Orbital Instability of Standing Waves for the Generalized 3D Nonlocal Nonlinear Schrödinger Equations *

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Abstract: The existence and orbital instability of standing waves for the generalized three-dimensional nonlocal nonlinear Schrödinger equations is studied. By defining some suitable functionals and a constrained variational problem, we first establish the existence of standing waves, which relys on the inner structure of the equations under consideration to overcome the drawback that nonlocal terms violate the space-scale invariance. We then show the orbital instability of standing waves. The arguments depend upon the conservation laws of the mass and of the energy.

Key words: Nonlocal Nonlinear Schrödinger Equations, Standing waves, Orbital instability

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1. Introduction

In this paper, we study the generalized three-dimensional nonlocal nonlinear Schrödinger equations :

$$\begin{aligned} &i\partial_t E_1 + \Delta E_1 + \left(|E_1|^2 + |E_2|^2 + |E_3|^2\right) E_1 \\ &\qquad + A_1(E_1, E_2, E_3) + A_2(E_1, E_2, E_3) = 0, \end{aligned} \tag{1.1}$$

$$\begin{aligned} &i\partial_t E_2 + \Delta E_2 + \left(|E_1|^2 + |E_2|^2 + |E_3|^2\right) E_2 \\ &\qquad + A_3(E_1, E_2, E_3) + A_4(E_1, E_2, E_3) = 0, \end{aligned} \tag{1.2}$$

$$\begin{aligned} &i\partial_t E_3 + \Delta E_3 + \left(|E_1|^2 + |E_2|^2 + |E_3|^2\right) E_3 \\ &\qquad + A_5(E_1, E_2, E_3) + A_6(E_1, E_2, E_3) = 0, \end{aligned} \tag{1.3}$$

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along with the initial data

$$E_1(0,x) = E_{10}(x), E_2(0,x) = E_{20}(x), E_3(0,x) = E_{30}(x). \tag{1.4}$$

Here,

$$\begin{aligned} A_1(E_1,E_2,E_3) = -E_2\mathcal{F}^{-1}\Big\{ \tfrac{\eta}{|\xi|^2-\delta} \big[& \xi_1\xi_3\mathcal{F}(E_2\overline{E_3} - \overline{E_2}E_3) \\ & - (\xi_1^2+\xi_2^2)\mathcal{F}(E_1\overline{E_2} - \overline{E_1}E_2) + \xi_2\xi_3\mathcal{F}(\overline{E_1}E_3 - E_1\overline{E_3})\big]\Big\}, \end{aligned} \tag{$I-1$}$$

$$\begin{aligned} A_2(E_1,E_2,E_3) = E_3\mathcal{F}^{-1}\Big\{ \tfrac{\eta}{|\xi|^2-\delta} \big[& \xi_2\xi_3\mathcal{F}(E_1\overline{E_2} - \overline{E_1}E_2) \\ & - (\xi_1^2+\xi_3^2)\mathcal{F}(\overline{E_1}E_3 - E_1\overline{E_3}) + \xi_1\xi_2\mathcal{F}(E_2\overline{E_3} - \overline{E_2}E_3)\big]\Big\}, \end{aligned} \tag{$I-2$}$$

$$\begin{aligned} A_3(E_1,E_2,E_3) = -E_3\mathcal{F}^{-1}\Big\{ \tfrac{\eta}{|\xi|^2-\delta} \big[& \xi_1\xi_2\mathcal{F}(\overline{E_1}E_3 - E_1\overline{E_3}) \\ & - (\xi_2^2+\xi_3^2)\mathcal{F}(E_2\overline{E_3} - \overline{E_2}E_3) + \xi_1\xi_3\mathcal{F}(E_1\overline{E_2} - \overline{E_1}E_2)\big]\Big\}, \end{aligned} \tag{$I-3$}$$

$$\begin{aligned} A_4(E_1,E_2,E_3) = E_1\mathcal{F}^{-1}\Big\{ \tfrac{\eta}{|\xi|^2-\delta} \big[& \xi_1\xi_3\mathcal{F}(E_2\overline{E_3} - \overline{E_2}E_3) \\ & - (\xi_1^2+\xi_2^2)\mathcal{F}(E_1\overline{E_2} - \overline{E_1}E_2) + \xi_2\xi_3\mathcal{F}(\overline{E_1}E_3 - E_1\overline{E_3})\big]\Big\}, \end{aligned} \tag{$I-4$}$$

$$\begin{aligned} A_5(E_1,E_2,E_3) = -E_1\mathcal{F}^{-1}\Big\{ \tfrac{\eta}{|\xi|^2-\delta} \big[& \xi_2\xi_3\mathcal{F}(E_1\overline{E_2} - \overline{E_1}E_2) \\ & - (\xi_1^2+\xi_3^2)\mathcal{F}(\overline{E_1}E_3 - E_1\overline{E_3}) + \xi_1\xi_2\mathcal{F}(E_2\overline{E_3} - \overline{E_2}E_3)\big]\Big\}, \end{aligned} \tag{$I-5$}$$

$$\begin{aligned} A_6(E_1,E_2,E_3) = E_2\mathcal{F}^{-1}\Big\{ \tfrac{\eta}{|\xi|^2-\delta} \big[& \xi_1\xi_2\mathcal{F}(\overline{E_1}E_3 - E_1\overline{E_3}) \\ & - (\xi_2^2+\xi_3^2)\mathcal{F}(E_2\overline{E_3} - \overline{E_2}E_3) + \xi_1\xi_3\mathcal{F}(E_1\overline{E_2} - \overline{E_1}E_2)\big]\Big\}, \end{aligned} \tag{$I-6$}$$

$\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and the Fourier inverse transform, respectively ([**?**, **?**, **?**, **?**]), $\eta > 0$ and $\delta \le 0$ are two constants, $(E_1, E_2, E_3)(t, x)$ are complex vector-valued functions from $\mathbb{R}^+ \times \mathbb{R}^3$ into $\mathbb{C}^3$, $\overline{E}_i (i = 1, 2, 3)$ denotes the complex conjugate of E_i. Due to rotational invariance of (1.1)-(1.3), let $\mathbf{E} = (E_1, E_2, E_3)$ and $\xi = (\xi_1, \xi_2, \xi_3)$, system (1.1)-(1.3) is equivalent to a vetor-valued nonlinear Schrödinger equations

$$i\mathbf{E}_t + \Delta\mathbf{E} + |\mathbf{E}|^2\mathbf{E} + i(\mathbf{E} \wedge \mathbf{B}) = 0, \tag{$M-S-1$}$$

$$\mathbf{B}(\mathbf{E}) = \mathcal{F}^{-1}\left[\frac{i\eta}{|\xi|^2 - \delta}(\xi \wedge (\xi \wedge \mathcal{F}(\mathbf{E} \wedge \overline{\mathbf{E}})))\right], \tag{$M-S-2$}$$

where $\wedge$ denotes the exterior product of vector-valued functions, and $\overline{\mathbf{E}}$ the complex conjugate of $\mathbf{E}$. (Indeed, a direct computation implies that equations (1.1)-(1.3) are the componential form of the equations (M-S-1)-(M-S-2). To understand the relationship of all the components E_1, E_2, E_3, we adopt the componential form (1.1)-(1.3) in the present paper.)

Equations (M-S-1)-(M-S-2) arise in the infinite ion acoustic speed limit of the self-generated magnetic field in a cold plasma, $\mathbf{E}$ denotes a slowly varying complex amplitude of the high-frequency electric field, and **B** the self-generated magnetic field [**?**, **?**, **?**, **?**]. Due to the gauge invariance $A_j(e^{i\omega t}E_1, e^{i\omega t}E_2, e^{i\omega t}E_3) = e^{i\omega t}A_j(E_1, E_2, E_3),\ j = 1, 2, 3, 4, 5, 6$, we can study the so-called standing wave solutions of the equations (1.1)-(1.3) in the form $E_i(t, x) = e^{i\omega t}u_i(x)$ $(i = 1, 2, 3)$ with the initial condition (1.4), where $\omega > 0$ is a real constant parameter called frequency and $u_i(x)(i = 1, 2, 3)$ is a complex-valued function. The search for standing waves of equations (1.1)-(1.3) leads to the following nonlinear elliptic equations (1.5)-(1.7):

$$\begin{aligned}&-\omega u_1 + \Delta u_1 + (|u_1|^2 + |u_2|^2 + |u_3|^2)u_1 - u_2\mathcal{F}^{-1}[\tfrac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(u_2\overline{u_3} - \overline{u_2}u_3)]\\ &+u_2\mathcal{F}^{-1}[\tfrac{\eta(\xi_1^2+\xi_2^2)}{|\xi|^2-\delta}\mathcal{F}(u_1\overline{u_2} - u_2\overline{u_1})] - u_2\mathcal{F}^{-1}[\tfrac{\eta\xi_2\xi_3}{|\xi|^2-\delta}\mathcal{F}(\overline{u_1}u_3 - u_1\overline{u_3})]\\ &+u_3\mathcal{F}^{-1}[\tfrac{\eta\xi_2\xi_3}{|\xi|^2-\delta}\mathcal{F}(u_1\overline{u_2} - \overline{u_1}u_2)] + u_3\mathcal{F}^{-1}[\tfrac{\eta(\xi_1^2+\xi_3^2)}{|\xi|^2-\delta}\mathcal{F}(u_1\overline{u_3} - \overline{u_1}u_3)]\\ &+u_3\mathcal{F}^{-1}[\tfrac{\eta\xi_1\xi_2}{|\xi|^2-\delta}\mathcal{F}(u_2\overline{u_3} - \overline{u_2}u_3)] = 0,\end{aligned} \tag{1.5}$$

$$\begin{aligned}&-\omega u_2 + \Delta u_2 + (|u_1|^2 + |u_2|^2 + |u_3|^2)u_2 - u_3\mathcal{F}^{-1}[\tfrac{\eta\xi_1\xi_2}{|\xi|^2-\delta}\mathcal{F}(\overline{u_1}u_3 - u_1\overline{u_3})]\\ &+u_3\mathcal{F}^{-1}[\tfrac{\eta(\xi_2^2+\xi_3^2)}{|\xi|^2-\delta}\mathcal{F}(u_2\overline{u_3} - \overline{u_2}u_3)] - u_3\mathcal{F}^{-1}[\tfrac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(u_1\overline{u_2} - \overline{u_1}u_2)]\\ &+u_1\mathcal{F}^{-1}[\tfrac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(u_2\overline{u_3} - \overline{u_2}u_3)] + u_1\mathcal{F}^{-1}[\tfrac{\eta(\xi_1^2+\xi_2^2)}{|\xi|^2-\delta}\mathcal{F}(\overline{u_1}u_2 - u_1\overline{u_2})]\\ &+u_1\mathcal{F}^{-1}[\tfrac{\eta\xi_2\xi_3}{|\xi|^2-\delta}\mathcal{F}(\overline{u_1}u_3 - u_1\overline{u_3})] = 0,\end{aligned} \tag{1.6}$$

$$\begin{aligned}&-\omega u_3 + \Delta u_3 + (|u_1|^2 + |u_2|^2 + |u_3|^2)u_3 - u_1\mathcal{F}^{-1}[\tfrac{\eta\xi_2\xi_3}{|\xi|^2-\delta}\mathcal{F}(u_1\overline{u_2} - \overline{u_1}u_2)]\\ &+u_1\mathcal{F}^{-1}[\tfrac{\eta(\xi_1^2+\xi_3^2)}{|\xi|^2-\delta}\mathcal{F}(\overline{u_1}u_3 - u_1\overline{u_3})] - u_1\mathcal{F}^{-1}[\tfrac{\eta\xi_1\xi_2}{|\xi|^2-\delta}\mathcal{F}(u_2\overline{u_3} - \overline{u_2}u_3)]\\ &+u_2\mathcal{F}^{-1}[\tfrac{\eta\xi_1\xi_2}{|\xi|^2-\delta}\mathcal{F}(\overline{u_1}u_3 - u_1\overline{u_3})] + u_2\mathcal{F}^{-1}[\tfrac{\eta(\xi_2^2+\xi_3^2)}{|\xi|^2-\delta}\mathcal{F}(\overline{u_2}u_3 - u_2\overline{u_3})]\\ &+u_2\mathcal{F}^{-1}[\tfrac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(u_1\overline{u_2} - \overline{u_1}u_2)] = 0.\end{aligned} \tag{1.7}$$

For the nonlinear Schrödinger equations without any nonlocal term, there have been many works on the stability results of their standing waves. Berestycki and Cazenave [**?**], Grillakis[**?**], Jones [**?**], Shatah and Strauss [**?**], Weinstein [**?**] and Zhang [**?**] investigated the instability of solitons. On the other hand, Cazenave and Lions [**?**], Weinstein [**?**], Grillakis, Shatah and Strauss [**?**] studied the stability of the standing waves.

In the study of equations (1.1)-(1.3), we still concentrate on the existence and orbital instability of the standing waves. For the nonlocal nonlinear Schrödinger equations, to our best knowledge, there have been no any works on the existence and instability of the standing waves other than those in our former papers [**?**, **?**, **?**], where we studied the similar topic for a general Davey-Stewartson system [**?**] and a simplified version for the nonlocal nonlinear Schrödinger equations (1.1)-(1.3) [**?**, **?**], in which $(E_1, E_2, E_3) = (E_1, E_2, 0)$, $(\xi_1, \xi_2, \xi_3) = (\xi_1, \xi_2, 0)$. To attain our goal in the present paper, the main difficulty is to deal with the nonlocal terms since the nonlocal terms may violate the space inner-scale invariance, and we are forced to make some additional arguments for them. Fortunately, by defining some suitable functionals and a constrained variational minimization problem, utilizing the monotonicity argument for some defined auxiliary functions to deal with these terms generated by the nonlocal effects, and applying the inner structure of the corresponding elliptic equations

(1.5)-(1.7), we first show the existence of standing waves for the equations (1.1)-(1.3). In addition, we establish the orbital instability of the standing waves for the equations under consideration. The arguments of the result rely on the conservation of energy and mass as well as the construction of a suitable invariant manifold of solution flows. However, it should be pointed out that the uniqueness of these ground states for (1.5)-(1.7) is a much different and difficult problem, and we do not intend to discuss it in the present paper.

This paper arranges as follows. In section 2, we give some preliminaries and state the main results. The existence of standing waves with ground states will be established in Section 3. At the last section, we will show the orbital instability of standing waves.

For simplicity, we denote any positive constant by C throughout the present paper.

2 Preliminaries and Main Results

In this section, we first establish the conservation laws of total mass and total energy. Then we define some functionals, a set and a constrained variational minimization problem. At the end of this section we state the main results of this paper.

2.1 Conservation Laws of the Mass and of the Energy

According to the inner structure of equations (1.1)-(1.3), making some estimates on the nonlocal terms and using the standard contraction mapping theorem, we can establish the local well-posedness in the energy space $H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$, the conservation laws of the total mass and of the total energy for the Cauchy problem (1.1)-(1.4).

Lemma 2.1 The Cauchy problem (1.1)-(1.4), for $\eta > 0$, $\delta \leq 0$ and

$$(E_{10}(x), E_{20}(x), , E_{30}(x)) \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3),$$

has a unique solution

$$(E_1, E_2, E_3) \in C\left([0, T); H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)\right)$$

for some $T \in (0, +\infty)$ with $T = +\infty$ or $T < +\infty$ and

$$\lim_{t \to T} \left(\|E_1\|_{H^1(\mathbb{R}^3)} + \|E_2\|_{H^1(\mathbb{R}^3)} + \|E_3\|_{H^1(\mathbb{R}^3)}\right) = +\infty.$$

In addition, the total mass and total energy are conserved:

$$\int_{\mathbb{R}^3} (|E_1|^2 + |E_2|^2 + |E_3|^2) dx = \int_{\mathbb{R}^3} (|E_{10}|^2 + |E_{20}|^2 + |E_{30}|^2) dx, \tag{2.1}$$

$$
\begin{aligned}
&\mathcal{H}(E_1,E_2,E_3)\\
&\quad=\int_{\mathbb{R}^3}(|\nabla E_1|^2+|\nabla E_2|^2+|\nabla E_3|^2)dx-\frac{1}{2}\int_{\mathbb{R}^3}(|E_1|^4+|E_2|^4+|E_3|^4)dx\\
&\qquad-\int_{\mathbb{R}^3}(|E_1|^2|E_2|^2+|E_1|^2|E_3|^2+|E_2|^2|E_3|^2)dx\\
&\qquad-\frac{1}{2}\int_{\mathbb{R}^3}\frac{\eta(\xi_1^2+\xi_2^2)}{|\xi|^2-\delta}(|\mathcal{F}(E_1\overline{E_2}-\overline{E_1}E_2)|^2d\xi\\
&\qquad-\frac{1}{2}\int_{\mathbb{R}^3}\frac{\eta(\xi_1^2+\xi_3^2)}{|\xi|^2-\delta}(|\mathcal{F}(\overline{E_1}E_3-E_1\overline{E_3})|^2d\xi\\
&\qquad-\frac{1}{2}\int_{\mathbb{R}^3}\frac{\eta(\xi_2^2+\xi_3^2)}{|\xi|^2-\delta}(|\mathcal{F}(E_2\overline{E_3}-\overline{E_2}E_3)|^2d\xi\\
&\qquad+Re\int_{\mathbb{R}^3}\frac{\eta\xi_1\xi_2}{|\xi|^2-\delta}\mathcal{F}(E_2\overline{E_3}-\overline{E_2}E_3)\overline{\mathcal{F}(\overline{E_1}E_3-E_1\overline{E_3})}d\xi\\
&\qquad+Re\int_{\mathbb{R}^3}\frac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(E_1\overline{E_2}-\overline{E_1}E_2)\overline{\mathcal{F}(E_2\overline{E_3}-\overline{E_2}E_3)}d\xi\\
&\qquad+Re\int_{\mathbb{R}^3}\frac{\eta\xi_2\xi_3}{|\xi|^2-\delta}\mathcal{F}(\overline{E_1}E_3-E_1\overline{E_3})\overline{\mathcal{F}(E_1\overline{E_2}-\overline{E_1}E_2)}d\xi\\
&\quad=\mathcal{H}(E_{10},E_{20},E_{30}).
\end{aligned}
\tag{2.2}
$$

To attain the conservation identities (2.1) and (2.2), besides using the standard arguments on the nonlinear Schrödinger equations without any nonlocal terms, we must give some extra discussions on the nonlocal terms, in which we will employ the Parseval identity, some properties of Fourier transform, suitable groupings and potential coupled arguments for these nonlocal terms. In Lemma 2.1 of [**?**], we established the detail of the proof for (2.1) and (2.2) □

2.2 Variational Structures

For $(u_1,u_2,u_3)\in H_r^1(\mathbb{R}^3)\times H_r^1(\mathbb{R}^3)\times H_r^1(\mathbb{R}^3)$ ($u_i(x)(i=1,2,3)$ is a complex-valued function), we define the following functionals:

$$
\begin{aligned}
S(u_1,u_2,u_3)=\;&\frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_3|^2)dx\\
&+\frac{\omega}{2}\int_{\mathbb{R}^3}(|u_1|^2+|u_2|^2+|u_3|^2)dx\\
&-\frac{1}{4}\int_{\mathbb{R}^3}(|u_1|^4+|u_2|^4+|u_3|^4)dx\\
&-\frac{1}{2}\int_{\mathbb{R}^3}(|u_1|^2|u_2|^2+|u_1|^2|u_3|^2+|u_2|^2|u_3|^2)dx\\
&-\frac{1}{4}\int_{\mathbb{R}^3}A(u_1,u_2,u_3)d\xi+\frac{1}{2}Re\int_{\mathbb{R}^3}B(u_1,u_2,u_3)d\xi,
\end{aligned}
\tag{2.3}
$$

$$
\begin{aligned}
R(u_1,u_2,u_3)=\;&\int_{\mathbb{R}^3}(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_3|^2)dx\\
&-\frac{3}{4}\int_{\mathbb{R}^3}(|u_1|^4+|u_2|^4+|u_3|^4)dx\\
&-\frac{3}{2}\int_{\mathbb{R}^3}(|u_1|^2|u_2|^2+|u_1|^2|u_3|^2+|u_2|^2|u_3|^2)dx\\
&-\frac{3}{4}\int_{\mathbb{R}^3}A(u_1,u_2,u_3)d\xi+\frac{1}{2}\delta\int_{\mathbb{R}^3}\frac{A(u_1,u_2,u_3)}{|\xi|^2-\delta}d\xi\\
&+\frac{3}{2}Re\int_{\mathbb{R}^3}B(u_1,u_2,u_3)d\xi-\delta Re\int_{\mathbb{R}^3}\frac{B(u_1,u_2,u_3)}{|\xi|^2-\delta}d\xi,
\end{aligned}
\tag{2.4}
$$

where

$$
\begin{aligned}
&H_r^1(\mathbb{R}^3)\times H_r^1(\mathbb{R}^3)\times H_r^1(\mathbb{R}^3)\\
&\quad=\big\{(f_1(x),f_2(x),f_3(x))\in H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3),\\
&\qquad\quad f_i(x)=f_i(|x|)\ \text{ is a function of } |x| \text{ alone}, i=1,2,3\big\},
\end{aligned}
$$

$$
\begin{aligned}
A(u_1,u_2,u_3)=\;&\frac{\eta}{|\xi|^2-\delta}\big[(\xi_1^2+\xi_2^2)(|\mathcal{F}(u_1\overline{u_2}-\overline{u_1}u_2)|^2)\\
&+(\xi_1^2+\xi_3^2)(|\mathcal{F}(u_1\overline{u_3}-\overline{u_1}u_3)|^2)\\
&+(\xi_2^2+\xi_3^2)(|\mathcal{F}(u_2\overline{u_3}-\overline{u_2}u_3)|^2)\big],
\end{aligned}
\tag{II-1}
$$

$$B(u_1,u_2,u_3) = \frac{\eta}{|\xi|^2-\delta}\Big[\xi_1\xi_2\mathcal{F}(\overline{u_2}u_3-u_2\overline{u_3})\overline{\mathcal{F}(u_1\overline{u_3}-\overline{u_1}u_3)} + \xi_1\xi_3\mathcal{F}(u_2\overline{u_3}-\overline{u_2}u_3)\overline{\mathcal{F}(u_1\overline{u_2}-\overline{u_1}u_2)} + \xi_2\xi_3\mathcal{F}(\overline{u_1}u_3-u_1\overline{u_3})\overline{\mathcal{F}(u_1\overline{u_2}-\overline{u_1}u_2)}\Big]. \qquad (II-2)$$

A natural attempt to find nontrivial solutions to (1.5)-(1.7) is to solve the constrained minimization problem

$$d := \inf_{(u_1,u_2,u_3)\in M} S(u_1,u_2,u_3), \tag{2.5}$$

where the set M is defined by

$$M = \left\{(u_1,u_2,u_3) \in H_r^1(\mathbb{R}^3)\times H_r^1(\mathbb{R}^3)\times H_r^1(\mathbb{R}^3)\setminus\{(0,0,0)\}, R(u_1,u_2,u_3)=0\right\}. \tag{2.6}$$

From $(u_1,u_2,u_3) \in H_r^1(\mathbb{R}^3)\times H_r^1(\mathbb{R}^3)\times H_r^1(\mathbb{R}^3)$ $\eta>0$, $\delta\le 0$, the Sobolev's embedding theorem and the properties of Fourier transform, it follows that functionals $S(u_1,u_2,u_3)$ and $R(u_1,u_2,u_3)$ are both well defined.

Remark 2.1. We note that for all $\theta\ge 1$, $j,k,l,m=1,2,3$,

$$\int_{\mathbb{R}^3}\frac{\eta\xi_l\xi_m}{(|\xi|^2-\delta)^\theta}|\mathcal{F}(u_j\overline{v_k})|^2d\xi = \int_{\mathbb{R}^3}\frac{\eta\xi_l\xi_m}{(|\xi|^2-\delta)^\theta}|\mathcal{F}(\overline{u_j}v_k)|^2d\xi.$$

We also note that if (u_1,u_2,u_3) is a critical point of $S(u_1,u_2,u_3)$ and hence a solution of (1.5)-(1.7), then $(E_1,E_2,E_3)=(e^{i\omega t}u_1,e^{i\omega t}u_2,e^{i\omega t}u_3)$ is a standing wave solution of (1.1)-(1.3).

2.3 Main Results

Here, we state the main results of this paper.

Theorem 2.1. For $\eta>0$ and $\delta\le 0$, there exists $(Q_1,Q_2,Q_3)\in M$ such that

(1) $S(Q_1,Q_2,Q_3) = d = \inf_{(u_1,u_2,u_3)\in M} S(u_1,u_2,u_3)$;

(2) (Q_1,Q_2,Q_3) is a ground state solution to (1.5)-(1.7).

(3) (Q_1,Q_2,Q_3) are functions of $|x|$ alone and decay exponentially at infinity.

From the physical viewpoint, an important role is played by the ground state solution of (1.5)-(1.7). A solution (Q_1,Q_2,Q_3) to (1.5)-(1.7) is termed as a ground state if it has some minimal action among all solutions to (1.5)-(1.7). Here, the action of solution (u_1,u_2,u_3) is defined by $S(u_1,u_2,u_3)$.

Concerning the dynamics of the ground state solution (Q_1,Q_2,Q_3), we have the following orbital instability result. Here, we assume that the ground state solution (Q_1,Q_2,Q_3) of (1.5)-(1.7) is unique.

Theorem 2.2. For $\eta>0$ and $\delta\le 0$, let $(Q_1,Q_2,Q_3)\in M$ be given by Theorem 2.1. For arbitrary $\varepsilon>0$, there exists $(E_{10},E_{20},E_{30})\in H_r^1(\mathbb{R}^3)\times H_r^1(\mathbb{R}^3)\times H_r^1(\mathbb{R}^3)$ with

$$\|E_{10}-Q_1\|_{H_r^1(\mathbb{R}^3)}<\varepsilon, \qquad \|E_{20}-Q_2\|_{H_r^1(\mathbb{R}^3)}<\varepsilon, \qquad \|E_{30}-Q_3\|_{H_r^1(\mathbb{R}^3)}<\varepsilon \tag{2.7}$$

such that the solution (E_1, E_2, E_3) of the equations (1.1)-(1.3) with the initial data (1.4) has the following property: For some finite time $T < \infty$, (E_1, E_2, E_3) exists on $[0, T)$, $(E_1, E_2, E_3) \in C([0, T); H_r^1(\mathbb{R}^3) \times H_r^1(\mathbb{R}^3) \times H_r^1(\mathbb{R}^3))$ and

$$\lim_{t \to T} \left(\|E_1\|_{H_r^1(\mathbb{R}^3)} + \|E_2\|_{H_r^1(\mathbb{R}^3)} + \|E_3\|_{H_r^1(\mathbb{R}^3)} \right) = +\infty. \tag{2.8}$$

3 Existence of standing waves

In this section, we prove Theorem 2.1, which concerns the existence of minimal energy standing waves of the system (1.1)-(1.3). For that purpose, we first give some key Propositions and Lemmas.

Remark 3.1. For $\eta > 0$, $\delta \le 0$, $\theta = 1, 2$, $j, k, l, m = 1, 2, 3$, since

$$\int_{\mathbb{R}^3} \frac{\eta \xi_l \xi_m}{(|\xi|^2 - \delta)^\theta} |\mathcal{F}(Q_j \overline{Q_k})|^2 d\xi = \int_{\mathbb{R}^3} \frac{\eta \xi_l \xi_m}{(|\xi|^2 - \delta)^\theta} |\mathcal{F}(\overline{Q_j} Q_k)|^2 d\xi, \tag{3.1}$$

$$\begin{aligned} &\left| Re \int_{\mathbb{R}^3} \frac{\eta \xi_l \xi_m}{(|\xi|^2 - \delta)^\theta} \mathcal{F}(Q_j \overline{Q_k}) \overline{\mathcal{F}(\overline{Q_j} Q_k)} d\xi \right| \\ &\le \frac{1}{4} \int_{\mathbb{R}^3} \frac{\eta(|\xi_l|^2 + |\xi_m|^2)}{(|\xi|^2 - \delta)^\theta} (|\mathcal{F}(Q_j \overline{Q_k})|^2 + |\mathcal{F}(\overline{Q_j} Q_k)|^2) d\xi \\ &= \frac{1}{2} \int_{\mathbb{R}^3} \frac{\eta(|\xi_l|^2 + |\xi_m|^2)}{(|\xi|^2 - \delta)^\theta} |\mathcal{F}(Q_j \overline{Q_k})|^2 d\xi, \end{aligned} \tag{3.2}$$

by Theorem 2.1, (2.3), (2.4) and (2.5), one has

$$\begin{aligned} &S(Q_1, Q_2, Q_3) \\ &= \frac{1}{6} \int_{\mathbb{R}^3} (|\nabla Q_1|^2 + |\nabla Q_2|^2 + |\nabla Q_3|^2) dx + \frac{w}{2} \int_{\mathbb{R}^3} (|Q_1|^2 + |Q_2|^2 + |Q_3|^2) dx \\ &- \frac{1}{6} \delta \int_{\mathbb{R}^3} \frac{\eta(\xi_1^2 + \xi_2^2)}{(|\xi|^2 - \delta)^2} (|\mathcal{F}(Q_1 \overline{Q_2} - \overline{Q_1} Q_2)|^2) d\xi \\ &- \frac{1}{6} \delta \int_{\mathbb{R}^3} \frac{\eta(\xi_1^2 + \xi_3^2)}{(|\xi|^2 - \delta)^2} (|\mathcal{F}(Q_1 \overline{Q_3} - \overline{Q_1} Q_3)|^2) d\xi \\ &- \frac{1}{6} \delta \int_{\mathbb{R}^3} \frac{\eta(\xi_2^2 + \xi_3^2)}{(|\xi|^2 - \delta)^2} (|\mathcal{F}(Q_2 \overline{Q_3} - \overline{Q_2} Q_3)|^2) d\xi \\ &+ \frac{1}{3} \delta Re \int_{\mathbb{R}^3} \frac{\eta \xi_1 \xi_2}{(|\xi|^2 - \delta)^2} \mathcal{F}(\overline{Q_2} Q_3 - Q_2 \overline{Q_3}) \overline{\mathcal{F}(Q_1 \overline{Q_3} - \overline{Q_1} Q_3)} d\xi \\ &+ \frac{1}{3} \delta Re \int_{\mathbb{R}^3} \frac{\eta \xi_1 \xi_3}{(|\xi|^2 - \delta)^2} \mathcal{F}(Q_2 \overline{Q_3} - \overline{Q_2} Q_3) \overline{\mathcal{F}(Q_1 \overline{Q_2} - \overline{Q_1} Q_2)} d\xi \\ &+ \frac{1}{3} \delta Re \int_{\mathbb{R}^3} \frac{\eta \xi_2 \xi_3}{(|\xi|^2 - \delta)^2} \mathcal{F}(\overline{Q_1} Q_3 - Q_1 \overline{Q_3}) \overline{\mathcal{F}(Q_1 \overline{Q_2} - \overline{Q_1}) Q_2} d\xi > 0. \end{aligned} \tag{3.3}$$

□

We continue to give some key facts.

Lemma 3.1 ([?, ?]). For $2 < \sigma < 6$, the embedding $H_r^1(\mathbb{R}^3) \hookrightarrow L_r^\sigma(\mathbb{R}^3)$ is compact, where

$$H_r^1(\mathbb{R}^3) = \left\{ f(x) \in H^1(\mathbb{R}^3) : f(x) = f(|x|) \textit{ is a function of } |x| \textit{ alone} \right\}.$$ □

Proposition 3.1. Let $\eta > 0$ and $\delta \leq 0$. Then the non-trivial solution to (1.5)-(1.7) belongs to M.

Proof. Let (u_1, u_2, u_3) be a non-trivial solution to (1.5)-(1.7). Multiplying (1.5) by $\overline{u_1}$, (1.6) by $\overline{u_2}$, (1.7) by $\overline{u_3}$, then integrating with respect to x on $\mathbb{R}^3$, we obtain

$$\begin{aligned} &-\int_{\mathbb{R}^3}(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_3|^2)dx-\omega\int_{\mathbb{R}^3}(|u_1|^2+|u_2|^2+|u_3|^2)dx \\ &+\int_{\mathbb{R}^3}(|u_1|^4+|u_2|^4+|u_3|^4)dx+2\int_{\mathbb{R}^3}(|u_1|^2|u_2|^2+|u_1|^2|u_3|^2+|u_2|^2|u_3|^2)dx \\ &+\int_{\mathbb{R}^3}A(u_1,u_2,u_3)d\xi-2Re\int_{\mathbb{R}^3}B(u_1,u_2,u_3)d\xi=0. \end{aligned} \tag{3.4}$$

We further attain the following identity:

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_3|^2)dx+\frac{3\omega}{2}\int_{\mathbb{R}^3}(|u_1|^2+|u_2|^2+|u_3|^2)dx \\ &-\frac{3}{4}\int_{\mathbb{R}^3}(|u_1|^4+|u_2|^4+|u_3|^4)dx-\frac{3}{2}\int_{\mathbb{R}^3}(|u_1|^2|u_2|^2+|u_1|^2|u_3|^2+|u_2|^2|u_3|^2)dx \\ &-\frac{3}{4}\int_{\mathbb{R}^3}A(u_1,u_2,u_3)d\xi+\frac{3}{2}Re\int_{\mathbb{R}^3}B(u_1,u_2,u_3)d\xi \\ &-\frac{\delta}{2}\int_{\mathbb{R}^3}\frac{1}{|\xi|^2-\delta}A(u_1,u_2,u_3)d\xi+\delta Re\int_{\mathbb{R}^3}\frac{1}{|\xi|^2-\delta}B(u_1,u_2,u_3)d\xi=0. \end{aligned} \tag{3.5}$$

Here, A and B are defined by (II-1) and (II-2) in Section 2, respectively. The identity (3.5) is obtained by multiplying (1.5) by $x\nabla\overline{u_1}$, (1.6) by $x\nabla\overline{u_2}$, (1.7) by $x\nabla\overline{u_3}$, then integrating with respect to x in $\mathbb{R}^3$ and taking the real parts for the resulting equations, finally using the following estimates:

$$\begin{aligned} &Re\int_{\mathbb{R}^3}(-wu_1x\nabla\overline{u}_1-wu_2x\nabla\overline{u}_2-wu_3x\nabla\overline{u}_3)dx \\ &=\frac{3w}{2}\int_{\mathbb{R}^3}(|u_1|^2+|u_2|^2+|u_3|^2)dx, \end{aligned} \tag{III-1}$$

$$\begin{aligned} &Re\int_{\mathbb{R}^3}(u_1x\nabla\overline{u}_1+u_2x\nabla\overline{u}_2+u_3x\nabla\overline{u}_3)dx \\ &=\frac{1}{2}\int_{\mathbb{R}^3}(|u_1|^2+|u_2|^2+|u_3|^2)dx, \end{aligned} \tag{III-2}$$

$$\begin{aligned} &Re\int_{\mathbb{R}^3}(|u_1|^2+|u_2|^2+|u_3|^2)(u_1x\nabla\overline{u}_1+u_2x\nabla\overline{u}_2+u_3x\nabla\overline{u}_3)dx \\ &=-\frac{3}{4}\int_{\mathbb{R}^3}(|u_1|^4+|u_2|^4+|u_3|^4)dx \\ &\quad-\frac{3}{2}\int_{\mathbb{R}^3}(|u_1|^2|u_2|^2+|u_1|^2|u_3|^2+|u_2|^2|u_3|^2)dx, \end{aligned} \tag{III-3}$$

$$\begin{aligned} &Re\int_{\mathbb{R}^3}u_1x\nabla\overline{u}_2\mathcal{F}^{-1}[\frac{\eta(\xi_1^2+\xi_2^2)}{|\xi|^2-\delta}\mathcal{F}(u_1\overline{u}_2-\overline{u}_1u_2)]dx \\ &-Re\int_{\mathbb{R}^3}u_2x\nabla\overline{u}_1\mathcal{F}^{-1}[\frac{\eta(\xi_1^2+\xi_2^2)}{|\xi|^2-\delta}\mathcal{F}(u_1\overline{u}_2-\overline{u}_1u_2)]dx \\ &=\frac{3}{4}\int_{\mathbb{R}^3}\frac{\eta(\xi_1^2+\xi_2^2)}{|\xi|^2-\delta}(|\mathcal{F}(u_1\overline{u}_2-\overline{u}_1u_2)|^2)d\xi \\ &+\frac{\delta}{2}\int_{\mathbb{R}^3}\frac{\eta(\xi_1^2+\xi_2^2)}{(|\xi|^2-\delta)^2}(|\mathcal{F}(u_1\overline{u}_2-\overline{u}_1u_2)|^2)d\xi, \end{aligned} \tag{III-4}$$

$$\begin{aligned} &Re\int_{\mathbb{R}^3}u_1x\nabla\overline{u}_3\mathcal{F}^{-1}[\frac{\eta(\xi_1^2+\xi_3^2)}{|\xi|^2-\delta}\mathcal{F}(u_1\overline{u}_3-\overline{u}_1u_3)]dx \\ &-Re\int_{\mathbb{R}^3}u_3x\nabla\overline{u}_1\mathcal{F}^{-1}[\frac{\eta(\xi_1^2+\xi_3^2)}{|\xi|^2-\delta}\mathcal{F}(u_1\overline{u}_3-\overline{u}_1u_3)]dx \\ &=\frac{3}{4}\int_{\mathbb{R}^3}\frac{\eta(\xi_1^2+\xi_3^2)}{|\xi|^2-\delta}(|\mathcal{F}(u_1\overline{u}_3-\overline{u}_1u_3)|^2)d\xi \\ &+\frac{\delta}{2}\int_{\mathbb{R}^3}\frac{\eta(\xi_1^2+\xi_3^2)}{(|\xi|^2-\delta)^2}(|\mathcal{F}(u_1\overline{u}_3-\overline{u}_1u_3)|^2)d\xi, \end{aligned} \tag{III-5}$$

$$\begin{aligned} &Re\int_{\mathbb{R}^3}u_2x\nabla\overline{u}_3\mathcal{F}^{-1}[\frac{\eta(\xi_2^2+\xi_3^2)}{|\xi|^2-\delta}\mathcal{F}(u_2\overline{u}_3-\overline{u}_2u_3)]dx \\ &-Re\int_{\mathbb{R}^3}u_3x\nabla\overline{u}_2\mathcal{F}^{-1}[\frac{\eta(\xi_2^2+\xi_3^2)}{|\xi|^2-\delta}\mathcal{F}(u_2\overline{u}_3-\overline{u}_2u_3)]dx \\ &=\frac{3}{4}\int_{\mathbb{R}^3}\frac{\eta(\xi_2^2+\xi_3^2)}{|\xi|^2-\delta}(|\mathcal{F}(u_2\overline{u}_3-\overline{u}_2u_3)|^2)d\xi \\ &+\frac{\delta}{2}\int_{\mathbb{R}^3}\frac{\eta(\xi_2^2+\xi_3^2)}{(|\xi|^2-\delta)^2}(|\mathcal{F}(u_2\overline{u}_3-\overline{u}_2u_3)|^2)d\xi, \end{aligned} \tag{III-6}$$

$$
\begin{aligned}
&Re \int_{\mathbb{R}^3} u_2 x\nabla\overline{u}_1\mathcal{F}^{-1}[\tfrac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(u_2\overline{u}_3-u_3\overline{u}_2)]dx\\
&\quad +Re \int_{\mathbb{R}^3} u_3 x\nabla\overline{u}_2\mathcal{F}^{-1}[\tfrac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(u_1\overline{u}_2-\overline{u}_1u_2)]dx\\
&\quad -Re \int_{\mathbb{R}^3} u_1 x\nabla\overline{u}_2\mathcal{F}^{-1}[\tfrac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(u_2\overline{u}_3-u_3\overline{u}_2)]dx\\
&\quad -Re \int_{\mathbb{R}^3} u_2 x\nabla\overline{u}_3\mathcal{F}^{-1}[\tfrac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(u_1\overline{u}_2-\overline{u}_1u_2)]dx\\
&= -\tfrac{3}{2}Re \int_{\mathbb{R}^3} \tfrac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(u_2\overline{u}_3-\overline{u}_2u_3)\overline{\mathcal{F}(u_1\overline{u}_2-\overline{u}_1u_2)}d\xi\\
&\quad -\delta Re \int_{\mathbb{R}^3} \tfrac{\eta\xi_1\xi_3}{(|\xi|^2-\delta)^2}\mathcal{F}(u_2\overline{u}_3-\overline{u}_2u_3)\overline{\mathcal{F}(u_1\overline{u}_2-\overline{u}_1u_2)}d\xi,
\end{aligned}
\tag{III-7}
$$

$$
\begin{aligned}
&Re \int_{\mathbb{R}^3} u_1 x\nabla\overline{u}_2\mathcal{F}^{-1}[\tfrac{\eta\xi_2\xi_3}{|\xi|^2-\delta}\mathcal{F}(u_1\overline{u}_3-\overline{u}_1u_3)]dx\\
&\quad +Re \int_{\mathbb{R}^3} u_1 x\nabla\overline{u}_3\mathcal{F}^{-1}[\tfrac{\eta\xi_2\xi_3}{|\xi|^2-\delta}\mathcal{F}(u_1\overline{u}_2-\overline{u}_1u_2)]dx\\
&\quad -Re \int_{\mathbb{R}^3} u_3 x\nabla\overline{u}_1\mathcal{F}^{-1}[\tfrac{\eta\xi_2\xi_3}{|\xi|^2-\delta}\mathcal{F}(u_1\overline{u}_2-\overline{u}_1u_2)]dx\\
&\quad -Re \int_{\mathbb{R}^3} u_2 x\nabla\overline{u}_1\mathcal{F}^{-1}[\tfrac{\eta\xi_2\xi_3}{|\xi|^2-\delta}\mathcal{F}(u_1\overline{u}_3-\overline{u}_1u_3)]dx\\
&= -\tfrac{3}{2}Re \int_{\mathbb{R}^3} \tfrac{\eta\xi_2\xi_3}{|\xi|^2-\delta}\mathcal{F}(u_1\overline{u}_3-\overline{u}_1u_3)\overline{\mathcal{F}(u_1\overline{u}_2-\overline{u}_1u_2)}d\xi\\
&\quad -\delta Re \int_{\mathbb{R}^3} \tfrac{\eta\xi_2\xi_3}{(|\xi|^2-\delta)^2}\mathcal{F}(u_1\overline{u}_3-\overline{u}_1u_3)\overline{\mathcal{F}(u_1\overline{u}_2-\overline{u}_1u_2)}d\xi,
\end{aligned}
\tag{III-8}
$$

$$
\begin{aligned}
&Re \int_{\mathbb{R}^3} u_3 x\nabla\overline{u}_1\mathcal{F}^{-1}[\tfrac{\eta\xi_1\xi_2}{|\xi|^2-\delta}\mathcal{F}(u_2\overline{u}_3-\overline{u}_2u_3)]dx\\
&\quad +Re \int_{\mathbb{R}^3} u_3 x\nabla\overline{u}_2\mathcal{F}^{-1}[\tfrac{\eta\xi_1\xi_2}{|\xi|^2-\delta}\mathcal{F}(u_1\overline{u}_3-\overline{u}_1u_3)]dx\\
&\quad -Re \int_{\mathbb{R}^3} u_1 x\nabla\overline{u}_3\mathcal{F}^{-1}[\tfrac{\eta\xi_1\xi_2}{|\xi|^2-\delta}\mathcal{F}(u_2\overline{u}_3-\overline{u}_2u_3)]dx\\
&\quad -Re \int_{\mathbb{R}^3} u_2 x\nabla\overline{u}_3\mathcal{F}^{-1}[\tfrac{\eta\xi_1\xi_2}{|\xi|^2-\delta}\mathcal{F}(u_1\overline{u}_3-\overline{u}_1u_3)]dx\\
&= -\tfrac{3}{2}Re \int_{\mathbb{R}^3} \tfrac{\eta\xi_1\xi_2}{|\xi|^2-\delta}\mathcal{F}(u_2\overline{u}_3-\overline{u}_2u_3)\overline{\mathcal{F}(u_1\overline{u}_3-\overline{u}_1u_3)}d\xi\\
&\quad -\delta Re \int_{\mathbb{R}^3} \tfrac{\eta\xi_1\xi_2}{(|\xi|^2-\delta)^2}\mathcal{F}(u_2\overline{u}_3-\overline{u}_2u_3)\overline{\mathcal{F}(u_1\overline{u}_3-\overline{u}_1u_3)}d\xi.
\end{aligned}
\tag{III-9}
$$

Combining (3.4) with (3.5), one easily verifies that $R(u_1, u_2, u_3) = 0$, and hence $(u_1, u_2, u_3) \in M$. □

Proposition 3.2. The functional S is bounded from below on M for $\eta > 0$ and $\delta \leqslant 0$.

Proof. Let A and B are defined by (II-1) and (II-2) in Section 2, respectively. According to (2.3), (2.4) and (2.6), for $(u_1, u_2, u_3) \in M$ one gets

$$
\begin{aligned}
&S(u_1, u_2, u_3)\\
&= \tfrac{1}{6}\int_{\mathbb{R}^3}(|\nabla u_1|^2 + |\nabla u_2|^2 + |\nabla u_3|^2)dx\\
&\quad +\tfrac{w}{2}\int_{\mathbb{R}^3}(|u_1|^2 + |u_2|^2 + |u_3|^2)dx\\
&\quad -\tfrac{\delta}{6}\int_{\mathbb{R}^3}\tfrac{A(u_1,u_2,u_3)}{|\xi|^2-\delta}d\xi + \tfrac{\delta}{3}Re\int_{\mathbb{R}^3}\tfrac{B(u_1,u_2,u_3)}{|\xi|^2-\delta}d\xi.
\end{aligned}
\tag{3.6}
$$

Noting that $\eta > 0$, $\delta \le 0$ and the inequality $2Re(ab) \le a^2 + b^2$, making some suitable rearrangements, we obtain

$$
-\tfrac{\delta}{6}\int_{\mathbb{R}^3}\tfrac{A(u_1,u_2,u_3)}{|\xi|^2-\delta}d\xi + \tfrac{\delta}{3}Re\int_{\mathbb{R}^3}\tfrac{B(u_1,u_2,u_3)}{|\xi|^2-\delta}d\xi \ge 0. \tag{3.7}
$$

(Indeed, recall that

$$
\begin{aligned}
A(u_1, u_2, u_3) = \;& \tfrac{\eta}{|\xi|^2-\delta}\big[(\xi_1^2+\xi_2^2)(|\mathcal{F}(u_1\overline{u_2}-\overline{u_1}u_2)|^2)\\
&+(\xi_1^2+\xi_3^2)(|\mathcal{F}(u_1\overline{u_3}-\overline{u_1}u_3)|^2)\\
&+(\xi_2^2+\xi_3^2)(|\mathcal{F}(u_2\overline{u_3}-\overline{u_2}u_3)|^2)\big],
\end{aligned}
\tag{II-1}
$$

$$
\begin{aligned}
B(u_1,u_2,u_3) = \quad & \tfrac{\eta}{|\xi|^2-\delta}\Big[\xi_1\xi_2\mathcal{F}(\overline{u_2}u_3-u_2\overline{u_3})\overline{\mathcal{F}(u_1\overline{u_3}-\overline{u_1}u_3)}\\
&\qquad +\xi_1\xi_3\mathcal{F}(u_2\overline{u_3}-\overline{u_2}u_3)\overline{\mathcal{F}(u_1\overline{u_2}-\overline{u_1}u_2)}\\
&\qquad +\xi_2\xi_3\mathcal{F}(\overline{u_1}u_3-u_1\overline{u_3})\overline{\mathcal{F}(u_1\overline{u_2}-\overline{u_1}u_2)}\Big].
\end{aligned}
\qquad (II-2)
$$

Since

$$
\begin{aligned}
&Re\xi_1\xi_2\mathcal{F}(\overline{u_2}u_3-u_2\overline{u_3})\overline{\mathcal{F}(u_1\overline{u_3}-\overline{u_1}u_3)}\\
&\quad\le \tfrac{1}{2}\xi_1^2|\mathcal{F}(u_1\overline{u_3}-\overline{u_1}u_3)|^2+\tfrac{1}{2}\xi_2^2|\mathcal{F}(\overline{u_2}u_3-u_2\overline{u_3})|^2,
\end{aligned}
$$

$$
\begin{aligned}
&Re\xi_1\xi_3\mathcal{F}(u_2\overline{u_3}-\overline{u_2}u_3)\overline{\mathcal{F}(u_1\overline{u_2}-\overline{u_1}u_2)}\\
&\quad\le \tfrac{1}{2}\xi_1^2|\mathcal{F}(u_1\overline{u_2}-\overline{u_1}u_2)|^2+\tfrac{1}{2}\xi_3^2|\mathcal{F}(u_2\overline{u_3}-\overline{u_2}u_3)|^2,
\end{aligned}
$$

$$
\begin{aligned}
&Re\xi_2\xi_3\mathcal{F}(\overline{u_1}u_3-u_1\overline{u_3})\overline{\mathcal{F}(u_1\overline{u_2}-\overline{u_1}u_2)}\\
&\quad\le \tfrac{1}{2}\xi_2^2|\mathcal{F}(u_1\overline{u_2}-\overline{u_1}u_2)|^2+\tfrac{1}{2}\xi_3^2|\mathcal{F}(\overline{u_1}u_3-u_1\overline{u_3})|^2,
\end{aligned}
$$

through regrouping and applying some properties of Fourier transform, we conclude that for $\eta>0$, and $\delta\le 0$,

$$
\begin{aligned}
&\tfrac{\delta}{3}Re\int_{\mathbb{R}^3}\tfrac{B(u_1,u_2,u_3)}{|\xi|^2-\delta}d\xi\\
&\quad\ge \tfrac{\delta}{6}\int_{\mathbb{R}^3}\tfrac{1}{|\xi|^2-\delta}\cdot\tfrac{\eta}{|\xi|^2-\delta}\big[(\xi_1^2+\xi_2^2)|\mathcal{F}(u_1\overline{u_2}-\overline{u_1}u_2)|^2\\
&\qquad\qquad (\xi_1^2+\xi_3^2)|\mathcal{F}(u_1\overline{u_3}-\overline{u_1}u_3)|^2\\
&\qquad\qquad (\xi_2^2+\xi_3^2)|\mathcal{F}(u_2\overline{u_3}-\overline{u_2}u_3)|^2\big]\\
&\quad= \tfrac{\delta}{6}\int_{\mathbb{R}^3}\tfrac{A(u_1,u_2,u_3)}{|\xi|^2-\delta}d\xi.
\end{aligned}
$$

Hence (3.7) is valid.)

Therefore, (3.6) and (3.7) imply that on M,

$$
\begin{aligned}
S(u_1,u_2,u_3) \ge \frac{1}{6}\int_{\mathbb{R}^3}(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_3|^2)dx\\
+\frac{w}{2}\int_{\mathbb{R}^3}(|u_1|^2+|u_2|^2+|u_3|^2)dx.
\end{aligned}
\qquad (3.8)
$$

This completes the proof of Proposition 3.2. □

Proposition 3.3. For $\eta>0$ and $\delta\le 0$, let $(u_1,u_2,u_3)\in H_r^1(\mathbb{R}^3)\times H_r^1(\mathbb{R}^3)\times H_r^1(\mathbb{R}^3)\setminus\{(0,0,0)\}$. For $\lambda>0$, we make the following scale transform:

$$
u_{1\lambda}(x)=\lambda^{\frac{3}{2}}u_1(\lambda x),\ u_{2\lambda}(x)=\lambda^{\frac{3}{2}}u_2(\lambda x),\ u_{3\lambda}(x)=\lambda^{\frac{3}{2}}u_3(\lambda x),
$$

then there exists a unique $\mu>0$ (relying on (u_1,u_2,u_3)) such that $R(u_{1\mu},u_{2\mu},u_{3\mu})=0$. Furthermore, the following three estimates will occur:

$$
\begin{aligned}
&R(u_{1\mu},u_{2\mu},u_{3\mu})>0, \quad for\ \ \lambda\in(0,\mu);\\
&R(u_{1\mu},u_{2\mu},u_{3\mu})<0, \quad for\ \ \lambda\in(\mu,\infty);\\
&S(u_{1\mu},u_{2\mu},u_{3\mu})\ge S(u_{1\lambda},u_{2\lambda},u_{3\lambda}), \quad for\ \ \forall\lambda>0.
\end{aligned}
$$

Proof. According to (2.3) and (2.4), $S(u_{1\lambda}, u_{2\lambda}, u_{3\lambda})$ and $R(u_{1\lambda}, u_{2\lambda}, u_{3\lambda})$ are of the following expressions:

$$
\begin{aligned}
&S(u_{1\lambda}, u_{2\lambda}, u_{3\lambda})\\
&= \tfrac{1}{2}\lambda^2 \int_{\mathbb{R}^3}(|\nabla u_1|^2 + |\nabla u_2|^2 + |\nabla u_3|^2)dx + \tfrac{w}{2}\int_{\mathbb{R}^3}(|u_1|^2 + |u_2|^2 + |u_3|^2)dx\\
&\quad -\tfrac{1}{4}\lambda^3 \int_{\mathbb{R}^3}(|u_1|^4 + |u_2|^4 + |u_3|^4)dx\\
&\quad -\tfrac{1}{2}\lambda^3 \int_{\mathbb{R}^3}(|u_1|^2|u_2|^2 + |u_1|^2|u_3|^2 + |u_2|^2|u_3|^2)dx\\
&\quad -\tfrac{1}{4}\lambda^3 \int_{\mathbb{R}^3} \tfrac{\eta\lambda^2(\xi_1^2+\xi_2^2)}{\lambda^2|\xi|^2-\delta}(|\mathcal{F}(u_1\overline{u}_2 - \overline{u}_1 u_2)|^2)d\xi\\
&\quad -\tfrac{1}{4}\lambda^3 \int_{\mathbb{R}^3} \tfrac{\eta\lambda^2(\xi_3^2+\xi_2^2)}{\lambda^2|\xi|^2-\delta}(|\mathcal{F}(u_2\overline{u}_3 - \overline{u}_2 u_3)|^2)d\xi\\
&\quad -\tfrac{1}{4}\lambda^3 \int_{\mathbb{R}^3} \tfrac{\eta\lambda^2(\xi_1^2+\xi_3^2)}{\lambda^2|\xi|^2-\delta}(|\mathcal{F}(u_1\overline{u}_3 - \overline{u}_1 u_3)|^2)d\xi\\
&\quad +\tfrac{1}{2}\lambda^3 Re\int_{\mathbb{R}^3} \tfrac{\eta\lambda^2(\xi_1\xi_2)}{\lambda^2|\xi|^2-\delta}\mathcal{F}(\overline{u_2}u_3 - u_2\overline{u_3})\overline{\mathcal{F}(u_1\overline{u_3} - \overline{u_1}u_3)}d\xi\\
&\quad +\tfrac{1}{2}\lambda^3 Re\int_{\mathbb{R}^3} \tfrac{\eta\lambda^2(\xi_2\xi_3)}{\lambda^2|\xi|^2-\delta}\mathcal{F}(\overline{u_1}u_3 - u_1\overline{u_3})\overline{\mathcal{F}(u_1\overline{u_2} - \overline{u_1}u_2)}d\xi\\
&\quad +\tfrac{1}{2}\lambda^3 Re\int_{\mathbb{R}^3} \tfrac{\eta\lambda^2(\xi_1\xi_3)}{\lambda^2|\xi|^2-\delta}|\mathcal{F}(u_2\overline{u_3} - \overline{u_2}u_3)\overline{\mathcal{F}(u_1\overline{u_2} - \overline{u_1}u_2)}d\xi,
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
&R(u_{1\lambda}, u_{2\lambda}, u_{3\lambda})\\
&= \lambda^2\Big\{\int_{\mathbb{R}^3}(|\nabla u_1|^2 + |\nabla u_2|^2 + |\nabla u_3|^2)dx\\
&\quad -\tfrac{3}{4}\lambda\int_{\mathbb{R}^3}(|u_1|^4 + |u_2|^4 + |u_3|^4)dx\\
&\quad -\tfrac{3}{2}\lambda\int_{\mathbb{R}^3}(|u_1|^2|u_2|^2 + |u_1|^2|u_3|^2 + |u_2|^2|u_3|^2)dx
\end{aligned}
$$

$$
R_{1\lambda}(u_1,u_2,u_3) := \left\{
\begin{aligned}
&-\tfrac{3}{4}\lambda\int_{\mathbb{R}^3}\tfrac{\eta\lambda^2(\xi_1^2+\xi_2^2)}{\lambda^2|\xi|^2-\delta}(|\mathcal{F}(u_1\overline{u}_2 - \overline{u}_1u_2)|^2)d\xi\\
&-\tfrac{3}{4}\lambda\int_{\mathbb{R}^3}\tfrac{\eta\lambda^2(\xi_3^2+\xi_2^2)}{\lambda^2|\xi|^2-\delta}(|\mathcal{F}(u_2\overline{u}_3 - \overline{u}_2u_3)|^2)d\xi\\
&-\tfrac{3}{4}\lambda\int_{\mathbb{R}^3}\tfrac{\eta\lambda^2(\xi_1^2+\xi_3^2)}{\lambda^2|\xi|^2-\delta}(|\mathcal{F}(u_1\overline{u}_3 - \overline{u}_1u_3)|^2)d\xi\\
&+\tfrac{\delta}{2}\lambda\int_{\mathbb{R}^3}\tfrac{\eta\lambda^2(\xi_1^2+\xi_2^2)}{(\lambda^2|\xi|^2-\delta)^2}(|\mathcal{F}(u_1\overline{u}_2 - \overline{u}_1u_2)|^2)d\xi\\
&+\tfrac{\delta}{2}\lambda\int_{\mathbb{R}^3}\tfrac{\eta\lambda^2(\xi_3^2+\xi_2^2)}{(\lambda^2|\xi|^2-\delta)^2}(|\mathcal{F}(u_2\overline{u}_3 - \overline{u}_2u_3)|^2)d\xi\\
&+\tfrac{\delta}{2}\lambda\int_{\mathbb{R}^3}\tfrac{\eta\lambda^2(\xi_1^2+\xi_3^2)}{(\lambda^2|\xi|^2-\delta)^2}(|\mathcal{F}(u_1\overline{u}_3 - \overline{u}_1u_3)|^2)d\xi\\
&+\tfrac{3}{2}\lambda^3 Re\int_{\mathbb{R}^3}\tfrac{\eta\lambda^2\xi_1\xi_2}{\lambda^2|\xi|^2-\delta}\mathcal{F}(\overline{u_2}u_3 - u_2\overline{u_3})\overline{\mathcal{F}(u_1\overline{u_3} - \overline{u_1}u_3)}d\xi\\
&+\tfrac{3}{2}\lambda Re\int_{\mathbb{R}^3}\tfrac{\eta\lambda^2\xi_2\xi_3}{\lambda^2|\xi|^2-\delta}\mathcal{F}(\overline{u_1}u_3 - u_1\overline{u_3})\overline{\mathcal{F}(u_1\overline{u_2} - \overline{u_1}u_2)}d\xi\\
&+\tfrac{3}{2}\lambda Re\int_{\mathbb{R}^3}\tfrac{\eta\lambda^2\xi_1\xi_3}{\lambda^2|\xi|^2-\delta}\mathcal{F}(u_2\overline{u_3} - \overline{u_2}u_3)\overline{\mathcal{F}(u_1\overline{u_2} - \overline{u_1}u_2)}d\xi\\
&-\lambda\delta Re\int_{\mathbb{R}^3}\tfrac{\eta\lambda^2\xi_1\xi_2}{(\lambda^2|\xi|^2-\delta)^2}\mathcal{F}(\overline{u_2}u_3 - u_2\overline{u_3})\overline{\mathcal{F}(u_1\overline{u_3} - \overline{u_1}u_3)}d\xi\\
&-\lambda\delta Re\int_{\mathbb{R}^3}\tfrac{\eta\lambda^2\xi_2\xi_3^2}{(\lambda^2|\xi|^2-\delta)^2}\mathcal{F}(\overline{u_1}u_3 - u_1\overline{u_3})\overline{\mathcal{F}(u_1\overline{u_2} - \overline{u_1}u_2)}d\xi\\
&-\lambda\delta Re\int_{\mathbb{R}^3}\tfrac{\eta\lambda^2\xi_1\xi_3}{(\lambda^2|\xi|^2-\delta)^2}\mathcal{F}(u_2\overline{u_3} - \overline{u_2}u_3)\overline{\mathcal{F}(u_1\overline{u_2} - \overline{u_1}u_2)}d\xi\Big\}
\end{aligned}
\right.
$$

$$
= \lambda^2 R^*(u_{1\lambda}, u_{2\lambda}, u_{3\lambda}). \tag{3.10}
$$

Making a preliminary estimate, we can verify that

$$
R_{1\lambda}(u_1, u_2, u_3) \le 0. \tag{3.11}
$$

((3.11) can be easily obtained by utilizing the method of verifying (3.7).)

First of all, we show that there exists $\mu > 0$ such that $R(u_{1\mu}, u_{2\mu}, u_{3\mu}) = 0$, and we divide the proof into two cases:

Case 1 $\quad R(u_1,u_2,u_3)>0$; $\qquad$ **Case 2** $\quad R(u_1,u_2,u_3)<0$.

♣ If **Case 1** occurs, and if there exists λ such that $R(u_{1\lambda},u_{2\lambda},u_{3\lambda})=0$, then $\lambda\in(1,\infty)$ according to (3.10) and (3.11). For $j,k=1,2,3$, let $G(\lambda^2)=\frac{3}{4}\frac{\eta\lambda^2(\xi_j^2+\xi_k^2)}{\lambda^2|\xi|^2-\delta}-\frac{\delta}{2}\frac{\eta\lambda^2(\xi_j^2+\xi_k^2)}{(\lambda^2|\xi|^2-\delta)^2}$. Then $G'(\lambda^2)=\frac{(-\delta\eta\lambda^2|\xi|^2+5\delta^2\eta)(\xi_j^2+\xi_k^2)}{4(\lambda^2|\xi|^2-\delta)^3}$, which together with $\eta>0$ and $\delta\le 0$ leads to $G'(\lambda^2)\ge 0$. Thus, $G(\lambda^2)$ is an increasing function of λ^2 ($\lambda^2\in(1,\infty)$). Thus there exists $\mu\in(1,\infty)$ such that $R(u_{1\mu},u_{2\mu},u_{3\mu})=0$.

♣ If **Case 2** occurs, and if there exists λ such that $R(u_{1\lambda},u_{2\lambda},u_{3\lambda})=0$, then $\lambda\in(0,1)$. Indeed, we consider functional $R^*(u_{1\lambda},u_{2\lambda},u_{3\lambda})$ defined by (3.10). Note that $R^*(u_{1\lambda},u_{2\lambda},u_{3\lambda})\to R(u_1,u_2,u_3)<0$ as $\lambda\to1$, and $R^*(u_{1\lambda},u_{2\lambda},u_{3\lambda})=\int_{\mathbb{R}^3}(|\nabla u_1|^2+|\nabla u_2|^2+|\nabla u_3|^2)dx>0$ as $\lambda\to0$, one can verifies that there exists $\mu\in(0,1)$ such that $R^*(u_{1\mu},u_{2\mu},u_{3\mu})=0$. The latter implies that $\mu^2R^*(u_{1\mu},u_{2\mu},u_{3\mu})=R(u_{1\mu},u_{2\mu},u_{3\mu})=0$.

In both cases as above, there always exists $\mu>0$ such that $R(u_{1\mu},u_{2\mu},u_{3\mu})=0$.

Furthermore, we can easily check that

$$\begin{aligned}&R(u_{1\lambda},u_{2\lambda},u_{3\lambda})>0 \quad for \quad \lambda\in(0,\mu),\\ &R(u_{1\lambda},u_{2\lambda},u_{3\lambda})<0 \quad for \quad \lambda\in(\mu,\infty).\end{aligned} \tag{3.12}$$

By a direct calculation, we achive for $j,k=1,2,3$,

$$\frac{d\big(\eta\lambda^5(\xi_j^2+\xi_k^2)/(\lambda^2|\xi|^2-\delta)\big)}{d\lambda}=\frac{3\eta\lambda^4(\xi_j^2+\xi_k^2)}{\lambda^2|\xi|^2-\delta}-\frac{2\eta\delta\lambda^4(\xi_j^2+\xi_k^2)}{(\lambda^2|\xi|^2-\delta)^2},$$

$$\frac{d\big(\eta\lambda^5\xi_j\xi_k/(\lambda^2|\xi|^2-\delta)\big)}{d\lambda}=\frac{3\eta\lambda^4\xi_j\xi_k}{\lambda^2|\xi|^2-\delta}-\frac{2\eta\delta\lambda^4\xi_j\xi_k}{(\lambda^2|\xi|^2-\delta)^2},$$

which together with (3.9) and (3.10) yield that

$$\frac{d}{d\lambda}S(u_{1\lambda},u_{2\lambda},u_{3\lambda})=\lambda^{-1}R(u_{1\lambda},u_{2\lambda},u_{3\lambda}). \tag{3.13}$$

By $R(u_{1\mu},u_{2\mu},u_{3\mu})=0$, (3.12) and (3.13) imply that

$$S(u_{1\mu},u_{2\mu},u_{3\mu})\ge S(u_{1\lambda},u_{2\lambda},u_{3\lambda}),\quad \forall\lambda>0.$$

This completes the proof of Proposition 3.3. □

Now, we begin to prove Theorem 2.1.

Step 1: Proof of (1).

Let $\{(Q_{1n},Q_{2n},Q_{3n}),n\in\mathbf{N}\}\subset M$ be a minimizing sequence for (2.5). There then has

$$R(Q_{1n},Q_{2n},Q_{3n})=0$$

and

$$S(Q_{1n},Q_{2n},Q_{3n})\to\inf_{(u_1,u_2,u_3)\in M}S(u_1,u_2,u_3)\ as\ n\to\infty. \tag{3.14}$$

In view of $\eta > 0$, $\delta \leq 0$ and Young's inequality, combining (3.6) with (3.14) yields that $\|Q_{1n}\|_{H_r^1(\mathbb{R}^3)}$, $\|Q_{2n}\|_{H_r^1(\mathbb{R}^3)}$, $\|Q_{3n}\|_{H_r^1(\mathbb{R}^3)}$ are all bounded for all $n \in \mathbb{N}$. Thus there exists a subsequence

$$\{(Q_{1nk}, Q_{2nk}, Q_{3nk}), k \in \mathbb{N}\} \subset \{(Q_{1n}, Q_{2n}, Q_{3n}), n \in \mathbb{N}\} \tag{3.15}$$

such that as $k \to \infty$,

$$Q_{1nk} \rightharpoonup Q_{1\infty} \textit{ weakly in } H_r^1(\mathbb{R}^3), \quad Q_{1nk} \to Q_{1\infty} \textit{ a.e. in } \mathbb{R}^3, \tag{3.16}$$

$$Q_{2nk} \rightharpoonup Q_{2\infty} \textit{ weakly in } H_r^1(\mathbb{R}^3), \quad Q_{2nk} \to Q_{2\infty} \textit{ a.e. in } \mathbb{R}^3, \tag{3.17}$$

and

$$Q_{3nk} \rightharpoonup Q_{3\infty} \textit{ weakly in } H_r^1(\mathbb{R}^3), \quad Q_{3nk} \to Q_{3\infty} \textit{ a.e. in } \mathbb{R}^3. \tag{3.18}$$

For simplicity, we still denote $\{(Q_{1nk}, Q_{2nk}, Q_{3nk}), k \in \mathbb{N}\}$ by $\{(Q_{1n}, Q_{2n}, Q_{3n}), n \in \mathbb{N}\}$. From Lemma 3.1, (3.16)-(3.18), it follows that

$$\begin{aligned} &Q_{1n} \to Q_{1\infty} \textit{ strongly in } L_r^4(\mathbb{R}^3),\\ &Q_{2n} \to Q_{2\infty} \textit{ strongly in } L_r^4(\mathbb{R}^3),\\ &Q_{3n} \to Q_{3\infty} \textit{ strongly in } L_r^4(\mathbb{R}^3). \end{aligned} \tag{3.19}$$

Since

$$\|Q_{1n}\|_{L_r^2(\mathbb{R}^3)} \leq C, \quad \|Q_{2n}\|_{L_r^2(\mathbb{R}^3)} \leq C, \|Q_{3n}\|_{L_r^2(\mathbb{R}^3)} \leq C,$$

the boundedness of $\{(Q_{1n}, Q_{2n}, Q_{3n}), n \in \mathbb{N}\}$ in $H_r^1(\mathbb{R}^3) \times H_r^1(\mathbb{R}^3) \times H_r^1(\mathbb{R}^3)$ and the Gagliardo-Nirenberg inequality

$$\|v\|^{p+1}_{L_r^{p+1}(\mathbb{R}^N)} \leq C \|\nabla v\|^{\frac{N}{2}(p-1)}_{L_r^2(\mathbb{R}^N)} \|v\|^{p+1-\frac{N}{2}(p-1)}_{L_r^2(\mathbb{R}^N)}, \quad v \in H_r^1(\mathbb{R}^N), \quad 1 \leq p < \frac{N+2}{N-2},$$

imply in particular that

$$\begin{aligned} &\textstyle\int_{\mathbb{R}^3} |Q_{1n}|^4 dx \leq C \left(\int_{\mathbb{R}^3} |\nabla Q_{1n}|^2 dx\right)^{\frac{3}{2}},\\ &\textstyle\int_{\mathbb{R}^3} |Q_{2n}|^4 dx \leq C \left(\int_{\mathbb{R}^3} |\nabla Q_{2n}|^2 dx\right)^{\frac{3}{2}},\\ &\textstyle\int_{\mathbb{R}^3} |Q_{3n}|^4 dx \leq C \left(\int_{\mathbb{R}^3} |\nabla Q_{3n}|^2 dx\right)^{\frac{3}{2}}. \end{aligned} \tag{3.20}$$

Here and henceforth, $C > 0$ denotes various positive constants. Via (3.20), $\eta > 0$, $\delta \leq 0$ and

$$R(Q_{1n}, Q_{2n}, Q_{3n}) = 0, \quad \int_{\mathbb{R}^3} |Q_{jn}|^2 |Q_{kn}|^2 dx \leq \frac{1}{2} \int_{\mathbb{R}^3} (|Q_{jn}|^4 + |Q_{kn}|^4) dx, \tag{3.21}$$

where $j, k = 1, 2, 3$, we thus obtain

$$\begin{aligned} &\textstyle\int_{\mathbb{R}^3} (|\nabla Q_{1n}|^2 + |\nabla Q_{2n}|^2 + |\nabla Q_{3n}|^2) dx\\ &\quad \leq C \left(\left(\int_{\mathbb{R}^3} |\nabla Q_{1n}|^2 dx\right)^{\frac{3}{2}} + \left(\int_{\mathbb{R}^3} |\nabla Q_{2n}|^2 dx\right)^{\frac{3}{2}} + \left(\int_{\mathbb{R}^3} |\nabla Q_{3n}|^2 dx\right)^{\frac{3}{2}}\right.\\ &\quad \leq C \left(\int_{\mathbb{R}^3} (|\nabla Q_{1n}|^2 + |\nabla Q_{2n}|^2 + |\nabla Q_{3n}|^2) dx\right)^{\frac{3}{2}}. \end{aligned} \tag{3.22}$$

(3.22) yields that $\|\nabla Q_{1n}\|^2_{L^2_r(\mathbb{R}^3)} + \|\nabla Q_{2n}\|^2_{L^2_r(\mathbb{R}^3)} + \|\nabla Q_{3n}\|^2_{L^2_r(\mathbb{R}^3)}$ is bounded away from 0. Furthermore, we claim that $(Q_{1\infty}, Q_{2\infty}, Q_{3\infty}) \neq (0,0,0)$. Assume to the contrary that $(Q_{1\infty}, Q_{2\infty}, Q_{3\infty}) \equiv (0,0,0)$, from (3.19), it follows that

$$\begin{aligned} &Q_{1n} \to 0 \textit{ strongly in } L^4_r(\mathbb{R}^3), \\ &Q_{2n} \to 0 \textit{ strongly in } L^4_r(\mathbb{R}^3), \\ &Q_{3n} \to 0 \textit{ strongly in } L^4_r(\mathbb{R}^3). \end{aligned} \tag{3.23}$$

Thus, (3.21) implies that as $n \to \infty$,

$$\begin{aligned} &\textstyle\int_{\mathbb{R}^3}(|Q_{1n}|^4 + |Q_{2n}|^4 + |Q_{3n}|^4)dx \to 0, \\ &\textstyle\int_{\mathbb{R}^3}(|Q_{1n}|^2|Q_{2n}|^2 + |Q_{1n}|^2|Q_{3n}|^2 + |Q_{2n}|^2|Q_{3n}|^2)dx \to 0, \\ &R_{1\lambda}(Q_{1n}, Q_{2n}, Q_{3n})|_{\lambda=1} \to 0, \end{aligned} \tag{3.24}$$

where $R_{1\lambda}(Q_{1n}, Q_{2n}, Q_{3n})$ is defined by (3.10) with replacing (u_1, u_2, u_3) by (Q_{1n}, Q_{2n}, Q_{3n}) . In view of $R(Q_{1n}, Q_{2n}, Q_{3n}) = 0$, one would then conclude that as $n \to \infty$,

$$\int_{\mathbb{R}^3}(|\nabla Q_{1n}|^2 + |\nabla Q_{2n}|^2 + |\nabla Q_{3n}|^2)dx \to 0,$$

which contradicts to (3.22). Thus $(Q_{1\infty}, Q_{2\infty}, Q_{3\infty}) \neq (0,0,0)$.

Let $Q_1 = (Q_{1\infty})_\mu$, $Q_2 = (Q_{2\infty})_\mu$, $Q_3 = (Q_{3\infty})_\mu$ with $\mu > 0$ uniquely determined by the condition $R(Q_1, Q_2, Q_3) = R[(Q_{1\infty})_\mu, (Q_{2\infty})_\mu, (Q_{3\infty})_\mu] = 0$, where $(Q_{1\infty})_\mu, (Q_{2\infty})_\mu$ and $(Q_{3\infty})_\mu$ are defined by Proposition 3.3. Thus Lemma 3.1 yields that, as $n \to \infty$,

$$\begin{cases} (Q_{1n})_\mu \to Q_1, \quad (Q_{2n})_\mu \to Q_2, \quad (Q_{3n})_\mu \to Q_3, \textit{ strongly in } L^4_r(\mathbb{R}^3), \\ \\ (Q_{1n})_\mu \to Q_1, \quad (Q_{2n})_\mu \to Q_2, \quad (Q_{3n})_\mu \to Q_3, \textit{ weakly in } H^1_r(\mathbb{R}^3), \end{cases} \tag{3.25}$$

whereas $R(Q_{1n}, Q_{2n}, Q_{3n}) = 0$ and Proposition 3.3 imply

$$S[(Q_{1n})_\mu, (Q_{2n})_\mu, (Q_{3n})_\mu] \leq S(Q_{1n}, Q_{2n}, Q_{3n}). \tag{3.26}$$

Hence, from (3.25) and (3.26) one concludes

$$\begin{aligned} S(Q_1, Q_2, Q_3) &\leq S[(Q_{1n})_\mu, (Q_{2n})_\mu, (Q_{3n})_\mu] \leq S(Q_{1n}, Q_{2n}, Q_{3n}) \\ &= \inf_{(u_1,u_2,u_3)\in M} S(u_1, u_2, u_3) \end{aligned} \tag{3.27}$$

which together with $R(Q_1, Q_2, Q_3) = 0$ yields that $(Q_1, Q_2, Q_3) \in M$. Therefore, (Q_1, Q_2, Q_3) solves the minimization problem

$$S(Q_1, Q_2, Q_3) = \min_{(u_1,u_2,u_3)\in M} S(u_1, u_2, u_3). \tag{3.28}$$

This completes the proof of (1) of Theorem 2.1 .

Step 2: Proofs of (2) and (3) of Theorem 2.1

We first prove (2). Since (Q_1, Q_2, Q_3) is a solution of the minimization problem (3.28), there exists a Lagrange multiplier Λ such that

$$\begin{aligned}
&\delta'_{\overline{Q_1}}[S(Q_1,Q_2,Q_3)+\Lambda R(Q_1,Q_2,Q_3)]=0,\\
&\delta'_{\overline{Q_2}}[S(Q_1,Q_2,Q_3)+\Lambda R(Q_1,Q_2,Q_3)]=0,\\
&\delta'_{\overline{Q_3}}[S(Q_1,Q_2,Q_3)+\Lambda R(Q_1,Q_2,Q_3)]=0.
\end{aligned} \tag{3.29}$$

Here, $\delta'_{u_1}T(u_1, u_2, u_3)$ denotes the variation of $T(u_1, u_2, u_3)$ with respect to u_1. By the formula $\delta'_{u_1}T(u_1, u_2, u_3) = \frac{\partial}{\partial\zeta}T(u_1 + \zeta\delta' u_1, u_2, u_3)|_{\zeta=0}$, and by taking $\delta'\overline{Q_1} = \overline{Q_1}$, $\delta'\overline{Q_2} = \overline{Q_2}$ and $\delta'\overline{Q_3} = \overline{Q_3}$, we obtain from (3.29) that

$$B_1(Q_1,Q_2,Q_3)=0, \quad B_2(Q_1,Q_2,Q_3)=0, \quad B_3(Q_1,Q_2,Q_3)=0, \tag{3.30}$$

where

$$\begin{aligned}
B_1(Q_1,Q_2,Q_3) \;=\; &-(1+2\Lambda)\Delta Q_1+\omega Q_1-(1+3\Lambda)|Q_1|^2Q_1\\
&-(1+3\Lambda)Q_1|Q_2|^2-(1+3\Lambda)Q_1|Q_3|^2\\
&-(1+3\Lambda)Q_2\mathcal{F}^{-1}[\tfrac{\eta(\xi_1^2+\xi_2^2)}{|\xi|^2-\delta}\mathcal{F}(Q_1\overline{Q_2}-\overline{Q_1}Q_2)]\\
&-(1+3\Lambda)Q_3\mathcal{F}^{-1}[\tfrac{\eta(\xi_1^2+\xi_3^2)}{|\xi|^2-\delta}\mathcal{F}(Q_1\overline{Q_3}-\overline{Q_1}Q_3)]\\
&+(1+3\Lambda)Q_2\mathcal{F}^{-1}[\tfrac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(Q_2\overline{Q_3}-\overline{Q_2}Q_3)]\\
&+(1+3\Lambda)Q_2\mathcal{F}^{-1}[\tfrac{\eta\xi_2\xi_3}{|\xi|^2-\delta}\mathcal{F}(\overline{Q_1}Q_3-Q_1\overline{Q_3})]\\
&-(1+3\Lambda)Q_3\mathcal{F}^{-1}[\tfrac{\eta\xi_2\xi_3}{|\xi|^2-\delta}\mathcal{F}(Q_1\overline{Q_2}-\overline{Q_1}Q_2)]\\
&+(1+3\Lambda)Q_3\mathcal{F}^{-1}[\tfrac{\eta\xi_1\xi_2}{|\xi|^2-\delta}\mathcal{F}(\overline{Q_2}Q_3-Q_2\overline{Q_3})]\\
&+2\Lambda\delta Q_2\mathcal{F}^{-1}[\tfrac{\eta(\xi_1^2+\xi_2^2)}{(|\xi|^2-\delta)^2}\mathcal{F}(Q_1\overline{Q_2}-\overline{Q_1}Q_2)]\\
&+2\Lambda\delta Q_3\mathcal{F}^{-1}[\tfrac{\eta(\xi_1^2+\xi_3^2)}{(|\xi|^2-\delta)^2}\mathcal{F}(Q_1\overline{Q_3}-\overline{Q_1}Q_3)]\\
&-2\Lambda\delta Q_2\mathcal{F}^{-1}[\tfrac{\eta\xi_1\xi_3}{(|\xi|^2-\delta)^2}\mathcal{F}(Q_2\overline{Q_3}-\overline{Q_2}Q_3)]\\
&-2\Lambda\delta Q_2\mathcal{F}^{-1}[\tfrac{\eta\xi_2\xi_3}{(|\xi|^2-\delta)^2}\mathcal{F}(\overline{Q_1}Q_3-Q_1\overline{Q_3})]\\
&+2\Lambda\delta Q_3\mathcal{F}^{-1}[\tfrac{\eta\xi_2\xi_3}{(|\xi|^2-\delta)^2}\mathcal{F}(Q_1\overline{Q_2}-\overline{Q_1}Q_2)]\\
&-2\Lambda\delta Q_3\mathcal{F}^{-1}[\tfrac{\eta\xi_1\xi_2}{(|\xi|^2-\delta)^2}\mathcal{F}(\overline{Q_2}Q_3-Q_2\overline{Q_3})],
\end{aligned} \tag{3.31}$$

$$
\begin{aligned}
B_2(Q_1,Q_2,Q_3) \;&= -(1+2\Lambda)\Delta Q_2+\omega Q_2-(1+3\Lambda)|Q_2|^2Q_2\\
&\quad -(1+3\Lambda)|Q_1|^2Q_2-(1+3\Lambda)|Q_3|^2Q_2\\
&\quad -(1+3\Lambda)Q_1\mathcal{F}^{-1}[\tfrac{\eta(\xi_1^2+\xi_2^2)}{|\xi|^2-\delta}\mathcal{F}(\overline{Q_1}Q_2-Q_1\overline{Q_2})]\\
&\quad -(1+3\Lambda)Q_3\mathcal{F}^{-1}[\tfrac{\eta(\xi_2^2+\xi_3^2)}{|\xi|^2-\delta}\mathcal{F}(Q_2\overline{Q_3}-\overline{Q_2}Q_3)]\\
&\quad -(1+3\Lambda)Q_1\mathcal{F}^{-1}[\tfrac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(Q_2\overline{Q_3}-\overline{Q_2}Q_3)]\\
&\quad -(1+3\Lambda)Q_1\mathcal{F}^{-1}[\tfrac{\eta\xi_2\xi_3}{|\xi|^2-\delta}\mathcal{F}(\overline{Q_1}Q_3-Q_1\overline{Q_3})]\\
&\quad -(1+3\Lambda)Q_3\mathcal{F}^{-1}[\tfrac{\eta\xi_1\xi_2}{|\xi|^2-\delta}\mathcal{F}(Q_1\overline{Q_3}-\overline{Q_1}Q_3)]\\
&\quad +(1+3\Lambda)Q_3\mathcal{F}^{-1}[\tfrac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(\overline{Q_1}Q_2-Q_1\overline{Q_2})]\\
&\quad +2\Lambda\delta Q_1\mathcal{F}^{-1}[\tfrac{\eta(\xi_1^2+\xi_2^2)}{(|\xi|^2-\delta)^2}\mathcal{F}(\overline{Q_1}Q_2-Q_1\overline{Q_2})]\\
&\quad +2\Lambda\delta Q_3\mathcal{F}^{-1}[\tfrac{\eta(\xi_2^2+\xi_3^2)}{(|\xi|^2-\delta)^2}\mathcal{F}(Q_2\overline{Q_3}-\overline{Q_2}Q_3)]\\
&\quad +2\Lambda\delta Q_1\mathcal{F}^{-1}[\tfrac{\eta\xi_1\xi_3}{(|\xi|^2-\delta)^2}\mathcal{F}(Q_2\overline{Q_3}-\overline{Q_2}Q_3)]\\
&\quad +2\Lambda\delta Q_1\mathcal{F}^{-1}[\tfrac{\eta\xi_2\xi_3}{(|\xi|^2-\delta)^2}\mathcal{F}(\overline{Q_1}Q_3-Q_1\overline{Q_3})]\\
&\quad +2\Lambda\delta Q_3\mathcal{F}^{-1}[\tfrac{\eta\xi_1\xi_2}{(|\xi|^2-\delta)^2}\mathcal{F}(Q_1\overline{Q_3}-\overline{Q_1}Q_3)]\\
&\quad +2\Lambda\delta Q_3\mathcal{F}^{-1}[\tfrac{\eta\xi_1\xi_3}{(|\xi|^2-\delta)^2}\mathcal{F}(\overline{Q_1}Q_2-Q_1\overline{Q_2})],
\end{aligned}
\tag{3.32}
$$

$$
\begin{aligned}
B_3(Q_1,Q_2,Q_3) \;&= -(1+2\Lambda)\Delta Q_3+\omega Q_3-(1+3\Lambda)|Q_3|^2Q_3\\
&\quad -(1+3\Lambda)|Q_1|^2Q_3-(1+3\Lambda)|Q_2|^2Q_3\\
&\quad -(1+3\Lambda)Q_1\mathcal{F}^{-1}[\tfrac{\eta(\xi_1^2+\xi_3^2)}{|\xi|^2-\delta}\mathcal{F}(\overline{Q_1}Q_3-Q_1\overline{Q_3})]\\
&\quad -(1+3\Lambda)Q_2\mathcal{F}^{-1}[\tfrac{\eta(\xi_2^2+\xi_3^2)}{|\xi|^2-\delta}\mathcal{F}(\overline{Q_2}Q_3-Q_2\overline{Q_3})]\\
&\quad -(1+3\Lambda)Q_1\mathcal{F}^{-1}[\tfrac{\eta\xi_1\xi_2}{|\xi|^2-\delta}\mathcal{F}(\overline{Q_2}Q_3-Q_2\overline{Q_3})]\\
&\quad +(1+3\Lambda)Q_1\mathcal{F}^{-1}[\tfrac{\eta\xi_2\xi_3}{|\xi|^2-\delta}\mathcal{F}(Q_1\overline{Q_2}-\overline{Q_1}Q_2)]\\
&\quad -(1+3\Lambda)Q_2\mathcal{F}^{-1}[\tfrac{\eta\xi_1\xi_3}{|\xi|^2-\delta}\mathcal{F}(\overline{Q_1}Q_2-Q_1\overline{Q_2})]\\
&\quad +(1+3\Lambda)Q_2\mathcal{F}^{-1}[\tfrac{\eta\xi_1\xi_2}{|\xi|^2-\delta}\mathcal{F}(Q_1\overline{Q_3}-\overline{Q_1}Q_3)]\\
&\quad +2\Lambda\delta Q_1\mathcal{F}^{-1}[\tfrac{\eta(\xi_1^2+\xi_3^2)}{(|\xi|^2-\delta)^2}\mathcal{F}(\overline{Q_1}Q_3-Q_1\overline{Q_3})]\\
&\quad +2\Lambda\delta Q_2\mathcal{F}^{-1}[\tfrac{\eta(\xi_2^2+\xi_3^2)}{(|\xi|^2-\delta)^2}\mathcal{F}(\overline{Q_2}Q_3-Q_2\overline{Q_3})]\\
&\quad +2\Lambda\delta Q_1\mathcal{F}^{-1}[\tfrac{\eta\xi_1\xi_2}{(|\xi|^2-\delta)^2}\mathcal{F}(\overline{Q_2}Q_3-Q_2\overline{Q_3})]\\
&\quad -2\Lambda\delta Q_1\mathcal{F}^{-1}[\tfrac{\eta\xi_2\xi_3}{(|\xi|^2-\delta)^2}\mathcal{F}(Q_1\overline{Q_2}-\overline{Q_1}Q_2)]\\
&\quad +2\Lambda\delta Q_2\mathcal{F}^{-1}[\tfrac{\eta\xi_1\xi_3}{(|\xi|^2-\delta)^2}\mathcal{F}(\overline{Q_1}Q_2-Q_1\overline{Q_2})]\\
&\quad -2\Lambda\delta Q_2\mathcal{F}^{-1}[\tfrac{\eta\xi_1\xi_2}{(|\xi|^2-\delta)^2}\mathcal{F}(Q_1\overline{Q_3}-\overline{Q_1}Q_3)].
\end{aligned}
\tag{3.33}
$$

Multiplying $B_1(Q_1,Q_2,Q_3)=0$ by $\overline{Q_1}$, $B_2(Q_1,Q_2,Q_3)=0$ by $\overline{Q_2}$ and $B_3(Q_1,Q_2,Q_3)=0$ by $\overline{Q_3}$, then integrating the resulting equations with respect to x on $\mathbb{R}^3$, we get

$$
K_1(Q_1,Q_2,Q_3)+\Lambda K_2(Q_1,Q_2,Q_3)=0, \tag{3.34}
$$

where

$$
\begin{aligned}
K_1(Q_1,Q_2,Q_3) \quad &= \int_{\mathbb{R}^3}(|\nabla Q_1|^2+|\nabla Q_2|^2+|\nabla Q_3|^2)dx \\
&+\omega \int_{\mathbb{R}^3}(|Q_1|^2+|Q_2|^2+|Q_3|^2)dx \\
&-\int_{\mathbb{R}^3}(|Q_1|^4+|Q_2|^4+|Q_3|^4)dx \\
&-2\int_{\mathbb{R}^3}(|Q_1|^2|Q_2|^2+|Q_1|^2|Q_3|^2+|Q_2|^2|Q_3|^2)dx \\
&-\int_{\mathbb{R}^3} A(Q_1,Q_2,Q_3)d\xi+2Re\int_{\mathbb{R}^3} B(Q_1,Q_2,Q_3)d\xi,
\end{aligned}
\tag{3.35}
$$

$$
\begin{aligned}
K_2(Q_1,Q_2,Q_3) \quad &= 2\int_{\mathbb{R}^3}(|\nabla Q_1|^2+|\nabla Q_2|^2+|\nabla Q_3|^2)dx \\
&-3\int_{\mathbb{R}^3}(|Q_1|^4+|Q_2|^4+|Q_3|^4)dx \\
&-6\int_{\mathbb{R}^3}(|Q_1|^2|Q_2|^2+|Q_1|^2|Q_3|^2+|Q_2|^2|Q_3|^2)dx \\
&-3\int_{\mathbb{R}^3} A(Q_1,Q_2,Q_3)d\xi+6Re\int_{\mathbb{R}^3} B(Q_1,Q_2,Q_3)d\xi \\
&+2\delta\int_{\mathbb{R}^3}\tfrac{A(Q_1,Q_2,Q_3)}{|\xi|^2-\delta}d\xi-4\delta Re\int_{\mathbb{R}^3}\tfrac{B(Q_1,Q_2,Q_3)}{|\xi|^2-\delta}d\xi,
\end{aligned}
\tag{3.36}
$$

$$
\begin{aligned}
A(Q_1,Q_2,Q_3) \quad &= \tfrac{\eta}{|\xi|^2-\delta}[(\xi_1^2+\xi_2^2)(|\mathcal{F}(Q_1\overline{Q_2}-\overline{Q_1}Q_2)|^2) \\
&+(\xi_1^2+\xi_3^2)(|\mathcal{F}(Q_1\overline{Q_3}-\overline{Q_1}Q_3)|^2) \\
&+(\xi_2^2+\xi_3^2)(|\mathcal{F}(Q_2\overline{Q_3}-\overline{Q_2}Q_3)|^2)],
\end{aligned}
\tag{3.37}
$$

$$
\begin{aligned}
B(Q_1,Q_2,Q_3) \quad &= \tfrac{\eta}{|\xi|^2-\delta}[\xi_1\xi_2\mathcal{F}(\overline{Q_2}Q_3-Q_2\overline{Q_3})\overline{\mathcal{F}(Q_1\overline{Q_3}-\overline{Q_1}Q_3)} \\
&+\xi_1\xi_3\mathcal{F}(Q_2\overline{Q_3}-\overline{Q_2}Q_3)\overline{\mathcal{F}(Q_1\overline{Q_2}-\overline{Q_1}Q_2)} \\
&+\xi_2\xi_3\mathcal{F}(\overline{Q_1}Q_3-Q_1\overline{Q_3})\overline{\mathcal{F}(Q_1\overline{Q_2}-\overline{Q_1}Q_2)}].
\end{aligned}
\tag{3.38}
$$

On the other hand, multiplying $B_1(Q_1,Q_2,Q_3)=0$ by $x\nabla\overline{Q_1}$, $B_2(Q_1,Q_2,Q_3)=0$ by $x\nabla\overline{Q_2}$, and $B_3(Q_1,Q_2,Q_3)=0$ by $x\nabla\overline{Q_3}$, then integrating the resulting equations with respect to x on $\mathbb{R}^3$ and taking the real part, one obtains

$$
K_3(Q_1,Q_2,Q_3)+\Lambda K_4(Q_1,Q_2,Q_3)=0,
\tag{3.39}
$$

where

$$
\begin{aligned}
K_3(Q_1,Q_2,Q_3) \quad &= -\tfrac{1}{2}\int_{\mathbb{R}^3}(|\nabla Q_1|^2+|\nabla Q_2|^2+|\nabla Q_3|^2)dx \\
&-\tfrac{3\omega}{2}\int_{\mathbb{R}^3}(|Q_1|^2+|Q_2|^2+|Q_3|^2)dx \\
&+\tfrac{3}{4}\int_{\mathbb{R}^3}(|Q_1|^4+|Q_2|^4+|Q_3|^4)dx \\
&+\tfrac{3}{2}\int_{\mathbb{R}^3}(|Q_1|^2|Q_2|^2+|Q_1|^2|Q_3|^2+|Q_2|^2|Q_3|^2)dx \\
&+\tfrac{3}{4}\int_{\mathbb{R}^3} A(Q_1,Q_2,Q_3)d\xi+\tfrac{\delta}{2}\int_{\mathbb{R}^3}\tfrac{A(Q_1,Q_2,Q_3)}{|\xi|^2-\delta}d\xi \\
&-\tfrac{3}{2}Re\int_{\mathbb{R}^3} B(Q_1,Q_2,Q_3)d\xi-\delta Re\int_{\mathbb{R}^3}\tfrac{B(Q_1,Q_2,Q_3)}{|\xi|^2-\delta}d\xi,
\end{aligned}
\tag{3.40}
$$

$$
\begin{aligned}
K_4(Q_1,Q_2,Q_3) \quad &= -\int_{\mathbb{R}^3}(|\nabla Q_1|^2+|\nabla Q_2|^2+|\nabla Q_3|^2)dx \\
&+\tfrac{9}{4}\int_{\mathbb{R}^3}(|Q_1|^4+|Q_2|^4+|Q_3|^4)dx \\
&+\tfrac{9}{2}\int_{\mathbb{R}^3}(|Q_1|^2|Q_2|^2+|Q_1|^2|Q_3|^2+|Q_2|^2|Q_3|^2)dx \\
&+\tfrac{9}{4}\int_{\mathbb{R}^3} A(Q_1,Q_2,Q_3)d\xi-\tfrac{9}{2}Re\int_{\mathbb{R}^3} B(Q_1,Q_2,Q_3)d\xi \\
&-\delta\int_{\mathbb{R}^3}\tfrac{A(Q_1,Q_2,Q_3)}{|\xi|^2-\delta}d\xi-2\delta^2\int_{\mathbb{R}^3}\tfrac{A(Q_1,Q_2,Q_3)}{(|\xi|^2-\delta)^2}d\xi \\
&+2\delta Re\int_{\mathbb{R}^3}\tfrac{B(Q_1,Q_2,Q_3)}{|\xi|^2-\delta}d\xi+4\delta^2 Re\int_{\mathbb{R}^3}\tfrac{B(Q_1,Q_2,Q_3)}{(|\xi|^2-\delta)^2}d\xi.
\end{aligned}
\tag{3.41}
$$

Here, $A(Q_1,Q_2,Q_3)$ and $B(Q_1,Q_2,Q_3)$ are defined by (3.37) and (3.38), respectively. Thus by (3.34), we obtain

$$\frac{3}{2}K_1(Q_1,Q_2,Q_3)+\frac{3}{2}\Lambda K_2(Q_1,Q_2,Q_3)=0. \tag{3.42}$$

Noting that

$$\begin{aligned}
&\tfrac{3}{2}K_1(Q_1,Q_2,Q_3)+K_3(Q_1,Q_2,Q_3)\\
&\quad=\int_{\mathbb{R}^3}(|\nabla Q_1|^2+|\nabla Q_2|^2+|\nabla Q_3|^2)dx-\tfrac{3}{4}\int_{\mathbb{R}^3}(|Q_1|^4+|Q_2|^4+|Q_3|^4)dx\\
&\qquad-\tfrac{3}{2}\int_{\mathbb{R}^3}(|Q_1|^2|Q_2|^2+|Q_1|^2|Q_3|^2+|Q_2|^2|Q_3|^2)dx-\tfrac{3}{4}\int_{\mathbb{R}^3}A(Q_1,Q_2,Q_3)d\xi\\
&\qquad+\tfrac{\delta}{2}\int_{\mathbb{R}^3}\tfrac{A(Q_1,Q_2,Q_3)}{|\xi|^2-\delta}d\xi+\tfrac{3}{2}Re\int_{\mathbb{R}^3}B(Q_1,Q_2,Q_3)d\xi-\delta Re\int_{\mathbb{R}^3}\tfrac{B(Q_1,Q_2,Q_3)}{|\xi|^2-\delta}d\xi\\
&\quad=R(Q_1,Q_2,Q_3)=0,
\end{aligned} \tag{3.43}$$

in view of (3.39), (3.42),(3.43), one can verify that

$$\frac{3}{2}\Lambda K_2(Q_1,Q_2,Q_3)+\Lambda K_4(Q_1,Q_2,Q_3)=0. \tag{3.44}$$

(3.44) is equivalent to

$$\Lambda[\frac{3}{2}K_2(Q_1,Q_2,Q_3)+K_4(Q_1,Q_2,Q_3)]=\Lambda K_5(Q_1,Q_2,Q_3)=0, \tag{3.45}$$

where

$$\begin{aligned}
K_5(Q_1,Q_2,Q_3)\ &=3R(Q_1,Q_2,Q_3)-\int_{\mathbb{R}^3}(|\nabla Q_1|^2+|\nabla Q_2|^2+|\nabla Q_3|^2)dx\\
&\quad+\tfrac{\delta}{2}\int_{\mathbb{R}^3}\tfrac{A(Q_1,Q_2,Q_3)}{|\xi|^2-\delta}d\xi-2\delta^2\int_{\mathbb{R}^3}\tfrac{A(Q_1,Q_2,Q_3)}{(|\xi|^2-\delta)^2}d\xi\\
&\quad-\delta Re\int_{\mathbb{R}^3}\tfrac{B(Q_1,Q_2,Q_3)}{|\xi|^2-\delta}d\xi+4\delta^2 Re\int_{\mathbb{R}^3}\tfrac{B(Q_1,Q_2,Q_3)}{(|\xi|^2-\delta)^2}d\xi.
\end{aligned} \tag{3.46}$$

Noting the expressions of $A(Q_1,Q_2,Q_3)$ and $B(Q_1,Q_2,Q_3)$ in (3.37) and (3.38), applying the Young's inequality, we have

$$\begin{aligned}
&\tfrac{\delta}{2}\int_{\mathbb{R}^3}\tfrac{A(Q_1,Q_2,Q_3)}{|\xi|^2-\delta}d\xi-2\delta^2\int_{\mathbb{R}^3}\tfrac{A(Q_1,Q_2,Q_3)}{(|\xi|^2-\delta)^2}d\xi\\
&-\delta Re\int_{\mathbb{R}^3}\tfrac{B(Q_1,Q_2,Q_3)}{|\xi|^2-\delta}d\xi+4\delta^2 Re\int_{\mathbb{R}^3}\tfrac{B(Q_1,Q_2,Q_3)}{(|\xi|^2-\delta)^2}d\xi\le 0.
\end{aligned}$$

Then, in view of $(Q_1,Q_2,Q_3)\neq(0,0,0)$, $R(Q_1,Q_2,Q_3)=0$, $\eta>0$ and $\delta\le 0$, there holds

$$K_5(Q_1,Q_2,Q_3)\le-\int_{\mathbb{R}^3}(|\nabla Q_1|^2+|\nabla Q_2|^2+|\nabla Q_3|^2)dx<0. \tag{3.47}$$

which implies that $K_5(Q_1,Q_2,Q_3)\neq 0$ and thus $\Lambda=0$ by (3.45). Hence, from (3.30), (3.31), (3.32) and (3.33), it follows that (Q_1,Q_2,Q_3) solves the following equations:

$$\begin{aligned}
&-\omega Q_1+\Delta Q_1+(|Q_1|^2+|Q_2|^2+|Q_3|^2)Q_1\\
&\quad-Q_2\mathcal{F}^{-1}[\tfrac{\eta(\xi_1^2+\xi_2^2)}{|\xi|^2-\delta}\mathcal{F}(\overline{Q_1}Q_2-Q_1\overline{Q_2})]+Q_3\mathcal{F}^{-1}[\tfrac{\eta(\xi_1^2+\xi_3^2)}{|\xi|^2-\delta}\mathcal{F}(Q_1\overline{Q_3}-\overline{Q_1}Q_3)]\\
&\quad-Q_2\mathcal{F}^{-1}\{\tfrac{\eta}{|\xi|^2-\delta}[\xi_1\xi_3\mathcal{F}(Q_2\overline{Q_3}-\overline{Q_2}Q_3)+\xi_2\xi_3\mathcal{F}(\overline{Q_1}Q_3-Q_1\overline{Q_3})]\}\\
&\quad+Q_3\mathcal{F}^{-1}\{\tfrac{\eta}{|\xi|^2-\delta}[\xi_2\xi_3\mathcal{F}(Q_1\overline{Q_2}-\overline{Q_1}Q_2)+\xi_1\xi_2\mathcal{F}(Q_2\overline{Q_3}-\overline{Q_2}Q_3)]\}\\
&\ =0,
\end{aligned} \tag{3.48}$$

$$
\begin{aligned}
&-\omega Q_2+\Delta Q_2+(|Q_1|^2+|Q_2|^2+|Q_3|^2)Q_2\\
&\quad -Q_1\mathcal{F}^{-1}Q_1[\tfrac{\eta(\xi_1^2+\xi_2^2)}{|\xi|^2-\delta}\mathcal{F}(Q_1\overline{Q_2}-\overline{Q_1}Q_2)]+Q_3\mathcal{F}^{-1}[\tfrac{\eta(\xi_2^2+\xi_3^2)}{|\xi|^2-\delta}\mathcal{F}(Q_2\overline{Q_3}-\overline{Q_2}Q_3)]\\
&\quad -Q_3\mathcal{F}^{-1}\{\tfrac{\eta}{|\xi|^2-\delta}[\xi_1\xi_2\mathcal{F}(\overline{Q_1}Q_3-Q_1\overline{Q_3})+\xi_1\xi_3\mathcal{F}(Q_1\overline{Q_2}-\overline{Q_1}Q_2)]\}\\
&\quad +Q_1\mathcal{F}^{-1}\{\tfrac{\eta}{|\xi|^2-\delta}[\xi_1\xi_3\mathcal{F}(Q_2\overline{Q_3}-\overline{Q_2}Q_3)+\xi_2\xi_3\mathcal{F}(\overline{Q_1}Q_3-Q_1\overline{Q_3})]\}\\
&=0,
\end{aligned}
\tag{3.49}
$$

$$
\begin{aligned}
&-\omega Q_3+\Delta Q_3+(|Q_1|^2+|Q_2|^2+|Q_3|^2)Q_3\\
&\quad -Q_1\mathcal{F}^{-1}[\tfrac{\eta(\xi_3^2+\xi_1^2)}{|\xi|^2-\delta}\mathcal{F}(Q_1\overline{Q_3}-\overline{Q_1}Q_3)]+Q_2\mathcal{F}^{-1}[\tfrac{\eta(\xi_2^2+\xi_3^2)}{|\xi|^2-\delta}\mathcal{F}(\overline{Q_2}Q_3-Q_2\overline{Q_3})]\\
&\quad -Q_1\mathcal{F}^{-1}\{\tfrac{\eta}{|\xi|^2-\delta}[\xi_2\xi_3\mathcal{F}(Q_1\overline{Q_2}-\overline{Q_1}Q_2)+\xi_1\xi_2\mathcal{F}(Q_2\overline{Q_3}-\overline{Q_2}Q_3)]\}\\
&\quad +Q_2\mathcal{F}^{-1}\{\tfrac{\eta}{|\xi|^2-\delta}[\xi_1\xi_2\mathcal{F}(\overline{Q_1}Q_3-Q_1\overline{Q_3})+\xi_1\xi_3\mathcal{F}(Q_1\overline{Q_2}-\overline{Q_1}Q_2)]\}\\
&=0.
\end{aligned}
\tag{3.50}
$$

That is, (Q_1,Q_2,Q_3) solves the equations (1.5)-(1.7). As (1.5)-(1.7) are the Euler-Lagrange equations of the functional $S(Q_1,Q_2,Q_3)$, applying Proposition 3.1, we conclude (Q_1,Q_2,Q_3) is a ground state solution of (1.5)-(1.7). Furthermore, it is obvious that (Q_1,Q_2,Q_3) are functions of $|x|$ alone. Motivated by the works [**?**, **?**], we can obtain that (Q_1,Q_2,Q_3) has exponential decay at infinity, which will be shown in the Appendix A for convenience.

This completes the proof of Theorem 2.1. □

4 Orbital Instability of Standing Waves in $\mathbb{R}^3$

In this section, we show the instability of standing waves of (1.1) – (1.3) in $\mathbb{R}^3$ obtained in Theorem 2.1 (Theorem 2.2). We first give a key proposition to show Theorem 2.2.

Proposition 4.1. Let $\delta\le 0$, $\eta>0$ and $u_{1\lambda}(x)=\lambda^{\frac{3}{2}}u_1(\lambda x)$, $u_{2\lambda}(x)=\lambda^{\frac{3}{2}}u_2(\lambda x)$, $u_{3\lambda}(x)=\lambda^{\frac{3}{2}}u_3(\lambda x)$ for $\lambda>0$. Suppose that $(u_1,u_2,u_3)\in H_r^1(\mathbb{R}^3)\times H_r^1(\mathbb{R}^3)\times H_r^1(\mathbb{R}^3)\setminus\{(0,0,0)\}$ and $(u_1,u_2,u_3)\in K$, where

$$
\begin{aligned}
K\ &=\big\{(u_1,u_2,u_3)\in H_r^1(\mathbb{R}^3)\times H_r^1(\mathbb{R}^3)\times H_r^1(\mathbb{R}^3),\\
&\qquad R(u_1,u_2,u_3)<0, S(u_1,u_2,u_3)<S(Q_1,Q_2,Q_3)\big\}.
\end{aligned}
\tag{4.1}
$$

Then there exists $0<\mu<1$ such that $R(u_{1\mu},u_{2\mu}u_{3\mu})=0$ and

$$
S(u_1,u_2,u_3)-S(u_{1\mu},u_{2\mu}u_{3\mu})\ge\frac{1}{2}R(u_1,u_2,u_3). \tag{4.2}
$$

Here, $S(u_1,u_2,u_3)$ and $R(u_1,u_2,u_3)$ are defined by (2.3) and (2.4), respectively.

Proof. By a direct calculation, we can easily show the estimate (4.2). □

Now, we begin to show Theorem 2.2.

Proof of Theorem 2.2. Let $(E_1,E_2,E_3)\in H_r^1(\mathbb{R}^3)\times H_r^1(\mathbb{R}^3)\times H_r^1(\mathbb{R}^3)$ be a solution to the equations (1.1)-(1.3) with (1.4) on $[0,T)$. By the conservation laws of the total mass and of the total energy (2.1) and (2.2), we get

$$
S(E_1(t),E_2(t),E_3(t))=S(E_{10},E_{20},E_{30}),t\in[0,T). \tag{4.3}
$$

Let

$$J(t) = \int_{\mathbb{R}^3} |x|^2(|E_1|^2 + |E_2|^2 + |E_3|^2)dx. \tag{4.4}$$

By a direct calculation, one achieves

$$\frac{d^2}{dt^2}J(t) = 8R(E_1, E_2, E_3). \tag{4.5}$$

We further need to show, for some initial data, that the right-hand side of (4.5) is strictly negative (that is, $R(E_1, E_2, E_3) < 0$). One first notices that

$$S(Q_1, Q_2, Q_3) = \min_{(u_1,u_2,u_3)\in M} S(u_1, u_2, u_3) > 0. \tag{4.6}$$

Let $(E_{10}, E_{20}, E_{30}) \in K$ such that

$$(|x|E_{10}, |x|E_{20}, |x|E_{30}) \in L^2_r(\mathbb{R}^3) \times L^2_r(\mathbb{R}^3) \times L^2_r(\mathbb{R}^3). \tag{4.7}$$

We shall see later that such (E_{10}, E_{20}, E_{30}) exists. We claim that there is a finite time T such that

$$\lim_{t\to T}(\|E_1\|_{H^1_r(\mathbb{R}^3)} + \|E_2\|_{H^1_r(\mathbb{R}^3)} + \|E_3\|_{H^1_r(\mathbb{R}^3)}) = +\infty. \tag{4.8}$$

Indeed, for such $(E_{10}, E_{20}, E_{30}) \in K$, one has

$$S(E_1, E_2, E_3) = S(E_{10}, E_{20}, E_{30}) < S(Q_1, Q_2, Q_3), t \in [0, T), \tag{4.9}$$

and

$$R(E_1, E_2, E_3) < 0, t \in [0, T). \tag{4.10}$$

The latter is true, for otherwise, by continuity, there would exist a $t_1 > 0$ such that $0 < t_1 < T$, and

$$R(E_1(t_1), E_2(t_1), E_3(t_1)) = 0, \tag{4.11}$$

which implies that $(E_1(t_1), E_2(t_1), E_3(t_1)) \in M$. This contradicts Theorem 2.1 and (4.9).

Next, for a fixed $t \in [0, T)$, $(E_1, E_2, E_3) = (E_1(t), E_2(t), E_3(t))$, and let $0 < \mu < 1$ be such that $R(E_{1\mu}, E_{2\mu}, E_{3\mu}) = 0$, $(E_{1\mu}(x), E_{2\mu}(x), E_{3\mu}(x)) = (\mu^{\frac{3}{2}}E_1(\mu x), \mu^{\frac{3}{2}}E_2(\mu x), \mu^{\frac{3}{2}}E_3(\mu x))$ (see Proposition 3.3). Since

$$S(E_{1\mu}, E_{2\mu}, E_{3\mu}) \geq S(Q_1, Q_2, Q_3), S(E_1, E_2, E_3) = S(E_{10}, E_{20}, E_{30}), \tag{4.12}$$

in view of Proposition 4.1, we have

$$\begin{aligned} R(E_1, E_2, E_3) &\leq 2[S(E_1, E_2, E_3) - S(E_{1\mu}, E_{2\mu}, E_{3\mu})] \\ &\leq 2[S(E_{10}, E_{20}, E_{30}) - S(Q_1, Q_2, Q_3)] \\ &=: \varphi < 0. \end{aligned} \tag{4.13}$$

(4.5) and (4.13) then yield that

$$\frac{d^2}{dt^2}J(t) \leq 8\varphi < 0, \tag{4.14}$$

which implies that T must be finite and that

$$\lim_{t\to T}(\|E_1\|_{H_r^1(\mathbb{R}^3)}+\|E_2\|_{H_r^1(\mathbb{R}^3)}+\|E_3\|_{H_r^1(\mathbb{R}^3)})=+\infty.$$

In order to complete the proof of Theorem 2.2, we need to show $(E_{10},E_{20},E_{30})\in K$ with (4.7). Let

$$E_{10}(x)=\lambda^{\frac{3}{2}}Q_1(\lambda x),E_{20}(x)=\lambda^{\frac{3}{2}}Q_2(\lambda x),E_{30}(x)=\lambda^{\frac{3}{2}}Q_3(\lambda x),\lambda>1. \tag{4.15}$$

By Proposition 3.3, the functions $E_{10}(x),E_{20}(x),E_{30}(x)$ verify

$$R(E_{10},E_{20},E_{30})<0,\ \ S(E_{10},E_{20},E_{30})<S(Q_1,Q_2,Q_3),\ \ \lambda>1. \tag{4.16}$$

In addition, by Theorem 2.1, one sees that $(Q_1(x),Q_2(x),Q_3(x))$ have exponential decays at infinity, and hence

$$(|x|E_{10},|x|E_{20},|x|E_{30})\in L_r^2(\mathbb{R}^3)\times L_r^2(\mathbb{R}^3)\times L_r^2(\mathbb{R}^3).$$

As $\lambda\to 1$,

$$\|E_{10}-Q_1\|_{H_r^1(\mathbb{R}^3)},\|E_{20}-Q_2\|_{H_r^1(\mathbb{R}^3)},\|E_{30}-Q_3\|_{H_r^1(\mathbb{R}^3)}$$

can be made arbitrarily small. We thus complete the proof of Theorem 2.2. □

Appendix A

In this Appendix, we will show the exponential decay at infinity of the solution (Q_1,Q_2,Q_3) to the equations (3.48)-(3.50).

Consider the equations (3.48)-(3.50):

$$\begin{aligned}
&-\omega Q_1+\Delta Q_1+(|Q_1|^2+|Q_2|^2+|Q_3|^2)Q_1\\
&\quad -Q_2\mathcal{F}^{-1}[\tfrac{\eta(\xi_1^2+\xi_2^2)}{|\xi|^2-\delta}\mathcal{F}(\overline{Q_1}Q_2-Q_1\overline{Q_2})]+Q_3\mathcal{F}^{-1}[\tfrac{\eta(\xi_1^2+\xi_3^2)}{|\xi|^2-\delta}\mathcal{F}(Q_1\overline{Q_3}-\overline{Q_1}Q_3)]\\
&\quad -Q_2\mathcal{F}^{-1}\{\tfrac{\eta}{|\xi|^2-\delta}[\xi_1\xi_3\mathcal{F}(Q_2\overline{Q_3}-\overline{Q_2}Q_3)+\xi_2\xi_3\mathcal{F}(\overline{Q_1}Q_3-Q_1\overline{Q_3})]\}\\
&\quad +Q_3\mathcal{F}^{-1}\{\tfrac{\eta}{|\xi|^2-\delta}[\xi_2\xi_3\mathcal{F}(Q_1\overline{Q_2}-\overline{Q_1}Q_2)+\xi_1\xi_2\mathcal{F}(Q_2\overline{Q_3}-\overline{Q_2}Q_3)]\}\\
&=0,
\end{aligned} \tag{3.48}$$

$$\begin{aligned}
&-\omega Q_2+\Delta Q_2+(|Q_1|^2+|Q_2|^2+|Q_3|^2)Q_2\\
&\quad -Q_1\mathcal{F}^{-1}Q_1[\tfrac{\eta(\xi_1^2+\xi_2^2)}{|\xi|^2-\delta}\mathcal{F}(Q_1\overline{Q_2}-\overline{Q_1}Q_2)]+Q_3\mathcal{F}^{-1}[\tfrac{\eta(\xi_2^2+\xi_3^2)}{|\xi|^2-\delta}\mathcal{F}(Q_2\overline{Q_3}-\overline{Q_2}Q_3)]\\
&\quad -Q_3\mathcal{F}^{-1}\{\tfrac{\eta}{|\xi|^2-\delta}[\xi_1\xi_2\mathcal{F}(\overline{Q_1}Q_3-Q_1\overline{Q_3})+\xi_1\xi_3\mathcal{F}(Q_1\overline{Q_2}-\overline{Q_1}Q_2)]\}\\
&\quad +Q_1\mathcal{F}^{-1}\{\tfrac{\eta}{|\xi|^2-\delta}[\xi_1\xi_3\mathcal{F}(Q_2\overline{Q_3}-\overline{Q_2}Q_3)+\xi_2\xi_3\mathcal{F}(\overline{Q_1}Q_3-Q_1\overline{Q_3})]\}\\
&=0,
\end{aligned} \tag{3.49}$$

$$
\begin{aligned}
&-\omega Q_3+\Delta Q_3+(|Q_1|^2+|Q_2|^2+|Q_3|^2)Q_3\\
&\quad -Q_1\mathcal{F}^{-1}[\tfrac{\eta(\xi_3^2+\xi_1^2)}{|\xi|^2-\delta}\mathcal{F}(Q_1\overline{Q_3}-\overline{Q_1}Q_3)]+Q_2\mathcal{F}^{-1}[\tfrac{\eta(\xi_2^2+\xi_3^2)}{|\xi|^2-\delta}\mathcal{F}(\overline{Q_2}Q_3-Q_2\overline{Q_3})]\\
&\quad -Q_1\mathcal{F}^{-1}\{\tfrac{\eta}{|\xi|^2-\delta}[\xi_2\xi_3\mathcal{F}(Q_1\overline{Q_2}-\overline{Q_1}Q_2)+\xi_1\xi_2\mathcal{F}(Q_2\overline{Q_3}-\overline{Q_2}Q_3)]\}\\
&\quad +Q_2\mathcal{F}^{-1}\{\tfrac{\eta}{|\xi|^2-\delta}[\xi_1\xi_2\mathcal{F}(\overline{Q_1}Q_3-Q_1\overline{Q_3})+\xi_1\xi_3\mathcal{F}(Q_1\overline{Q_2}-\overline{Q_1}Q_2)]\}\\
&=0.
\end{aligned}
\tag{3.50}
$$

Let

$$
\begin{aligned}
&g_1(Q_1,Q_2,Q_3)\\
&\quad=-\omega Q_1+(|Q_1|^2+|Q_2|^2+|Q_3|^2)Q_1\\
&\qquad-Q_2\mathcal{F}^{-1}[\tfrac{\eta(\xi_1^2+\xi_2^2)}{|\xi|^2-\delta}\mathcal{F}(\overline{Q_1}Q_2-Q_1\overline{Q_2})]\\
&\qquad+Q_3\mathcal{F}^{-1}[\tfrac{\eta(\xi_1^2+\xi_3^2)}{|\xi|^2-\delta}\mathcal{F}(Q_1\overline{Q_3}-\overline{Q_1}Q_3)]\\
&\qquad-Q_2\mathcal{F}^{-1}\{\tfrac{\eta}{|\xi|^2-\delta}[\xi_1\xi_3\mathcal{F}(Q_2\overline{Q_3}-\overline{Q_2}Q_3)+\xi_2\xi_3\mathcal{F}(\overline{Q_1}Q_3-Q_1\overline{Q_3})]\}\\
&\qquad+Q_3\mathcal{F}^{-1}\{\tfrac{\eta}{|\xi|^2-\delta}[\xi_2\xi_3\mathcal{F}(Q_1\overline{Q_2}-\overline{Q_1}Q_2)+\xi_1\xi_2\mathcal{F}(Q_2\overline{Q_3}-\overline{Q_2}Q_3)]\},
\end{aligned}
\tag{A-1}
$$

$$
\begin{aligned}
&g_2(Q_1,Q_2,Q_3)\\
&\quad=-\omega Q_2+(|Q_1|^2+|Q_2|^2+|Q_3|^2)Q_2\\
&\qquad-Q_1\mathcal{F}^{-1}Q_1[\tfrac{\eta(\xi_1^2+\xi_2^2)}{|\xi|^2-\delta}\mathcal{F}(Q_1\overline{Q_2}-\overline{Q_1}Q_2)]\\
&\qquad+Q_3\mathcal{F}^{-1}[\tfrac{\eta(\xi_2^2+\xi_3^2)}{|\xi|^2-\delta}\mathcal{F}(Q_2\overline{Q_3}-\overline{Q_2}Q_3)]\\
&\qquad-Q_3\mathcal{F}^{-1}\{\tfrac{\eta}{|\xi|^2-\delta}[\xi_1\xi_2\mathcal{F}(\overline{Q_1}Q_3-Q_1\overline{Q_3})+\xi_1\xi_3\mathcal{F}(Q_1\overline{Q_2}-\overline{Q_1}Q_2)]\}\\
&\qquad+Q_1\mathcal{F}^{-1}\{\tfrac{\eta}{|\xi|^2-\delta}[\xi_1\xi_3\mathcal{F}(Q_2\overline{Q_3}-\overline{Q_2}Q_3)+\xi_2\xi_3\mathcal{F}(\overline{Q_1}Q_3-Q_1\overline{Q_3})]\}.
\end{aligned}
\tag{A-2}
$$

$$
\begin{aligned}
&g_3(Q_1,Q_2,Q_3)\\
&\quad=-\omega Q_3+(|Q_1|^2+|Q_2|^2+|Q_3|^2)Q_3\\
&\qquad-Q_1\mathcal{F}^{-1}[\tfrac{\eta(\xi_3^2+\xi_1^2)}{|\xi|^2-\delta}\mathcal{F}(Q_1\overline{Q_3}-\overline{Q_1}Q_3)]\\
&\qquad+Q_2\mathcal{F}^{-1}[\tfrac{\eta(\xi_2^2+\xi_3^2)}{|\xi|^2-\delta}\mathcal{F}(\overline{Q_2}Q_3-Q_2\overline{Q_3})]\\
&\qquad-Q_1\mathcal{F}^{-1}\{\tfrac{\eta}{|\xi|^2-\delta}[\xi_2\xi_3\mathcal{F}(Q_1\overline{Q_2}-\overline{Q_1}Q_2)+\xi_1\xi_2\mathcal{F}(Q_2\overline{Q_3}-\overline{Q_2}Q_3)]\}\\
&\qquad+Q_2\mathcal{F}^{-1}\{\tfrac{\eta}{|\xi|^2-\delta}[\xi_1\xi_2\mathcal{F}(\overline{Q_1}Q_3-Q_1\overline{Q_3})+\xi_1\xi_3\mathcal{F}(Q_1\overline{Q_2}-\overline{Q_1}Q_2)]\}.
\end{aligned}
\tag{A-3}
$$

Then equations (3.48)-(3.50) reduce to the following system:

$$-\Delta Q_1=g_1(Q_1,Q_2,Q_3), \tag{A-4}$$

$$-\Delta Q_2=g_2(Q_1,Q_2,Q_3), \tag{A-5}$$

$$-\Delta Q_3=g_3(Q_1,Q_2,Q_3). \tag{A-6}$$

By the expressions of $g_i(Q_1,Q_2,Q_3)$ $(i=1,2,3)$, for $\eta>0$ and $\delta\le 0$, we conclude the following properties:

$$
\begin{aligned}
-\infty\ &<\underline{lim}_{(Q_1,Q_2,Q_3)\to(0^+,0^+,0^+)}\quad g_i(Q_1,Q_2,Q_3)/Q_i\\
&\le\overline{lim}_{(Q_1,Q_2,Q_3)\to(0^+,0^+,0^+)}\quad g_i(Q_1,Q_2,Q_3)/Q_i\\
&=-\omega<0,
\end{aligned}
\tag{A-7}
$$

$$-\infty \le \overline{lim}_{(Q_1,Q_2,Q_3)\to(+\infty,+\infty,+\infty)} \quad g_i(Q_1,Q_2,Q_3)/Q_i^5 \le 0. \qquad (A-8)$$

Furthermore, motivated by Lemma 1 and Radial lemma A.II in [?, ?], we can establish the following two lemmas.

Lemma A1. Under the properties (A-7)-(A-8), if (Q_1,Q_2,Q_3) is a spherically symmetric solution of (A-4)-(A-6) then $(Q_1,Q_2,Q_3)\in\mathcal{C}^2(\mathbb{R}^3)\times\mathcal{C}^2(\mathbb{R}^3)\times\mathcal{C}^2(\mathbb{R}^3)$.

Lemma A2. Let $N=3$. The radial function $Q_i\in H^1(\mathbb{R}^3)$ $(i=1,2,3)$ is almost everywhere equal to a function $U_i(x)$, continuous for $x\neq 0$ and such that

$$|U_i(x)| \le C_3|x|^{-1}\|Q_i\|_{H^1(\mathbb{R}^3)} \;\; for \;\; |x|\ge\alpha_3, \qquad (A-9)$$

where C_3 and α_3 are two constants depend only on the dimension N (N=3).

We now begin to prove the exponential decay at infinity of the solution (Q_1,Q_2,Q_3) to (A-4)-(A-6). That is, we need to show the following proposition.

Proposition A1. Under the properties (A-7)-(A-8), if (Q_1,Q_2,Q_3) is a spherically symmetric solution of (A-4)-(A-6) then

$$|D^\alpha Q_i(x)| \le Ce^{-\beta|x|}, \quad x\in\mathbb{R}^3 \qquad (A-10)$$

for some C, $\beta>0$ and for $|\alpha|\le 2$.

Proof. By lemma A1 $Q_i(i=1,2,3)$ is of class $\mathcal{C}^2(\mathbb{R}^3)$, and it satisfies the equations below:

$$-\frac{\partial^2 Q_i}{\partial r^2}-\frac{2}{r}\frac{\partial Q_i}{\partial r}=g_i(Q_1,Q_2,Q_3), \qquad (A-11)$$

where $i=1,2,3$. Let $P_i=rQ_i$ $(i=1,2,3)$, then P_i satisfies

$$\frac{\partial^2 P_i}{\partial r^2}=\left[\frac{-g_i(Q_1,Q_2,Q_3)}{Q_i}\right]P_i.$$

For r large enough, say $r\ge r_0$, one gets $-g_i(Q_1,Q_2,Q_3)/Q_i\ge\omega/\varepsilon \; for \; any \; \varepsilon\ge 1$.

(Indeed, Lemma A2 yields that $Q_i(r)\to 0$ as $r\to+\infty$).

Next, let $W_i=P_i^2$ $(i=1,2,3)$, then W_i solves

$$\frac{1}{2}\frac{\partial^2 W_i}{\partial r^2}=\left(\frac{\partial P_i}{\partial r}\right)^2+\left(-g_i(Q_1,Q_2,Q_3)/Q_i\right)W_i.$$

Thus for $r\ge r_0$ one has $\frac{\partial^2 W_i}{\partial r^2}\ge\frac{2\omega}{\varepsilon}W_i$, and $W_i\ge 0$.

Further, let

$$Z_i=e^{-\sqrt{2\omega/\varepsilon}\;r}\left(\frac{\partial W_i}{\partial r}+\sqrt{2\omega/\varepsilon}W_i\right). \qquad (A-12)$$

Direct calculation yields that

$$\frac{\partial Z_i}{\partial r}=e^{-\sqrt{2\omega/\varepsilon}\;r}\left(\frac{\partial^2 W_i}{\partial r^2}-\frac{2\omega}{\varepsilon}W_i\right)\ge 0. \qquad (A-13)$$

This implies that Z_i is nondecreasing on $(r_0,+\infty)$.

We now claim that

$$Z_i(r)\le 0 \;\; for \;\; r\ge r_1>r_0. \qquad (A-14)$$

(Otherwise, if there exists $r_1 > r_0$ such that $Z_i(r_1) > 0$, then $Z_i(r) \geq Z_i(r_1) > 0$ for all $r \geq r_1$. In view of (A-12),

$$\frac{\partial W_i}{\partial r} + \sqrt{2\omega/\varepsilon} W_i \geq Z_i(r_1) e^{\sqrt{2\omega/\varepsilon}\ r}, \qquad (A-15)$$

which implies that $\frac{\partial W_i}{\partial r} + \sqrt{2\omega/\varepsilon} W_i$ is not integrable on $(r_1, +\infty)$. But P_i^2 and $P_i \frac{\partial P_i}{\partial r}$ are integrable near ∞ $(P_i = rQ_i, Q_i \in H^1(\mathbb{R}^3))$, so that $\frac{\partial W_i}{\partial r}$, and W_i are also integrable $(W_i = P_i^2)$, a contradiction).

(A-14) then implies that

$$\frac{\partial(e^{\sqrt{2\omega/\varepsilon r}} W_i)}{\partial r} = e^{2\sqrt{2\omega/\varepsilon r}} Z_i \leq 0 \quad for \quad r \geq r_1. \qquad (A-16)$$

Hence $W_i(r) \leq Ce^{-\sqrt{2\omega/\varepsilon r}}$ and in turn

$$|Q_i(r)| \leq Cr^{-1} e^{(-\sqrt{2\omega/\varepsilon}/2)r} \quad for \quad r \geq r_1, \qquad (A-17)$$

for certain positive constants C, r_1, $\omega > 0$ and $\varepsilon \geq 1$.

Next, we show the exponential decay of $\frac{\partial Q_i}{\partial r}$ $(i = 1, 2, 3)$ at infinity.

Note that $\frac{\partial Q_i}{\partial r}$ satisfies

$$\frac{r^2 \frac{\partial Q_i}{\partial r}}{\partial r} = -r^2 g_i(Q_1, Q_2, Q_3). \qquad (A-18)$$

Applying (A-7) and the exponential decay of Q_i, it is easily verified that for r large enough, say $r \geq r_0$, one has

$$\omega_1 |Q_i| \leq |g_i(Q_1, Q_2, Q_3)| \leq \omega_2 |Q_i|, \qquad (A-19)$$

where $\omega_2 \geq \omega_1 \geq 0$. Hence integrating (A-18) on (r, R), using (A-17) and letting $r, R \to +\infty$ shows that $r^2 \frac{\partial Q_i}{\partial r}$ has a limit as $r \to +\infty$. This limit can only be zero by (A-17).

Now, integrating (A-18) on $(r, +\infty)$ then yields that $\frac{\partial Q_i}{\partial r}$ has exponential decay.

Finally, the exponential decay of $\frac{\partial^2 Q_i}{\partial r^2}$ (and thus $|D^\alpha Q_i(x)|$ for $|\alpha| \leq 2$) follows immediately from equations (A-11).

The proof of the exponential decay of Q_i at infinity is completed. □

References

[1] Berestycki H., Cazenave T. Instabilit$\acute{e}$ des $\acute{e}$tats stationnaires dans les $\acute{e}$quations de Schrödinger et de Klein-Gordon non lin$\acute{e}$airees. C. R. Acad. Sci. Paris, 1981, **293**:489-492

[2] Berestycki H., Lions P L. Nonlinear scalar field equations, I. Existence of a ground state. Arch. Rat. Mech. Anal., 1983, **82**:313-345

[3] Berestycki H., Lions P L. Nonlinear scalar field equations, II. Existence of infinitely many solutions. Arch. Rat. Mech. Anal., 1983, **82**:347-375

[4] Cazenave T., Lions P L. Orbital stability of standing waves for some nonlinear Schrödinger equations. Commun. Math. Phys., 1982, **85**:549-561

[5] Dendy R O. Plasma Dynamics. Oxford University Press, 1990

[6] Gan Z H, Zhang J. Sharp threshold of global existence and instability of stangding wave for a Davey-Stewartson System. Commun. Math. Phys., 2008, **283**:93-125

[7] Gan Z H, Zhang J. Blow-up, global existence and standing waves for the magnetic nonlinear Schrödinger equations. Discrete and Continuous Dynamical Systems, 2012, **32**(3):827-846

[8] Gan Z H, Zhang J. Nonlocal nonlinear Schrödinger equations in $\mathbb{R}^3$. Arch. Rational Mech. Anal., 2013, **209**:1-39

[9] Grillakis M. Linearized instability for nonlinear Schrödinger and Klein-Gordon equations . Comm. Pure Appl. Math., 1988, **XLI**:747-774

[10] Grillakis M, Shatah J, Strauss W. Stability theory of solitary waves in the presence of symmetry, I*. J. Funct. Anal., 1987, **74**:160-197

[11] Jiang X, Gan ZH. Collapse for the generalized three-dimensional nonlocal nonlinear Schrödinger equations. Advanced Nonlinear Studies, 2014, **14**:777-790

[12] Jones C. An instability mechanism for radically symmetric standing waves of a nonlinear Schrödinger equation. J. Differential Equations, 1988, **71**:34-62

[13] Kono M, Skoric M M, Ter Haar D. Spontaneous excitation of magnetic fields and collapse dynamics in a Langmuir plasma. J. Plasma Phys., 1981, **26**:123-146

[14] Laurey C. The Cauchy problem for a generalized Zakharov system. Diffe. Integral Equ., 1995, **8**(1):105-130

[15] Miao C X. Harmonic Analysis and Applications to Partial Differential Equations. Monographs on Modern Pure Mathematics No. 89. Second Edition. Beijing: Science Press, 2004

[16] Miao C X. The Modern Method of Nonlinear Wave Equations. Lectures in Contemporary Mathematics, No. 2. Beijing: Science Press, 2005

[17] Miao C X, Zhang B. Harmonic Analysis Method of Partial Differential Equations. Monographs on Modern Pure Mathematics, No. 117. Second Edition. Science Press, Beijing, 2008

[18] Shatah J, Strauss W. Instability of nonlinear bound states. Commun. Math. Phys., 1985, **100**:173-190

[19] Strauss W A. Existence of solitary waves in high dimensions. Commun. Math. Phys., 1977, **55**:149-162

[20] Weinstein M I. Nonlinear Schrödinger equations and sharp interpolation estimates. Commun. Math. Phys., 1983, **87**:567-576

[21] Weinstein M I. Lyapunov stability of ground states of nonlinear dispersive evolution equations. Commun. Pure Appl.Math., 1986, **39**:51-68

[22] Zhang J. Sharp threshold for blowup and global existence in nonlinear Schrödinger equations under a harmonic potentia. Commun. in PDE, 2005, **30**:1429-1443

[23] Zakharov V E. The collapse of Langmuir waves. Soviet Phys., JETP, 1972,**35**:908-914

[24] Zakharov V E, Musher S L, Rubenchik A M. Hamiltonian approach to the description of nonlinear plasma phenomena. Physics Reports, 1985, **129**(5):285-366