An Andrews–Gordon Type Identity for Overpartitions

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Abstract. In 1961, Gordon found a combinatorial generalization of the Rogers–Ramanujan identities, which has been called the Rogers–Ramanujan–Gordon theorem. In 1974, Andrews derived an identity which can be considered as the generating function counterpart of the Rogers–Ramanujan–Gordon theorem, and it has been called the Andrews–Gordon identity. The Andrews–Gordon identity is an analytic generalization of the Rogers–Ramanujan identities with odd moduli. In 1979, Bressoud obtained a Rogers–Ramanujan–Gordon type theorem and the corresponding Andrews–Gordon type identity with even moduli. In 2004, Lovejoy proved two overpartition analogues of two special cases of the Rogers–Ramanujan–Gordon theorem. In 2012, Chen, Sang and Shi found the overpartition analogue of the Rogers–Ramanujan–Gordon theorem in general cases and the corresponding Andrews-Gordon type identity with even moduli. In 2008, Corteel, Lovejoy, and Mallet found an overpartition analogue of a special case of Bressoud's theorem of the Rogers–Ramanujan–Gordon type. In 2012, Chen, Sang and Shi obtained the overpartition analogue of Bressoud's theorem in the general case. In this paper, we obtain an Andrew–Gordon type identity corresponding to this overpartition theorem with odd moduli by using the Gordon marking representation of an overpartition.

Keywords: Rogers–Ramanujan–Gordon Theorem, Andrews–Gordon identity, overpartition, Gordon marking

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1 Introduction

In this paper, we present an Andrews–Gordon type identity for overpartition with odd moduli. This identity can be considered as the generating function form of an overpartition analogue of Bressoud's theorem of the Rogers–Ramanujan–Gordon type.

Let us give an overview of some definitions. A partition λ of a positive integer n is a non-increasing sequence of positive integers $\lambda_1 \geq \cdots \geq \lambda_s > 0$ such that $n = \lambda_1 + \cdots + \lambda_s$.

An overpartition λ of a positive integer n is also a non-increasing sequence of positive integers $\lambda_1 \geq \cdots \geq \lambda_s > 0$ such that $n = \lambda_1 + \cdots + \lambda_s$ and the first occurrence of each integer may be overlined, see Corteel and Lovejoy [8]. Given a partition or an overpartition λ, let $f_l(\lambda)$ ($f_{\bar{l}}(\lambda)$) denote the number of occurrences of non-overlined (overlined) l in λ. Let $V_\lambda(l)$ denote the number of overlined parts in λ that are less than or equal to l.

We adopt the common notation in q-series as used in Andrews [3]. Let

$$(a)_\infty = (a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i),$$

and

$$(a_1,\ldots,a_k;q)_\infty = (a_1;q)_\infty \cdots (a_k;q)_\infty.$$

We also write

$$(a)_n = (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

Gordon [10] found the following combinatorial generalization of the Rogers–Ramanujan identities, which has been called the Rogers–Ramanujan–Gordon theorem, see Andrews [1].

Theorem 1.1 *For $k \geq i \geq 1$, let $B_{k,i}(n)$ denote the number of partitions of n of the form $\lambda_1 + \lambda_2 + \cdots + \lambda_s$, where $\lambda_j \geq \lambda_{j+1}$, $\lambda_i - \lambda_{i+2} \geq 2$ and at most $i-1$ of the λ_j are equal to 1. Let $A_{k,i}(n)$ denote the number of partitions of n into parts not congruent to $0, \pm i$ modulo $2k+1$. Then for all $n \geq 0$, we have*

$$A_{k,i}(n) = B_{k,i}(n).$$

In 1974, Andrews [2] derived an identity which can be considered as the generating function counterpart of the Rogers–Ramanujan–Gordon theorem. It has been called the Andrews–Gordon identity, and it is an analytic generalization of the Rogers–Ramanujan identities with odd moduli.

Theorem 1.2 *For $k \geq i \geq 1$, we have*

$$\sum_{N_1 \geq N_2 \geq \cdots \geq N_{k-1} \geq 0} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_i+\cdots+N_{k-1}}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}} = \frac{(q^i, q^{2k+1-i}, q^{2k+1}; q^{2k+1})_\infty}{(q)_\infty}. \tag{1.1}$$

Andrews showed that both sides of (1.1) satisfy the same recurrence relation with the same initial condition. In 2009, Kurşungöz [11] proved that the sum on the left-hand side of (1.1) can be viewed as the generating function for $B_{k,i}(n)$ by using the Gordon markings of partitions.

Bressoud [4, 5] obtained a Rogers–Ramanujan–Gordon type theorem and the corresponding Andrews–Gordon type identity with even moduli.

Theorem 1.3 *For $k \geq i \geq 1$, let $\widetilde{B}_{k,i}(n)$ denote the number of partitions of n of the form $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_s$, such that*

(i) $f_1(\lambda) \leq i-1$,

(ii) $f_l(\lambda)+f_{l+1}(\lambda)\leq k-1$,

(iii) *if the equality in condition (ii) is attained at* l, *then* $lf_l(\lambda)+(l+1)f_{l+1}(\lambda)\equiv i-1 \pmod 2$.

Let $\widetilde{A}_{k,i}(n)$ *denote the number of partitions of* n *whose parts are not congruent to* $0,\pm i$ *modulo* $2k$. *Then for all* $n\geq 0$, *we have*

$$\widetilde{A}_{k,i}(n)=\widetilde{B}_{k,i}(n).$$

The generating function form of the above theorem can be stated as follows.

Theorem 1.4 *For* $k\geq i\geq 1$, *we have*

$$\sum_{N_1\geq N_2\geq\cdots\geq N_{k-1}\geq 0}\frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_{i+1}+\cdots+N_{k-1}}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q^2;q^2)_{N_{k-1}}}=\frac{(q^i,q^{2k-i},q^{2k};q^{2k})_\infty}{(q)_\infty}. \tag{1.2}$$

In 2004, Lovejoy [12] obtained the overpartition analogues of Theorem 1.1 for $i=k$ and $i=1$.

Theorem 1.5 *Let* $\overline{B}_k(n)$ *denote the number of overpartitions of* n *of the form* $\lambda_1+\lambda_2+\cdots+\lambda_s$, *such that* $\lambda_j-\lambda_{j+k-1}\geq 1$ *if* λ_j *is overlined and* $\lambda_j-\lambda_{j+k-1}\geq 2$ *otherwise. Let* $\overline{A}_k(n)$ *denote the number of overpartitions of* n *into parts not divisible by* k. *Then for all* $n\geq 0$, *we have*

$$\overline{A}_k(n)=\overline{B}_k(n).$$

Theorem 1.6 *Let* $\overline{D}_k(n)$ *denote the number of overpartitions of* n *of the form* $\lambda_1+\lambda_2+\cdots+\lambda_s$, *such that* 1 *can not occur as a non-overlined part, and where* $\lambda_j-\lambda_{j+k-1}\geq 1$ *if* λ_j *is overlined and* $\lambda_j-\lambda_{j+k-1}\geq 2$ *otherwise. Let* $\overline{C}_k(n)$ *denote the number of overpartitions of* n *whose non-overlined parts are not congruent to* $0,\pm 1$ *modulo* $2k$. *Then for all* $n\geq 0$, *we have*

$$\overline{C}_k(n)=\overline{D}_k(n).$$

In 2012, Chen, Sang and Shi [6] found an overpartition analogue of the Rogers–Ramanujan–Gordon theorem in the general case for $k\geq i\geq 1$.

Theorem 1.7 *For* $k\geq i\geq 1$, *let* $D_{k,i}(n)$ *denote the number of overpartitions of* n *of the form* $\lambda_1+\lambda_2+\cdots+\lambda_s$, *such that* 1 *can occur as a non-overlined part at most* $i-1$ *times, and where* $\lambda_j-\lambda_{j+k-1}\geq 1$ *if* λ_j *is overlined and* $\lambda_j-\lambda_{j+k-1}\geq 2$ *otherwise. For* $k-1\geq i\geq 1$, *let* $C_{k,i}(n)$ *denote the number of overpartitions of* n *whose non-overlined parts are not congruent to* $0,\pm i$ *modulo* $2k$ *and let* $C_{k,k}(n)$ *denote the number of overpartitions of* n *with parts not divisible by* k. *Then for all* $n\geq 0$ *and* $k\geq i\geq 1$, *we have*

$$C_{k,i}(n)=D_{k,i}(n).$$

Theorem 1.5 and 1.6 are special cases of Theorem 1.7 for $i=k$ and $i=1$. The generating function form of Theorem 1.7 is given by Chen, Sang and Shi [6].

Theorem 1.8 *For $k \geq i \geq 1$, we have*

$$\sum_{N_1 \geq N_2 \cdots \geq N_{k-1} \geq 0} \frac{q^{\frac{(N_1+1)N_1}{2}+N_2^2+\cdots+N_{k-1}^2+N_{i+1}+\cdots+N_{k-1}}(-q)_{N_1-1}(1+q^{N_i})}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q)_{N_{k-1}}}$$

$$= \frac{(-q)_\infty(q^i, q^{2k-i}, q^{2k}; q^{2k})_\infty}{(q)_\infty}. \tag{1.3}$$

As an overpartition analogue of Bressoud's theorem for the case $i = 1$, Corteel, Lovejoy, and Mallet [9] obtained the following overpartition theorem.

Theorem 1.9 *For $k \geq 2$, let $\overline{A}_k^3(n)$ denote the number of overpartitions whose non-overlined parts are not congruent to $0, \pm 1$ modulo $2k-1$. Let $\overline{B}_k^3(n)$ denote the number of overpartitions λ of n such that*

(i) $f_1(\lambda) = 0$,

(ii) $f_l(\lambda) + f_{\bar{l}}(\lambda) + f_{l+1}(\lambda) \leq k-1$,

(iii) *if the equality in Condition (ii) is attained at l, then* $lf_l(\lambda) + lf_{\bar{l}}(\lambda) + (l+1)f_{l+1}(\lambda) \equiv V_\lambda(l) (\mathrm{mod}\ 2)$.

Then for all $n \geq 0$, we have

$$\overline{A}_k^3(n) = \overline{B}_k^3(n).$$

In 2013, Chen, Sang and Shi [7] obtained the overpartition analogue of the Bressoud's theorem in the general case.

Theorem 1.10 *For $k-1 \geq i \geq 1$, let $\widetilde{D}_{k,i}(n)$ denote the number of overpartitions of n of the form $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_s$, such that*

(i) $f_1(\lambda) \leq i-1$,

(ii) $f_l(\lambda) + f_{\bar{l}}(\lambda) + f_{l+1}(\lambda) \leq k-1$,

(iii) *if the equality in Condition (ii) is attained at l, then* $lf_l(\lambda) + lf_{\bar{l}}(\lambda) + (l+1)f_{l+1}(\lambda) \equiv V_\lambda(l) + i - 1 (\mathrm{mod}\ 2)$.

Let $\widetilde{C}_{k,i}(n)$ denote the number of overpartitions of n whose non-overlined parts are not congruent to $0, \pm i$ modulo $2k-1$. Then for all $n \geq 0$, we have

$$\widetilde{C}_{k,i}(n) = \widetilde{D}_{k,i}(n).$$

It should be noticed that $\overline{A}_k^3(n)$ and $\overline{B}_k^3(n)$ in Theorem 1.9 are $\widetilde{C}_{k,1}(n)$ and $\widetilde{D}_{k,1}(n)$ in Theorem 1.10.

In this paper, we obtain the generating function formula for Theorem 1.10, which is an Andrews–Gordon type identity for overpartitions with odd modulo.

Theorem 1.11 *For $k-1 \geq i \geq 1$, we have*

$$\sum_{N_1\geq N_2\geq\cdots\geq N_{k-1}\geq 0} \frac{q^{\frac{(N_1+1)N_1}{2}+N_2^2+\cdots+N_{k-1}^2+N_{i+1}+\cdots+N_{k-1}}(-q)_{N_1-1}(1+q^{N_i})}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q^2;q^2)_{N_{k-1}}}$$
$$= \frac{(-q)_\infty(q^i,q^{2k-1-i},q^{2k-1};q^{2k-1})_\infty}{(q)_\infty}. \quad (1.4)$$

To prove the above identity, we use the Gordon marking for overpartitions introduced in [6] to show that the left hand side of (1.4) can be interpreted in terms of overpartitions in $\widetilde{D}_{k,i}(n)$. Let $\widetilde{D}_{k,i}(m,n)$ denote the number of overpartitions enumerated by $\widetilde{D}_{k,i}(n)$ with m parts. The generating function of $\widetilde{D}_{k,i}(m,n)$ is given below.

Theorem 1.12 *For $m,n \geq 0$ and $k-1 \geq i \geq 1$, we have*

$$\sum_{m,n\geq 0} \widetilde{D}_{k,i}(m,n)x^m q^n$$
$$= \sum_{N_1\geq N_2\geq\cdots\geq N_{k-1}\geq 0} \frac{q^{\frac{(N_1+1)N_1}{2}+N_2^2+\cdots+N_{k-1}^2+N_{i+1}+\cdots+N_{k-1}}(-q)_{N_1-1}(1+q^{N_i})x^{N_1+\cdots+N_{k-1}}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q^2;q^2)_{N_{k-1}}}, \quad (1.5)$$

where assume that $N_k = 0$.

By setting $x = 1$ in (1.5), we see that the generating function for $D_{k,i}(n)$ becomes the left hand side of (1.4). On the other hand, it is clear that the generating function for $C_{k,i}(n)$ equals the right-hand side of (1.4). Hence identity (1.4) follows from Theorem 1.10.

Let $\widetilde{T}_{k,i}(m,n)$ denote the set of overpartitions enumerated by $\widetilde{D}_{k,i}(m,n)$. Let $\widetilde{U}_{k,i}(m,n)$ denote the subset of $\widetilde{T}_{k,i}(m,n)$ in which the overpartitions with the smallest parts has an overlined part, and let $\widetilde{F}_{k,i}(m,n)$ denote the number of overpartitions in $\widetilde{U}_{k,i}(m,n)$. We shall establish a connection between $\widetilde{D}_{k,i}(m,n)$ and $\widetilde{F}_{k,i}(m,n)$ so that the generating function of $\widetilde{D}_{k,i}(m,n)$ can be expressed by the generating function of $\widetilde{F}_{k,i}(m,n)$.

This paper is organized as follows. In Section 2, we establish a connection between $\widetilde{D}_{k,i}(m,n)$ and $\widetilde{F}_{k,i}(m,n)$. In Section 3, we give the generating function formula of $\widetilde{F}_{k,i}(m,n)$ and to give the proof of this formula we introduce the definition of the Gordon marking of an overpartition. In Section 4, we use the first reduction operation and the first dilation operation to describe the first bijection for the proof of the formula of $\widetilde{F}_{k,i}(m,n)$. In Section 5, we employ the second reduction operation and the second dilation operation on Gordon markings to describe the second bijection for the proof of the formula of $\widetilde{F}_{k,i}(m,n)$ and complete the proof.

2 Connection between $\widetilde{D}_{k,i}(m,n)$ and $\widetilde{F}_{k,i}(m,n)$

For the purpose of computing the generating function of $\widetilde{D}_{k,i}(m,n)$, we consider the connection between $\widetilde{D}_{k,i}(m,n)$ and $\widetilde{F}_{k,i}(m,n)$. In the next section, we present a formula for $\widetilde{F}_{k,i}(m,n)$, which leads to the formula for $\widetilde{D}_{k,i}(m,n)$ as given in Theorem 1.12. The detailed proof of the formula for $\widetilde{F}_{k,i}(m,n)$ depends on two bijections which will be presented in Sections 4 and 5.

Recall that $\widetilde{T}_{k,i}(m,n)$ is the set of overpartitions enumerated by $\widetilde{D}_{k,i}(m,n)$ and $\widetilde{U}_{k,i}(m,n)$ is the subset of $\widetilde{T}_{k,i}(m,n)$ in which the overpartitions with the smallest parts has an overlined part. Let $\widetilde{W}_{k,i}(m,n)=\widetilde{T}_{k,i}(m,n)\setminus\widetilde{U}_{k,i}(m,n)$. In other words, $\widetilde{W}_{k,i}(m,n)$ is the set of overpartitions enumerated by $\widetilde{D}_{k,i}(m,n)$ in which none of the smallest parts is overlined. Let $\widetilde{G}_{k,i}(m,n)$ denote the number of overpartitions in $\widetilde{W}_{k,i}(m,n)$. So we have

$$\widetilde{D}_{k,i}(m,n)=\widetilde{F}_{k,i}(m,n)+\widetilde{G}_{k,i}(m,n). \tag{2.6}$$

The following theorem gives a connection between $\widetilde{F}_{k,i}(m,n)$ and $\widetilde{G}_{k,i}(m,n)$ which enables us to deduce the generating function of $\widetilde{D}_{k,i}(m,n)$ from the generating function of $\widetilde{F}_{k,i}(m,n)$.

Theorem 2.1 *For $k\geq 2$ and $m\leq n$, we have*

$$\widetilde{G}_{k,1}(m,n)=\widetilde{F}_{k,k-1}(m,n-m). \tag{2.7}$$

For $k\geq 3$, $k-1\geq i\geq 2$ and $m\leq n$, we have

$$\widetilde{G}_{k,i}(m,n)=\widetilde{F}_{k,i-1}(m,n). \tag{2.8}$$

Proof. To prove (2.7), we give a bijection between $\widetilde{W}_{k,1}(m,n)$ and $\widetilde{U}_{k,k-1}(m,n-m)$. For an overpartition λ in $\widetilde{W}_{k,1}(m,n)$, there are no parts equal to 1 in λ, that is, each part is greater than or equal to 2, so we can substract 1 from each part of λ and set one of the smallest parts to an overlined part to obtain an overpartition λ'. We aim to show that λ' belongs to $\widetilde{U}_{k,k-1}(m,n-m)$. It suffices to verify that λ' satisfies Condition (iii) in Theorem 1.10 with $i=k-1$, namely, if

$$f_l(\lambda')+f_{\overline{l}}(\lambda')+f_{l+1}(\lambda')=k-1,$$

then

$$lf_l(\lambda')+lf_{\overline{l}}(\lambda')+(l+1)f_{l+1}(\lambda')\equiv V_{\lambda'}(l)+k-2 \pmod 2.$$

From the definition of the bijection it can be seen that

$$f_l(\lambda')+f_{\overline{l}}(\lambda')+f_{l+1}(\lambda')=k-1,$$

if and only if

$$f_{l+1}(\lambda)+f_{\overline{l+1}}(\lambda)+f_{l+2}(\lambda)=k-1.$$

Since λ is an overpartition in $\widetilde{T}_{k,1}(m,n)$ and the equality in Condition (ii) in Theorem 1.10 is attained at $l+1$, we have

$$(l+1)f_{l+1}(\lambda)+(l+1)f_{\overline{l+1}}(\lambda)+(l+2)f_{l+2}(\lambda)\equiv V_\lambda(l+1)+1-1 \pmod 2. \tag{2.9}$$

Again, by the construction of the bijection, we see that

$$f_{l+1}(\lambda)+f_{\overline{l+1}}(\lambda)=f_l(\lambda')+f_{\overline{l}}(\lambda'), \tag{2.10}$$

$$f_{l+2}(\lambda)=f_{l+1}(\lambda') \tag{2.11}$$

and

$$V_\lambda(l+1) = V_{\lambda'}(l) - 1. \tag{2.12}$$

By (2.9), (2.10), (2.11) and (2.12), we find that

$$lf_l(\lambda') + lf_{\bar{l}}(\lambda') + (l+1)f_{l+1}(\lambda') \equiv V_{\lambda'}(l) + k - 2 \pmod 2.$$

So λ' is an overpartition in $\widetilde{U}_{k,k-1}(m, n-m)$.

Conversely, for an overpartition in $\widetilde{U}_{k,k-1}(m, n-m)$, we can add 1 to each part and change the smallest overlined part to a non-overlined part to get an overpartition in $\widetilde{W}_{k,1}(m,n)$. So $\widetilde{G}_{k,1}(m,n) = \widetilde{F}_{k,k-1}(m, n-m)$ for all $k \geq 2$.

For the case $k-1 \geq i \geq 2$, there is a simple bijection between $\widetilde{U}_{k,i-1}(m,n)$ and $\widetilde{W}_{k,i}(m,n)$. Let λ be an overpartition in $\widetilde{U}_{k,i-1}(m,n)$. Switching the smallest overlined part of λ to a non-overlined part, we get an overpartition λ' with non-overlined smallest parts. It can be checked that λ' satisfies Condition (i) and Condition (ii) in the definition of $\widetilde{D}_{k,i}(m,n)$ given in Theorem 1.10. It remains to verify that λ' satisfies Condition (iii). From the definition of this bijection, the equality in Condition (ii) is attained at l in λ' if and only if the equality in Condition (ii) is attained at l in λ. Since λ is an overpartition in $\widetilde{U}_{k,i-1}(m,n)$, we have

$$lf_l(\lambda) + lf_{\bar{l}}(\lambda) + (l+1)f_{l+1}(\lambda) \equiv V_\lambda(l) + i - 2 \pmod 2. \tag{2.13}$$

From the definition of this map, it can be easily seen that

$$f_l(\lambda) + f_{\bar{l}}(\lambda) = f_l(\lambda') + f_{\bar{l}}(\lambda'), \tag{2.14}$$

$$f_{l+1}(\lambda) = f_{l+1}(\lambda') \tag{2.15}$$

and

$$V_{\lambda'}(l) = V_\lambda(l) - 1. \tag{2.16}$$

Combining (2.13), (2.14), (2.15) and (2.16), we get

$$lf_l(\lambda') + lf_{\bar{l}}(\lambda') + (l+1)f_{l+1}(\lambda') \equiv V_{\lambda'}(l) + i - 2 \pmod 2.$$

So we have shown that λ' is an overpartition in $\widetilde{W}_{k,i}(m,n)$.

To see that this map is a bijection, we give the inverse map. For an overpartition in $\widetilde{W}_{k,i}(m,n)$, one can change one of the smallest part to an overlined part to get an overpartition belonging to $\widetilde{U}_{k,i-1}(m,n)$. It follows that $\widetilde{F}_{k,i-1}(m,n) = \widetilde{G}_{k,i}(m,n)$ for $k \geq 3$ and $k-1 \geq i \geq 2$. This completes the proof. ∎

By the relation (2.7) and (2.8), we can express $\widetilde{D}_{k,i}(m,n)$ in terms of $\widetilde{F}_{k,i}(m,n)$.

Theorem 2.2 *For $k \geq 2$ and $m \leq n$, we have*

$$\widetilde{D}_{k,1}(m,n) = \widetilde{F}_{k,1}(m,n) + \widetilde{F}_{k,k-1}(m, n-m). \tag{2.17}$$

For $k \geq 3$, $k-1 \geq i \geq 2$ and $m \leq n$, we have

$$\widetilde{D}_{k,i}(m,n) = \widetilde{F}_{k,i-1}(m,n) + \widetilde{F}_{k,i}(m,n). \tag{2.18}$$

3 The generating function for $\widetilde{F}_{k,i}(m,n)$

In this section, we give a formula for the generating function for $\widetilde{F}_{k,i}(m,n)$. By Theorem 2.2 we obtain a formula for the generating function for $\widetilde{D}_{k,i}(m,n)$. We outline the proof of the generating function formula for $\widetilde{F}_{k,i}(m,n)$. The detailed proof relies on two bijections that will be given in Section 4 and Section 5.

The generating function formula for $\widetilde{F}_{k,i}(m,n)$ is stated as follows.

Theorem 3.1 *For $k-1\geq i\geq 1$, we have*

$$\sum_{n,m\geq 0}\widetilde{F}_{k,i}(m,n)x^mq^n$$
$$=\sum_{N_1\geq N_2\geq\cdots\geq N_{k-1}\geq 0}\frac{q^{\frac{(N_1+1)N_1}{2}+N_2^2+\cdots+N_{k-1}^2+N_{i+1}+\cdots+N_{k-1}}(-q)_{N_1-1}x^{N_1+\cdots+N_{k-1}}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q^2;q^2)_{N_{k-1}}}. \tag{3.19}$$

By the generating function of $\widetilde{F}_{k,i}(m,n)$ and Theorem 2.2, we obtain the generating function of $\widetilde{D}_{k,i}(m,n)$.

Proof of Theorem 1.12. Using the generating function of $\widetilde{F}_{k,i}(m,n)$ and relation (2.18), we find that for $k-1\geq i\geq 2$,

$$\sum_{m,n\geq 0}\widetilde{D}_{k,i}(m,n)x^mq^n$$
$$=\sum_{m,n\geq 0}\widetilde{F}_{k,i}(m,n)x^mq^n+\sum_{m,n\geq 0}\widetilde{F}_{k,i-1}(m,n)x^mq^n$$
$$=\sum_{N_1\geq N_2\geq\cdots\geq N_{k-1}\geq 0}\frac{q^{\frac{(N_1+1)N_1}{2}+N_2^2+\cdots+N_{k-1}^2+N_{i+1}+\cdots+N_{k-1}}(-q)_{N_1-1}(1+q^{N_i})x^{N_1+\cdots+N_{k-1}}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q^2;q^2)_{N_{k-1}}}. \tag{3.20}$$

By relation (2.17), we find that for $i=1$ and $k-1\geq i$,

$$\sum_{m,n\geq 0}\widetilde{D}_{k,1}(m,n)x^mq^n$$
$$=\sum_{m,n\geq 0}\widetilde{F}_{k,1}(m,n)x^mq^n+\sum_{m,n\geq 0}\widetilde{F}_{k,k-1}(m,n)x^mq^n$$
$$=\sum_{N_1\geq N_2\geq\cdots\geq N_{k-1}\geq 0}\frac{q^{\frac{(N_1+1)N_1}{2}+N_2^2+\cdots+N_{k-1}^2+N_2+\cdots+N_{k-1}}(-q)_{N_1-1}(1+q^{N_2})x^{N_1+\cdots+N_{k-1}}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q^2;q^2)_{N_{k-1}}}. \tag{3.21}$$

Observe that formula (3.20) for $i\geq 2$ and formula (3.21) for $i=1$ take the same form as (1.5) in Theorem 1.12. This completes the proof. ∎

To prove Theorem 3.1, we need to use the Gordon marking of an overpartition as defined by Chen, Sang and Shi [6], which is an overpartition analogue of the Gordon marking of an ordinary partition introduced by Kurşungöz [11]. Recall that the Gordon marking of an overpartition λ is an assignment of positive integers, called marks, to parts of λ, subject to certain conditions. More precisely, we assign the marks to parts in the following order

$$\bar{1} < 1 < \bar{2} < 2 < \cdots$$

such that the marks are as small as possible subject to the following conditions:

(i) If $\overline{j+1}$ is not a part of λ, then all the parts j, $\bar{j}$, and $j+1$ are assigned different integers.

(ii) If λ contains an overlined part $\overline{j+1}$, then the smallest mark assigned to a part j or $\bar{j}$ can be used as the mark of $j+1$ or $\overline{j+1}$.

For example, let

$$\lambda = (16, 13, 12, 12, 11, \overline{10}, \bar{8}, 8, 8, 7, \bar{6}, 6, 5, 5, \bar{4}, 2, 2, \bar{1}).$$

Then the Gordon marking of λ is

$$(\bar{1}_1, 2_2, 2_3, \bar{4}_1, 5_2, 5_3, \bar{6}_1, 6_2, 7_3, \bar{8}_1, 8_2, 8_3, \overline{10}_1, 11_2, 12_1, 12_3, 13_2, 16_1),$$

where the subscripts stand for marks. The Gordon marking of λ can also be illustrated as follows

$$\lambda = \left[\begin{array}{cccccccccccc} & 2 & & 5 & & 7 & 8 & & & 12 & & \\ & 2 & & 5 & 6 & & 8 & & 11 & & 13 & \\ \bar{1} & & \bar{4} & & \bar{6} & & \bar{8} & \overline{10} & & 12 & & 16 \end{array}\right] \begin{array}{c} 3 \\ 2 \\ 1 \end{array},$$

where the parts in the third row are marked by 1, the parts in the second row are marked by 2, and the parts in the first row are marked by 3.

Let $\lambda^{(r)}$ denote the overpartition that consists of all r-marked parts of λ. Let N_r be the number of r-marked parts, namely, the number of parts in $\lambda^{(r)}$, and let $n_r = N_r - N_{r-1}$ for any positive integer r. For an overpartition in $\widetilde{U}_{k,i}(n)$, any part l and non-overlined part $l+1$ occur at most a total number of $k-1$ times, so there are no parts with marks greater than or equal to k. Using the parameters $N_1, \ldots, N_{k-1}$ we can further classify the set $\widetilde{U}_{k,i}(m,n)$. Let $\widetilde{U}_{N_1,N_2,\ldots,N_{k-1};i}(n)$ denote the set of overpartitions in $\widetilde{U}_{k,i}(m,n)$ that have N_r r-marked parts for $1 \leq r \leq k-1$, where $N_1 + N_2 + \cdots + N_{k-1} = m$, and let $\widetilde{P}_{N_1,N_2,\ldots,N_{k-1};i}(n)$ denote the set of overpartitions in $\widetilde{U}_{N_1,N_2,\ldots,N_{k-1};i}(n)$ with all the 1-marked parts overlined.

The proof of Theorem 3.1 also involves restrictions of two bijections given by Chen, Sang and Shi [6] to subsets of $\widetilde{U}_{N_1,N_2,\ldots,N_{k-1};i}(n)$ and $\widetilde{P}_{N_1,N_2,\ldots,N_{k-1};i}(n)$ respectively. Let us recall the notation used in [6]. We use $U_{k,i}(m,n)$ to denote the set of overpartitions enumerated by $D_{k,i}(m,n)$ for which the smallest 1-marked part is overlined. Let $U_{N_1,N_2,\ldots,N_{k-1};i}(n)$ denote the set of overpartitions in $U_{k,i}(m,n)$ that have N_r r-marked parts for $1 \leq r \leq k-1$, and let $P_{N_1,N_2,\ldots,N_{k-1};i}(n)$ denote the set of overpartitions in $U_{N_1,N_2,\ldots,N_{k-1};i}(n)$ such that the 1-marked parts are overlined. Define $Q_{N_1,N_2,\ldots,N_{k-1};i}(n)$ to be the set of overpartitions λ in $P_{N_1,N_2,\ldots,N_{k-1};i}(n)$ satisfying the following conditions:

(1) $f_1(\lambda)=i-1$;

(2) $f_t(\lambda)+f_{\bar{t}}(\lambda)+f_{t+1}(\lambda)=k-1$ for any positive integer t that is smaller than the greatest $(k-1)$-marked part.

It should be noticed that $\widetilde{U}_{N_1,N_2,\ldots,N_{k-1};i}(n)$ is the subset of overpartitions in $U_{N_1,N_2,\ldots,N_{k-1};i}(n)$ that satisfy Condition (iii) in Theorem 1.10.

We also define the following sets

$$\begin{aligned}\widetilde{U}_{N_1,N_2,\ldots,N_{k-1};i}&=\bigcup_{n\geq 0}\widetilde{U}_{N_1,N_2,\ldots,N_{k-1};i}(n),\\ U_{N_1,N_2,\ldots,N_{k-1};i}&=\bigcup_{n\geq 0}U_{N_1,N_2,\ldots,N_{k-1};i}(n),\\ \widetilde{P}_{N_1,N_2,\ldots,N_{k-1};i}&=\bigcup_{n\geq 0}\widetilde{P}_{N_1,N_2,\ldots,N_{k-1};i}(n),\\ P_{N_1,N_2,\ldots,N_{k-1};i}&=\bigcup_{n\geq 0}P_{N_1,N_2,\ldots,N_{k-1};i}(n),\end{aligned}$$

and

$$Q_{N_1,N_2,\ldots,N_{k-1};i}=\bigcup_{n\geq 0}Q_{N_1,N_2,\ldots,N_{k-1};i}(n).$$

From the definition of $\widetilde{U}_{N_1,N_2,\ldots,N_{k-1};i}$, for $1\leq i\leq k-1$ and $m\geq 0$, we have

$$\sum_{n\geq 0}\widetilde{F}_{k,i}(m,n)q^n=\sum_{N_1\geq N_2\geq\cdots\geq N_{k-1}\geq 0}\ \sum_{\lambda\in\widetilde{U}_{N_1,N_2,\ldots,N_{k-1};i}}q^{|\lambda|},\tag{3.22}$$

where $m=N_1+N_2+\cdots+N_{k-1}$.

The following theorem gives the generating function of $\widetilde{U}_{N_1,N_2,\ldots,N_{k-1};i}$. Theorem 3.1 can be derived from the generating function of $\widetilde{U}_{N_1,N_2,\ldots,N_{k-1};i}$ and identity (3.22). The proof of the following theorem will be presented in Sections 4 and 5.

Theorem 3.2 *For $k-1\geq i\geq 1$ and $N_1\geq N_2\geq\cdots\geq N_{k-1}\geq 0$, we have*

$$\sum_{\lambda\in\widetilde{U}_{N_1,N_2,\ldots,N_{k-1};i}}q^{|\lambda|}=\frac{q^{\frac{(N_1+1)N_1}{2}+N_2^2+\cdots+N_{k-1}^2+N_{i+1}+\cdots+N_{k-1}}(-q)_{N_1-1}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q^2;q^2)_{N_{k-1}}}.\tag{3.23}$$

We conclude this section with a sketch of the proof of Theorem 3.2. Let D_{N_1} denote the set of ordinary partitions with distinct parts such that each part is less than N_1, and let $R_{N_{k-1}}$ denote the set of ordinary partitions with at most N_{k-1} parts. Let $E_{N_{k-1}}$ denote the set of partitions with at most N_{k-1} even parts. In Section 4, we show that a bijection between $U_{N_1,N_2,\ldots,N_{k-1};i}$ and $P_{N_1,N_2,\ldots,N_{k-1};i}\times D_{N_1}$ given by Chen, Sang and Shi remains a bijection when restricted to $\widetilde{U}_{N_1,N_2,\ldots,N_{k-1};i}$ and $\widetilde{P}_{N_1,N_2,\ldots,N_{k-1};i}\times D_{N_1}$. Then we express the generating function for $\widetilde{U}_{N_1,N_2,\ldots,N_{k-1};i}$ in terms of the generating function for $\widetilde{P}_{N_1,N_2,\ldots,N_{k-1};i}$.

In Section 5, we show that $Q_{N_1,N_2,\ldots,N_{k-1};i}$ is a subset of $\widetilde{P}_{N_1,N_2,\ldots,N_{k-1};i}$. Then we restrict the bijection between $P_{N_1,N_2,\ldots,N_{k-1};i}$ and $Q_{N_1,N_2,\ldots,N_{k-1};i} \times R_{N_{k-1}}$ given by Chen, Sang and Shi to $\widetilde{P}_{N_1,N_2,\ldots,N_{k-1};i}$ and $\widetilde{Q}_{N_1,N_2,\ldots,N_{k-1};i} \times E_{N_{k-1}}$. This enables us to compute the generating function for $\widetilde{P}_{N_1,N_2,\ldots,N_{k-1};i}$ by using the generating function for $Q_{N_1,N_2,\ldots,N_{k-1};i}$. Hence we can derive the generating function for $\widetilde{U}_{N_1,N_2,\ldots,N_{k-1};i}$ from the generating function of $Q_{N_1,N_2,\ldots,N_{k-1};i}$.

4 The first bijection for the proof of Theorem 3.2

In this section, we establish a connection between the generating function for $\widetilde{U}_{N_1,N_2,\ldots,N_{k-1};i}$ and the generating function for $\widetilde{P}_{N_1,N_2,\ldots,N_{k-1};i}$ by giving a bijection between $\widetilde{U}_{N_1,N_2,\ldots,N_{k-1};i}$ and $\widetilde{P}_{N_1,N_2,\ldots,N_{k-1};i} \times D_{N_1}$.

Theorem 4.1 *For $k-1 \geq i \geq 1$ and $N_1 \geq N_2 \geq \cdots \geq N_{k-1}$, we have*

$$\sum_{\lambda \in \widetilde{U}_{N_1,N_2,\ldots,N_{k-1};i}} q^{|\lambda|} = (-q)_{N_1-1} \sum_{\alpha \in \widetilde{P}_{N_1,N_2,\ldots,N_{k-1};i}} q^{|\alpha|}. \tag{4.24}$$

In [6], Chen, Sang and Shi introduced the first reduction operation and the first dilation operation in order to construct a bijection between $U_{N_1,N_2,\cdots,N_{k-1};i}$ and $P_{N_1,N_2,\cdots,N_{k-1};i} \times D_{N_1}$. We show that this bijection induces a bijection between $\widetilde{U}_{N_1,N_2,\ldots,N_{k-1};i}$ and $\widetilde{P}_{N_1,N_2,\cdots,N_{k-1};i} \times D_{N_1}$.

Let us recall the definition of the first reduction operation. Let $\lambda = (\lambda_1, \ldots, \lambda_m) \in U_{N_1,N_2,\ldots,N_{k-1};i}$ be an overpartition of n containing at least one non-overlined part with mark 1 and let a_s be the underlying part of λ_s for $1 \leq s \leq m$. Assume that λ_j is the greatest non-overlined part with mark 1. Here are two cases.

Case 1: There is a non-overlined part $a_j + 1$ of λ but there is no overlined 1-marked part $\overline{a_j+1}$. First, we change the part λ_j to a 1-marked overlined part $\overline{a}_j$. Then we choose the part $a_j + 1$ with the smallest mark, say r, and replace this r-marked part $a_j + 1$ with an r-marked part a_j. Moreover, if there is a 1-marked overlined part to the right of $\overline{a_j}$, we switch it to a non-overlined part.

Case 2: Either there is a 1-marked overlined part $\overline{a_j+1}$ or there are no parts with underlying part $a_j + 1$. In this case, we may change the part λ_j to a 1-marked overlined part with underlying part $a_j - 1$. Moreover, if there are 1-marked parts larger than a_j, we switch the overlined 1-marked part next to λ_j to a non-overlined part.

For example, let λ be the following overpartition in $U_{7,6,5;1}(135)$

$$\left[\begin{array}{cccccccccccc} & 2 & & 5 & & 7 & 8 & & & 12 & & \\ & 2 & & 5 & 6 & & 8 & & 11 & & \mathbf{13} & \\ \overline{1} & & 4 & & \overline{6} & & \overline{8} & \overline{10} & & \mathbf{12} & & \overline{\mathbf{15}} \end{array}\right]\begin{array}{c} 3 \\ 2 \\ 1 \end{array}.$$

The part **12** with mark 1 is the λ_j as in the description of the reduction operation, since it is the rightmost non-overlined part with mark 1. Notice that 13 is not a 1-marked part of λ, but **13** is a 2-marked part. By the operation in Case 1, we change the 1-marked part **12** to a part

$\overline{\mathbf{12}}$, then we change the 2-marked part $\mathbf{13}$ to $\mathbf{12}$ and place it in a position with mark 2. Then we switch $\overline{\mathbf{15}}$ to $\mathbf{15}$ to get an overpartition μ in $U_{7,6,5;1}(134)$

$$\left[\begin{array}{ccccccccccccccc}
 & 2 & & & 5 & & 7 & 8 & & & & 12 & & & \\
 & 2 & & & 5 & 6 & & 8 & & & 11 & \mathbf{12} & & & \\
\bar{1} & & & 4 & & \bar{6} & & \bar{8} & & \overline{10} & & \overline{\mathbf{12}} & & & \mathbf{15}
\end{array}\right]\begin{array}{c}3\\2\\1\end{array}.$$

The first reduction operation possesses the following property.

Proposition 4.2 *Let λ be an overpartition in $\widetilde{U}_{N_1,N_2,\ldots,N_{k-1};i}(n)$. Let λ' be the overpartition obtained from λ by applying the first reduction operation. Then $\lambda' \in \widetilde{U}_{N_1,N_2,\ldots,N_{k-1};i}(n-1)$.*

Proof. Let λ be an overpartition in $\widetilde{U}_{N_1,N_2,\ldots,N_{k-1};i}(n)$. Applying the first reduction operation to λ, we get an overpartition λ' in $U_{N_1,N_2,\ldots,N_{k-1};i}(n-1)$. To prove $\lambda' \in \widetilde{U}_{N_1,N_2,\ldots,N_{k-1};i}(n-1)$, it suffices to verify that λ' satisfies Condition (iii) in the definition of $\widetilde{D}_{k,i}(n)$ in Theorem 1.10. To be more specific, we shall show that if

$$f_l(\lambda') + f_{\bar{l}}(\lambda') + f_{l+1}(\lambda') = k-1 \tag{4.25}$$

for some l, then

$$lf_l(\lambda') + lf_{\bar{l}}(\lambda') + (l+1)f_{l+1}(\lambda') \equiv V_{\lambda'}(l) + i - 1 \pmod 2. \tag{4.26}$$

Suppose that λ_j is the greatest non-overlined part with mark 1 whose underlying part is a_j.

For the first case of the reduction operation, there is a non-overlined part a_j+1 of λ but there is no overlined 1-marked part $\overline{a_j+1}$. In this case, one can check that λ' satisfies Condition (iii) for l not equal to a_j-1 or a_j. So it suffices to verify that the Condition (iii) in Theorem 1.10 is satisfied for $l = a_j - 1$ and $l = a_j$.

For $l = a_j$, we show that if relation (4.25) holds, then relation (4.26) also holds for $l = a_j$. According to the definition of the first reduction operation, we have

$$f_l(\lambda') + f_{\bar{l}}(\lambda') = f_l(\lambda) + f_{\bar{l}}(\lambda) + 1, \tag{4.27}$$

and

$$f_{l+1}(\lambda') = f_{l+1}(\lambda) - 1. \tag{4.28}$$

Under the assumption that $f_l(\lambda') + f_{\bar{l}}(\lambda') + f_{l+1}(\lambda') = k-1$, from (4.27) and (4.28) we deduce that

$$f_l(\lambda) + f_{\bar{l}}(\lambda) + f_{l+1}(\lambda) = k-1. \tag{4.29}$$

Since $\lambda \in \widetilde{U}_{N_1,N_2,\ldots,N_{k-1};i}(n)$ and λ satisfies the relation (4.29) for $l = a_j$, we have

$$lf_l(\lambda) + lf_{\bar{l}}(\lambda) + (l+1)f_{l+1}(\lambda) \equiv i - 1 + V_\lambda(l) \pmod 2$$

for $l = a_j$. In view of (4.27) and (4.28), we find that

$$lf_l(\lambda') + lf_{\bar{l}}(\lambda') + (l+1)f_{l+1}(\lambda') = lf_l(\lambda) + lf_{\bar{l}}(\lambda) + (l+1)f_{l+1}(\lambda) - 1.$$

Noticing that $V_{\lambda'}(l) = V_\lambda(l) + 1$, we obtain that

$$l f_l(\lambda') + l f_{\bar{l}}(\lambda') + (l+1) f_{l+1}(\lambda') \equiv i - 1 + V_{\lambda'}(l) \pmod 2.$$

We have proved that Condition (iii) is satisfied for $l = a_j$, now we prove that also satisfied for $l = a_j - 1$.

For $l = a_j - 1$, we show that if relation (4.25) holds, then relation (4.26) also holds for $l = a_j - 1$. It can be seen from the definition of the first reduction operation that

$$f_l(\lambda') = f_l(\lambda), \tag{4.30}$$

$$f_{\bar{l}}(\lambda') = f_{\bar{l}}(\lambda) \tag{4.31}$$

and

$$f_{l+1}(\lambda') = f_{l+1}(\lambda). \tag{4.32}$$

By the above relations (4.30), (4.31) and (4.32), the assumption

$$f_l(\lambda') + f_{\bar{l}}(\lambda') + f_{l+1}(\lambda') = k - 1$$

can be rewritten as

$$f_l(\lambda) + f_{\bar{l}}(\lambda) + f_{l+1}(\lambda) = k - 1. \tag{4.33}$$

Since $\lambda \in \widetilde{U}_{N_1,N_2,\cdots,N_{k-1};i}(n)$ and λ satisfies (4.33), from the definition of $\widetilde{U}_{N_1,N_2,\cdots,N_{k-1};i}(n)$, we have

$$l f_l(\lambda) + l f_{\bar{l}}(\lambda) + (l+1) f_{l+1}(\lambda) \equiv i - 1 + V_\lambda(l) \pmod 2. \tag{4.34}$$

Notice that $V_{\lambda'}(l) = V_\lambda(l)$. Substituting (4.30), (4.31) and (4.32) into (4.34), we find that

$$l f_l(\lambda') + l f_{\bar{l}}(\lambda') + (l+1) f_{l+1}(\lambda') \equiv i - 1 + V_{\lambda'}(l) \pmod 2,$$

which implies that the Condition (iii) in Theorem 1.10 is satisfied for $l = a_j - 1$. So we have proved that Condition (iii) is satisfied for all integers l in λ'. Thus, λ' is an overpartition in $\widetilde{U}_{N_1,N_2,\ldots,N_{k-1};i}(n-1)$. This completes the proof for the first case of the first reduction operation.

For the second case, it can be verified that λ' also satisfies Condition (iii) in Theorem 1.10. The proof is similar to that in the first case, and hence it is omitted. ∎

We are now ready to complete the proof of Theorem 4.1.

Proof of Theorem 4.1. Based on the first reduction operation and Proposition 4.2, we give the following bijection φ between $\widetilde{U}_{N_1,N_2,\ldots,N_{k-1};i}$ and $\widetilde{P}_{N_1,N_2,\ldots,N_{k-1};i} \times D_{N_1}$ which is induced from the bijection between $U_{N_1,N_2,\ldots,N_{k-1};i}$ and $P_{N_1,N_2,\ldots,N_{k-1};i} \times D_{N_1}$ obtained by Chen, Sang and Shi in [6].

Let λ be an overpartition in $\widetilde{U}_{N_1,N_2,\ldots,N_{k-1};i}(n)$. We proceed to construct an overpartition α and an ordinary partition β such that $\alpha \in \widetilde{P}_{N_1,N_2,\ldots,N_{k-1};i}$, $\beta \in D_{N_1}$ and $|\lambda| = |\alpha| + |\beta|$.

If there are no non-overlined 1-marked parts in λ, then $\lambda \in \widetilde{P}_{N_1,N_2,\ldots,N_{k-1};i}$, we just set $\alpha = \lambda$ and $\beta = \emptyset$.

We now consider the case when λ contains at least one non-overlined 1-marked part, say, $\lambda^{(1)}_{s_1} < \lambda^{(1)}_{s_2} < \cdots < \lambda^{(1)}_{s_t}$, where $s_1 < s_2 < \cdots < s_t$. We have the following two cases.

Case 1. If the largest 1-marked part is overlined, that is, $s_t < N_1$, then we get an overpartition η in $\widetilde{U}_{N_1,N_2,\ldots,N_{k-1};i}(n-1)$ with t non-overlined 1-marked parts by applying the first reduction operation to λ. Moreover, it can be seen that the 1-marked part to the right of $\lambda^{(1)}_{s_t}$ in λ is overlined. Hence $\eta^{(1)}_{s_t}$ is an overlined part in η, but $\eta^{(1)}_{s_t+1}$ is a non-overlined part. It follows that the t non-overlined 1-marked parts in η are $\eta^{(1)}_{s_1} < \eta^{(1)}_{s_2} < \cdots < \eta^{(1)}_{s_t+1}$.

Case 2. If the largest 1-marked part is non-overlined, that is, $s_t = N_1$, then we get an overpartition η in $\widetilde{U}_{N_1,N_2,\ldots,N_{k-1};i}(n-1)$ with $t-1$ non-overlined 1-marked parts by applying the first reduction operation to λ. Moreover, the $t-1$ non-overlined 1-marked parts in η are $\eta^{(1)}_{s_1} < \eta^{(1)}_{s_2} < \cdots < \eta^{(1)}_{s_{t-1}}$.

From the above discussions, we see that applying the first reduction operation $N_1 - s_t$ times to λ for the first case , we get an overpartition in $\widetilde{U}_{N_1,N_2,\ldots,N_{k-1};i}(n-N_1+s_t)$ with t 1-marked non-overlined parts and the largest 1-marked part is non-overlined. Applying the first reduction operation to the resulting overpartition for the second case, we get an overpartition in $\widetilde{U}_{N_1,N_2,\ldots,N_{k-1};i}(n-N_1+s_t-1)$ with $t-1$ 1-marked non-overlined parts.

In conclusion, to get an overpartition with $t-1$ non-overlined 1-marked parts, we need to apply the first reduction operation N_1-s_t+1 times to λ. Similarly, to get an overpartition with $t-2$ non-overlined 1-marked parts, we need to apply the first reduction operation $N_1-s_{t-2}+1$ times to the resulting overpartition. Hence one needs to apply the first reduction operation $N_1-s_{t-1}+1, N_1-s_{t-2}+1, \ldots, N_1-s_2+1$ and N_1-s_1+1 times to λ successively to generate an overpartition α with no non-overlined 1-marked parts. Meanwhile, we also get an ordinary partition $\beta = (N_1-s_{t-1}+1, N_1-s_{t-2}+1, \ldots, N_1-s_1+1)$.

We define the map φ to be that $\varphi(\lambda) = (\alpha, \beta)$. One can check that α is an overpartition in $\widetilde{P}_{N_1,N_2,\ldots,N_{k-1};i}$, $\beta \in D_{N_1}$ and $|\lambda| = |\alpha| + |\beta|$.

The inverse map φ^{-1} can be described based on the first dilation operation, and the details can be found in [6]. It can be checked that φ^{-1} maps a pair (α, β) in $\widetilde{P}_{N_1,N_2,\ldots,N_{k-1};i} \times D_{N_1}$ to an overpartition λ in $\widetilde{U}_{N_1,N_2,\ldots,N_{k-1};i}$. So we come to the conclusion that φ is a bijection between $\widetilde{U}_{N_1,N_2,\ldots,N_{k-1};i}$ and $\widetilde{P}_{N_1,N_2,\ldots,N_{k-1};i} \times D_{N_1}$. ∎

Let us give an example to demonstrate the above bijection. Let

$$\lambda = (14, 14, 14, 12, 11, 11, 10, \overline{9}, \overline{7}, 7, 7, 6, \overline{5}, 5, 4, 4, 3, 2, 2, \overline{1}),$$

which is an overpartition in $\widetilde{U}_{7,7,6;1}(148)$. The Gordon marking of λ is

$$\lambda = \left[\begin{array}{cccccccccccccc} & 2 & & 4 & & 6 & 7 & & & & 11 & & & 14 \\ & 2 & & 4 & 5 & & 7 & & & 10 & & 12 & & 14 \\ \overline{1} & & 3 & & \overline{5} & & \overline{7} & & \overline{9} & & 11 & & & 14 \end{array}\right] \begin{array}{c} 3 \\ 2 \\ 1 \end{array}.$$

There are 3 non-overlined 1-marked parts in λ and the largest 1-marked part is non-overlined. By applying the first reduction operation for the second case to λ, we get the following overpartition with two non-overlined 1-marked parts

$$\eta = \left[\begin{array}{ccccccccccccc} & 2 & & 4 & & 6 & 7 & & & 11 & & & 14 \\ & 2 & & 4 & 5 & & 7 & & 10 & & 12 & & 14 \\ \overline{1} & & 3 & & \overline{5} & & \overline{7} & \overline{9} & & 11 & & \overline{13} & \end{array}\right]\begin{array}{c} 3 \\ 2 \\ 1 \end{array}.$$

For the above overpartition η, we have

$$f_{\overline{13}}(\eta) + f_{13}(\eta) + 14f_{14}(\eta) = 3.$$

Consequently,

$$13f_{\overline{13}}(\eta) + 13f_{13}(\eta) + 14f_{14}(\eta) \equiv V_\eta(13) + i - 1 \pmod 2.$$

So η is an overpartition in $\widetilde{U}_{7,7,6;1}(147)$.

Since the largest 1-marked non-overlined part in η is 11 and there is one 1-marked part larger than 11, one needs to use the first reduction operation twice to η to get the following overpartition with one non-overlined 1-marked part

$$\theta = \left[\begin{array}{cccccccccccc} & 2 & & 4 & & 6 & 7 & & & 11 & & 14 \\ & 2 & & 4 & 5 & & 7 & & 10 & 11 & 13 & \\ \overline{1} & & 3 & & \overline{5} & & \overline{7} & \overline{9} & & \overline{11} & \overline{13} & \end{array}\right]\begin{array}{c} 3 \\ 2 \\ 1 \end{array}.$$

It can be checked that θ is an overpartition in $\widetilde{U}_{7,7,6;1}(145)$.

The largest non-overlined 1-marked part in θ is 3 and there are five 1-marked parts larger than 3. So it requires six times of the first reduction operation in order to get the following overpartition with no non-overlined 1-marked parts,

$$\alpha = \left[\begin{array}{ccccccccccc} & 2 & & 4 & 5 & & 7 & & & 11 & 13 \\ & 2 & 3 & & 5 & & 7 & 9 & & 11 & 13 \\ \overline{1} & & \overline{3} & & 5 & \overline{6} & & \overline{9} & \overline{10} & & \overline{13} \end{array}\right]\begin{array}{c} 3 \\ 2 \\ 1 \end{array}$$

which is an overpartition in $\widetilde{P}_{7,7,6;1}(141)$. Meanwhile, we get a partition $\beta = (6, 2, 1)$ such that $|\lambda| = |\alpha| + |\beta|$.

Conversely, one can recover the overpartition $\lambda \in \widetilde{U}_{7,7,6;1}(148)$ from (α, β) via the inverse map φ^{-1}.

5 The second bijection for the proof of Theorem 3.2

In this section, we give a relation between the generating function for $\widetilde{P}_{N_1,N_2,\ldots,N_{k-1};i}$ and the generating function for $Q_{N_1,N_2,\ldots,N_{k-1};i}$ by establishing a bijection between $\widetilde{P}_{N_1,N_2,\ldots,N_{k-1};i}$ and $Q_{N_1,N_2,\ldots,N_{k-1};i} \times E_{N_{k-1}}$.

Theorem 5.1 *For $k - 1 \geq i \geq 1$ and $N_1 \geq N_2 \geq \cdots \geq N_{k-1} \geq 0$, we have*

$$\sum_{\alpha \in \widetilde{P}_{N_1,N_2,\ldots,N_{k-1};i}} q^{|\alpha|} = \frac{1}{(q^2;q^2)_{N_{k-1}}} \sum_{\gamma \in Q_{N_1,N_2,\ldots,N_{k-1};i}} q^{|\gamma|}. \tag{5.35}$$

We first show that $Q_{N_1,N_2,\ldots,N_{k-1};i}$ is a subset of $\widetilde{P}_{N_1,N_2,\ldots,N_{k-1};i}$. To this end, we need the following proposition which says that for any overpartition $\alpha \in \widetilde{P}_{N_1,N_2,\ldots,N_{k-1};i}$ if the equality in Condition (ii) in Theorem 1.10 is attained for integers $l-1$ and l, then Condition (iii) in Theorem 1.10 is satisfied for l if and only if it satisfied for $l-1$. Recall that $V_\lambda(l)$ stands for the number of overlined parts in λ that are less than or equal to l.

Proposition 5.2 *Let α be an overpartition in $P_{N_1,N_2,\ldots,N_{k-1};i}$. Assume that*

$$f_{l-1}(\alpha) + f_{\overline{l-1}}(\alpha) + f_l(\alpha) = k-1 \tag{5.36}$$

and

$$f_l(\alpha) + f_{\bar{l}}(\alpha) + f_{l+1}(\alpha) = k-1. \tag{5.37}$$

Then

$$(l-1)f_{l-1}(\alpha) + (l-1)f_{\overline{l-1}}(\alpha) + lf_l(\alpha) \equiv V_\alpha(l-1) + i - 1 \pmod 2 \tag{5.38}$$

if and only if

$$lf_l(\alpha) + lf_{\bar{l}}(\alpha) + (l+1)f_{l+1}(\alpha) \equiv V_\alpha(l) + i - 1 \pmod 2. \tag{5.39}$$

Proof. We consider the following two cases.

Case 1: There is an overlined part $\bar{l}$, that is, $f_{\bar{l}}(\alpha) = 1$. In this case, we have

$$V_\alpha(l) - V_\alpha(l-1) = 1, \tag{5.40}$$

Consequently, (5.38) can be rewritten as

$$(l-1)f_{l-1}(\alpha) + (l-1)f_{\overline{l-1}}(\alpha) + lf_l(\alpha) + 1 \equiv V_\alpha(l) + i - 1 \pmod 2. \tag{5.41}$$

To express (5.41) in the form of (5.39), it remains to show that

$$lf_l(\alpha) + lf_{\bar{l}}(\alpha) + (l+1)f_{l+1}(\alpha) \equiv (l-1)f_{l-1}(\alpha) + (l-1)f_{\overline{l-1}}(\alpha) + lf_l(\alpha) + 1 \pmod 2. \tag{5.42}$$

Comparing (5.36) and (5.37), we see that

$$f_{l-1}(\alpha) + f_{\overline{l-1}}(\alpha) + f_l(\alpha) = f_l(\alpha) + f_{\bar{l}}(\alpha) + f_{l+1}(\alpha). \tag{5.43}$$

But $f_{\bar{l}}(\alpha) = 1$, hence we get

$$f_{l-1}(\alpha) + f_{\overline{l-1}}(\alpha) = 1 + f_{l+1}(\alpha).$$

It then follows that

$$\begin{aligned} &lf_l(\alpha) + lf_{\bar{l}}(\alpha) + (l+1)f_{l+1}(\alpha) - [(l-1)f_{l-1}(\alpha) + (l-1)f_{\overline{l-1}}(\alpha) + lf_l(\alpha) + 1] \\ &\quad = 2f_{l+1}(\alpha) \equiv 0 \pmod 2. \end{aligned}$$

This proves (5.42), so that (5.38) is equivalent to (5.39).

Case 2: There is no overlined part equal to l, that is, $f_{\bar{l}}(\alpha) = 0$. In this case, we have

$$V_\alpha(l) = V_\alpha(l-1). \tag{5.44}$$

Hence (5.39) is equivalent to

$$l f_l(\alpha) + l f_{\bar{l}}(\alpha) + (l+1) f_{l+1}(\alpha) \equiv V_\alpha(l-1) + i - 1 \pmod 2. \tag{5.45}$$

To prove that (5.45) is equivalent to (5.38), it suffices to show that

$$l f_l(\alpha) + l f_{\bar{l}}(\alpha) + (l+1) f_{l+1}(\alpha) \equiv (l-1) f_{l-1}(\alpha) + (l-1) f_{\overline{l-1}}(\alpha) + l f_l(\alpha) \pmod 2. \tag{5.46}$$

Combining (5.36) and (5.37), we arrive at

$$f_{l-1}(\alpha) + f_{\overline{l-1}}(\alpha) + f_l(\alpha) = f_l(\alpha) + f_{\bar{l}}(\alpha) + f_{l+1}(\alpha),$$

which simplifies to

$$f_{l-1}(\alpha) + f_{\overline{l-1}}(\alpha) = f_{l+1}(\alpha), \tag{5.47}$$

since $f_{\bar{l}}(\alpha) = 0$. By using (5.47), we get

$$l f_l(\alpha) + l f_{\bar{l}}(\alpha) + (l+1) f_{l+1}(\alpha) - [(l-1) f_{l-1}(\alpha) + (l-1) f_{\overline{l-1}}(\alpha) + l f_l(\alpha)] = 2 f_{l+1}(\alpha) \equiv 0 \pmod 2,$$

which leads to (5.46). This completes the proof. ∎

We can now show that $Q_{N_1,N_2,\ldots,N_{k-1};i}$ is a subset of $\widetilde{P}_{N_1,N_2,\ldots,N_{k-1};i}$.

Theorem 5.3 *For $k-1 \geq i \geq 1$ and $N_1 \geq N_2 \geq \cdots \geq N_{k-1} \geq 0$, we have*

$$Q_{N_1,N_2,\ldots,N_{k-1};i} \subset \widetilde{P}_{N_1,N_2,\ldots,N_{k-1};i}.$$

Proof. Recall that $\widetilde{P}_{N_1,N_2,\ldots,N_{k-1};i}$ is a set of overpartitions in which the overpartitions satisfy the Condition (i), (ii) and (iii) in Theorem 1.10 with an additional requirement that all 1-marked parts are overlined. The set $Q_{N_1,N_2,\ldots,N_{k-1};i}$ consists of overpartitions α subject to the following conditions:

1. Condition (i) and Condition (ii) in Theorem 1.10 hold.
2. All 1-marked parts in α are overlined.
3. $f_1(\lambda) = i-1$, for $1 \leq l \leq \alpha^{(k-1)}_{N_{k-1}} - 1$,

$$f_l(\alpha) + f_{\bar{l}}(\alpha) + f_{l+1}(\alpha) = k-1, \tag{5.48}$$

and for $l > \alpha^{(k-1)}_{N_{k-1}}$,

$$f_l(\alpha) + f_{\bar{l}}(\alpha) + f_{l+1}(\alpha) < k-1. \tag{5.49}$$

Let α be an overpartition in $Q_{N_1,N_2,\ldots,N_{k-1};i}$, we proceed to show $\alpha \in \widetilde{P}_{N_1,N_2,\ldots,N_{k-1};i}$. We shall use the conditions (5.48) and (5.49) to demonstrate that α also satisfies Condition (iii) in Theorem 1.10.

To prove α satisfies Condition (iii) in Theorem 1.10, we need to show that for all l satisfying (5.48), we have

$$l f_l(\alpha) + l f_{\bar{l}}(\alpha) + (l+1) f_{l+1}(\alpha) \equiv V_\alpha(l) + i - 1 \pmod 2. \tag{5.50}$$

By definition, (5.48) holds for $1 \leq l < \alpha^{(k-1)}_{N_{k-1}}$, but it does not hold for $l > \alpha^{(k-1)}_{N_{k-1}}$. As for the case $l = \alpha^{(k-1)}_{N_{k-1}}$, there are two cases:
Case 1:

$$f_l(\alpha) + f_{\bar{l}}(\alpha) + f_{l+1}(\alpha) = k-1;$$

Case 2:

$$f_l(\alpha) + f_{\bar{l}}(\alpha) + f_{l+1}(\alpha) < k-1.$$

For the first case, we show that (5.50) holds for $l = \alpha^{(k-1)}_{N_{k-1}}$. Using Proposition 5.2, it suffices to prove that if (5.50) holds for $l = \alpha^{(k-1)}_{N_{k-1}} - 1$, then it also holds for $l = \alpha^{(k-1)}_{N_{k-1}}$. Since (5.48) holds for $1 \leq l \leq \alpha^{(k-1)}_{N_{k-1}} - 1$, we need to show that (5.50) holds for $1 \leq l \leq \alpha^{(k-1)}_{N_{k-1}} - 1$.

If $N_{k-1} = 0$, there are no integers l satisfying that

$$f_l(\alpha) + f_{\bar{l}}(\alpha) + f_{l+1}(\alpha) = k-1.$$

Hence (5.50) is not required.

If $N_{k-1} \geq 1$, we use induction to show that (5.50) holds for $1 \leq l \leq \alpha^{(k-1)}_{N_{k-1}} - 1$. From the definition of $Q_{N_1,N_2,\ldots,N_{k-1};i}$, we have $f_1(\alpha) = i-1$, $f_{\bar{1}}(\alpha) = 1$ and $f_l(\alpha)+f_{\bar{l}}(\alpha)+f_{l+1}(\alpha) = k-1$. It follows that $f_2(\alpha) = k-i-1$ and $V_\alpha(1) = 1$. Moreover, we get

$$f_1(\alpha) + f_{\bar{1}}(\alpha) + 2f_2(\alpha) \equiv V_\alpha(1) + i - 1 \pmod 2,$$

so that (5.50) is valid for $l = 1$.

Assume that (5.50) holds for l with $1 \leq l \leq \alpha^{(k-1)}_{N_{k-1}} - 2$, that is,

$$f_l(\alpha) + f_{\bar{l}}(\alpha) + 2f_{l+1}(\alpha) \equiv V_\alpha(l) + i - 1 \pmod 2, \tag{5.51}$$

we aim to prove that (5.50) is satisfied for $l+1$, that is,

$$f_{l+1}(\alpha) + f_{\overline{l+1}}(\alpha) + 2f_{l+2}(\alpha) \equiv V_\alpha(l+1) + i - 1 \pmod 2, \tag{5.52}$$

Since α is an overpartition in $Q_{N_1,N_2,\ldots,N_{k-1};i}$, we have

$$f_l(\alpha) + f_{\bar{l}}(\alpha) + f_{l+1}(\alpha) = k-1$$

and

$$f_{l+1}(\alpha) + f_{\overline{l+1}}(\alpha) + f_{l+2}(\alpha) = k-1$$

for $1 \leq l \leq \alpha^{(k-1)}_{N_{k-1}} - 2$. Combining the assumption (5.51) and Proposition 5.2, we obtain (5.52). This complete the proof. ∎

We are now ready to construct a bijection between $\widetilde{P}_{N_1,N_2,\ldots,N_{k-1};i}$ and $\widetilde{Q}_{N_1,N_2,\ldots,N_{k-1};i} \times E_{N_{k-1}}$. We shall use the second reduction operation and the second dilation operation introduced by Chen, Sang and Shi [6].

Let us recall the definition of the second reduction operation. Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be an overpartition in $P_{N_1,N_2,\ldots,N_{k-1};i} \setminus Q_{N_1,N_2,\ldots,N_{k-1};i}$. From the definition of $P_{N_1,N_2,\ldots,N_{k-1};i}$ and $Q_{N_1,N_2,\ldots,N_{k-1};i}$, it can be seen that among the $(k-1)$-marked parts in the Gordon marking representation of α, there exists a part α_j whose underlying part, denoted by t, satisfies one of the following conditions:

1. There are no parts with underlying part $t-1$;
2. There is a part with underlying part $t-1$, and $f_1(\alpha) < i-1$ for $t=2$,

$$f_{t-2}(\alpha) + f_{\overline{t-2}}(\alpha) + f_{t-1}(\alpha) < k-1 \text{ for } t > 2.$$

So we may choose α_j as a $(k-1)$-marked part with the smallest underlying part t such that one of the above conditions holds. Now we can define the second reduction operation on α with respect to a selected part α_j by considering the following three cases.

Case 1: α_j satisfies the above Condition 1. There is an overlined part $\bar{t}$ in α. We replace $\bar{t}$ with a 1-marked overlined part $\overline{t-1}$.

Case 2: α_j satisfies the above Condition 2 and $t=2$. Let $s = f_1(\alpha)$ and replace the $(s+2)$-marked part 2 with an $(s+2)$-marked part 1.

Case 3: α_j satisfies the above Condition 2 and $t>2$. In this case, there exists an r-marked part with underlying part t such that $r \geq 2$ and

$$\sum_{j=1}^{r} (f_{t-2}(\alpha^{(j)}) + f_{\overline{t-2}}(\alpha^{(j)}) + f_{t-1}(\alpha^{(j)})) < r, \tag{5.53}$$

where $\alpha^{(j)}$ is the overpartition consisting of the j-marked parts of α. Assume that r is the smallest integer for the above choice of $\alpha^{(j)}$. We replace this r-marked part t with an r-marked part $t-1$.

After applying a second reduction operation to an overpartition α in $\widetilde{P}_{N_1,N_2,\ldots,N_{k-1};i}(n)$, we get an overpartition α' in $P_{N_1,N_2,\ldots,N_{k-1};i}(n-1)$. Although α' is not necessarily an overpartition in $\widetilde{P}_{N_1,N_2,\ldots,N_{k-1};i}(n-1)$, as will be seen, applying the second reduction operation one more time, we are led to an overpartition in $\widetilde{P}_{N_1,N_2,\ldots,N_{k-1};i}(n-2)$.

Proposition 5.4 *Let α be an overpartition in $\widetilde{P}_{N_1,N_2,\ldots,N_{k-1};i}(n) \backslash Q_{N_1,N_2,\ldots,N_{k-1};i}(n)$. Applying the second reduction operation twice to α, we get an overpartition β in $\widetilde{P}_{N_1,N_2,\ldots,N_{k-1};i}(n-2)$.*

Proof. From the construction of the second reduction operation, it is easy to see that β belongs to $P_{N_1,N_2,\ldots,N_{k-1};i}(n-2)$. To prove that $\beta \in \widetilde{P}_{N_1,N_2,\ldots,N_{k-1};i}(n-2)$, we are required to verify that for any integer l satisfying

$$f_l(\beta) + f_{\bar{l}}(\beta) + f_{l+1}(\beta) = k-1, \tag{5.54}$$

we have

$$l f_l(\beta) + l f_{\bar{l}}(\beta) + (l+1) f_{l+1}(\beta) \equiv V_\beta(l) + i - 1 \pmod 2. \tag{5.55}$$

Let

$$c_l(\beta) = l f_l(\beta) + l f_{\bar{l}}(\beta) + (l+1) f_{l+1}(\beta) - V_\beta(l),$$

so that (5.55) can be rewritten as

$$c_l(\beta) \equiv i-1 \pmod 2. \tag{5.56}$$

In fact, from the construction of the Gordon marking of an overpartition, it can be seen that (5.54) holds for l if and only if either l or $l+1$ is a $(k-1)$-marked part in the Gordon marking of β. Let $\beta_j^{(k-1)}$ be the j-th $(k-1)$-marked part in β with the underlying part equal to t, that is, $\beta_j^{(k-1)} = t$.

We claim that $t-1$ satisfies (5.54) or t satisfies (5.54). Assume that $t-1$ does not satisfy (5.54). We aim to show that t satisfies (5.54). Note that there is a $(k-1)$-marked part equal to t in β. Since $t-1$ does not satisfy (5.54), we deduce that there are no parts equal to $t-1$ in β. From the construction of the Gordon marking of β, it can be seen that there are $k-1$ parts equal to t in β, and hence

$$f_t(\beta) + f_{\bar{t}}(\beta) = k-1,$$

that is, t satisfies (5.54). This proves the claim.

For notational convenience, to each $(k-1)$-marked part $\beta_j^{(k-1)}$ of β we associate a number $d_j(\beta)$ which is defined as follows. Let $\beta_j^{(k-1)} = t$. From the above claim, we see that either $t-1$ or t satisfies (5.54). If $t-1$ satisfies (5.54), we set $d_j(\beta) = c_{t-1}(\beta)$. Otherwise, we set $d_j(\beta) = c_t(\beta)$.

With the above notation, it is enough to show that for all $(k-1)$-marked parts of β,

$$d_j(\beta) \equiv i-1 \pmod 2, \tag{5.57}$$

where $j = 1, 2, \ldots, N_{k-1}$.

From the definition of β, we see that β is obtained from α by applying the second reduction operation twice. More precisely, for fixed h, we first obtain γ by applying the second reduction operation to α with respect to $\alpha_h^{(k-1)}$. Then β is obtained from γ by applying the second reduction operation with respect to $\gamma_h^{(k-1)}$. We now establish a connection between $d_j(\beta)$ and $d_j(\alpha)$.

We proceed to show that for $j \neq h$,

$$d_j(\gamma) = d_j(\alpha), \tag{5.58}$$

and

$$d_h(\gamma) \not\equiv d_h(\alpha) \pmod 2. \tag{5.59}$$

Since β is obtained from γ by applying the second reduction operation, as a consequence of the above claim, we find that for $j \neq h$,

$$d_j(\beta) = d_j(\gamma), \tag{5.60}$$

and

$$d_h(\beta) \not\equiv d_h(\gamma) \pmod 2. \tag{5.61}$$

Combining (5.58), (5.59), (5.60), and (5.61), we deduce that

$$d_j(\beta) \equiv d_j(\alpha) \pmod 2, \tag{5.62}$$

where $j = 1, 2, \ldots, N_{k-1}$. Since $\alpha \in \widetilde{P}_{N_1, N_2, \ldots, N_{k-1}; i}$, we have

$$d_j(\alpha) \equiv i-1 \pmod 2, \tag{5.63}$$

where $j = 1, 2, \ldots, N_{k-1}$. From (5.62) and (5.63), we arrive at (5.57). So we conclude that $\beta \in \widetilde{P}_{N_1,N_2,\ldots,N_{k-1};i}(n-2)$.

Up to now, we have shown that if (5.58) and (5.59) hold, then $\beta \in \widetilde{P}_{N_1,N_2,\ldots,N_{k-1};i}(n-2)$. It remains to show that (5.58) and (5.59) hold. From the construction of the second reduction operation, it is easy to see that (5.58) holds. To prove (5.59), we consider the following two cases. Suppose that $\alpha_h^{(k-1)} = t$.

Case 1: There are no parts with underlying part $t-1$ in α. In this case, $t-1$ does not satisfy (5.54), but t satisfies (5.54). From the definition of $d_h(\alpha)$, we have

$$d_h(\alpha) = c_t(\alpha) = tf_t(\alpha) + tf_{\overline{t}}(\alpha) + (t+1)f_{t+1}(\alpha) - V_\alpha(t).$$

Furthermore, from the construction of the second reduction operation, we see that γ is obtained from α by replacing a 1-marked $\overline{t}$ by a 1-marked $\overline{t-1}$. Consequently, there is a $\overline{t-1}$ in γ, so that $t-1$ satisfies (5.54). From the definition of $d_h(\gamma)$, we find that

$$d_h(\gamma) = c_{t-1}(\gamma) = (t-1)f_{t-1}(\gamma) + (t-1)f_{\overline{t-1}}(\gamma) + tf_t(\gamma) - V_\gamma(t-1).$$

It is obvious that $f_t(\alpha) = f_t(\gamma)$, $f_{\overline{t}}(\alpha) = f_{\overline{t-1}}(\gamma) = 1$, $f_{t-1}(\gamma) = f_{t+1}(\alpha) = 0$ and $V_\alpha(t) = V_\gamma(t-1)$. So we deduce that

$$d_h(\alpha) \not\equiv d_h(\gamma) \pmod 2.$$

Case 2: There are at least one part equal to $t-1$ in α. In this case, $t-1$ satisfies (5.54). Hence

$$d_h(\alpha) = c_{t-1}(\alpha) = (t-1)f_{t-1}(\alpha) + (t-1)f_{\overline{t-1}}(\alpha) + tf_t(\alpha) - V_\alpha(t-1).$$

We consider the following two subcases.

Case 2.1: $t = 2$. Let $s = f_1(\alpha)$, from the construction of the second reduction operation, we see that γ is obtained from α by replacing $(s+2)$-marked 2 with $(s+2)$-marked 1. It follows that

$$d_h(\gamma) = c_1(\gamma) = f_1(\gamma) + f_{\overline{1}}(\gamma) + 2f_2(\gamma) - V_\gamma(1).$$

Since $f_1(\gamma) = f_1(\alpha) + 1$, $f_{\overline{1}}(\gamma) = f_{\overline{1}}(\alpha)$, $f_2(\gamma) = f_2(\alpha) - 1$ and $V_\gamma(1) = V_\alpha(1)$, we deduce that

$$d_h(\gamma) = c_1(\alpha) - 1 = d_h(\alpha) - 1,$$

which implies that

$$d_h(\alpha) \not\equiv d_h(\gamma) \pmod 2.$$

Case 2.2: $t > 2$. Let r be the smallest integer such that there is an r-marked part equal to t in α and

$$\sum_{m=1}^{r} (f_{t-2}(\alpha^{(m)}) + f_{\overline{t-2}}(\alpha^{(m)}) + f_{t-1}(\alpha^{(m)})) < r. \tag{5.64}$$

In this case, γ is obtained from α by replacing an r-marked t with an r-marked $t-1$.

If $1 \leq r < k-1$, then $t-1$ satisfies (5.54) in γ, so that

$$d_h(\gamma) = c_{t-1}(\gamma) = (t-1)f_{t-1}(\gamma) + (t-1)f_{\overline{t-1}}(\gamma) + tf_t(\gamma) - V_\gamma(t-1).$$

It is easily seen that $V_\gamma(t-1) = V_\alpha(t-1)$, $f_{t-1}(\gamma) = f_{t-1}(\alpha)+1$, $f_{\overline{t-1}}(\gamma) = f_{\overline{t-1}}(\alpha)$ and $f_t(\gamma) = f_t(\alpha)-1$. Thus we deduce that

$$d_h(\gamma) = d_h(\alpha) - 1,$$

which yields

$$d_h(\alpha) \not\equiv d_h(\gamma) \pmod 2.$$

If $r = k-1$ and there are no parts equal to $t-2$ in α, then $\gamma_h^{k-1} = t-1$ and

$$d_h(\gamma) = c_{t-1}(\gamma) = (t-1)f_{t-1}(\gamma) + (t-1)f_{\overline{t-1}}(\gamma) + tf_t(\gamma) - V_\gamma(t-1).$$

From the construction of γ, we have $V_\gamma(t-1) = V_\alpha(t-1)$, $f_{t-1}(\gamma) = f_{t-1}(\alpha)+1$, $f_{\overline{t-1}}(\gamma) = f_{\overline{t-1}}(\alpha)$ and $f_t(\gamma) = f_t(\alpha)-1$. It follows that

$$d_h(\gamma) = d_h(\alpha) - 1,$$

which implies that

$$d_h(\alpha) \not\equiv d_h(\gamma) \pmod 2.$$

If $r = k-1$ and there is at least one part equal to $t-2$ in α, then $\gamma_h^{k-1} = t-1$, It follows that $t-2$ satisfies (5.54) in γ, so that

$$d_h(\gamma) = c_{t-2}(\gamma) = (t-2)f_{t-2}(\gamma) + (t-2)f_{\overline{t-2}}(\gamma) + (t-1)f_{t-1}(\gamma) - V_\gamma(t-2).$$

Since $r = k-1$ is the smallest integer subject to condition (5.64) holds, we have

$$f_{t-2}(\alpha) + f_{\overline{t-2}}(\alpha) + f_{t-1}(\alpha) = k-2. \tag{5.65}$$

On the other hand,

$$f_{t-1}(\alpha) + f_{\overline{t-1}}(\alpha) + f_t(\alpha) = k-1. \tag{5.66}$$

Combining (5.65) and (5.66), we get

$$f_{\overline{t-1}}(\alpha) + f_t(\alpha) = f_{t-2}(\alpha) + f_{\overline{t-2}}(\alpha) + 1. \tag{5.67}$$

Now, we have $f_{t-2}(\gamma) = f_{t-2}(\alpha)$ and $f_{\overline{t-2}}(\gamma) = f_{\overline{t-2}}(\alpha)$. Thus, (5.67) becomes

$$f_{\overline{t-1}}(\alpha) + f_t(\alpha) = f_{t-2}(\gamma) + f_{\overline{t-2}}(\gamma) + 1. \tag{5.68}$$

By the second reduction operation, we also have

$$f_{t-1}(\gamma) = f_{t-1}(\alpha) + 1 \tag{5.69}$$

and

$$V_\alpha(t-1) = V_\gamma(t-2) + f_{\overline{t-1}}(\alpha). \tag{5.70}$$

Combining (5.68), (5.69) and (5.70), we find that

$$\begin{aligned}
d_h(\gamma) &= c_{t-2}(\gamma) \\
&= (t-2)f_{t-2}(\gamma) + (t-2)f_{\overline{t-2}}(\gamma) + (t-1)f_{t-1}(\gamma) - V_\gamma(t-2) \\
&= (t-2)(f_{\overline{t-1}}(\alpha) + f_t(\alpha) - 1) + (t-1)(f_{t-1}(\alpha)+1) - (V_\alpha(t-1) - f_{\overline{t-1}}(\alpha)) \\
&= tf_t(\alpha) + (t-1)f_{\overline{t-1}}(\alpha) + (t-1)f_{t-1}(\alpha) - 2f_t(\alpha) - V_\alpha(t-1) + 1 \\
&\equiv c_{t-1}(\alpha) + 1 \pmod 2.
\end{aligned}$$

Noticing that $d_h(\alpha) = c_{t-1}(\alpha)$, we obtain

$$d_h(\alpha) \not\equiv d_h(\gamma) \pmod 2,$$

as required. ∎

For example, let α be an overpartition in $\widetilde{P}_{7,6,5;1}(133)$ as given below.

$$\alpha=\left[\begin{array}{ccccccccccc} & 2 & & 5 & & 7 & 8 & & & 12 & \\ & 2 & & 5 & 6 & & 8 & & 11 & 12 & \\ \overline{1} & & \overline{4} & & \overline{6} & & \overline{8} & \overline{10} & & \overline{12} & \overline{14} \end{array}\right]\begin{array}{l} 3\\ 2\\ 1 \end{array}.$$

Choosing the 3-marked part 5 to be $\alpha_h^{(k-1)}$, we see that it satisfies Condition 2 of the second reduction operation and the integer r is 2. Replacing the 2-marked 5 by 2-marked 4 in α, we get

$$\gamma=\left[\begin{array}{ccccccccccc} & 2 & & 5 & & 7 & 8 & & & 12 & \\ & 2 & 4 & & 6 & & 8 & & 11 & 12 & \\ \overline{1} & & \overline{4} & & \overline{6} & & \overline{8} & \overline{10} & & \overline{12} & \overline{14} \end{array}\right]\begin{array}{l} 3\\ 2\\ 1 \end{array}.$$

This overpartition is not in $\widetilde{P}_{7,6,5;1}(132)$ since $4f_4(\gamma)+4f_{\overline{4}}(\gamma)+5f_5(\gamma)\not\equiv V_\gamma(4)+1-1 \pmod 2$.

In the above overpartition, we still apply the second reduction operation to γ with respect to the 3-marked part 5 to be $\gamma_h^{(k-1)}$. It can be seen that it satisfies Condition 2 of the second reduction operation. Clearly, 3 is the smallest mark r such that (5.53) holds. So we replace the 3-marked part 5 with a 3-marked part 4 to form an overpartition:

$$\beta=\left[\begin{array}{ccccccccccc} & 2 & 4 & & & 7 & 8 & & & 12 & \\ & 2 & 4 & & 6 & & 8 & & 11 & 12 & \\ \overline{1} & & \overline{4} & & \overline{6} & & \overline{8} & \overline{10} & & \overline{12} & \overline{14} \end{array}\right]\begin{array}{l} 3\\ 2\\ 1 \end{array}.$$

After applying the second reduction operation twice to $\alpha_2^{(3)}$, it can be checked that $\beta \in \widetilde{P}_{7,6,5;1}(131)$.

Based on the second reduction operation and Proposition 5.4, we give the following bijection ψ between $\widetilde{P}_{N_1,N_2,\ldots,N_{k-1};i}$ and $Q_{N_1,N_2,\ldots,N_{k-1};i}\times E_{N_{k-1}}$ which induces from the bijection between $P_{N_1,N_2,\cdots,N_{k-1};i}$ and $Q_{N_1,N_2,\cdots,N_{k-1};i}\times R_{N_{k-1}}$ obtained by Chen, Sang and Shi in [6]. Using the bijection ψ, we are now ready to finish the proof of Theorem 5.1.

Proof of Theorem 5.1. Let α be an overpartition in $\widetilde{P}_{N_1,N_2,\cdots,N_{k-1};i}$. We wish to construct a pair of partitions $(\mu,\delta)=\psi(\alpha)$ where $\mu\in Q_{N_1,N_2,\ldots,N_{k-1};i}$ and $\delta\in E_{N_{k-1}}$. Assume that

$$\alpha_1^{(k-1)}<\alpha_2^{(k-1)}<\cdots<\alpha_{N_{k-1}}^{(k-1)}$$

are the $(k-1)$-marked parts of α.

We choose the smallest integer s, such that the underlying part of $\alpha_s^{(k-1)}$ satisfies either Condition 1 or Condition 2 in the construction of the second reduction operation. For each j from s to N_{k-1}, we repeatedly apply the second reduction operation with respect to $\alpha_j^{(k-1)}$ until the underlying part of $\alpha_j^{(k-1)}$ satisfies neither Condition 1 nor Condition 2. At last, there are no parts in α can be applied the second reduction operation, so we get an overpartition $\mu\in Q_{N_1,N_2,\ldots,N_{k-1};i}$. Let t_j be the number of the second reduction operation that have been applied with respect to $\alpha_j^{(k-1)}$, we get a partition $\delta=(t_{N_{k-1}},t_{N_{k-1}-1},\ldots,t_s)$. By the Proposition 5.4, it can be checked that $\delta\in E_{N_{k-1}}$. ∎

We conclude this section with an example to demonstrate the above bijection. For $k = 4$ and $i = 1$, let α be an overpartition in $\widetilde{P}_{7,6,5;1}(131)$ as given below:

$$\alpha = \left[\begin{array}{cccccccccccccc} & 2 & & 4 & & & 7 & 8 & & & & 12 & & \\ & 2 & & 4 & & 6 & & 8 & & & 11 & 12 & & \\ \overline{1} & & & \overline{4} & & \overline{6} & & \overline{8} & & \overline{10} & & \overline{12} & & \overline{14} \end{array}\right]\begin{matrix} 3 \\ 2 \\ 1 \end{matrix}.$$

We can see that for the 3-marked 2, there are no second reduction operations can be applied, so we choose the 3-marked 4. We apply the second reduction operation on it for two time to obtain the following overpartition in $\widetilde{P}_{7,6,5;1}(129)$:

$$\alpha = \left[\begin{array}{cccccccccccccc} & 2 & & 4 & & & 7 & 8 & & & & 12 & & \\ & 2 & 3 & & & 6 & & 8 & & & 11 & 12 & & \\ \overline{1} & & \overline{3} & & & \overline{6} & & \overline{8} & & \overline{10} & & \overline{12} & & \overline{14} \end{array}\right]\begin{matrix} 3 \\ 2 \\ 1 \end{matrix}.$$

We see that one cannot apply the second reduction operation with respect to $\alpha_2^{(3)} = 4$, and so we have $\delta = (2)$. Next we choose $\alpha_h^{(3)}$ to be $\alpha_3^{(3)} = 7$. Then we can apply the reduction operation four times to get an overpartition α in $P_{7,6,5;1}(125)$ along with $\delta = (4, 2)$:

$$\alpha = \left[\begin{array}{cccccccccccccc} & 2 & & 4 & 5 & & & 8 & & & & 12 & & \\ & 2 & 3 & & 5 & & & 8 & & & 11 & 12 & & \\ \overline{1} & & \overline{3} & & \overline{5} & & & \overline{8} & & \overline{10} & & \overline{12} & & \overline{14} \end{array}\right]\begin{matrix} 3 \\ 2 \\ 1 \end{matrix}.$$

We continue to consider $\alpha_4^{(3)} = 8$ as a choice of $\alpha_h^{(3)}$. Applying the second reduction operation four times, α becomes an overpartition in $P_{7,6,5;1}(121)$ as given below:

$$\alpha = \left[\begin{array}{cccccccccccccc} & 2 & & 4 & 5 & & 7 & & & & & 12 & & \\ & 2 & 3 & & 5 & & 7 & & & & 11 & 12 & & \\ \overline{1} & & \overline{3} & & \overline{5} & \overline{6} & & & & \overline{10} & & \overline{12} & & \overline{14} \end{array}\right]\begin{matrix} 3 \\ 2 \\ 1 \end{matrix}.$$

Meanwhile, adding 4 to δ as a new part, we get $\delta = (4, 4, 2)$.

For the remaining 3-marked part 12, applying the reduction operation eight times with respect to $\alpha_j = 12$, we get a partition $\delta = (8, 4, 4, 2) \in E_5$ and an overpartition $\mu \in Q_{7,6,5;1}(113)$ as given by

$$\mu = \left[\begin{array}{cccccccccccccc} & 2 & & 4 & 5 & & 7 & & 9 & & & & & \\ & 2 & 3 & & 5 & & 7 & 8 & & & & 12 & & \\ \overline{1} & & \overline{3} & & \overline{5} & \overline{6} & & \overline{8} & & & & \overline{12} & & \overline{14} \end{array}\right]\begin{matrix} 3 \\ 2 \\ 1 \end{matrix}.$$

Finally, we see that Theorem 3.2 follows from Theorem 4.1, Theorem 5.1 and the following generating function for $Q_{N_1,\ldots,N_{k-1};i}$ given by Chen, Sang and Shi [6]. For $k \geq 2$ and $1 \leq i \leq k - 1$, we have

$$\sum_{\mu \in Q_{N_1,\ldots,N_{k-1};i}} x^{l(\mu)} q^{|\mu|} = \frac{q^{\frac{(N_1+1)N_1}{2} + N_2^2 + \cdots + N_{k-1}^2 + N_{i+1} + \cdots + N_{k-1}} x^{N_1 + \cdots + N_{k-1}}}{(q)_{N_1 - N_2} \cdots (q)_{N_{k-2} - N_{k-1}}}, \tag{5.71}$$

where $l(\mu)$ denotes the number of parts of μ.

Proof of Theorem 3.2. Combining Theorems 4.1, 5.1, and the generating function (5.71), we obtain

$$\sum_{\lambda\in\widetilde{U}_{N_1,N_2,\ldots,N_{k-1};i}} q^{|\lambda|}$$
$$= \frac{(-q)_{N_1-1}}{(q^2;q^2)_{N_{k-1}}} \sum_{\lambda\in Q_{N_1,\ldots,N_{k-1};i}} q^{|\lambda|}$$
$$= \frac{q^{\frac{(N_1+1)N_1}{2}+N_2^2+\cdots+N_{k-1}^2+N_{i+1}+\cdots+N_{k-1}}(-q)_{N_1-1}}{(q)_{N_1-N_2}\cdots(q)_{N_{k-2}-N_{k-1}}(q^2;q^2)_{N_{k-1}}}.$$

A comparison with (3.23) completes the proof. ∎

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