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Solution to a problem on hamiltonicity of graphs under Ore- and Fan-type heavy subgraph conditions

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Abstract A graph G is called *claw-o-heavy* if every induced claw ($K_{1,3}$) of G has two end-vertices with degree sum at least $|V(G)|$. For a given graph S, G is called *S-f-heavy* if for every induced subgraph H of G isomorphic to S and every pair of vertices $u,v \in V(H)$ with $d_H(u,v)=2$, there holds $\max\{d(u),d(v)\} \geq |V(G)|/2$. In this paper, we prove that every 2-connected claw-o-heavy and Z_3-f-heavy graph is hamiltonian (with two exceptional graphs), where Z_3 is the graph obtained by identifying one end-vertex of P_4 (a path with 4 vertices) with one vertex of a triangle. This result gives a positive answer to a problem proposed in [B. Ning, S. Zhang, Ore- and Fan-type heavy subgraphs for Hamiltonicity of 2-connected graphs, Discrete Math. 313 (2013) 1715–1725], and also implies two previous theorems of Faudree et al. and Chen et al., respectively.

Keywords Induced subgraphs · Claw-o-heavy graphs · f-Heavy subgraphs · Hamiltonicity

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1 Introduction

Throughout this paper, the graphs considered are simple, finite and undirected. For terminology and notation not defined here, we refer the reader to Bondy and Murty [2].

Let G be a graph. For a vertex $v \in V(G)$, we use $N_G(v)$ to denote the set, and $d_G(v)$ the number, of neighbors of v in G. When there is no danger of ambiguity, we use $N(v)$ and $d(v)$ instead of $N_G(v)$ and $d_G(v)$. If H and H' are two subgraphs of G, then we set $N_H(H') = \{v \in V(H) : N_G(v) \cap V(H') \neq \emptyset\}$. For two vertices $u, v \in V(H)$, the *distance* between u and v in H, denoted by $d_H(u,v)$, is the length of a shortest path connecting u and v in H. In particular, when we use the notation G to denote a graph, then for some subgraph H of G, we set $N_H(v) = N_G(v) \cap V(H)$ and $d_H(v) = |N_H(v)|$ (so, if G' is another graph defined on the same vertex set $V(G)$ and H is a subgraph of G', we will not use $N_H(v)$ to denote $N_{G'}(v) \cap V(H)$).

We call H an *induced subgraph* of G, if for every $x, y \in V(H)$, $xy \in E(G)$ implies that $xy \in E(H)$. For a given graph S, G is called *S-free* if G contains no induced subgraph isomorphic to S. Following [8], G is called *S-o-heavy* if every induced subgraph of G isomorphic to S contains two nonadjacent vertices with degree sum at least $|V(G)|$ in G. Following [9], G is called *S-f-heavy* if for every induced subgraph H isomorphic to S and any two vertices $u, v \in V(H)$ such that $d_H(u,v) = 2$, there holds $\max\{d(u), d(v)\} \geq |V(G)|/2$. Note that an S-free graph is S-o-heavy (S-f-heavy).

The *claw* is the bipartite graph $K_{1,3}$. Note that a claw-f-heavy graph is also claw-o-heavy. Further graphs that will be often considered as forbidden subgraphs are shown in Fig. 1.

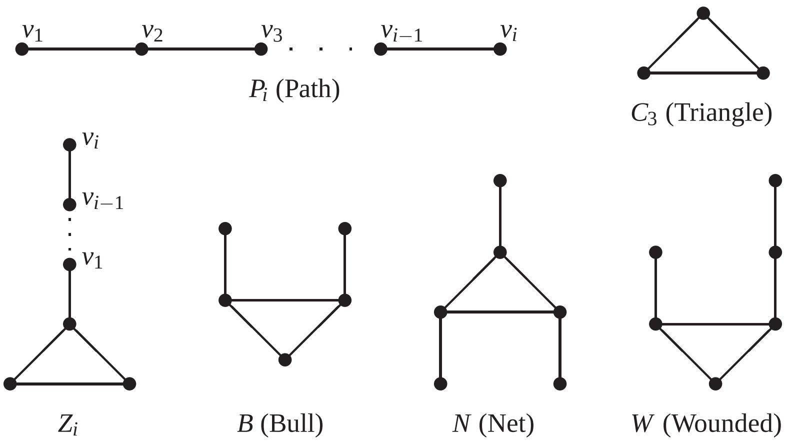

Fig. 1. Graphs P_i, C_3, Z_i, B, N and W.

Bedrossian [1] characterized all connected forbidden pairs for a 2-connected graph to be hamiltonian.

Theorem 1 (Bedrossian [1]) *Let G be a 2-connected graph and let R and S be connected graphs other than P_3. Then G being R-free and S-free implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = C_3, P_4, P_5, P_6, Z_1, Z_2, B, N$ or W.*

Faudree and Gould [6] extended Bedrossian's result by giving a proof of the 'only if' part based on infinite families of non-hamiltonian graphs.

Theorem 2 (Faudree and Gould [6]) *Let G be a 2-connected graph of order at least 10 and let R and S be connected graphs other than P_3. Then G being R-free and S-free implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = C_3, P_4, P_5, P_6, Z_1, Z_2, Z_3, B, N$ or W.*

Li et al. [8] extended Bedrossian's result by restricting Ore's condition to pairs of induced subgraphs of a graph. Ning and Zhang [9] gave another extension of Bedrossian's theorem by restricting Ore's condition to induced claws and Fan's condition to other induced subgraphs of a graph.

Theorem 3 (Ning and Zhang [9]) *Let G be a 2-connected graph and S be a connected graph other than P_3. Suppose that G is claw-o-heavy. Then G being S-f-heavy implies G is hamiltonian if and only if $S = P_4, P_5, P_6, Z_1, Z_2, B, N$ or W.*

Motivated by Theorems 2 and 3, Ning and Zhang [9] proposed the following problem.

Problem 1 (Ning and Zhang [9]) Is every claw-*o*-heavy and Z_3-*f*-heavy graph of order at least 10 hamiltonian?

The main goal of this paper is to give an affirmative solution to this problem. Our answer is the following theorem, where the graphs L_1 and L_2 are shown in Fig. 2.

Theorem 4 *Let G be a 2-connected graph. If G is claw-o-heavy and Z_3-f-heavy, then G is either hamiltonian or isomorphic to L_1 or L_2.*

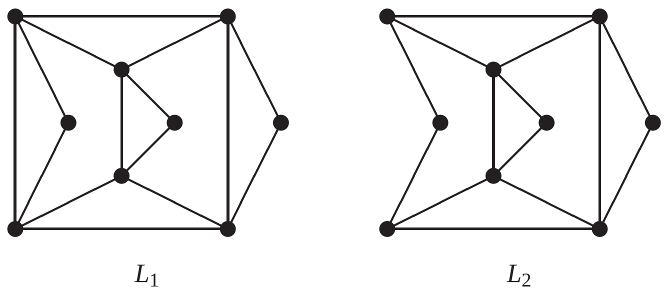

Fig. 2. Graphs L_1 and L_2.

Theorem 4 extends the following two previous theorems.

Theorem 5 (Faudree et al. [7]) *If G is a 2-connected claw-free and Z_3-free graph, then G is either hamiltonian or isomorphic to L_1 or L_2.*

Theorem 6 (Chen et al. [5]) *If G is a 2-connected claw-f-heavy and Z_3-f-heavy graph, then G is either hamiltonian or isomorphic to L_1 or L_2.*

We remark that there are infinite 2-connected claw-o-heavy and Z_3-o-heavy graphs which are non-hamiltonian, see [8].

Together with Theorem 3 and Theorem 4, we can obtain the following result which generalizes Theorem 2.

Theorem 7 *Let G be a 2-connected graph of order at least 10 and S be a connected graph other than P_3. Suppose that G is claw-o-heavy. Then G being S-f-heavy implies G is hamiltonian if and only if $S = P_4, P_5, P_6, Z_1, Z_2, Z_3, B, N$ or W.*

2 Preliminaries

In this section, we will list some necessary preliminaries. First, we will introduce the closure theory of claw-o-heavy graphs proposed by Čada [4], which is an extension of the closure theory of claw-free graphs due to Ryjáček [10].

Let G be a graph of order n. A vertex $x \in V(G)$ is called *heavy* if $d(x) \geq n/2$; otherwise, it is called *light*. A pair of nonadjacent vertices $\{x,y\} \subset V(G)$ is called a *heavy pair* of G if $d(x)+d(y) \geq n$.

Let G be a graph and $x \in V(G)$. Define $B_x^o(G) = \{uv : \{u,v\} \subset N(x),\ d(u)+d(v) \geq |V(G)|\}$. Let G_x^o be a graph with vertex set $V(G_x^o) = V(G)$ and edge set $E(G_x^o) = E(G) \cup B_x^o(G)$. Suppose that $G_x^o[N(x)]$ consists of two disjoint cliques C_1 and C_2. For a vertex $y \in V(G) \backslash (N(x) \cup \{x\})$, if $\{x,y\}$ is a heavy pair in G and there are two vertices $x_1 \in C_1$ and $x_2 \in C_2$ such that $x_1y, x_2y \in E(G)$, then y is called a *join vertex* of x in G. If $N(x)$ is not a clique and $G_x^o[N(x)]$ is connected, or $G_x^o[N(x)]$ consists of two disjoint cliques and there is some join vertex of x, then the vertex x is called an *o-eligible vertex* of G. The *locally completion of G at x*, denoted by G_x', is the graph with vertex set $V(G_x') = V(G)$ and edge set $E(G_x') = E(G) \cup \{uv : u,v \in N(x)\}$.

Let G be a claw-o-heavy graph. The *closure* of G, denoted by $cl_o(G)$, is the graph such that:
(1) there is a sequence of graphs $G_1, G_2, \ldots, G_t$ such that $G = G_1$, $G_t = cl_o(G)$, and for any $i \in \{1,2,\ldots,t-1\}$, there is an o-eligible vertex x_i of G_i, such that $G_{i+1} = (G_i)'_{x_i}$; and
(2) there is no o-eligible vertex in G_t.

Theorem 8 (Čada [4]) *Let G be a claw-o-heavy graph. Then*
(1) *the closure $cl_o(G)$ is uniquely determined;*
(2) *there is a C_3-free graph H such that $cl_o(G)$ is the line graph of H; and*
(3) *the circumferences of $cl_o(G)$ and G are equal.*

Now we introduce some new terminology and notations. Let G be a claw-o-heavy graph and C be a maximal clique of $cl_o(G)$. We call $G[C]$ a *region* of G. For a vertex v of G, we call v an *interior vertex* if it is contained in only one region, and a *frontier vertex* if it is contained in two distinct regions. For two vertices $u,v \in V(G)$, we say u and v are *associated* if u,v are contained in a common region of G; otherwise u and v are *dissociated*. For a region R of G, we denote by I_R the set of interior vertices of R, and by F_R the set of frontier vertices of R.

From the definition of the closure, it is not difficult to get the following lemma.

Lemma 1 *Let G be a claw-o-heavy graph. Then*
(1) *every vertex is either an interior vertex of a region or a frontier vertex of two regions;*
(2) *every two regions are either disjoint or have only one common vertex; and*
(3) *every pair of dissociated vertices have degree sum less than $|V(G)|$ in $cl_o(G)$ (and in G).*

Proof In the proof of the lemma, we let $G' = cl_o(G)$.

(1) Let v be an arbitrary vertex of G. Since G' is closed, $N_{G'}(v)$ is either a clique or a disjoint union of two cliques in G'. Thus v is contained in one or two regions of G, and the assertion is true.

(2) Let R and R' be two regions of G, and C and C' be the two maximal cliques of G' corresponding to R and R', respectively. If C and C' have two common vertices, say u and v, then u and v will be o-eligible vertices of G', contradicting the definition of the closure of G. This implies that C and C' (and then, R and R') have at most one common vertex.

(3) Let u,v be two nonadjacent vertices with $d_{G'}(u)+d_{G'}(v) \geq n = |V(G)|$. Then u,v have at least two common neighbors in G'. Suppose that u and v are not in a common clique of G'. Let x be a common neighbor of u and v in G'. Since $N_{G'}(x)$ is not a clique in G', it is the disjoint union of two cliques, one containing u and the other containing v. Since $uv \in B_x^o(G')$, x is an o-eligible vertex of G', a contradiction. Thus we conclude that u,v are in a common clique of G', i.e., u and v are associated. □

The next lemma provides some structural information on regions.

Lemma 2 *Let G be a claw-o-heavy graph and R be a region of G. Then*
(1) *R is nonseparable;*
(2) *if v is a frontier vertex of R, then v has an interior neighbor in R or R is complete and has no interior vertices;*
(3) *for any two vertices $u,v \in R$, there is an induced path of G from u to v such that every internal vertex of the path is an interior vertex of R; and*
(4) *for two vertices u,v in R, if $\{u,v\}$ is a heavy pair of G, then u,v have two common neighbors in I_R.*

Proof Let $G_1,G_2,\ldots,G_t$ be the sequence of graphs, and $x_1,x_2,\ldots,x_{t-1}$ the sequence of vertices in the definition of $cl_o(G)$.

(1) Suppose that R has a cut-vertex y. We prove by induction that y would be a cut-vertex of $G_i[V(R)]$ for all $i \in [1,t]$. Since y is a cut-vertex of $G_1[V(R)] = R$, we assume that $2 \leq i \leq t$. By the induction hypothesis, y is a cut-vertex of $G_{i-1}[V(R)]$. Let R' and R'' be two components of $G_{i-1}[V(R)]-y$, u be a vertex of R' and v be a vertex of R''. Then u and v have at most one common neighbor y in R. Note that each two maximal cliques of $cl_o(G)$ is either disjoint or have only one common vertex (see Lemma 1 (1)). This implies that u and v have no common neighbors in $G_{i-1}-V(R)$. Hence $\{u,v\}$ is not a heavy pair of G. Note that an o-eligible vertex of G_{i-1} will be an interior vertex of $cl_o(G)$. This implies that y is not an o-eligible vertex of G_{i-1}. Thus $x_{i-1} \neq y$. Note that x_{i-1} has no neighbors in R' or has no neighbors in R''. This

implies that there are no new edges in G_i between R' and R''. Thus y is also a cut-vertex of $G_i[V(R)]$. By induction, we can see that y is a cut-vertex of $cl_o(G)[V(R)]$, contradicting the fact that $V(R)$ is a clique in $cl_o(G)$.

(2) Note that $cl_o(G)[V(R)]$ is complete. If R has no interior vertex, then R contains no o-eligible vertex of G. Since the locally completion of G at every o-eligible vertex does not add an edge in R, $R = cl_o(G)[V(R)]$ is complete.

Now we assume that R has at least one interior vertex. Suppose that v has no interior neighbors in R, i.e., $N(v)\cap I_R = \emptyset$. Using induction, we will prove that $N_{G_i}(v)\cap I_R = \emptyset$. Since $N_{G_1}(v)\cap I_R = \emptyset$, we assume that $2 \le i \le t$. By the induction hypothesis, $N_{G_{i-1}}(v)\cap I_R = \emptyset$. Note that x_{i-1} is either nonadjacent to v or nonadjacent to every vertex in $N_{G_{i-1}}(v)\cap V(R)$. This implies that there are no new edges of G_i between v and $G_i[V(R)]-v$. Hence $N_{G_i}(v)\cap I_R = \emptyset$. Thus by the induction hypothesis, we can see that $N_{cl_o(G)}(v)\cap I_R = \emptyset$, a contradiction.

(3) We use induction on $t-i$ (t is the subscript of $G_t = cl_o(G)$) to prove that there is an induced path of $G_i[V(R)]$ from u to v such that every internal vertex of the path is an interior vertex of R. Note that uv is an edge in $G_t[V(R)]$. We are done if $i = t$. Now suppose that there is an induced path P of $G_i[V(R)]$ from u to v such that every internal vertex of the path is an interior vertex of R. We will prove that there is an induced path of $G_{i-1}[V(R)]$ from u to v such that every internal vertex of the path is an interior vertex of R. If P is also a path of $G_{i-1}[V(R)]$, then we are done. So we assume that there is an edge $u'v' \in E(P)$ such that $u'v' \notin E(G_{i-1})$. This implies that $u',v' \in N(x_{i-1})$. Since P is an induced path of G_i, x_{i-1} has the only two neighbors u',v' on P. We also note that $x_{i-1} \in V(R)$ is an interior vertex. Thus $P' = (P-u'v')\cup u'xv'$ (with the obvious meaning) is an induced path of $G_{i-1}[V(R)]$ from u to v such that every internal vertex of the path is an interior vertex of R. Thus by the induction hypothesis, the proof is complete.

(4) Since every vertex in F_R has at least one neighbor in $G-R$ and every vertex in $G-R$ has at most one neighbor in F_R, we have $|N_{G-R}(F_R\backslash\{u,v\})| \ge |F_R\backslash\{u,v\}|$. Furthermore, we have $n = |I_R\backslash\{u,v\}| + |F_R\backslash\{u,v\}| + |V(G-R)| + 2$. Thus, we get

$$
\begin{aligned}
n &\le d(u)+d(v)\\
&= d_{I_R}(u)+d_{I_R}(v)+d_{F_R}(u)+d_{F_R}(v)+d_{G-R}(u)+d_{G-R}(v)\\
&\le d_{I_R}(u)+d_{I_R}(v)+2|F_R\backslash\{u,v\}|+d_{G-R}(u)+d_{G-R}(v)\\
&\le d_{I_R}(u)+d_{I_R}(v)+|F_R\backslash\{u,v\}|+|N_{G-R}(F_R\backslash\{u,v\})|+|N_{G-R}(u)|+|N_{G-R}(v)|\\
&= d_{I_R}(u)+d_{I_R}(v)+|F_R\backslash\{u,v\}|+|N_{G-R}(F_R)|\\
&\le d_{I_R}(u)+d_{I_R}(v)+|F_R\backslash\{u,v\}|+|V(G-R)|,
\end{aligned}
$$

and

$$d_{I_R}(u)+d_{I_R}(v) \ge n-|F_R\backslash\{u,v\}|-|V(G-R)| = |I_R\backslash\{u,v\}|+2.$$

This implies that u,v have two common neighbors in I_R. □

Let G be a graph and Z be an induced copy of Z_3 in G. We denote the vertices of Z as in Fig. 3, and say that Z is *center-heavy* in G if a_1 is a heavy vertex of G. If every induced copy of Z_3 in G is center-heavy, then we say that G is Z_3-*center-heavy*.

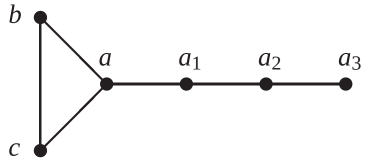

Fig. 3. The Graph Z_3.

Lemma 3 *Let G be a claw-o-heavy and Z_3-f-heavy graph. Then $cl_o(G)$ is Z_3-center-heavy.*

Proof Let Z be an arbitrary induced copy of Z_3 in $G' = cl_o(G)$. We denote the vertices of Z as in Fig. 3, and will prove that a_1 is heavy in G'.

Let R be the region of G containing $\{a,b,c\}$. Recall that I_R is the set of interior vertices of R, and F_R is the set of frontier vertices of R.

Claim 1 $|N_R(a_2) \cup N_R(a_3)| \leq 1$.

Proof Note that every vertex in $G-R$ has at most one neighbor in R. If $N_R(a_2) = \emptyset$, then the assertion is obviously true. Now we assume that $N_R(a_2) \neq \emptyset$. Let x be the vertex in $N_R(a_2)$. Clearly $x \neq a$ and $a_1x \notin E(G')$. If $a_3x \notin E(G')$, then $\{a_2,a_1,a_3,x\}$ induces a claw in G', a contradiction. This implies that $a_3x \in E(G')$, and x is the unique vertex in $N_{G'}(a_3) \cap V(R)$. Thus $N_R(a_2) \cup N_R(a_3) = \{x\}$. □

Claim 2 *Let x,y be two vertices in $I_R \cup \{a\}$. If $xy \in E(G)$ and $d(x)+d(y) \geq n$, then x,y have a common neighbor in I_R.*

Proof Note that every vertex in F_R has at least one neighbor in $G-R$ and every vertex in $G-R$ has at most one neighbor in R. By Claim 1, $|V(G-R)| \geq |F_R|+1$. Moreover, since a is not the neighbor of a_2 and a_3 in R, $|V(G-R)| \geq |F_R \backslash \{a\}| + |N_{G-R}(a)| + 1$.

If $x,y \in I_R$, then

$$\begin{aligned} n &\leq d(x)+d(y) \\ &= d_{I_R}(x)+d_{I_R}(y)+d_{F_R}(x)+d_{F_R}(y) \\ &\leq d_{I_R}(x)+d_{I_R}(y)+2|F_R| \\ &\leq d_{I_R}(x)+d_{I_R}(y)+|F_R|+|V(G-R)|-1, \end{aligned}$$

and

$$d_{I_R}(x)+d_{I_R}(y) \geq n-|F_R|-|V(G-R)|+1 = |I_R|+1.$$

This implies that x,y have a common neighbor in I_R.

If one of x,y, say y is a, then

$$\begin{aligned} n &\leq d(x)+d(a) \\ &= d_{I_R}(x)+d_{I_R}(a)+d_{F_R}(x)+d_{F_R}(a)+d_{G-R}(a) \\ &\leq d_{I_R}(x)+d_{I_R}(a)+|F_R|+|F_R \backslash \{a\}|+d_{G-R}(a) \\ &\leq d_{I_R}(x)+d_{I_R}(a)+|F_R|+|V(G-R)|-1, \end{aligned}$$

and

$$d_{I_R}(x)+d_{I_R}(a)\geq n-|F_R|-|V(G-R)|+1=|I_R|+1.$$

This implies that x,a have a common neighbor in I_R. $\square$

By Lemma 2 (3), G has an induced path P from a to a_3 such that every vertex of P is either in $\{a,a_1,a_2,a_3\}$ or an interior vertex outside R. Let a,a_1',a_2',a_3' be the first four vertices of P.

Note that a_1' is either a_1 or an interior vertex in the region containing $\{a,a_1\}$. This implies that $d_{G'}(a_1)\geq d_{G'}(a_1')\geq d(a_1')$. If a_1' is heavy in G, then a_1 is heavy in G' and we are done. So we assume that a_1' is not heavy in G.

If $abca$ is also a triangle in G, then the subgraph induced by $\{a,b,c,a_1',a_2',a_3'\}$ is a Z_3. Since G is Z_3-f-heavy and a_1' is not heavy in G, b and a_3' are heavy in G. By Lemma 1 (3), b and a_3' are associated, a contradiction. Thus we conclude that one edge of $\{ab,ac,bc\}$ is not in $E(G)$.

Note that R is not complete. By Lemma 2 (2), a has a neighbor in I_R.

Claim 3 $d_{I_R}(a)=1$.

Proof Suppose that $d_{I_R}(a)\geq 2$. Let x,y be two arbitrary vertices in $N_{I_R}(a)$. If $xy\in E(G)$, then $\{a,x,y,a_1',a_2',a_3'\}$ induces a Z_3 in G. Note that a_1' is not heavy in G. Thus x and a_3' are heavy in G. Note that x and a_3' are dissociated, a contradiction. This implies that $N_{I_R}(a)$ is an independent set.

Since $\{a,x,y,a_1'\}$ induces a claw in G, and $\{a_1',x\}$, $\{a_1',y\}$ are not heavy pairs of G by Lemma 1 (3), we have $\{x,y\}$ is a heavy pair of G. We assume without loss of generality that x is heavy in G.

If a is also heavy in G, then by Claim 2, a,x have a common neighbor in I_R, contradicting the fact that $N_{I_R}(a)$ is an independent set. So we conclude that a is not heavy in G.

Since $\{x,y\}$ is a heavy pair of G, by Lemma 2 (4), x,y have two common neighbors in I_R. Let x',y' be two vertices in $N_{I_R}(x)\cap N_{I_R}(y)$. Clearly $ax',ay'\notin E(G)$. If $x'y'\in E(G)$, then $\{x,x',y',a,a_1',a_2'\}$ induces a Z_3 in G. Since a is light, x',a_2' are heavy. Note that x' and a_2' are dissociated, a contradiction. Thus we obtain that $x'y'\notin E(G)$.

Note that $\{x,x',y',a\}$ induces a claw in G, and a is light in G. So one vertex of $\{x',y'\}$, say x', is heavy in G. By Claim 2, x,x' have a common neighbor x'' in I_R. Clearly $ax''\notin E(G)$. Thus $\{x,x',x'',a,a_1',a_2'\}$ induces a Z_3. Since a is not heavy in G, x',a_2' are heavy in G, a contradiction. $\square$

Now let $N_{I_R}(a)=\{x\}$.

Claim 4 $N_R(a)=V(R)\backslash\{a\}$.

Proof Suppose that $V(R)\backslash(\{a\}\cup N_R(a))\neq\emptyset$. By Lemma 2 (1), $R-x$ is connected. Let y be a vertex in $V(R)\backslash(\{a\}\cup N_R(a))$ such that a,y have a common neighbor z in $R-x$. Note that z is a frontier vertex of R. Let z' be a vertex in $N_{G-R}(z)$. Then $\{z,y,a,z'\}$ induces a claw in G. Since $\{a,z'\}$, $\{y,z'\}$ are not heavy pairs of G, $\{a,y\}$ is a heavy pair of G. By Lemma 2 (4), a,y have two common neighbors in I_R, contradicting Claim 3. $\square$

By Claims 3 and 4, we can see that $|I_R| = 1$. Recall that one edge of $\{ab, bc, ac\}$ is not in $E(G)$. By Claim 4, $ab, ac \in E(G)$. This implies that $bc \notin E(G)$, and $\{a, b, c, a'_1\}$ induces a claw in G. Since $\{b, a'_1\}$, $\{c, a'_1\}$ are not heavy pairs of G, $\{b, c\}$ is a heavy pair of G. By Lemma 2 (4), b and c have two common neighbors in I_R, contradicting the fact that $|I_R| = 1$. □

Following [3], we define $\mathscr{P}$ to be the class of graphs obtained by taking two vertex-disjoint triangles $a_1a_2a_3a_1$, $b_1b_2b_3b_1$ and by joining every pair of vertices $\{a_i, b_i\}$ by a path $P_{k_i} = a_ic_i^1c_i^2 \cdots c_i^{k_i-2}b_i$, for $k_i \geq 3$ or by a triangle $a_ib_ic_ia_i$. We denote the graphs in $\mathscr{P}$ by P_{l_1,l_2,l_3}, where $l_i = k_i$ if a_i, b_i are joined by a path P_{k_i}, and $l_i = T$ if a_i, b_i are joined by a triangle. Note that $L_1 = P_{T,T,T}$ and $L_2 = P_{3,T,T}$.

Theorem 9 (Brousek [3]) *Every non-hamiltonian 2-connected claw-free graph contains an induced subgraph $H \in \mathscr{P}$.*

3 Proof of Theorem 4

Let $G' = cl_o(G)$. If G' is hamiltonian, then so is G by Theorem 8, and we are done. Now we assume that G' is not hamiltonian. By Theorem 9, G' contains an induced subgraph $H = P_{l_1,l_2,l_3} \in \mathscr{P}$. We denote the vertices of H by a_i, b_i, c_i and c_i^j as in Section 2. By Lemma 3, G' is Z_3-center-heavy.

Claim 1 *For $i \in \{1, 2, 3\}$, $l_i = 3$ or T; and at most one of $\{l_1, l_2, l_3\}$ is 3.*

Proof If one of $\{l_1, l_2, l_3\}$ is at least 4, say $l_1 \geq 4$, then the subgraph of G' induced by $\{a_1, a_2, a_3, c_1^1, c_1^2, c_1^3\}$ is a Z_3 (we set $c_1^3 = b_1$ if $l_1 = 4$). Thus c_1^1 is heavy in G'. If $l_2 = T$, then the subgraph of G' induced by $\{a_2, a_1, a_3, b_2, b_1, c_1^{l_1-2}\}$ is a Z_3, implying b_2 is heavy in G'. But c_1^1 and b_2 are dissociated, a contradiction. If $l_2 \neq T$, then the subgraph of G' induced by $\{a_2, a_1, a_3, c_2^1, \ldots, c_2^{l_2-2}, b_2, b_1\}$ is a Z_r with $r \geq 3$, implying c_2^1 is heavy in G'. But c_1^1 and c_2^1 are dissociated, a contradiction again. Thus we conclude that $l_i = 3$ or T for all $i = 1, 2, 3$.

If two of $\{l_1, l_2, l_3\}$ equal 3, say $l_1 = l_2 = 3$, then the subgraphs of G' induced by $\{a_1, a_2, a_3, c_1^1, b_1, b_2\}$ and by $\{a_2, a_1, a_3, c_2^1, b_2, b_1\}$ are Z_3's. This implies that c_1^1 and c_2^1 are heavy in G'. But c_1^1 and c_2^1 are dissociated, a contradiction. Thus we conclude that at most one of $\{l_1, l_2, l_3\}$ is 3. □

By Claim 1, we assume without loss of generality that $l_2 = l_3 = T$ and $l_1 = 3$ or T. If G' has only the nine vertices in H, then $G' = L_1$ or L_2, and G has no o-eligible vertices. This implies that $G = L_1$ or L_2. Now we assume that G' has a tenth vertex.

Let A be the region containing $\{a_1, a_2, a_3\}$ and B be the region containing $\{b_1, b_2, b_3\}$. For $l_i = T$, let C_i be the region containing $\{a_i, b_i, c_i\}$; and if $l_1 = 3$, then let C_1^1 and C_1^2 be the regions containing $\{a_1, c_1^1\}$ and $\{b_1, c_1^1\}$, respectively.

Claim 2 $|V(A)| = |V(B)| = |V(C_i)| = 3$*; and if $l_1 = 3$, then $|V(C_1^1)| = |V(C_1^2)| = 2$.*

Proof Suppose that $|V(A)| \geq 4$. Let x be a vertex in $V(A)\backslash\{a_1,a_2,a_3\}$. Then the subgraphs of G' induced by $\{a_2,a_1,x,b_2,b_3,c_3\}$ and by $\{a_3,a_1,x,b_3,b_2,c_2\}$ are Z_3's. This implies that b_2 and b_3 are heavy in G'. Since there are two vertices a_1,x nonadjacent to b_2 and b_3, b_2 and b_3 have at least two common neighbors in G'. Let y be a common neighbor of b_2 and b_3 in G' other than b_1. Then $y \in V(B)$, and the subgraphs of G' induced by $\{b_2,b_1,y,a_2,a_3,c_3\}$ is a Z_3. Thus a_2 is heavy in G'. By Lemma 1 (3), a_2 and b_3 are associated, a contradiction. Thus we conclude that $|V(A)| = 3$, and similarly, $|V(B)| = 3$.

Suppose that $|V(C_i)| \geq 4$ for $l_i = T$. We assume up to symmetry that $|V(C_2)| \geq 4$. Let x be a vertex in $V(C_2)\backslash\{a_2,b_2,c_2\}$. Then the subgraph of G' induced by $\{a_2,c_2,x,a_3,b_3,b_1\}$ is a Z_3, implying that a_3 is heavy in G. If $l_1 = T$, then the subgraph of G' induced by $\{b_2,c_2,x,b_1,a_1,a_3\}$ is a Z_3; if $l_1 = 3$, then the subgraph of G' induced by $\{b_2,c_2,x,b_1,c_1,a_1\}$ is a Z_3. In any case, we have b_1 is heavy in G'. But a_3 and b_1 are dissociated in G, a contradiction.

Suppose that $l_1 = 3$ and $|V(C_1^1)| \geq 3$. Let x be a vertex in $V(C_1^1)\backslash\{a_1,c_1^1\}$. Then the subgraphs of G' induced by $\{a_1,c_1^1,x,a_2,b_2,b_3\}$ and by $\{c_1^1,a_1,x,b_1,b_2,c_2\}$ are Z_3's. This implies that a_2 and b_1 are heavy in G'. But a_2 and b_1 are dissociated, a contradiction. Thus we conclude that $|V(C_1^1)| = 2$, and similarly, $|V(C_1^2)| = 2$. □

In the following, we set $S = \{v \in V(G') : N_{G'}(v) \cap V(H) \neq \emptyset\}$.

Claim 3 $l_1 = 3$, *and for* $x \in S$, $xc_2, xc_3 \in E(G')$.

Proof By Claim 2, all the neighbors of a_1,a_2,a_3,b_1,b_2,b_3 and c_1^1 (if $l_1 = 3$) are in H. Note that G' has at least 10 vertices. The vertices a_1,a_2,a_3,b_1,b_2,b_3 and c_1^1 (if $l_1 = 3$) are not heavy in G'.

Let x be a vertex in S. Suppose that $l_1 = T$. Note that x cannot be adjacent to all the three vertices c_1,c_2,c_3. We assume up to symmetry that $xc_1 \in E(G')$ and $xc_2 \notin E(G')$. Then the subgraph of G' induced by $\{a_2,b_2,c_2,a_1,c_1,x\}$ is a Z_3, implying a_1 is heavy in G', a contradiction. Thus we conclude that $l_1 = 3$.

Suppose that one edge of xc_2,xc_3 is not in $E(G')$, say $xc_2 \notin E(G')$. Then the subgraph of G' induced by $\{a_2,b_2,c_2,a_3,c_3,x\}$ is a Z_3, implying a_3 is heavy in G', a contradiction. Thus we conclude that $xc_2,xc_3 \in E(G')$. □

Let x be a vertex in S. By Claim 3, $xc_2,xc_3 \in E(G')$. If G' has only ten vertices, then $C = a_1a_2a_3c_3xc_2b_2b_3b_1c_1^1a_1$ is a Hamilton cycle of G', a contradiction. Suppose now that G' has an eleventh vertex. Since G' is 2-connected, let x' be a vertex in $S\backslash\{x\}$. By Claim 3, $x'c_2,x'c_3 \in E(G')$. Thus $xx' \in E(G')$. Note that $N_{G'}(x)$ is neither a clique nor a disjoint union of two cliques of G'. This implies that x is an o-eligible vertex of G', a contradiction.

The proof is complete. □

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