Sun's log-concavity conjecture on the Catalan-Larcombe-French sequence

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Abstract

Let $\{P_n\}_{n\geq 0}$ denote the Catalan-Larcombe-French sequence, which naturally came from the series expansion of the complete elliptic integral of the first kind. In this paper, we prove the strict log-concavity of the sequence $\{\sqrt[n]{P_n}\}_{n\geq 1}$, which was originally conjectured by Z.W. Sun. We also obtain the strict log-concavity of the sequence $\{\sqrt[n]{V_n}\}_{n\geq 1}$, where $\{V_n\}_{n\geq 0}$ is the Fennessey-Larcombe-French sequence arising from the series expansion of the complete elliptic integral of the second kind.

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1 Introduction

The main purpose of this paper is to prove the log-concavity of the sequences $\{\sqrt[n]{P_n}\}_{n\geq 1}$ and $\{\sqrt[n]{V_n}\}_{n\geq 1}$, where $\{P_n\}_{n\geq 0}$ and $\{V_n\}_{n\geq 0}$, known as the Catalan-Larcombe-French sequence and the Fennessey-Larcombe-French sequence respectively, are given by

$$(n+1)^2 P_{n+1} = 8(3n^2 + 3n + 1)P_n - 128n^2 P_{n-1}, \tag{1.1}$$

$$n(n+1)^2 V_{n+1} = 8n(3n^2 + 5n + 1)V_n - 128(n-1)(n+1)^2 V_{n-1}, \tag{1.2}$$

with the initial values $P_0 = V_0 = 1$ and $P_1 = V_1 = 8$. These two sequences came naturally from the series expansions of the complete elliptic integrals. For more information on $\{P_n\}_{n\geq 0}$ and $\{V_n\}_{n\geq 0}$, see [1, 3, 4, 5, 6].

Let us first give an overview of some background. Recall that a sequence $\{a_n\}_{n\geq 0}$ of real numbers is said to be log-concave (resp. log-convex) if

$$a_n^2 \geq a_{n-1}a_{n+1} \ (\text{resp. } a_n^2 \leq a_{n-1}a_{n+1})$$

for all $n \geq 1$, and it is strictly log-concave (resp. strictly log-convex) if the inequality is strict. Clearly, for some integer $N > 0$, the positive sequence $\{a_n\}_{n\geq N}$ is log-concave (resp. log-convex) if and only if the sequence $\{a_{n+1}/a_n\}_{n\geq N}$ is decreasing (resp. increasing).

The strict log-concavity of $\{\sqrt[n]{P_n}\}_{n\geq1}$ was first conjectured by Sun [10], who also conjectured that the sequences $\{P_{n+1}/P_n\}_{n\geq0}$ and $\{\sqrt[n]{P_n}\}_{n\geq1}$ are strictly increasing. In fact, motivated by Firoozbakht's conjecture on the strictly decreasing property of the sequence $\{\sqrt[n]{p_n}\}_{n\geq1}$, where p_n is the n-th prime number (see [7, p. 185]), Sun [10] studied the monotonicity of many number theoretical and combinatorial sequences.

The strictly increasing property of the sequence $\{P_{n+1}/P_n\}_{n\geq0}$ and $\{\sqrt[n]{P_n}\}_{n\geq1}$ has been confirmed by Xia and Yao [12], and independently by Zhao [14]. We would like to point out that Zhao [14] only proved the log-convexity of $\{P_n\}_{n\geq0}$, which also implies the monotonicity of $\{\sqrt[n]{P_n}\}_{n\geq1}$ by using a result due to Wang and Zhu [11].

For a positive sequence $\{a_n\}_{n\geq0}$ satisfying a three-term recurrence relation, Chen, Guo and Wang [2] obtained a useful criterion to determine the log-concavity of $\{\sqrt[n]{a_n}\}_{n\geq N}$ for some positive integer N. While, their criterion does not apply to the the Catalan-Larcombe-French sequence, namely not on Sun's conjecture. The first main result of this paper is as follows.

Theorem 1.1. *The sequence $\{\sqrt[n]{P_n}\}_{n\geq1}$ is strictly log-concave, that is, for $n \geq 2$,*

$$\left(\sqrt[n]{P_n}\right)^2 > \sqrt[n-1]{P_{n-1}} \cdot \sqrt[n+1]{P_{n+1}}. \tag{1.3}$$

Since the Fennessey-Larcombe-French sequence $\{V_n\}_{n\geq0}$ is closely related to the sequence $\{P_n\}_{n\geq0}$, we are led to study the log-behavior of $\{\sqrt[n]{V_n}\}_{n\geq1}$. The second main result of this paper is as follows.

Theorem 1.2. *The sequence $\{\sqrt[n]{V_n}\}_{n\geq1}$ is strictly log-concave, that is, for $n \geq 2$,*

$$\left(\sqrt[n]{V_n}\right)^2 > \sqrt[n-1]{V_{n-1}} \cdot \sqrt[n+1]{V_{n+1}}. \tag{1.4}$$

This paper is organized as follows. In Section 2, we give a proof of Theorem 1.1 by establishing a lower bound and an upper bound for the ratio P_n/P_{n-1}. The proof of Theorem 1.2 is similar to that of Theorem 1.1, which will be given in Section 3. Finally, we give a new proof of the monotonicity of $\{\sqrt[n]{P_n}\}_{n\geq1}$ by using Theorem 1.1, and obtain the monotonicity of $\{\sqrt[n]{V_n}\}_{n\geq1}$ by using Theorem 1.2.

2 Proof of Theorem 1.1

In this section we aim to prove Theorem 1.1. To this end, we first establish a lower bound and an upper bound for the ratio P_n/P_{n-1} that will lead to the strict log-concavity of the sequence $\{\sqrt[n]{P_n}\}_{n\geq1}$.

Lemma 2.1. *For any integer $n \geq 1$, let*

$$f(n) = \frac{16(n-1)}{n}. \tag{2.1}$$

Then for $n \geq 5$, we have

$$f(n-1) < \frac{P_n}{P_{n-1}} < f(n). \tag{2.2}$$

Proof   For notational convenience, let $v(n) = P_n/P_{n-1}$. We first use induction on n to prove $v(n) > f(n-1)$ for $n \geq 5$. By the recurrence (1.1), we have

$$v(n+1) = \frac{8(3n^2 + 3n + 1)}{(n+1)^2} - \frac{128n^2}{(n+1)^2 v(n)}, \quad n \geq 1, \tag{2.3}$$

with the initial value $v(1) = 8$. Clearly, $v(5) = 2152/169 > 12 = f(4)$. Assume that $v(n) > f(n-1)$, and we proceed to prove that $v(n+1) > f(n)$ for $n \geq 5$. Observe that

$$
\begin{aligned}
v(n+1) - f(n) &= \frac{8(3n^2 + 3n + 1)}{(n+1)^2} - \frac{128n^2}{(n+1)^2 v(n)} - \frac{16(n-1)}{n} \\
&= \frac{8(n^3 + n^2 + 3n + 2)v(n) - 128n^3}{n(n+1)^2 v(n)}.
\end{aligned}
$$

By the induction hypothesis, we have $v(n) > f(n-1) > 0$ for $n \geq 5$, therefore

$$v(n+1) - f(n) > \frac{8(n^3 + n^2 + 3n + 2)f(n-1) - 128n^3}{n(n+1)^2 v(n)} = \frac{128(n^2 - 4n - 4)}{(n-1)n(n+1)^2 v(n)} > 0$$

for $n \geq 5$, since $n^2 - 4n - 4 = (n+1)(n-5) + 1 > 0$. This proves that $v(n) > f(n-1)$ for $n \geq 5$. Moreover, it is clear that $v(n)$ is positive for $n \geq 1$.

The inequality $v(n) < f(n)$ for $n \geq 5$ can be obtained in the same manner, so the detailed proof is omitted here. This completes the proof. $\square$

With the bounds given in Lemma 2.1 we are now able to prove Theorem 1.1.

Proof of Theorem 1.1   Note that for $n \geq 2$, the inequality (1.3) can be rewritten as

$$\frac{\sqrt[n]{P_n}}{\sqrt[n-1]{P_{n-1}}} > \frac{\sqrt[n+1]{P_{n+1}}}{\sqrt[n]{P_n}},$$

or equivalently,

$$\left(\frac{P_n}{P_{n-1}}\right)^{n(n+1)} > P_n^2 \left(\frac{P_{n+1}}{P_n}\right)^{n(n-1)}. \tag{2.4}$$

By the recurrence (1.1), it is easy to verify that (2.4) holds for $2 \leq n \leq 6$. We proceed to prove that (2.4) is true for $n \geq 7$. By Lemma 2.1 we have

$$\left(\frac{P_n}{P_{n-1}}\right)^{n(n+1)} - P_n^2 \left(\frac{P_{n+1}}{P_n}\right)^{n(n-1)} > (f(n-1))^{n(n+1)} - P_n^2 (f(n+1))^{n(n-1)}$$

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$$= \left(\frac{16(n-2)}{n-1} \right)^{n(n+1)} - P_n^2 \left(\frac{16n}{n+1} \right)^{n(n-1)}$$

$$= 16^{n(n-1)} \left(16^{2n} \left(\frac{n-2}{n-1} \right)^{n(n+1)} - P_n^2 \left(\frac{n}{n+1} \right)^{n(n-1)} \right).$$

It suffices to prove that

$$16^{2n} \left(\frac{n-2}{n-1} \right)^{n(n+1)} - P_n^2 \left(\frac{n}{n+1} \right)^{n(n-1)} > 0,$$

or equivalently

$$\frac{P_n^2}{16^{2n}} < \left(\frac{(n-2)(n+1)}{(n-1)n} \right)^{n(n-1)} \left(\frac{n-2}{n-1} \right)^{2n}.$$

Thus we only need to show that, for $n \geq 7$,

$$\frac{P_n}{16^n} < \left(\frac{(n-2)(n+1)}{(n-1)n} \right)^{\frac{n(n-1)}{2}} \left(\frac{n-2}{n-1} \right)^n. \tag{2.5}$$

Let l_n denote the term on the left hand side, and r_n denote the term on the right hand side. We claim that

(i) the sequence $\{l_n\}_{n \geq 5}$ is strictly decreasing, and

(ii) the sequence $\{r_n\}_{n \geq 5}$ is strictly increasing.

By Lemma 2.1, we see that

$$0 < \frac{l_n}{l_{n-1}} = \frac{P_n/P_{n-1}}{16} < 1$$

for $n \geq 5$, which implies (i).

We proceed to prove (ii). Note that

$$r_n = \left(1 - \frac{1}{\binom{n}{2}} \right)^{\binom{n}{2}} \cdot \left(1 - \frac{1}{n-1} \right)^{n-1} \cdot \left(1 - \frac{1}{n-1} \right).$$

The increasing property of the sequence $\{r_n\}_{n \geq 5}$ immediately follows from the well-known fact that the sequence $\{ \left(1 - \frac{1}{n} \right)^n \}_{n \geq 1}$ is strictly increasing.

It is easy to verify that $l_7 < r_7$. Combining (i) and (ii), we get that $l_n < r_n$ for $n \geq 7$, namely (2.5) holds. This completes the proof. $\qquad \square$

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3 Proof of Theorem 1.2

In this section, we complete the proof of Theorem 1.2 in a similar way of that of Theorem 1.1. For this purpose, we need a lower bound and an upper bound for the ratio V_n/V_{n-1}. For integer $n \geq 2$, let

$$h(n) = \frac{16(n^3 - n^2 + 1)}{n^3 - n^2}, \tag{3.1}$$

which was introduced by Yang and Zhao [13] in their study of the log-concavity of the sequence $\{V_n\}_{n\geq 1}$. They [13] showed that $h(n)$ is a lower bound for the ratio V_n/V_{n-1}, precisely,

$$\frac{V_n}{V_{n-1}} > h(n), \tag{3.2}$$

for $n \geq 4$. We further show that $h(n-1)$ is an upper bound for the ratio V_n/V_{n-1}.

Lemma 3.1. *Let $h(n)$ be given by (3.1). Then for $n \geq 11$, we have*

$$\frac{V_n}{V_{n-1}} < h(n-1). \tag{3.3}$$

Proof   Let $g(n) = V_n/V_{n-1}$, by (1.2) it is clear that

$$g(n+1) = \frac{8(3n^2 + 5n + 1)}{(n+1)^2} - \frac{128(n-1)}{ng(n)}, \quad n \geq 1, \tag{3.4}$$

with $g(1) = 8$. The inequality (3.3) can be proved inductively based on the recurrence (3.4). The proof is similar to that of Lemma 2.1, and hence is omitted here. $\qquad\square$

We proceed to prove Theorem 1.2.

Proof of Theorem 1.2   Note that for $n \geq 2$, the inequality (1.4) can be rewritten as

$$\frac{\sqrt[n]{V_n}}{\sqrt[n-1]{V_{n-1}}} > \frac{\sqrt[n+1]{V_{n+1}}}{\sqrt[n]{V_n}},$$

or equivalently,

$$\left(\frac{V_n}{V_{n-1}}\right)^{n(n+1)} > V_n^2 \left(\frac{V_{n+1}}{V_n}\right)^{n(n-1)}. \tag{3.5}$$

By the recurrence (1.2), it is easy to verify that (3.5) holds for $2 \leq n \leq 9$. We proceed to prove that (3.5) is true for $n \geq 10$. By (3.2) we have

$$\frac{V_n}{V_{n-1}} > h(n),$$

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for $n \geq 4$. By Lemma 3.1, we have

$$\frac{V_{n+1}}{V_n} < h(n),$$

for $n \geq 10$. Then for $n \geq 10$, it follows that

$$\left(\frac{V_n}{V_{n-1}}\right)^{n(n+1)} - V_n^2 \left(\frac{V_{n+1}}{V_n}\right)^{n(n-1)} > (h(n))^{n(n+1)} - V_n^2 (h(n))^{n(n-1)}$$

$$= (h(n))^{n(n-1)} \left((h(n))^n + V_n\right) \left((h(n))^n - V_n\right).$$

Clearly, both $(h(n))^{n(n-1)}$ and $((h(n))^n + V_n)$ are positive for $n \geq 10$. Thus it suffices to show that for $n \geq 10$,

$$(h(n))^n - V_n > 0. \tag{3.6}$$

Note that the relation (24) in [5] gave an upper bound of V_n, that is, for $n \geq 1$,

$$V_n < (2n+1)\binom{2n}{n}^2. \tag{3.7}$$

Sasvári [9, Corollary 1] showed that for $n \geq 1$,

$$\binom{2n}{n} < \frac{4^n}{\sqrt{\pi n}} e^{-\frac{1}{8n} + \frac{1}{192n^3}}. \tag{3.8}$$

It is clear that for $n \geq 1$,

$$0 < e^{-\frac{1}{8n} + \frac{1}{192n^3}} < 1.$$

Then it follows from (3.8) that for $n \geq 1$

$$\binom{2n}{n} < \frac{4^n}{\sqrt{\pi n}}. \tag{3.9}$$

Combining (3.7) and (3.9), for $n \geq 1$ we have

$$V_n < \frac{2n+1}{\pi n} 16^n < 16^n. \tag{3.10}$$

Note that for $n \geq 2$

$$h(n) = \frac{16(n^3 - n^2 + 1)}{n^3 - n^2} > 16. \tag{3.11}$$

Thus by (3.10) and (3.11), we obtain (3.6). This completes the proof. □

4 The monotonicity of $\{\sqrt[n]{P_n}\}_{n\geq 1}$ and $\{\sqrt[n]{V_n}\}_{n\geq 1}$

In this section, we aim to derive the monotonicity of both $\{\sqrt[n]{P_n}\}_{n\geq 1}$ and $\{\sqrt[n]{V_n}\}_{n\geq 1}$ from their log-concavity obtained above. The main result of this section is as follows.

Proposition 4.1. *Both $\{\sqrt[n]{P_n}\}_{n\geq 1}$ and $\{\sqrt[n]{V_n}\}_{n\geq 1}$ are strictly increasing.*

Proof   Recall that for a real sequence $\{a_n\}_{n\geq 0}$ with positive numbers, it was shown that

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n\to\infty} \sqrt[n]{a_n},$$

and

$$\limsup_{n\to\infty} \sqrt[n]{a_n} \leq \limsup_{n\to\infty} \frac{a_{n+1}}{a_n},$$

see Rudin [8, §3.37]. These two inequalities imply a well-known criterion, that is, if $\lim_{n\to\infty} \frac{a_n}{a_{n-1}} = c$, then $\lim_{n\to\infty} \sqrt[n]{a_n} = c$, where c is a real number.

Note that it was proved in [5, Eq. (30)] that

$$\lim_{n\to\infty} \frac{P_n}{P_{n-1}} = \lim_{n\to\infty} \frac{V_n}{V_{n-1}} = 16.$$

Then it follows that

$$\lim_{n\to\infty} \sqrt[n]{P_n} = \lim_{n\to\infty} \sqrt[n]{V_n} = 16.$$

Consequently,

$$\lim_{n\to\infty} \sqrt[n+1]{P_{n+1}}/\sqrt[n]{P_n} = \lim_{n\to\infty} \sqrt[n+1]{V_{n+1}}/\sqrt[n]{V_n} = 1.$$

By Theorems 1.1 and 1.2, we see that both $\{\sqrt[n+1]{P_{n+1}}/\sqrt[n]{P_n}\}_{n\geq 1}$ and $\{\sqrt[n+1]{V_{n+1}}/\sqrt[n]{V_n}\}_{n\geq 1}$ are strictly decreasing. Thus for all $n \geq 1$, we have

$$\sqrt[n+1]{P_{n+1}}/\sqrt[n]{P_n} > 1, \quad \text{and} \quad \sqrt[n+1]{V_{n+1}}/\sqrt[n]{V_n} > 1.$$

This completes the proof. $\qquad\square$

Remark. By employing a criterion due to Wang and Zhu [11, Theorem 2.1], Yang and Zhao [13] showed that the sequence $\{\sqrt[n]{V_{n+1}}\}_{n\geq 1}$ is strictly decreasing.

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