|_\infty&\le c_5\sum_{m=1}^\infty\frac{\mathrm{e}^{-(\lambda_m-\lambda_0)t^{-\beta}}}{[B(\lambda_m)-B(\lambda_0)](\lambda_m-\lambda_0)}m^{\frac{3d+4}{2d}}\\
&\le c_6\sum_{m=1}^\infty \mathrm{e}^{-\alpha_0^{-1}m^{\frac2d}t^{-\beta}}m^{\frac{3d+4}{2d}-\frac{2(1+\alpha)}{d}}\\
&\le c_7t^{\frac{\beta(5d-4\alpha)}{4}},\quad t\ge1.
\end{aligned}
$$

The proof is completed. □

In the following lemma, we present the lower bound estimate for $\mathbb{W}_2(\tilde{\mu}_{t,\beta}^{B,\nu},\mu_0)$. With Lemma 4.9 in hand, the proof can be achieved by the same approach employed for [27, Lemma 4.2]. So we omit the details here.

Lemma 4.10. For any $\alpha\in(0,1]$, $\beta\in(0,\frac{1}{4(5d+2-4\alpha)}]$, there exist constants $c,t_0>0$ such that

$$t^2\mathbb{W}_2(\tilde{\mu}_{t,\beta}^{B,\nu},\mu_0)^2\geq J_\beta-ct^{-\frac{1}{4}},\quad t\geq t_0,\,\nu\in\mathcal{P}_0,$$

where

$$J_\beta:=\frac{1}{\{\mu(\phi_0)\nu(\phi_0)\}^2}\sum_{m=1}^{\infty}\frac{[\mu(\phi_0)\nu(\phi_m)+\nu(\phi_0)\mu(\phi_m)]^2}{(\lambda_m-\lambda_0)[B(\lambda_m)-B(\lambda_0)]^2}\mathrm{e}^{-2(\lambda_m-\lambda_0)t^{-\beta}},\quad t>0,\,\nu\in\mathcal{P}_0. \tag{4.32}$$

The main result of this subsection, which is the lower bound in Theorem 1.3.

Proposition 4.11. Let $\alpha\in(0,1]$, $B\in\mathbf{B}^\alpha$ and $p\in(p_0,\infty]$. Then for any $\nu=h\mu\in\mathcal{P}_0$ with $h\phi_0^{-1}\in L^p(\mu_0)$,

$$\liminf_{t\to\infty}\{t^2\mathbb{W}_2(\mu_t^{B,\nu},\mu_0)^2\}\geq\frac{1}{\{\mu(\phi_0)\nu(\phi_0)\}^2}\sum_{m=1}^{\infty}\frac{[\mu(\phi_0)\nu(\phi_m)+\nu(\phi_0)\mu(\phi_m)]^2}{(\lambda_m-\lambda_0)[B(\lambda_m)-B(\lambda_0)]^2}.$$

Proof. By the triangle inequality of $\mathbb{W}_2$, we have for any $\beta\in(0,\frac{2}{2d-2\alpha+1})$, there exists a constant $t_0>0$ such that $\tilde{\mu}_{t,\beta}^{B,\nu}\in\mathcal{P}_0$ for every $t\geq t_0$ and

$$\mathbb{W}_2(\mu_t^{B,\nu},\mu_0)\geq\mathbb{W}_2(\tilde{\mu}_{t,\beta}^{B,\nu},\mu_0)-\mathbb{W}_2(\tilde{\mu}_{t,\beta}^{B,\nu},\mu_{t,\beta}^{B,\nu})-\mathbb{W}_2(\mu_{t,\beta}^{B,\nu},\mu_t^{B,\nu}),\quad t\geq t_0. \tag{4.33}$$

Combining Lemmas 4.5, 4.10 and 4.6 together, for any $\beta\in(0,\frac{1}{4(5d+2-4\alpha)}]$ and any $p\in(p_0,\infty]$, we can find some constants $c,t_0>0$ such that, for every $\nu=h\mu\in\mathcal{P}_0$ with $h\phi_0^{-1}\in L^p(\mu_0)$ and every $t\geq t_0$,

$$\begin{aligned}&t\mathbb{W}_2(\tilde{\mu}_{t,\beta}^{B,\nu},\mu_{t,\beta}^{B,\nu})\leq ct\mathrm{e}^{-[B(\lambda_1)-B(\lambda_0)]t/2}\|h\phi_0^{-1}\|_{L^p(\mu_0)}^{\frac{1}{2}},\\&t\mathbb{W}_2(\tilde{\mu}_{t,\beta}^{B,\nu},\mu_0)\geq\big([J_\beta-ct^{-\frac{1}{4}}]^+\big)^{\frac{1}{2}},\\&t\mathbb{W}_2(\mu_{t,\beta}^{B,\nu},\mu_t^{B,\nu})\leq c\|h\phi_0^{-1}\|_{L^p(\mu_0)}t^{-\frac{\beta}{2}},\end{aligned}$$

where J_β is defined in (4.32). Substituting these estimates into (4.33), we immediately obtain that

$$\begin{aligned}t\mathbb{W}_2(\mu_t^{B,\nu},\mu_0)\geq{}&\big([J_\beta-ct^{-\frac{1}{4}}]^+\big)^{\frac{1}{2}}-ct\mathrm{e}^{-[B(\lambda_1)-B(\lambda_0)]t/2}\|h\phi_0^{-1}\|_{L^p(\mu_0)}^{\frac{1}{2}}\\&-c\|h\phi_0^{-1}\|_{L^p(\mu_0)}t^{-\frac{\beta}{2}},\quad t\geq t_0.\end{aligned}$$

Since $\|h\phi_0^{-1}\|_{L^p(\mu_0)}<\infty$, by letting $t\to\infty$, we prove the desired result. □

Appendix. Let $B\in\mathbf{B}^\alpha$ for some $\alpha\in(0,1]$. Recall that $\mu_0=\phi_0^2\mu$, and μ_0 is called a quasi-ergodic distribution of the B-subordinated Dirichlet diffusion process $(X_t^B)_{t\geq0}$ if for every $\nu\in\mathcal{P}$ supported on $\mathring{M}$ and every Borel set $E\subset M$,

$$\lim_{t\to\infty}\mathbb{E}^\nu\Big[\frac{1}{t}\int_0^t 1_E(X_s^B)\,\mathrm{d}s\Big|\sigma_\tau^B>t\Big]=\mu_0(E).$$

The following result implies that μ_0 is the unique quasi-ergodic distribution of $(X_t^B)_{t\geq0}$.

Proposition A.1. Let $\alpha \in (0,1]$ and $B \in \mathbf{B}^\alpha$. Then, for every $\nu \in \mathcal{P}$ supported on $\mathring{M}$,

$$\lim_{t\to\infty} \|\mu_t^{B,\nu} - \mu_0\|_{\rm var} = 0,$$

and

$$\lim_{t\to\infty} \mathbb{E}^\nu\Big[\frac{1}{t}\int_0^t f(X_s^B)\,\mathrm{d}s\Big|\sigma_\tau^B > t\Big] = \int_M f\,\mathrm{d}\mu_0, \quad f \in \mathcal{B}_b(M).$$

Proof. The proof is a direct application of [8, Theorem 2.1]. So we only need to check the assumptions (A_1) and (A_2) on [8, page 185]. (A_1) is clearly satisfied. By (2.15), Fubini's theorem, $\mu(\phi_m^2) = 1$ for each $m \in \mathbb{N}_0$, (2.1) and (2.18),

$$\begin{aligned}\int_M p_t^{D,B}(x,x)\,\mu(\mathrm{d}x) &= \int_M \sum_{m=0}^\infty \mathrm{e}^{-B(\lambda_m)t}\phi_m^2(x)\,\mu(\mathrm{d}x) = \sum_{m=0}^\infty \mathrm{e}^{-B(\lambda_m)t}\\ &= \mathrm{e}^{-B(\lambda_0)t}\Big(1+\sum_{m=1}^\infty \mathrm{e}^{-[B(\lambda_m)-B(\lambda_0)]t}\Big)\\ &\le c_1\Big(1+\sum_{m=1}^\infty \mathrm{e}^{-c_2 t m^{2\alpha/d}}\Big) < \infty, \quad t>0,\end{aligned}$$

for some constants $c_1, c_2 > 0$. By (2.17), (2.5), a similar argument as in (3.19), there exist constants $c_3, c_4 > 0$ such that

$$\begin{aligned}\|P_t^{D,B} f\|_{L^\infty(\mu)} &= \Big\|\int_0^\infty P_u^D f\,\mathbb{P}(S_t^B \in \mathrm{d}u)\Big\|_{L^\infty(\mu)} \le \int_0^\infty \|P_u^D f\|_{L^\infty(\mu)}\,\mathbb{P}(S_t^B\in\mathrm{d}u)\\ &\le c_3\|f\|_{L^2(\mu)}\mathbb{E}\Big[\mathrm{e}^{-\lambda_0 S_t^B}(1\wedge S_t^B)^{-\frac{d}{4}}\Big] \le c_3\|f\|_{L^2(\mu)}\mathbb{E}\Big[(1\wedge S_t^B)^{-\frac{d}{4}}\Big]\\ &\le c_4\|f\|_{L^2(\mu)}(1+t^{-\frac{d}{4\alpha}}), \quad t>0,\ f\in L^2(\mu),\end{aligned}$$

i.e., $\|P_t^{D,B}\|_{L^2(\mu)\to L^\infty(\mu)} \le c_4(1+t^{-\frac{d}{4\alpha}})$, $t>0$. Thus, (A_2) is satisfied. □

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